Complex Analysis

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1. The algebra of complex numbers.

Let $(a, b) \in \mathbb{R} \times \mathbb{R}$ be pairs of real numbers with addition defined by $(a, b) + (c, d) = (a + c, b + d)$ and multiplication defined by $(a, b) \cdot (c, d) = (ac - bd, bc + ad)$.

Proposition 1.1. *$\mathbb{R} \times \mathbb{R}$ with these operations forms a field. The additive identity is $(0, 0)$ and multiplicative identity is $(1, 0)$.*

Proof. The only thing to check is the existence of a multiplicative inverse. If $(a, b) \neq (0, 0)$, then a calculation shows that $(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$ times (a, b) yields $(1, 0)$. $\square$

We write $a + bi$ for (a, b) and let $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$. We can identify $\mathbb{R}$ with $\{a + 0i\}$. Note also that $i^2 = (0, 1) \cdot (0, 1) = -1$. We usually write wz for $w \cdot z$ when $w, z \in \mathbb{C}$.

We define $\overline{z} = a - bi$ if $z = a + bi$, and this is called the complex conjugate of z. We let $\operatorname{Re} z = a, \operatorname{Im} z = b$, for $z = a + bi$. It is easy to check that $\overline{w + z} = \overline{w} + \overline{z}$ and $\overline{wz} = \overline{w}\,\overline{z}$.

We define $|z| = \sqrt{a^2 + b^2}$. It is then apparent that $|z| \geq 0$ and $|z|^2 = z\overline{z}$. Note that $|ab|^2 = ab\overline{ab} = a\overline{a}b\overline{b} = |a|^2|b|^2$, or $|ab| = |a|\,|b|$.

Note also that $|a + b|^2 = (a + b)(\overline{a} + \overline{b}) = a\overline{a} + b\overline{b} + 2\operatorname{Re} a\overline{b}$. It is clear that $-|z| \leq \operatorname{Re} z \leq |z|$ and $-|z| \leq \operatorname{Im} z \leq |z|$. Using this, we obtain $|a + b|^2 \leq (|a| + |b|)^2$, or

$$|a + b| \leq |a| + |b|.$$

Cauchy's inequality is the following.

Proposition 1.2.
$$\Big|\sum_{i=1}^n a_i b_i\Big|^2 \leq \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2.$$

Proof. The result if obvious if $\sum |b_i|^2 = 0$, so suppose not. We have

$$0 \leq \sum_{i=1}^n |a_i - \lambda\overline{b}_i|^2 = \sum |a_i|^2 + |\lambda|^2 \sum |b_i|^2 - 2\operatorname{Re} \overline{\lambda} \sum a_i b_i$$

for any complex number λ. Taking $\lambda = \sum a_i b_i / \sum |b_i|^2$ and doing some algebra, we obtain the inequality we seek. $\square$

2. The geometry of complex numbers.

A complex number $z = a + bi$ is identified with $(a, b) \in \mathbb{R}^2$. We can thus identify a complex number with a point in the plane. If we let $r = |z| = \sqrt{a^2 + b^2}$ and φ the angle between the x axis and the ray from 0 to z, measured in a counterclockwise direction from the x axis, then z in polar coordinates is (r, φ), and we have $a = r \cos \varphi$, $b = r \sin \varphi$. We define $\arg z$, the argument of z, to be φ.

If $z_1 = r_1(\cos \varphi_1 + i \sin \varphi_1)$ and $z_2 = r_2(\cos \varphi_2 + i \sin \varphi_2)$, then a calculation together with some trigonometry identities shows that

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [(\cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2) + i(\sin \varphi_1 \cos \varphi_2 + \sin \varphi_2 \cos \varphi_1)] \\ &= r_1 r_2 [\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)]. \end{aligned}$$

We deduce

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2.$$

We will later give an analytic proof of this fact. However this is useful in doing calculations. For example, the above plus induction shows that

$$z^n = r^n (\cos(n\varphi) + i \sin(n\varphi)).$$

To find the nth root of $a = \rho(\cos \theta + i \sin \theta)$, we set $z^n = a$, and comparing we find that $r = \rho^{1/n} = |a|^{1/n}$ and $\varphi = \arg z = \theta/n = (\arg a)/n$. For example, i has modulus 1 and argument $\pi/2$. So $\sqrt{i}$ will have modulus 1 and argument $\pi/4$, hence $\sqrt{i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$.

Sometimes we want to introduce a point at infinity ∞, and we refer to $\mathbb{C} \cup \{\infty\}$ as the extended complex plane. We can give a geometric interpretation of the extended plane through what is known as the stereographic projection. Consider the unit sphere $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. To every point of S with the exception of $(0, 0, 1)$ (the north pole), we associate the complex number $z = (x_1 + ix_2)/(1 - x_3)$. Some algebra shows that

$$|z|^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3},$$

and so

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

More algebra shows that

$$x_1 = \frac{z + \overline{z}}{1 + |z|^2}, \qquad x_2 = \frac{z - \overline{z}}{i(1 + |z|^2)}.$$

We then associate ∞ to the point $(0,0,1)$.

The line through $(0,0,1)$ and (z_1,z_2,z_3) is $\{(tz_1,tz_2,t(z_3-1)+1 : t\in\mathbb{R}\}$, and this line crosses the plane $x-y$ plane when $t=1/(1-z_3)$, so

$$x=\frac{z_1}{1-z_3},\qquad y=\frac{z_2}{1-z_3}.$$

This agrees with the equations above, so if we identify the complex plane with the (x_1,x_2) plane, and $z=x+iy$, then the points $(x,y,0)$, (x_1,x_2,x_3), and $(0,0,1)$ are on a straight line. Thus drawing a straight line from the north pole of S through the point (x_1,x_2,x_3) on S and extending it to the (x_1,x_2) plane, the intersection with the (x_1,x_2) plane is the point z.

3. Analytic functions.

The definition of limit and of continuous function needs no modification from the real case. We define the derivative of a function $f:\mathbb{C}\to\mathbb{C}$ to be

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}.$$

The difference with the real case is that we require the limit to exist as $h\to 0$ in $\mathbb{C}$. In other words, given $\varepsilon>0$ there exists δ such that $|\frac{f(z+h)-f(z)}{h}-f'(z)|<\varepsilon$ whenever $h\in\mathbb{C}$ and $|h|<\delta$.

A function f is analytic or holomorphic if $f'(z)$ exists whenever f is defined.

Proposition 3.1. (The Cauchy-Riemann equations) *Suppose $f(z)=u(z)+iv(z)$ and f' exists at z. Then*

$$\frac{\partial u}{\partial x}(z)=\frac{\partial v}{\partial y}(z),\qquad \frac{\partial u}{\partial y}(z)=-\frac{\partial v}{\partial x}(z).$$

Proof. If $f(z)=u(z)+iv(z)$, $f'(z)$ exists, and we take the limit as $h\to 0$ with h real, then

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}.$$

If we let $h=ik$ with k real and let $k\to 0$, we similarly obtain

$$f'(z)=\lim_{k\to 0}\frac{f(z+ik)-f(z)}{ik}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}.$$

Equating the two expressions for $f'(z)$, we obtain our result. □

Note

$$|f'(z)|^2=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x},$$

or $|f'(z)|^2$ is the Jacobian of the mapping $z = (x, y) \to f(z) = (u(z), v(z))$.

If f' is itself differentiable (and we will see later that the derivative of an analytic function is again analytic), then the Cauchy-Riemann equations imply that

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

We can give a converse to Proposition 3.1.

Proposition 3.2. *Suppose u and v are continuously differentiable and satisfy the Cauchy-Riemann equations. Then $f = u + iv$ is analytic.*

Proof. The assumptions on u and v imply that

$$u(x+h, y+k) - u(x, y) = \frac{\partial u}{\partial x} h + \frac{\partial u}{\partial y} k + R_u(h, k)$$

and similarly for v, where $R_u(h, k)/(h + ik) \to 0$ as $h + ik \to 0$. Then

$$f(z + h + ik) - f(z) = \Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big)(h + ik) + R_u(h, k) + iR_v(h, k),$$

so

$$\lim_{h+ik \to 0} \frac{f(z + h + ik) - f(z)}{h + ik} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

□

An expression of the form

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

is called a polynomial. The standard rules of calculus show that the sum and products of analytic functions are analytic, so polynomials are analytic.

A ratio of polynomials is called a rational function. The simplest nontrivial rational functions are the linear fractional transformations or Möbius transformations. These are the rational functions of the form

$$S(z) = \frac{az + b}{cz + d}.$$

Note that if

$$T(w) = \frac{dw - b}{-cw + a},$$

then S and T are inverses of each other.

Finally we discuss the partial fractions expansion of a rational function. A zero of a rational function $R(z) = P(z)/Q(z)$ is a value of z such that $P(z) = 0$, a pole is a value of z where $Q(z) = 0$, and we assume that P and Q have no factors in common.

Proposition 3.3. *A rational function R is equal to*

$$R(z) = G(z) + \sum_{i=1}^{q} G_j\Big(\frac{1}{z-\beta_j}\Big),$$

where G and G_j are polynomials and the β_j are the poles of R.

Proof. We carry out the division of P by Q until the degree of the remainder is at most that of the denominator. So $R(z) = G(z) + H(z)$, where G is a polynomial without constant term.

The rational function $R(\beta_j + \frac{1}{\zeta})$ has a pole at $\zeta = \infty$; this means that the degree of the numerator is larger than the degree of the denominator. We carry out a division, and we can write

$$R(\beta_j + \frac{1}{\zeta}) = G_j(\zeta) + H_j(\zeta),$$

which becomes

$$R(z) = G_j\Big(\frac{1}{z-\beta_j}\Big) + H_j\Big(\frac{1}{z-\beta_j}\Big).$$

The function $H_j(\frac{1}{z-\beta_j})$ is finite for $z = \beta_j$.

The function

$$R(z) - G(z) - \sum_{j=1}^{q} G_j\Big(\frac{1}{z-\beta_j}\Big)$$

is a rational function with no poles other than $\beta_1, \dots, \beta_q, \infty$. At $z = \beta_j$, $R(z) - G_j(\frac{1}{z-\beta_j}) = H_j(\frac{1}{z-\beta_j})$, which is finite at $z = \beta_j$, and the same is true at ∞. So this function has no finite poles nor a pole at ∞. Such a rational function must be constant. We add this constant to $G(z)$, and we have our representation. □

4. Power series.

A power series is an expression of the form

$$a_0 + a_1 z + \cdots + a_n z^n + \cdots.$$

Define the radius of convergence R by

$$1/R = \limsup_{n\to\infty} |a_n|^{1/n}.$$

We have $0 \le R \le \infty$, with both endpoints possible.

Theorem 4.1. *(a) The series converges absolutely if $|z| < R$. If $0 \le \rho < R$, the convergence is uniform for $|z| \le \rho$.*

(b) If $|z| > R$, the series is divergent.

(c) For $|z| < R$ the series is an analytic function, the derivative can be obtained by termwise differentiation, and the derived series has the same radius of convergence.

Proof. If $|z| < R$, choose ρ so that $|z| < \rho < R$. Then $1/\rho > 1/R$, and so for n large $|a_n|^{1/n} < 1/\rho$, or $|a_n z^n| \le (|z|/\rho)^n$. So the power series is dominated by the terms of a convergent geometric series. To show uniform convergence, find ρ' so that $\rho < \rho' < R$. Then $|a_n z^n| \le (\rho'/\rho)^n$, which is a term of a convergent geometric series.

If $|z| > R$, choose $\rho \in (R, |z|)$. So for infinitely many n we have $|a_n|^{1/n} > 1/\rho$, or for infinitely many n we have $|a_n z^n| > (|z|/\rho)^n \ge 1$, hence the series diverges.

Note $n^{1/n} \ge 1$, so let $\delta_n = n^{1/n} - 1$. By the binomial theorem, $n = (1+\delta_n)^n > 1 + \frac{1}{2}n(n-1)\delta_n^2$. It follows that $n - 1 > \frac{1}{2}n(n-1)\delta_n^2$ or $\delta_n^2 < 2/n \to 0$. Thus $n^{1/n} \to 1$, which implies that the derived series has the same radius of convergence.

Let $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=1}^\infty n a_n z^{n-1}$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} - g(z_0) = \sum_{n=0}^{m} a_n \Big(\frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1} \Big) + \sum_{n=m+1}^{\infty} a_n \frac{z^n - z_0^n}{z - z_0} - \sum_{m=n+1}^{\infty} n a_n z^{n-1}.$$

Suppose $|z_0| < R$ and z is close enough to z_0 so that $|z|$ is also less than R. The third sum can be made small by taking m large since the series for g has the same radius of convergence as f. The terms in the second series are equal to

$$a_n(z^{n-1} + z^{n-2} z_0 + \cdots + z z_0^{n-2} + z_0^{n-1}).$$

If $|z_0|, |z| \le \rho$, this is bounded by $|a_n| n \rho^{n-1}$, which we know is a term in a convergent series, so the second sum can be made small by taking m large. Finally, given m, we can make the first sum small by taking z sufficiently close to z_0. □

5. Exponential and trigonometric series.

Define

$$e^z = \exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots.$$

Since $n! \ge m^{n-m}$, then $\limsup_{n\to\infty} (n!)^{1/n} \ge \limsup_{n\to\infty} m^{1-m/n} = m$. Since this is true for every m, then $R = \infty$. By differentiating e^z, we see that its derivative is again e^z. Note also that $e^0 = 1$ by setting $z = 0$.

Proposition 5.1 $e^{a+b} = e^a \cdot e^b$**.**

Proof. Observe by the product rule that

$$\frac{d}{dz}(e^z e^{c-z}) = e^z e^{c-z} + e^z(-e^{c-z}) = 0,$$

or $e^z e^{c-z}$ is constant. Setting $z = 0$, the constant must be e^c. Now set $z = a$ and $c = a+b$. $\square$

Note that taking conjugates, $\overline{e^z} = e^{\overline{z}}$. So $|e^z|^2 = e^z\overline{e^z} = e^z e^{\overline{z}} = e^{z+\overline{z}} = e^{2\mathrm{Re}\, z}$, or $|e^z| = e^{\mathrm{Re}\, z}$. In particular, $|e^{ik}| = 1$ when k is real.

Define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Substituting in the series definition of $\exp z$, we have

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

and

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots.$$

We see from this that the derivative of $\sin z$ is $\cos z$ and the derivative of $\cos z$ is $-\sin z$. Euler's formula

$$e^{iz} = \cos z + i \sin z$$

and the identity

$$\sin^2 z + \cos^2 z = 1$$

follow from the definitions of sine and cosine. The addition theorems

$$\cos(a+b) = \cos a \cos b - \sin a \sin b,$$

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$

also follow from the definitions.

A function f has period c if $f(z+c) = f(z)$ for all c.

Proposition 5.2. *There exists* ω_0 *that is a period for* $\sin z$ *and* $\cos z$ *and* $i\omega_0$ *is a period for* $\exp z$.

Proof. From the series expansion of sine and cosine, both are real if z is real. Since $\sin^2 z + \cos^2 z = 1$, this shows that $\cos x \leq 1$ if x is real. Since $(\sin y)' = \cos y$ and

$\sin 0 = 0$, then $\sin y \le y$ for all y. As $(\cos y)' = -\sin y \ge -y$ and $\cos 0 = 1$, we deduce $\cos y \ge 1 - y^2/2$. Repeating, $\sin y \ge y - y^3/6$, and $\cos y \le 1 - y^2/2 + y^4/24$. Hence $\cos\sqrt{3} < 0$, and so there exists $y_0 \in (0, \sqrt{3})$ with $\cos y_0 = 0$. This forces $\sin y_0 = \pm 1$, so $e^{iy_0} = \pm i$, and then $e^{4iy_0} = 1$ It follows that $4y_0$ is a period for sine and cosine.

We claim that $\omega_0 = 4y_0$ is the smallest positive period. If $y \in (0, y_0)$, then $\sin y > y(1 - y^2/6) > y/2 > 0$, and $\cos y$ is strictly decreasing. Because $\sin y$ is positive and $\sin^2 y + \cos^2 y = 1$, then $\sin y$ is strictly increasing, and therefore $\sin y < \sin y_0 = 1$. Because $0 < \sin y < 1$, then e^{iy} cannot equal either ± 1 nor $\pm i$, and thus $e^{4iy} \ne 1$.

If ω is a period, there exists an integer n such that $n\omega_0 \le \omega < (n+1)\omega$. Then $\omega - n\omega_0$ would be a positive period less than ω_0 unless ω is an integer multiple of ω_0. So every period is an integral multiple of ω_0. □

We define $\pi = \omega_0/2$.

Since $e^z e^{-z} = 1$, then e^z never equals 0. If $w \ne 0$, the logarithm of w is a complex number z such that $e^z = w$. Since e^z has period $2\pi i$, if z is a logarithm of w, then so is $z + 2\pi k i$ for any integer k. We define the argument of w by $\arg w = \operatorname{Im} \log w$. So we have $\log w = \log |w| + i \arg w$, and if $|z| = r$ and $\arg z = \theta$, we write $z = re^{i\theta}$.

The inverses of the trigonometric functions can be written in terms of the logarithm. For example, if $w = \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$, multiplying by e^{iz} and solving the resultant quadratic in e^{iz}. we obtain

$$e^{iz} = w \pm \sqrt{w^2 - 1},$$

and consequently

$$z = \arccos w = -i \log(w \pm \sqrt{w^2 - 1}).$$

From the addition formula $e^{a+b} = e^a e^b$ we have $\log(ab) = \log a + \log b$, in the sense that the set of points on both sides are equal. From this follows

$$\arg(ab) = \arg a + \arg b.$$

6. Conformality.

A region is an open connected set. An arc is a function $z : [a, b] \to \mathbb{C}$. If a nondecreasing function $t = \varphi(\tau)$ maps an interval $[\alpha, \beta]$ onto $[a, b]$, then $z(\varphi(\tau))$ defines the same succession of points as $z(t)$, and we say the first arc arises from the second by a change of parameter. An arc is differentiable if $z'(t)$ exists and is continuous. An arc is simple if $z(t_1) = z(t_2)$ only for $t_1 = t_2$. If $z(a) = z(b)$, the arc is a closed curve. A Jordan curve is a simple closed curve.

A complex valued function $f(z)$ is analytic in a region if has a derivative at each point of the region. It is analytic on a set A if it is analytic on some region which contains A.

Analytic functions are single valued, so to define some well known functions, it is necessary to restrict the domain of definition. For example, let Ω be $\mathbb{C}$ with the negative real axis removed, and define $\sqrt{z}$ to be the value which has positive real part. Then $\sqrt{z}$ is single-valued.

For the function $\log z$, use the same domain Ω and define the principal branch of $\log z$ by the condition $|\text{Im} \log z| < \pi$.

Recall $\arccos z = i \log(z + \sqrt{z^2 - 1})$. Let Ω' be $\mathbb{C}$ minus the half lines $\{x \leq 0, y = 0\}$ and $\{x \geq 1, y = 0\}$. In Ω' the function $1 - z^2$ never takes negative real values, so we can define $\sqrt{1 - z^2}$ as the value with positive real part, and set $\sqrt{z^2 - 1} = i\sqrt{1 - z^2}$. Since $z + \sqrt{z^2 - 1}$ and $z - \sqrt{z^2 - 1}$ are reciprocals whose sum is $2z$, which is not negative, then $z + \sqrt{z^2 - 1}$ cannot be negative. We can thus define a single-valued branch of $\log(z + \sqrt{z^2 - 1})$.

We will use the following proposition.

Proposition 6.1. *Let f be analytic in a region. If the derivative of f vanishes, if the real or imaginary parts of f are constant, if the modulus of f is constant, or the argument of f is constant, then f is constant.*

Proof. If the derivative vanishes, then $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y$ are all 0. So u and v are constant on any line segment in the region which is parallel to the coordinate axes. Since any two points in a region can be joined by a polygonal path, each straight line segment of which is parallel to either the x or y axes, then u and v are constant in the region.

If the real part of f is constant, then $\partial u/\partial x, \partial u/\partial y$ are both 0, and by the Cauchy-Riemann equations so are $\partial v/\partial x, \partial v/\partial y$. We now apply the argument above.

Suppose $u^2 + v^2$ is constant. If it is 0, then f is identically 0. If not, differentiating gives $u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$. Similarly, if we differentiate with respect to y and use the Cauchy-Riemann equations,

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0.$$

Solving these equations under the condition $u^2 + v^2 = 0$ implies $\partial u/\partial x = \partial v/\partial x = 0$. By the Cauchy-Riemann equations again, this implies f' vanishes.

Finally, if $a + bi = z = re^{i\theta} = r\cos\theta + ir\sin\theta$, then $a/b = \cot\theta$. So if f has constant argument, $u = kv$ for some constant k, unless v is identically 0. So $0 = u - kv = \text{Re}\,((1 + ik)f)$, which implies by the above that f is constant. $\square$

If $z = re^{i\theta}$, then $\arg z = \theta$, or the argument of z can be thought of as the angle the ray from 0 to z makes with the x axis. Thus $\arg z'(t)$ may be thought of as the direction of the tangent to z.

Suppose $z_1(t), z_2(t)$ are two arcs starting at a point a, $t \in [0,1]$, and let f be analytic at a. Let $w_i(t) = f(z_i(t))$, $i = 1, 2$. Since $w_i'(t) = f'(z_i(t))z_i'(t)$, then

$$\arg w_i'(0) = \arg f'(a) + \arg z_i'(0),$$

or

$$\arg w_1'(0) - \arg w_2'(0) = \arg z_1'(0) - \arg z_2'(0).$$

This means that the angle between $w_1(t)$ and $w_2(t)$ at $t = 0$ is the same as the angle between $z_1(t)$ and $z_2(t)$ at $t = 0$. In other words, f preserves angles, or f is conformal.

7. Linear fractional transformations.

Let

$$w = S(z) = Sz = \frac{az + b}{cz + d}.$$

If $c \neq 0$, this can be written as

$$Sz = \frac{a}{c}\frac{z + b/a}{z + d/c} = \frac{a}{c}\Big(1 + \frac{(b/a) - (d/c)}{z + d/c}\Big).$$

This means that S can be viewed as a translation $z \to z + d/c$, an inversion $z \to 1/z$, followed by a rotation/scaling $z \to ((b/a) - (d/c))z$, then another translation $z \to z + 1$, and finally another scaling/rotation $z \to (a/c)z$. Since each of these operations performed on a linear fractional transformation gives rise to another linear fractional transformation, then the composition of two linear fractional transformations is another linear fractional transformation. If $c = 0$, the linear fractional transformation is also of the same form, but simpler.

Given three points z_2, z_3, z_4 in the extended complex plane, there is a linear fractional transformation S which maps them to $1, 0, \infty$ respectively. If none of them are ∞, S is given by

$$Sz = \frac{z - z_3}{z - z_4}\frac{z_2 - z_4}{z_2 - z_3}. \tag{7.1}$$

If $z_2, z_3, z_4 = \infty$, use

$$\frac{z - z_3}{z - z_4}, \qquad \frac{z_2 - z_4}{z - z_4}, \qquad \frac{z - z_3}{z_2 - z_3}$$

respectively. If T is another such linear fractional transformation, then ST^{-1} leaves $1, 0, \infty$ invariant, and a calculation shows that the only linear fractional transformation that does that is the identity. So S is uniquely determined.

The cross ratio (z_1, z_2, z_3, z_4) is the image of z_1 under the linear fractional transformation that carries z_2, z_3, z_4 into $1, 0, \infty$.

Proposition 7.1. *If z_1, z_2, z_3, z_4 are distinct points in the extended plane, and T is a linear fractional transformation, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.*

Proof. If $Sz = (z, z_2, z_3, z_4)$, then ST^{-1} maps (Tz_2, Tz_3, Tz_4) into $(1, 0, \infty)$. Then

$$(Tz_1, Tz_2, Tz_3, Tz_4) = ST^{-1}(Tz_1) = Sz_1 = (z_1, z_2, z_3, z_4).$$

□

Proposition 7.2. *The cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle or a straight line.*

Proof. First suppose z_1, z_2, z_3, z_4 are on a line. By a linear fractional transformation S (in fact a linear one), we can map these points to w_1, w_2, w_3, w_4 on the real axis. So using (7.1) with z_i replaced by w_i and z by w_1, (or the other formulas when one of the points is ∞), we see that $(z_1, z_2, z_3, z_4) = (Sz_1, Sz_2, Sz_3, Sz_4)$ is real.

Next suppose z_1, z_2, z_3, z_4 lie on a circle. By first mapping the circle to a circle having center at the origin, and then multiplying by a complex number, we have mapped the points onto the boundary of the unit circle. The map

$$z \to \frac{1+z}{1-z} = \frac{(1+z)(1-\overline{z})}{|1-z|^2} = \frac{2i\mathrm{Im}\, z}{|1-z|^2}$$

takes the unit circle to the line $x = 0$, and then multiplying by i takes it to the real axis. So there exists a linear fractional transformation mapping the four points into the real axis, and as before we see the corss ratio is real.

Now suppose the cross ratio is real. Take a linear fractional transformation T mapping z_2, z_3, z_4 to $1, 0, \infty$. $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$ is real, and from the formula (7.1) we conclude Tz_1 is real. So if $S = T^{-1}$ maps the real axis onto a line or a circle, then z_1, z_2, z_3, z_4 will be on a line or a circle. It therefore suffices to show that the image of the real axis under any linear fractional transformation is either a circle or a straight line. Let z be real, T a linear fractional transformation, and $S = T^{-1}$. $w = Tz = S^{-1}z$, so

$$\frac{aw+b}{cw+d} = Sw = z = \overline{z} = \overline{Sw} = \frac{\overline{a}\overline{w}+\overline{b}}{\overline{c}\overline{w}+\overline{d}}.$$

Cross multiplying,

$$(a\overline{c} - c\overline{a})|w|^2 + (a\overline{d} - c\overline{b})w + (b\overline{c} - d\overline{a})\overline{w} + b\overline{d} - d\overline{b} = 0.$$

If $a\overline{c} - c\overline{a} = 0$, this is the equation of a straight line: $\mathrm{Re}\,(\alpha w) = \beta$. If not, after some algebra we obtain

$$\left|w - \frac{\overline{a}d - \overline{c}b}{\overline{a}c - \overline{c}a}\right| = \left|\frac{ad - bc}{\overline{a}c - \overline{c}a}\right|,$$

which is the equation of a circle. □

A corollary of this is that linear fractional transformations map circles and lines onto circle and lines.

8. Some elementary mappings.

If $w = z^\alpha$ for real $\alpha > 0$, $|w| = |z|^\alpha$ and $\arg w = \alpha \arg z$. So circles centered about the origin are mapped into circles about the origin, and rays from the origin are mapped into other rays about the origin.

The sector $\{z = re^{i\theta} : 0 < \theta < \theta_0\}$ is mapped into the sector $\{z = re^{i\theta} : 0 < \theta < \alpha\theta_0\}$ as long as $0 < \theta_0 < 2\pi/\alpha$.

The mapping $w = e^z$ maps the strip $-\pi/2 < \operatorname{Im} z < \pi/2$ onto the half plane $\operatorname{Re} w > 0$.

The mapping $w = \frac{z-1}{z+1}$ maps $\operatorname{Re} z > 0$ onto $|w| < 1$.

Consider a region whose boundary consists of two circular arcs with common end points. If the end points are a, b, the mapping $z_1 = (z-a)/(z-b)$ maps the region onto an angular sector. An appropriate power mapping maps the sector onto a half plane.

Consider Ω equal to $\mathbb{C}$ minus the line segment $-1 \le x \le 1, y = 0$. $z_1 = (z+1)/(z-1)$ maps this onto $\mathbb{C}$ minus the negative real axis. $z_2 = \sqrt{z_1}$ maps this to the right half plane, and $w = (z_2 - 1)(z_2 + 1)$ maps the half plane onto the unit disk. some algebra leads to

$$z = \tfrac{1}{2}\Big(w + \frac{1}{w}\Big), \qquad w = z - \sqrt{z^2 - 1}.$$

If $w = \rho e^{i\theta}$, then $x = \frac{1}{2}(\rho + \rho^{-1})\cos\theta$, $y = \frac{1}{2}(\rho - \rho^{-1})\sin\theta$, and so

$$\frac{x^2}{(\frac{1}{2}(\rho + \rho^{-1}))^2} + \frac{y^2}{(\frac{1}{2}(\rho - \rho^{-1}))^2} = 1,$$

the equation of an ellipse.

9. Cauchy's theorem.

We define the integral of $f : [a, b] \to \mathbb{C}$ by $\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$ if $f = u + iv$. The usual properties of the integral hold. To show $|\int_a^b f(t)\,dt| \le \int_a^b |f(t)|dt$, let $\theta = \arg \int_a^b f(t)\,dt$. Then

$$\Big|\int_a^b f(t)\,dt\Big| = \operatorname{Re}\Big[e^{-i\theta}\int_a^b f(t)\,dt\Big] = \int_a^b \operatorname{Re}\,[e^{-i\theta} f(t)]\,dt \le \int_a^b |f(t)|\,dt.$$

If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise differentiable arc with $\gamma(t) = z(t)$, we define

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt.$$

We note that this integral is invariant under change of parameter. If $t = t(\tau)$ maps $[\alpha, \beta]$ one-to-one and piecewise differentiably onto $[a, b]$, then

$$\int_a^b f(z(t))z'(t)\,dt = \int_\alpha^\beta f(z(\tau(t)))z'(\tau(t))\tau'(t)\,dt$$

by the change of variables formula for integrals. Since $\frac{d}{dt}z(\tau(t)) = z'(\tau(t))\tau'(t)$, we see the value for $\int_\gamma f(z)\,dz$ is the same whichever parameterization we use.

A simple change of variables shows that $\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$. If we divide an arc into subarcs $\gamma_1, \dots, \gamma_n$, we write $\gamma = \gamma_1 + \dots + \gamma_n$ and it is easy to check that $\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \cdots + \int_{\gamma_n} f(z)\,dz$. Finally the integral over a closed curve is invariant under a shift of parameter. The old and new initial points determine two subarcs γ_1 and γ_2; one integral is the integral over the arc $\gamma_1 + \gamma_2$ and the other is the integral over the arc $\gamma_2 + \gamma_1$. By the formula for a sum, these two values are equal.

We define

$$\int_\gamma f(z)\,ds = \int_a^b f(z(t))|z'(t)|\,dt.$$

We also define

$$\int_\gamma p\,dx + q\,dy = \int_a^b [p(z(t))x'(t) + q(z(t))y'(t)]\,dt,$$

where $\gamma(t) = (x(t), y(t))$.

We say $p\,dx + q\,dy$ is an exact differential if $\int_\gamma p\,dx + \,dy$ depends only on the starting and ending points of γ. Note that this is equivalent to saying the integral over a closed curve is 0, for if γ is a closed curve, then γ and $-\gamma$ have the same endpoints, or $\int_\gamma = \int_{-\gamma} = -\int_\gamma$, or $\int_\gamma = 0$. Conversely, if γ_1 and γ_2 have the same endpoints, then $\gamma = \gamma_1 - \gamma_2$ is a closed curve, and $\int_{\gamma_1} - \int_{\gamma_2} = \int_{\gamma_1 - \gamma_2} = 0$.

The following theorem characterizes exact differentials.

Theorem 9.1. *$p\,dx + q\,dy$ is exact if and only if there exists $U(x, y)$ such that $\partial U/\partial x = p$ and $\partial U/\partial y = q$.*

Proof. If such as U exists, then by the chain rule for partial derivatives

$$\begin{aligned}\int_\gamma p\,dx + q\,dy &= \int_a^b \frac{\partial U}{\partial x}(x(t), y(t))x'(t) + \frac{\partial U}{\partial y}(x(t), y(t))y'(t)]dt \\ &= \int_a^b \frac{dU}{dt}(x(t), y(t))dt \\ &= U(x(b), y(b)) - U(x(a), y(a)),\end{aligned}$$

which depends only on the end points.

To prove necessity, fix (x_0, y_0) and define $U(x,y) = \int_\gamma p\,dx + q\,dy$, where γ is a curve in Ω which connects (x_0, y_0) and (x, y). By our assumption that $p\,dx + q\,dy$ is exact, U is well defined. If we choose γ so that the last part of it is a line segment that is horizontal and y is fixed, then as we let x vary we see

$$U(x,y) = \int^x p(x,y)\,dx + constant,$$

so $\partial U/\partial x = p$. Similarly if we arrange it so the last part of γ is vertical, then $\partial U/\partial y = q$. □

Corollary 9.2. *$\int_\gamma f\,dz$ with f continuous depends only on the endpoints of γ if and only if f is the derivative of an analytic function.*

Proof. $f(z) = f(z)\,dx + if(z)\,dy$ is an exact differential if and only if there exists a function F with $\partial F/\partial x = f(z)$ and $\partial F/\partial y = if(z)$. This happens if and only if $\partial F/\partial x = -i\partial F/\partial y$. If $F = U + iV$, then $\partial F/\partial x = U_x + iV_x$ and $\partial F/\partial y = U_y + iV_y$, which implies we have $\partial F/\partial x = -\partial F/\partial y$ if and only if F satisfies the Cauchy-Riemann equations. Since f is continuous, this implies F is analytic. □

Corollary 9.3. *If $n \geq 0$, then $\int_\gamma (z-a)^n dz = 0$ for all closed curves γ. If γ does not pass through a, then it also holds for n integer not equal to -1.*

Proof. $(z-a)^n$ is the derivative of $(z-a)^{n+1}/(n+1)$ for $n \neq -1$. □

In the case $n = -1$, it might not be the case that the integral is 0. Suppose γ is a circle about a of radius r. Then

$$\int_\gamma \frac{1}{z-a} dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it} dt = 2\pi i.$$

We now proceed by a series of steps to Cauchy's theorem.

Let R be a closed rectangle, and let ∂R denote its boundary. Saying a function is analytic on R means that it is analytic on a region containing R.

We will use the following several times. If $\gamma : [a,b] \to \mathbb{C}$, then

$$\begin{aligned} \Big|\int_\gamma f(z)\,dz\Big| &= \Big|\int_a^b f(\gamma(t))\gamma'(t)\,dt\Big| \\ &\leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt \\ &\leq \Big(\sup_{z\in\gamma} |f(z)|\Big) \int_a^b |\gamma'(t)|\,dt. \end{aligned}$$

Observe that $\int_a^b |\gamma'(t)|\,dt$ is the arc length of γ.

Proposition 9.4. *If f is analytic on R, then*

$$\int_{\partial R} f(z)\,dz = 0.$$

Proof. Given any rectangle S, let $\eta(S) = \int_{\partial S} f(z)\,dz$. If we divide R into 4 congruent rectangles R_1, R_2, R_3, R_4, then $\eta(R) = \eta(R_1) + \cdots + \eta(R_4)$, since integrals over common sides cancel each other. For at least one rectangle R_i we must have $|\eta(R_i)| \ge \frac{1}{4}|\eta(R)|$. We call this rectangle S_1 and we now divide S_1 into 4 congruent rectangles and repeat the process. We thus arrive at a sequence $S_0 = R, S_1, S_2, \ldots$ such that $S_{i+1} \subset S_i$ and $|\eta(S_{i+1}| \ge \frac{1}{4}|\eta(S_i)|$, and hence $|\eta(S_n)| \ge 4^{-n}|\eta(R)|$.

Let z^* be the point of intersection of the S_i. Given $\varepsilon > 0$ we can choose $\delta > 0$ such that $f(z)$ is analytic in $|z - z^*| < \delta$ and also

$$\Big|\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*)\Big| < \varepsilon$$

on $|z - z^*| < \delta$. If n is large enough, then S_n is contained in $|z - z^*| < \delta$. Now $\int_{\partial S_n} dz = 0$ and $\int_{\partial S_n} z\,dz = 0$ by Corollary 9.3, so

$$\eta(S_n) = \int_{\partial S_n} [f(z) - f(z^*) - f'(z^*)(z - z^*)]\,dz,$$

and it follows that

$$|\eta(S_n)| \le \varepsilon \int_{\partial S_n} |z - z^*| \cdot |dz|.$$

But in this last integral, $|z - z^*| \le c_1 2^{-n}$ and $|\int_{\partial S_n} |dz|\ |$, which is the length of the perimeter of S_n, is $\le c_2 2^{-n}$, so the last integral is bounded by $c_1 c_2 4^{-n}\varepsilon$. This implies $|\eta(R)| \le c_1 c_2 \varepsilon$; since ε is arbitrary, $\eta(R)$ must be 0. □

Here is Cauchy's theorem in a disk. It is sometimes referred to as the Cauchy-Goursat theorem.

Theorem 9.5. *Let Δ be the open disk $|z - z_0| < \rho$. If f is analytic in Δ, then $\int_\gamma f(z)\,dz = 0$ for every closed curve γ in Δ.*

Proof. Define $F(z)$ by $F(z) = \int_\sigma f\,dz$, where σ consists of the horizontal line segment from the center (x_0, y_0) to (x, y_0) and then the vertical segment from (x, y_0) to (x, y). It is clear that $\partial F/\partial y = if(z)$. By Proposition 9.4, σ can be replaced by a path consisting of a vertical segment followed by a horizontal segment. This gives rise to the same function F and $\partial F/\partial x = f(z)$. Hence F is analytic in Δ with derivative $f(z)$ and $f(z)\,dz$ is an exact differential. □

It is worthwhile reviewing the proof. The steps are

1. An exact differential $p\,dx + q\,dy$ is one where the line integral depends only on the starting and ending points
2. To be an exact differential, we must have $p = \partial U/\partial x$ and $q = \partial U/\partial y$ for some U.
3. The derivative of an analytic function is an exact differential.
4. The integral of an analytic function around a rectangle is 0.
5. If we define F in terms of a line integral of f, F is analytic and its derivative is f.
6. Since f is the derivative of an analytic function, $f\,dz$ is an exact differential, and so its integral along a closed curve is 0.

Proposition 9.6. *Suppose R' is obtained from a rectangle R by omitting a finite number of points ζ_j. Suppose f is analytic on R' and $\lim_{z\to\zeta_j}(z-\zeta_j)f(z) = 0$ for all j. Then $\int_{\partial R} f(z)\,dz = 0$.*

Proof. By dividing R into subrectangles and adding, it suffices to suppose there is a single exceptional point ζ. Let $\varepsilon > 0$. Divide R into 9 subrectangles by dividing each side of R into three pieces. Let the division be done so that ζ is the center of the central rectangle R_0 and R_0 is small enough that $|f(z)| \le \varepsilon/|z-\zeta|$ on R_0. Since the integral around the boundary of the subrectangles is 0 by Proposition 9.4 for each rectangle except R_0, we have $\int_{\partial R} f(z)\,dz = \int_{\partial R_0} f(z)\,dz$. But $|\int_{\partial R_0} f\,dz| \le \varepsilon \int_{\partial R_0} \frac{|dz|}{|z-\zeta|}$, which is less than $c\varepsilon$ by elementary estimates. Since ε is arbitrary, this proves the assertion. □

Proposition 9.7. *Let f be analytic in the region Δ' obtained by omitting a finite number of points ζ_j from Δ and suppose $\lim_{z\to\zeta_j}(z-\zeta_j)f(z) = 0$ for all j. The $\int_\gamma f(z)\,dz = 0$ for any closed curve in Δ'.*

Proof. Assume first that no ζ_j lies on the lines $x = x_0$ and $y = y_0$. We can define $F(z) = \int_\sigma f(z)\,dz$, where σ consists of four lines segments, 2 vertical ones and 2 horizontal ones, that lead from the center to z and such that none pass through any ζ_j. By Proposition 9.6 the value of F does not depend on the choice of the middle segment. With this change, the proof of Theorem 9.5. goes through. □

10. Index of a point.

Lemma 10.1. *If a piecewise differentiable closed curve γ does not pass through the point a, then the value of the integral*

$$\int_\gamma \frac{dz}{z-a}$$

is a multiple of $2\pi i$.

Proof. Suppose the equation of γ is $z = z(t), \alpha \leq t \leq \beta$. Let

$$h(t) = \int_\alpha^t \frac{z'(t)}{z(t) - a} dt.$$

Its derivative is $z'(t)/(z(t) - a)$ whenever z' is continuous. Therefore the derivative of $e^{-h(t)}(z(t)-a)$ is 0 except possibly at finitely many points. Since the function is continuous, it must be constant. Therefore

$$e^{h(t)} = \frac{z(t) - a}{z(\alpha) - a}.$$

Since $z(\beta) = z(\alpha)$, then $e^{h(\beta)} = 1$, which proves that $h(\beta)$ is a multiple of $2\pi i$. □

We define the index of the point a with respect to γ or the winding number of γ with respect to a by

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - a}.$$

Proposition 10.2. *(a) If γ lies inside a circle, then $n(\gamma, a) = 0$ for all points a outside the circle.*

(b) As a function of a the index $n(\gamma, a)$ is constant in each of the regions determined by γ and zero in the unbounded region.

As a point set γ is closed and bounded, so its complement consists of the union of disjoint regions. There can be only one which contains the point at ∞ and we refer to that as the unbounded region determined by γ.

Proof. For (a), if a lies outside the circle, then $1/(z - a)$ is analytic inside the circle, and hence the integral over γ must be 0.

For (b), any two points in the same region can be joined by a polygonal path which does not intersect γ. So it is sufficient to prove that $n(\gamma, a) = n(\gamma, b)$ if γ does not intersect the line segment connecting a and b. Outside the segment the function $(z - a)/(z - b)$ is never real and ≤ 0; to see this, note that the function is a linear fractional transformation mapping a to 0, b to ∞, and $(a + b)/2$ to -1. Since linear fractional transformations map lines to either lines or circles, and the image contains ∞, this transformation must map this line segment to the negative real axis. So we can define $\log[(z-a)/(z-b)]$ an a single-valued analytic function in the complement of this segment. The derivative is $\frac{1}{z-a} - \frac{1}{z-b}$, and if γ does not intersect the segment, then

$$\int_\gamma \Big(\frac{1}{z - a} - \frac{1}{z - b}\Big) dz = 0,$$

or $n(\gamma, a) = n(\gamma, b)$. If $|a|$ is sufficiently large, then by (a) we have $n(\gamma, a) = 0$. Hence n is 0 in the unbounded region. □

11. Cauchy integral formula.

Theorem 11.1. *Suppose that f is analytic in an open disk Δ and γ is a closed curve in Δ. If $a \notin \gamma$,*

$$n(\gamma, a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-a}.$$

Proof. Let us apply Cauchy's theorem to $F(z) = (f(z) - f(z))/(z-a)$. This is analytic for $z \neq a$. For $z = a$ it satisfies the condition $\lim_{z\to a} F(z)(z-a) = \lim(f(z) - f(a)) = 0$. Hence

$$\int_\gamma \frac{f(z) - f(a)}{z-a}dz = 0,$$

which is the same as

$$\int_\gamma \frac{f(z)}{z-a}dz = f(a)\int_\gamma \frac{dz}{z-a}.$$

□

The above formula also remains valid when there are exceptional points $\zeta_1, \ldots, \zeta_n$ as in Proposition 9.7, as long as none of them are equal to a.

Corollary 11.2. *If $n(\gamma, a) = 1$, then*

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-a}.$$

12. Higher derivatives.

Proposition 12.1. *Suppose that $\varphi(\zeta)$ is continuous on the arc γ. Then the function*

$$F_n(z) = \int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta - z)^n}$$

is analytic in each of the regions determined by γ and its derivative is $F_n'(z) = nF_{n+1}(z)$.

Proof. We first show $F_1(z)$ is continuous. If $z_0 \notin \gamma$, choose δ so that $|z - z_0| < \delta$ does not meet γ. If $|z - z_0| < \delta/2$, then $|\zeta - z| > \delta/2$ if $\zeta \in \gamma$. Since

$$F_1(z) - F_1(z_0) = (z - z_0)\int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta - z)(\zeta - z_0)},$$

we have

$$|F_1(z) - F_1(z_0)| \le |z - z_0| \frac{2}{\delta^2} \int_\gamma |\varphi| \, |d\zeta|,$$

hence F_1 is continuous also.

Next let

$$\Phi_1(\zeta) = \frac{\varphi(\zeta)}{\zeta - z_0},$$

and we see that

$$\frac{F_1(z) - F_1(z_0)}{z - z_0} = \int_\gamma \frac{\varphi(\zeta) d\zeta}{(\zeta - z)(\zeta - z_0)} = \int_\gamma \frac{\Phi_1(\zeta) d\zeta}{\zeta - z}$$

is continuous, so tends to the limit $F_2(z_0)$ as $z \to z_0$. Therefore $F_1'(z) = F_2(z)$.

We now use induction. From

$$F_n(z) - F_n(z_0) = \Big[\int_\gamma \frac{\varphi d\zeta}{(\zeta - z)^{n-1}(\zeta - z_0)} - \int_\gamma \frac{\varphi d\zeta}{(\zeta - z_0)^n} \Big] + (z - z_0) \int_\gamma \frac{\varphi d\zeta}{(\zeta - z)^n (\zeta - z_0)}$$

we see that $F_n(z)$ is continuous: by the induction hypothesis applied to $\varphi(\zeta)/(\zeta - z_0)$ the first term goes to 0, while in the second term the factor $z - z_0$ is bounded in a neighborhood of z_0.

Now divide the above identity by $z - z_0$ and let $z \to z_0$. The quotient in the first term tends to the derivative of $\int_\gamma \frac{\Phi_1(\zeta) d\zeta}{(\zeta - z_0)^{n-1}}$, which by the induction hypothesis is

$$(n-1) \int_\gamma \frac{\Phi_1(\zeta) d\zeta}{(\zeta - z_0)^n} = (n-1) \int_\gamma \frac{\varphi(\zeta) d\zeta}{(\zeta - z_0)^{n+1}} = (n-1) F_{n+1}(z).$$

The second term, we just showed, is continuous, and has the limit $F_{n+1}(z_0)$.

Theorem 12.2. *Let Δ be an open region such that f is analytic on $\overline{\Delta}$. Let C be the boundary of Δ. Then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{\zeta - z},$$

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^2},$$

and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}}.$$

Proof. The first follows from Cauchy's integral formula with a replaced by z, and the other two from applying Proposition 12.1. □

Theorem 12.3. (Morera's theorem) *If f is continuous in a region Ω and $\int_\gamma f\,dz = 0$ for all closed curves in Ω, then f is analytic in Ω.*

Proof. From the hypothesis, we conclude by Corollary 9.2 that f is the derivative of an analytic function F. We have just shown that the derivative of an analytic function exists (and is differentiable, hence continuous). So $F' = f$ is analytic. □

Theorem 12.4. (Liouville's theorem) *If f is analytic and bounded in the plane, then f is constant.*

Proof. If $|f|$ is bounded by M, then

$$|f'(z)| \leq Mr^{-1}$$

from Theorem 12.2 with C the circle of radius r about z. Let $r \to \infty$ to see $f' \equiv 0$. □

Theorem 12.5. (Fundamental theorem of algebra) *If $P(z)$ is a polynomial of degree greater than 0, then P has a zero.*

Proof. If not, $1/P(z)$ would be analytic in the whole plane. Since the degree of P is larger than 0, then $|P(z)| \to \infty$ as $|z| \to \infty$, hence $1/P(z)$ is bounded. But by Theorem 12.4 this implies $1/P$ is constant, a contradiction. □

Corollary 12.6. (Cauchy's estimate) $|f^{(n)}(a)| \leq Mn!r^{-n}$ *if $|f|$ is bounded on a circle C of radius r about a.*

Proof. Apply the last formula in the statement of Theorem 12.2. □

13. Removable singularities and Taylor's theorem.

Theorem 13.1. *Suppose f is analytic in a region Ω' obtained by omitting a point from a region Ω. A necessary and sufficient condition that there exists an analytic function in Ω which agrees with f in Ω' is that $\lim_{z\to a}(z-a)f(z) = 0$. The extended function is uniquely determined.*

Proof. Since the extended function must be continuous at a, the necessity and uniqueness are obvious. To prove sufficiency, take a circle C centered at a that is contained in Ω. If we set

$$\widetilde{f}(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{\zeta - z},$$

then $\widetilde{f}$ will equal $f(z)$ for $z \neq a$ and will be analytic inside C. □

We denote the extension of f by f also.

Theorem 13.2. (Taylor's theorem) *If f is analytic in a region Ω containing a, then*

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n, \quad (13.1)$$

where $f_n(z)$ is analytic in Ω. Moreover, if C is a circle centered at a and contained in Ω, then

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-a)^n(\zeta-z)}. \quad (13.2)$$

Proof. Applying Theorem 13.1 to $F(z) = (f(z) - f(a))/(z-a)$, there exists an analytic function $f_1(z)$ that is equal to F for $z \neq a$ and equals $f'(a)$ for $z = a$. Repeating, there exists an analytic function f_2 that equals $(f_1(z) - f_1(a))/(z-a)$ for $z \neq a$ and $f_1'(a)$ for $z = a$. Continuing, we have

$$\begin{aligned} f(z) &= f(a) + (z-a)f_1(z), \\ f_1(z) &= f_1(a) + (z-a)f_2(z), \\ &\cdots \\ f_{n-1}(z) &= f_{n-1}(a) + (z-a)f_n(z). \end{aligned}$$

These equations also hold for $z = a$. Combining,

$$f(z) = f(a) + (z-a)f_1(a) + (z-a)^2 f_2(a) + \cdots + (z-a)^{n-1} f_{n-1}(a) + (z-a)^n f_n(z).$$

Differentiating n times and setting $z = a$ we obtain $f^{(n)}(a) = n! f_n(a)$, and substituting gives the first assertion of the theorem.

Let C be a circle centered at a and contained in Ω. Then

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\zeta)d\zeta}{\zeta - z}. \quad (13.3)$$

Solve (13.1) for $f_n(z)$ and substitute the result in (13.3). We obtain

$$f_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta-a)^n(\zeta-z)} - \frac{f'(a)}{2\pi i}F_{n-1}(a) - \cdots - \frac{f^{(n-1)}(a)}{(n-1)!2\pi i}F_1(a),$$

where

$$F_j(a) = \int_C \frac{d\zeta}{(\zeta-a)^j(\zeta-z)}, \qquad j \geq 1.$$

But

$$F_1(a) = \frac{1}{z-a}\int_C \Big(\frac{1}{\zeta-z} - \frac{1}{\zeta-a}\Big)d\zeta = 0$$

for all a inside C, so that F_1 is identically zero there. By Proposition 12.1 with $\varphi(\zeta) = 1/(\zeta - z)$, we obtain that $F_{j+1}(a) = F_1^{(j)}(a)/j!$ is also 0 for all $j \geq 0$. □

14. Taylor series.

Theorem 14.1. *If f is analytic in a region Ω containing z_0, then the representation*

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

is valid in the largest open disk of center z_0 contained in Ω.

Saying the representation is valid means that the series converges absolutely there and uniformly in any smaller disk.

Proof. We already know

$$f(z) = f(z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + f_{n+1}(z)(z - z_0)^{n+1}, \tag{13.4}$$

where

$$f_{n+1}(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}(\zeta - z)}$$

and C is any circle $|z - z_0| = \rho$ such that $|z - z_0| \leq \rho$ is contained in Ω.

If M is the maximum of $|f|$ on C, then the last term in (13.4) is bounded by

$$\frac{M|z - z_0|^{n+1}}{\rho^n(\rho - |z - z_0|)}.$$

This converges to zero uniformly in every disk $|z - z_0| \leq r < \rho$. Thus the series expansion is valid in every closed disk contained in Ω. We can take ρ to be as close as we like to the distance from z_0 to the boundary of Ω.

Therefore the radius of convergence R must be at least the distance from z_0 to the boundary of Ω. Now apply Theorem 4.1 to get the assertion about the absolute and uniform convergence. □

Let us find the Taylor series for a few familiar functions. We have

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

by taking derivatives (or by integrating $1/(1 + z)$). The radius of convergence is seen to be 1 by taking the limsup of $n^{-1/n}$. We also have

$$(1 + z)^a = 1 + az + \binom{a}{2} z^2 + \cdots + \binom{a}{n} z^n + \cdots,$$

where

$$\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}.$$

We obtain this expansion by taking derivatives of f. We show that the radius of convergence for this series is 1 as follows. $(1+z)^a$ is single-valued and analytic in $|z|<1$ so the radius of convergence is at least 1. If it were more than 1, $(1+z)^a$ and all its derivatives would be analytic and bounded in some disk $|z|<\rho$ for some $\rho>1$. But unless a is a positive integer, some derivative will have a term $(1+z)$ to a negative power, which is not bounded in that disk. So the radius of convergence can be at most 1.

If we expand $1/(1+z)$ in a geometric series and replace z by z^2, we obtain

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - \cdots.$$

Integrating yields

$$\arctan z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots.$$

If we use the binomial expansion for $(1+z)^{1/2}$ and replace z by $-z^2$, we obtain

$$\frac{1}{\sqrt{1-z^2}} = 1 = \frac{1}{2}z^2 + \frac{1}{2}\frac{3}{4}z^4 + \frac{1}{2}\frac{3}{4}\frac{5}{6}z^6 + \cdots.$$

Integrating gives the Taylor series for $\arcsin z$.

Here are some general techniques. Let R_n denote the remainder of any power series containing all terms of order $n+1$ and larger. If $f(z) = P_n(z) + R_n$, where P_n is a polynomial of degree n, and similarly $g(z) = Q_n(z) + R_n$, then

$$f(z)g(z) = P_n(z)Q_n(z) + R_n.$$

In the case of division, if $f = P_n + R_n$, $g = Q_n + R_n$, and $P_n = Q_nT_n + R_n$, where T_n is a polynomial of degree n, and if $g(0) \neq 0$, then

$$f - T_ng = P_n - T_nQ_n + R_n.$$

Because $g(0) \neq 0$, dividing the higher order terms by g leaves an expression that consists of terms of order $n+1$ and higher. Therefore

$$f/g = T_n + \text{ higher order terms.}$$

For the composition of two functions, it is easy to see that

$$f(g(z)) = P_n(Q_n(z)) + \text{ higher order terms.}$$

Finally we look at the inverse of a function. We suppose that $g(0) = 0$ and in order to have an inverse, at 0, it is necessary and sufficient that $g'(0) = a_1 \neq 0$. So $g(z) = Q_n(z) + S_n(z)$, where $Q_n(z) = a_1z + a_2a^2 + \cdots + a_nz^n$. Then if $f = g^{-1}$, we have

$$z = f(g(z)) = P_n(Q_n) + \text{ higher order terms.}$$

Clearly $P_1(w) = w/a_1$. If P_{n-1} is determined, then $P_n = P_{n-1} + b_n z^n$, and

$$\begin{aligned} P_n(Q_n(z)) &= P_{n-1}(Q_n(z)) + b_n a_1^n z^n + \text{ higher order terms} \\ &= P_{n-1}(Q_{n-1}(z) + a_n z^n) + b_n a_1^n z^n + \text{ higher order terms} \\ &= P_{n-1}(Q_{n-1}(z)) + P'_{n-1}(Q_{n-1}(z)) a_n z^n + b_n a_1^n z^n + \text{ higher order terms.} \end{aligned}$$

The first two terms can be written as $z + c_n z^n$ where c_n is determined, and we then solve for $b_n n = -c_n/a_1^n$.

In practice, it is easier to use successive substitution. For example,

$$w = \arctan z \approx z - \frac{z^3}{3} + \frac{z^5}{5},$$

so

$$z \approx w + \frac{z^3}{3} - \frac{z^5}{5}.$$

Here "≈" means "up to terms of order higher than fifth." Substituting the expression for z in the right,

$$\begin{aligned} z &\approx w + \frac{1}{3}\Big(w + \frac{z^3}{3}\Big)^3 - \frac{1}{5}\Big(w\Big)^5 \\ &\approx w + \frac{1}{3}w^3 + \frac{1}{3}w^2 z^3 - \frac{1}{5}w^5 \\ &\approx w + \frac{1}{3}w^3 + \frac{1}{3}w^2\Big(w + \frac{1}{3}z^3\Big)^3 - \frac{1}{5}w^5 \\ &= w + \frac{1}{3}w^3 + \frac{2}{15}w^5. \end{aligned}$$

15. Zeros and poles.

Proposition 15.1. *Suppose f is analytic in a connected domain Ω and $a \in \Omega$. If $f(a)$ and $f^{(n)}(a)$ are zero for all n, then f is identically 0.*

Proof. By Taylor's theorem,

$$f(z) = f_n(z)(z-a)^n,$$

where

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - a)^n(\zeta - z)}.$$

Here C is the circumference of a disk contained in Ω. Let M be the maximum of f on C and R the radius of C. Then

$$|f_n(z)| \le \frac{M}{R^{n-1}(R - |z-a|)}$$

if $|z-a|<R$. So

$$|f(z)| \le \Big(\frac{|z-a|}{R}\Big)^n \frac{MR}{R-|z-a|}$$

for every n. Since $|z-a|/<1$, letting $n\to\infty$ shows tht $f(z)=0$ inside of C.

If E_1 is the set on which $f(z)$ and all its derivatives vanish and $E_2=\Omega-E_1$, then E_1 is open by the above paragraph. If E_2 is not open, there exists $z\in E_2$ such that every disk about z that is contained in Ω also contains a point of E_1; hence we can find a sequence $z_m\in E_1$ with $z_m\to z$. Since f and all its derivatives are continuous and f and all its derivatives are zero on each z_m, then f and all its derivatives are zero at z, or $z\in E_1$ a contradiction. Therefore E_2 is open. Since Ω is connected and E_1 is nonempty, then E_1 must be all of Ω. □

If $f(z)$ is not identically 0 and $f(a)=0$, there exists a first derivative $f^{(h)}(a)$ that is not zero. We say that a is a zero of order h. We can write $f(z)=(z-a)^h f_h(z)$ where f_h is analytic and $f_h(a)\ne 0$ by Taylor's theorem. Since f_h is continuous, it is nonzero in a neighborhood of a, and so $z=a$ is the only zero of $f(z)$ in this neighborhood. Therefore the zeros of an analytic function which does not vanish identically are isolated.

If there is a sequence $z_m\in\Omega$ tending to $z\in\Omega$ with $f(z_m)=0$, then by continuity $f(z)=0$, contradicting the fact that the zeros are isolated. Therefore the set of zeros of f cannot have a limit point in Ω.

Corollary 15.2. *Suppose f and g are analytic in a connected domain Ω and $f(z)=g(z)$ on a set which has a limit point in Ω. Then $f(z)=g(z)$ for all z.*

Proof. If $f-g$ is not identically zero, then the zeros of $f-g$ are isolated, a contradiction. □

Suppose f is analytic in a neighborhood of a, except possibly at a itself. So f is analytic in the region $0<|z-a|<\delta$. a is called an isolated singularity. Suppose $\lim_{z\to a} f(z)=\infty$. Then a is said to be a pole of f. There exists $\delta'<\delta$ such that $f(z)\ne 0$ for $0<|z-a|<\delta'$. Then $g(z)=1/f(z)$ is defined and analytic. Moreover the singularity of g at a is removable. Since g does not vanish identically, then g has a zero of order h at a. We say the pole of f at a has order h.

f is said to have a zero or pole of order h at ∞ if $g(z)=f(1/z)$ has a zero or pole of order h at 0.

A function which is analytic in a region Ω except for poles is called meromorphic. The sum, difference, product, and difference of meromorphic functions is again meromorphic.

Proposition 15.3. *Consider the conditions*

(1) $\lim_{z\to a} |z-a|^\alpha f(z) = 0$,

(2) $\lim_{z\to a} |z-a|^\alpha f(z) = \infty$.

Either (i) condition (1) holds for all α; (ii) there exists an integer h such tht (1) holds for $\alpha > h$ and (2) holds for $\alpha < h$; or (iii) neither (1) nor (2) holds for any α.

Proof. Suppose (1) holds for some α. Then it holds for all larger α, and so it holds for some integer m. Then $(z-a)^m f(z)$ has a removable singularity and vanishes for $z = a$. Either $f(z)$ is identically zero, and (i) holds, or else $(z-a)^m f(z)$ has a zero of finite order k. In the latter case (1) holds for all $\alpha > h = m - k$ and (2) holds for all $\alpha < h$.

Suppose (2) holds for some α. It then holds for all smaller α and hence for some integer n. $(z-a)^n f(z)$ has a pole of finite order l, and if $h = n + l$, then (1) holds for $\alpha > h$ and (2) for $\alpha < h$. □

In case (iii) a is called an essential isolated singularity.

Theorem 15.4. (Weierstrass) *An analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.*

Proof. If not, there exists A and $\delta > 0$ such that $|f(z) - A| > \delta$ in a neighborhood of a except possibly for a itself. Then for any $\alpha < 0$ we have $\lim_{z\to a} |z-a|^\alpha (f(z) - A) = \infty$. Hence a cannot be an essential singularity of $f(z) - A$. So there exists β such that $\lim_{z\to a} |z-a|^\beta |f(z) - A| = 0$, and we may take $\beta > 0$. Since $\lim_{z\to a} |z-a|^\beta |A| = 0$, then $\lim_{z\to a} |z-a|^\beta |f(z)| = 0$, and a is not an essential singularity of f, a contradiction. □

16. The local mapping.

Proposition 16.1. *Suppose γ is a closed curve in a disk Δ such that $f(z) \neq 0$ on γ. Then if $z_1, z_2, \ldots$ are the zeros of f inside γ, repeated if the multiplicity is greater than one, then*

$$n(\gamma, z_1) + n(\gamma, z_2) + \cdots = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz. \tag{16.1}$$

Proof. First suppose that f has only finitely many zeros in Δ. We can write $f(z) = (z-z_1)(z-z_2)\cdots(z-z_n)g(z)$, where g is analytic and nonzero in Δ. Taking the logarithm and then taking the derivative,

$$\frac{f'(z)}{f(z)} = \frac{1}{z-z_1} + \cdots + \frac{1}{z-z_n} + \frac{g'(z)}{g(z)}$$

as long as z does not equal any z_j. This is true if $z \in \gamma$. Since $g(z) \neq 0$ in Δ, by Cauchy's theorem $\int_\gamma (g'(z)/g(z))dz = 0$. Therefore, multiplying by $1/(2\pi i)$ and integrating along γ, we have (16.1).

Suppose now that f has infinitely many zeros in Δ. There will exist a concentric disk Δ' smaller than Δ that contains γ. Unless f is identically 0, then f can only have finitely many zeros in Δ'. So (16.1) holds if we restrict attention to those zeros inside γ. However $n(\gamma, z_i) = 0$ for any zero outside γ, so does not contribute to the sum in (16.1). □

If γ is a circle, then $n(z, \gamma)$ is either zero or one, and then the integral is simply the number of zeros of f enclosed by γ.

The function $w = f(z)$ maps γ onto a closed curve Γ and

$$\int_\Gamma \frac{dw}{w} = \int_\gamma \frac{f'(z)}{f(z)} dz.$$

Therefore (16.1) says that

$$n(\Gamma, 0) = \sum_j n(\gamma, z_j).$$

If we apply the above to the function $f(z) - a$, we have

$$\sum_j n(\gamma, z_j(a)) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a} dz$$

and

$$n(\Gamma, a) = \sum_j n(\gamma, z_j(a)), \tag{16.2}$$

where $z_j(a)$ are the zeros of $f(z) - a$, i.e., the places where $f(z) = a$, and we assume none of them lie on γ.

Theorem 16.2. *Suppose f is analytic at z_0, $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order n at z_0. If $\varepsilon > 0$ is sufficiently small, there exists $\delta > 0$ such that for all a with $|a - w_0| < \delta$ the equation $f(z) = a$ has exactly n roots in the disk $|z - z_0| < \varepsilon$.*

Proof. Choose ε so that f is defined and analytic for $|z - z_0| \leq \varepsilon$ and so that z_0 is the only zero of $f(z) - w_0$ in this disk. Let γ be the circle $|z - z_0| = \varepsilon$ and Γ its image under f. Since $w_0 \notin \Gamma$ and Γ is closed, choose δ small enough so that $|w - w_0| < \delta$ does not intersect Γ. The function $f(z) - a$ has n roots at z_0, so by (16.2) $n(\Gamma, a) = n$. But for every b inside $|w - w_0| < \delta$, we have $n(\Gamma, b) = n(\Gamma, a)$. Therefore $f(z) - b$ has only n roots. □

Corollary 16.3. *A nonconstant analytic function maps open sets into open sets.*

Proof. If z_0 is in a domain and $w_0 = f(z_0)$, then for ε sufficiently small, there is $\delta > 0$ such that for all a with $|a - w_0| < \delta$, the equation $f(z) = a$ has at least one root in the disk $|z - z_0| < \varepsilon$, i.e., $|a - w_0| < \delta$ is contained in the image under f of $|z - z_0| < \varepsilon$. Thus the image of this disk under f contains a disk about w_0, and this proves the image of open sets is open. □

17. The maximum principle.

Theorem 17.1. *If f is analytic and nonconstant in a region Ω, then $|f(z)|$ has no maximum inside Ω.*

If f is continuous on $\overline{\Omega}$, the maximum must take place on $\partial\Omega$.

Proof. If $w_0 = f(z_0)$ with $z_0 \in \Omega$, there is a neighborhood $|w - w_0| < \varepsilon$ contained in the image of Ω. In this neighborhood there are points of modulus larger than $|w_0|$, and hence $|f(z_0)|$ is not the maximum of $|f(z)|$. □

An application of the maximum principle is the Schwarz lemma.

Theorem 17.2. *If f is analytic for $|z| < 1$ and $|f(z)| \leq 1$ with $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality holds only if $f(z) = cz$ for some constant c with modulus 1.*

Proof. Let $f_1(z) = f(z)/z$ for $z \neq 0$ and $f_1(0) = f'(0)$. On $|z| = r$, the absolute value of $|f_1|$ is bounded by $1/r$, and so $|f_1|$ is bounded by $1/r$ on $|z| \leq r$. Now let $r \to 1$.

If equality holds at a single point, then $|f_1|$ attains its maximum, and hence must be a constant. □

18. Chains, cycles, and simple connectedness.

A chain is a finite union of arcs. If the arcs are $\gamma_1, \ldots, \gamma_n$, we write the chain as $\gamma_1 + \cdots + \gamma_n$. We define

$$\int_\gamma f = \int_{\gamma_1} f + \cdots + \int_{\gamma_n} f.$$

We say two chains γ, γ' are equal if $\int_\gamma f = \int_{\gamma'} f$ for all functions f. A cycle is a chain that can be represented as the sum of closed curves.

It should be clear that if a cycle is contained in a domain, then the theorems that hold for closed curves are also valid for cylces. In particular, the integral of an exact differential over any cycle is 0.

The index of a point with respect to a cycle is defined in the same way as in the case of a single closed curve, and we have $n(\gamma_1 + \gamma_2, a) = n(\gamma_1, a) + n(\gamma_2, a)$.

We say a region is simply connected if its complement with respect to the extended plane is connected. Looking at the infinite strip shows why the complement must be with respect to the extended plane. Unless stated otherwise, we will only look at domains lying in the finite plane.

Proposition 18.1. *A region Ω is simply connected if and only if $n(\gamma, a) = 0$ for all cycles γ in Ω and all points a which do not belong to Ω.*

Proof. Suppose Ω is simply connected and γ is a cycle in Ω. Since the complement of Ω is connected, it must be contained in one of the regions determined by γ, and since ∞ is in the component, this must be the unbounded region. Therefore $n(\gamma, a)$ must be 0 for all a in the complement.

Now suppose that $n(\gamma, a) = 0$ when γ is a cycle in Ω and a is in the complement of Ω. Suppose the complement of Ω can be written as the union of two disjoint closed sets A and B. Without loss of generality, suppose $\infty \in B$, so that A is bounded. Let δ be the smallest distance between points of A and B and cover the plane with a net of squares Q of side length less than $\delta/\sqrt{2}$. We can do this so we can find a point $a \in A$ that is the center of one of the squares. We take the squares Q to be closed with the interior of a square to the left of the directed line segments that make up its boundary ∂Q.

Let $\gamma = \sum_j \partial Q_j$, where the sum is over all squares which have a point in common with A. (We do not know that A is connected, so γ could possibly be a cycle without being a closed curve.) Since a is in one and only one of these squares, $n(\gamma, a) = 1$. We claim that γ does not intersect A. If it did, there would exist a Q_j which has a point a_0 of A on its boundary. But then the neighboring square Q_i would also have a_0 on its boundary, and so ∂Q_i would also contribute to the sum. However, the line segment in common of Q_i and Q_j would cancel in the sum making up γ, contradicting that $a_0 \in \gamma$. A similar argument holds when the point of A is on the corner of one of the Q_j. It is clear that γ does not meet B by the definition of δ. Therefore γ is contained in Ω, a contradiction to $n(\gamma, a) = 0$. □

18. Multiply connected regions and Cauchy's theorem.

Let us say that a cycle γ in Ω is homologous to zero with respect to Ω if $n(\gamma, a) = 0$ for all points a not in Ω. We write $\gamma \sim 0$.

Theorem 18.1. *If f is analytic in Ω and $\gamma \sim 0$, then*

$$\int_\gamma f(z)\, dz = 0.$$

Proof. Let δ be the distance from γ to the complement of Ω, and cover γ by finitely many disks of radius $\gamma/2$. Within each disk, we can replace the subarc γ_i of γ that is contained in the disk by a polygonal path σ_i connecting the endpoints of the subarc. Here the polygonal path is made up of line segments parallel to the axes. By Cauchy's theorem for the disk, the integral of f over γ_i and σ_i is the same. Therefore, the integral of f over γ is the same as the integral of f over $\sigma = \sum \sigma_i$, and we have reduced the problem to the situation where γ is a polygonal path.

Extend each line segment making up γ into an infinite line. As a result we will have divided $\mathbb{C}$ into some finite rectangles $\{R_j\}$ and some infinite rectangles $\{R'_j\}$. Without loss of generality we can modify γ slightly so that we have at least one finite rectangle; this insures that two infinite rectangles do not have a finite line segment as their common side.

Choose $a_j \in R_j$ and define $\gamma_0 = \sum_j n(\gamma, a_j)\partial R_j$. Our sum is only over the finite rectangles. Note $n(\gamma_0, a_i) = n(\gamma, a_i)$ because $n(\partial R_j, a_i)$ is 1 if $i = j$ and 0 otherwise. If we choose $a'_j \in R'_j$, then $n(\partial R_i, a'_j) = 0$ so $n(\gamma_0, a'_j) = 0$, while $n(\gamma, a'_j) = 0$ because a'_j must be in the unbounded component determined by γ. Now suppose R_j is to the left of R_k and σ is their common boundary. If $\gamma - \gamma_0$ contains a multiple $c\sigma$, then $\gamma - \gamma_0 - c\partial R_j$ will not contain σ. Therefore the index of this cycle will be the same about a_j as about a_k. But $n(\gamma - \gamma_0 - c\partial R_j, a_j) = -c$ while $n(\gamma - \gamma_0 - c\partial R_j, a_k) = 0$. Hence $c = 0$. A similar argument holds if R_j is above R_k or if R_j is finite and is next to some R'_k. We conclude $\gamma_0 = \gamma$.

If $n(\gamma, a_j) \neq 0$, then we claim R_j is contained in Ω. If not, there would be $a \in R_j$ with $a \notin \Omega$. Since $\gamma \sim 0$, then $n(\gamma, a) = 0$. But then $0 = n(\gamma, a) = n(\gamma_0, a) = n(\gamma_0, a_j) = n(\gamma, a_j)$, a contradiction. Therefore R_j is contained in Ω. By Cauchy's theorem for rectangles, $\int_{\partial R_j} f = 0$. Therefore $\int_\gamma f = \int_{\gamma_0} f = \sum n(\gamma, a_j) \int_{\partial R_j} f = 0$. □

As a useful application, we have

Corollary 18.2. *If f is analytic and nonzero in a simply connected domain, then a single valued analytic branch of $\log f(z)$ can be defined in Ω.*

Proof. $f'(z)/f(z)$ is analytic, and define F to be its indefinite integral: $F(z) = \int_\sigma f'/f\, dz$, where σ is a polygonal path connecting a fixed point $z_0 \in \Omega$ to z. Then $f(z)e^{-F(z)}$ has derivative 0, hence must be constant. Choose an arbitrary value of $\log f(z_0)$ and define $\log f(z) = F(z) - F(z_0) + \log f(z_0)$. □

If f is analytic, then it is not hard to see that the image under f of a simply connected domain Ω is again simply connected. The image will not contain 0 and is simply connected, so it is not possible to wind around zero within $f(\Omega)$. This gives some intuition as to why Corollary 18.2 is valid.

18. The residue theorem.

Once we have Cauchy's theorem, we can extend all the other results to the more general situation. For example,

Theorem 18.1. *If f is analytic in a region Ω and γ in Ω is homologous to 0, then*

$$n(\gamma, a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz.$$

We define the residue of f at an isolated singularity a to be the complex number R which makes $f(z) - R/(z-a)$ the derivative of a single-valued analytic function in an annulus $0 < |z-a| < \delta$. We write $R = \text{Res}_{z=a} f(z)$. If f has a pole at a, then we can write

$$f(z) = B_h(z-a)^{-h} + \cdots B_1(z-a)^{-1} + \varphi(z)$$

and we see $\text{Res}_{z=a} f(z) = B_1$. If a is a simple pole of f, then

$$\text{Res}_{z=a} f(z) = \lim_{z\to a} f(z)(z-a),$$

but this formula holds only in the case when the pole is simple.

The residue theorem is the following.

Theorem 18.2. *Let f be analytic except for isolated singularities a_j in a region Ω. Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_j n(\gamma, a_j)\text{Res}_{z=a_j} f(z)$$

for any cycle γ in Ω which is homologous to zero and does not pass through any of the a_j's.

Proof. First suppose there are only finitely many singularities. Let Ω' be the region Ω with the points $\{a_j\}$ omitted. To each a_j there exists δ_j such that $0 < |z-a_j| < \delta_j$ is contained in Ω'. Let C_j be the circle about a_j of radius $\delta_j/2$ and let R_j be the residue at a_j. Then

$$\int_{C_j} f(z)\,dz = \int_{C_j} \Big[f(z) - \frac{R_j}{z-a_j}\Big]\,dz + R_j\int_{C_j}\frac{dz}{z-a_j}.$$

The first integral on the right is zero, while the second is $2\pi i R_j$.

Let γ be a cycle in Ω' which is homologous to zero with respect to Ω. Then

$$\gamma \sim \sum_j n(\gamma, a_j)C_j$$

with respect to Ω' since all points outside of Ω and also the a_j have the same order with respect to both cycles. Then

$$\int_\gamma f(z)\,dz = \sum_j n(\gamma, a_j) \int_{C_j} f(z)\,dz = \sum_j n(\gamma, a_j) 2\pi i R_j.$$

In the general case, we need only prove that $n(\gamma, a_j)$ is nonzero for a finite number of a_j's. The set of all points a with $n(\gamma, a) = 0$ is open and contains all points outside of a large circle. Its complement is compact, and cannot contain more than finitely many of the a_j's. □

21. The argument principle.

Theorem 21.1. *(The argument principle) Suppose f is meromorphic in Ω with zeros a_j and poles b_k and γ is a cycle which is homologous to zero in Ω and does not pass through any zeros or poles. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k). \tag{21.1}$$

Zeros and poles are repeated according to their multiplicity.

Proof. There will only be finitely many zeros and poles inside $\Omega' \subset \overline{\Omega'} \subset \Omega$, where $\gamma \subset \Omega'$. We can write $f(z) = (z-a_1)^{\alpha_1} \cdots (z-a_j)^{\alpha_j} (z-b_1)^{-\beta_1} \cdots (z-b_k)^{-\beta_k} g(z)$, where g is analytic inside Ω' and nonzero. Then

$$\frac{f'(z)}{f(z)} = \sum_j \alpha_j (z-a_j)^{-1} - \sum_k \beta_k (z-b_k)^{-1} + \frac{g'(z)}{g(z)}.$$

The integral of the last ratio over γ is zero since g'/g is analytic, and the result follows by the definition of n. □

The reason for the name is that the left hand side of (21.1) is $n(\Gamma, 0)$, where Γ is the image under f of γ.

Corollary 21.2. *(Rouché's theorem) Let γ be homologous to zero in Ω and such that $n(\gamma, z)$ is either 0 or 1 for any point z not in γ. Suppose f and g are analytic in Ω and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on γ. Then f and g have the same number of zeros enclosed by γ.*

Proof. Since we have strict inequality, we cannot have $f(z) = 0$ (or else $0 > 0$) or $g(z) = 0$ (or else $|f(z)| < |f(z)|$) if $z \in \gamma$. Also, on γ we have the inequality

$$\left|\frac{g(z)}{f(z)} - 1\right| < 1$$

on γ. So the values of $F(z) = g(z)/f(z)$ on γ are contained in the disk of radius 1 with center 1, hence does not contain the origin. If Γ is the image of γ under F, then $n(\Gamma, 0) = 0$. By the change of variables formula, this says

$$\frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)}dz = \frac{1}{2\pi i}\int_\Gamma \frac{dw}{w} = 0.$$

But $F'(z)/F(z) = g'(z)/g(z) - f'(z)/f(z)$ by a calculation. So the integral of g'/g and f'/f on γ is the same. Now by Theorem 21.1, we have our conclusion. □

Corollary 21.2. *Suppose the equation $f(z) = w$ has one root $z(w)$ in the disk $|z - z_0| < \varepsilon$. Then*

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon} \frac{f'(z)}{f(z) - w} z\, dz.$$

Proof. If f is meromorphic, g is analytic, and g has no zeros at any poles or zeros of f, then $g(z)f'(z)/f(z)$ has the residue $hg(a)$ at a zero a of order h and a residue of $-hg(a)$ at a pole of order h. So

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}dz = \sum_j n(\gamma, a_j)g(a_j) - \sum_k n(\gamma, b_k)g(b_k).$$

If f has no zero at 0 and there is only one zero of $f(z) = w$ in the disk $|z - z_0| = \varepsilon$, we apply this with $g(z) = z$ and $f(z)$ replaced by $f(z) - w$. The formula says that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - w} z\, dz = g(a_1) = a_1,$$

where a_1 is the zero of $f(z) - w$, namely, $f^{-1}(w)$. If f has a zero at 0, we let $g(z) = z + \delta$ and let $\delta \to 0$. □

22. Evaluation of definite integrals.

One of the important applications of complex analysis is to evaluate some definite *real* integrals, and the method used is the residue theorem. This can best be explained by examples.

Example 1. Let us compute

$$\int_0^\pi \frac{d\theta}{2 + \cos\theta}.$$

Since $\cos\theta$ takes the same values in $(0, \pi)$ and $(\pi, 2\pi)$, let us compute the integral over 0 to 2π; our answer will be half of what we obtain. Let us make the substitution $z = e^{i\theta}$, so $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$ and $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$. Also, $dz/z = i\, d\theta$. So our integral becomes

$$-i\int_{|z|=1} \frac{dz}{z}\frac{1}{2 + \frac{1}{2}(z + \frac{1}{z})} = -i\int_{|z|=1} \frac{dz}{z^2 + 4z + 1}.$$

The denominator can be factored as $(z-\alpha)(z-\beta)$ with

$$\alpha = -2+\sqrt{3}, \qquad \beta = -2-\sqrt{3}.$$

So the only pole of the integrand is at α, and the residue is $1/(\alpha-\beta)$. Therefore the value of the integral is $\pi/\sqrt{3}$.

Example 2. Let us compute

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}.$$

We will integrate $f(z) = 1/(z-i)(z+i)(z-2i)(z+2i)$ over the curve γ consisting of the semicircle of radius ρ in the upper half plane and the part of the real axis between $-\rho$ and ρ and then let $\rho \to \infty$. The poles inside this semicircle are $z = i$ and $z = 2i$, with residues $-i/6$ and $i/12$. So

$$\int_\gamma f(z)\,dz = 2\pi i(-\tfrac{i}{6}+\tfrac{i}{12}) = \frac{\pi}{12}.$$

This will be our answer provided we can show that the integral of $f(z)$ over the semicircle tends to 0 as $\rho \to \infty$. We need to show

$$\int_0^\pi \frac{1}{(\rho^2 e^{2i\theta}+1)(\rho^2 e^{2i\theta}+4)} \rho i e^{i\theta}\,d\theta \to 0.$$

Now the modulus of $\rho^2 e^{2i\theta}$ is ρ^2, so as long as $\rho \geq 2$, we have the modulus of $1/(\rho^2 e^{2i\theta}+1)$ is at most $2/\rho^2$. Similarly for the other factor, so the integral is bounded by

$$\int_0^\pi \frac{4}{\rho^2}\frac{4}{\rho^2}\rho\,d\theta,$$

which clearly goes to 0 as $\rho \to \infty$.

Example 3. Let us look at

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4}dx.$$

This is done very similarly to the preceding. We will compute the integral of

$$\frac{e^{iz}}{z^2+4}$$

over γ, which is made up of the semicircle of radius ρ and the part of the real axis from $-\rho$ to ρ and then let $\rho \to \infty$. We will then take the real part of what remains. The poles are $\pm 2i$, of which only $2i$ is in the domain. The residue at $z = 2i$ is $1/(4i)$, and so $\int_\gamma f(z)\,dz = e^{-2}\pi/2$. As in the preceding example, the integral over the semicircle tends to

0. So the integral over the real axis is $e^{-2}\pi/2$. Take the real parts (which doesn't change anything) and we have our answer.

We mention that if we had, say, $\int_{-\infty}^{\infty} \sin^2 x/(x^2+4)\,dx$, we could reduce it to a problem like Example 3 by writing $\sin^2 x = \frac{1}{2}(1-\cos(2x))$, and then proceeding as in Example 3.

Example 4. Let us look at

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx.$$

Here the $\sin x$ is crucial for the existence of the integral because $x/(x^2+1)$ is not integrable. We look at

$$f(z) = \frac{ze^{iz}}{z^2+1}$$

and this time we cannot use the semicircular domain. We let γ be the outside of the rectangle with sides $y=0, c$, $x=-a, b$. The pole inside this domain, provided c is big, is $z=i$, and the residue is $ie^{-1}/(2i) = e^{-1}/2$. Let us first hold a, b fixed and let $c \to \infty$. On the line $y=c$, we have $z = x+ic$ and so $e^{iz} = e^{ix}e^{-c}$ which has modulus bounded by e^{-c}. On the line $y=c$, we have $|z|/|z^2+1| \leq 2/|z| \leq 2/c$. Since $\int_{-a}^{b} 2e^{-c}/c\,dx \to 0$ as $c \to \infty$, we now need to look at the integrals on the lines $x=-a, b$. We do $x=b$, the other line being similar. On $x=b$ we have $z = b+iy$, so $e^{iz} = e^{ib}e^{-y}$, and the modulus is bounded by e^{-y}. Similarly to before, $|z|/|z^2+1| \leq 2/b$. So the integral along the line $x=b$ is bounded by $\int_0^{\infty} 2e^{-y}/b\,dy$, which tends to 0 as $b \to \infty$. We are left with the integral on the real axis being equal to $2\pi i e^{-1}/2 = \pi i/e$. If we now take the imaginary parts, we get π/e for our answer.

We need to take the rectangle with sides $x=-a, b$ where a is not necessarily equal to b. Otherwise we would be finding

$$\lim_{a\to\infty} \int_{-a}^{a} \frac{x \sin x}{x^2+1} dx,$$

which is a little different from what we wanted.

Example 5. For our last example we look at

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx.$$

It turns out that this particular integral has lots of uses. We have to be a little more clever here: $\sin x/x$ is fine near 0 (recall that the limit is 1) but $\cos x/x$ blows up near 0. On the other hand, $\cos x/x$ is an odd function, and although it is not integrable, in some sense its integral should be 0. We proceed as follows: we define (Cauchy's) principal value integral by

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \lim_{a,b\to\infty}\lim_{\delta\to 0}\Big[\int_{-a}^{-\delta} \frac{e^{ix}}{x}\,dx + \int_{\delta}^{b} \frac{e^{ix}}{x}\,dx\Big].$$

We cannot have a pole on γ for the residue theorem to work, so we let D be the union of the rectangle with sides $x = -a, b$, $y = 0, c$ and the semicircle centered at 0 in the lower half plane with radius δ, and we then let γ be the curve bounding D. The pole $z = 0$ is inside γ, and the residue is 1, so the residue theorem says that the integral over γ is $2\pi i$. As in Example 4, we can show that the contributions from $y = c$, $x = -a, b$ can be made arbitrarily small if we let a, b, c be big. In a neighborhood of 0, $e^{iz}/z = 1/z + g(z)$, where $g(z)$ is analytic and $g(0) = i$. If σ represents the boundary of the semicircle of radius δ, then

$$\int_{\sigma} g(z)\,dz = \int_{\pi}^{2\pi} g(\delta e^{i\theta}) i\delta e^{i\theta}\,d\theta,$$

which is bounded in modulus by π times the maximum of g on the semicircle. This will be small when δ is small, because $g(0) = i$, so g is bounded in a neighborhood of 0. We evaluate similarly

$$\int_{\sigma} \frac{1}{z}\,dz = \int_{\pi}^{2\pi} \frac{1}{\delta e^{i\theta}} i\delta e^{i\theta}\,d\theta == \pi i.$$

Therefore

$$\lim_{\delta\to 0}\Big[\int_{-\infty}^{-\delta} \frac{e^{ix}}{x}\,d + \int_{\delta}^{\infty} \frac{e^{ix}}{x}\,dx\Big] = 2\pi i - \pi i = \pi i.$$

Taking the imaginary parts,

$$\lim_{\delta\to 0}\Big[\int_{-\infty}^{-\delta} \frac{\sin x}{x}\,dx + \int_{\delta}^{\infty} \frac{\sin x}{x}\,dx\Big] = \pi.$$

But $\int_{-\delta}^{\delta} \sin x/x\,dx \to 0$ as $\delta \to 0$, so we conclude that the integral over the whole real line is π.

23. Basic properties of harmonic functions.

A function u is harmonic in a region Ω if it is C^2 there (this means that u and its first and second partial derivatives are continuous) and

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in Ω. This equation, called Laplace's equation, can be written in polar coordinates as

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

It follows that $u = \log r$ is harmonic and $v = \theta$ are harmonic (for v we need to specify a domain so that v is single valued). This isn't surprising: u and v are the real and imaginary parts of $f(z) = \log z$.

If u is harmonic, v is its conjugate harmonic function if $u+iv$ is analytic. In general, there is no single-valued conjugate harmonic function, but there will be in a disk. To show this, let

$$g(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

g is analytic because the Cauchy-Riemann equations are satisfied (we need the fact that u satisfies Laplaces's equation to show this.). Therefore $\int_\gamma g(z)\,dz = 0$ for every closed curve γ in the disk. Now we can write

$$g(z)\,dz = \Big(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\Big) + i\Big(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\Big).$$

The real part is the differential of u, and the integral over every closed curve of du is 0. Therefore the integral over every closed curve of the imaginary part is 0. Define $v(z)$ to be

$$\int_\gamma \Big(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\Big)$$

where γ is any curve starting from 0 and ending at z and contained in the unit disk. So v is well defined, and has partials $-\partial u/\partial y$ and $\partial u/\partial x$. We check that v satisfies Laplace's equation, and that v is the conjugate harmonic function to u.

Harmonic functions have the mean value property.

Theorem 23.1. *If u is harmonic in a disk and $|z - z_0| = r$ is contained in that disk, then*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta.$$

Proof. By a change of coordinates, we may suppose $z_0 = 0$. Let f be the analytic function which has u as its real part. By Cauchy's integral formula, we have

$$f(0) = \frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)}{z}\,dz = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(re^{i\theta})}{re^{i\theta}}ire^{i\theta}\,d\theta.$$

The result follows by taking the real parts. □

A corollary is the maximum principle for harmonic functions.

Theorem 23.2. *A nonconstant harmonic function has neither a maximum nor a minimum in its region of definition.*

Proof. If M is the maximum of the harmonic function u in a connected domain Ω and $u(z_0) = M$ for some $z_0 \in \Omega$, then for any r less than the distance from z_0 to the boundary of Ω,

$$M = u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta \leq M. \tag{23.1}$$

If u is strictly less than M anywhere on the circle $|z - z_0| = r$, then by continuity it will be strictly less than M for an interval of values of θ, and then we will have strict inequality in (23.1), a contradiction. Therefore u is identically equal to M on the boundary of the circle. This is true for any r less than the distance from z_0 to the boundary of Ω, hence u is equal to M in a neighborhood of z_0. It follows that $\{z \in \Omega : u(z) = M\}$ is an open set. Since $\{z \in \Omega : u(z) < M\}$ is open by the continuity of u and Ω is connected, then either u is identically equal to M or $\{z \in \Omega : u(z) = M\}$ is empty. The proof for the minimum follows by considering $-u$. □

24. Poisson's formula.

If u_1 and u_2 are two harmonic functions with the same boundary values on a disk, then $u_1 - u_2$ has boundary values 0 and is harmonic. By the maximum principle, $u_1 - u_2 = 0$ in the disk. So a harmonic function is determined by its values on the boundary of the disk. The Poisson formula gives an explicit representation.

Theorem 24.1. *Suppose u is harmonic in the open disk of radius R and continuous on the closed disk. Then*

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z - a|^2} u(z)\, d\theta$$

for all $|a| < R$.

Proof. Let us first suppose that u is harmonic in a neighborhood of the closed disk. Let

$$z = S\zeta = \frac{R(R\zeta + a)}{R + \bar{a}\zeta}.$$

Then S maps $|\zeta| \leq 1$ onto $|z| \leq R$ with $\zeta = 0$ mapping to $z = a$. Since S is analytic, $u(S(\zeta))$ is harmonic in $|\zeta| \leq 1$ and so

$$u(a) = u(S(0)) = \frac{1}{2\pi} \int_{|\zeta|=1} u(S(\zeta))\, d\arg\zeta.$$

We have
$$\zeta = \frac{R(z-a)}{R^2 - \overline{a}z}.$$
We also have $\arg\zeta = \operatorname{Im}\,\log\zeta = \operatorname{Re}\,(-i\log\zeta)$, while the real part of $\log\zeta$ is 0 on the circle $|\zeta| = 1$, hence
$$\begin{aligned} d\arg\zeta &= -i\frac{d\zeta}{\zeta} \\ &= -i\Big(\frac{1}{z-a} + \frac{\overline{a}}{R^2 - \overline{a}z}\Big)\, dz \\ &= \Big(\frac{z}{z-a} + \frac{\overline{a}z}{R^2 - \overline{a}z}\Big)\, d\theta, \end{aligned}$$
where $z = Re^{i\theta}$ implies $dz = iz\, d\theta$. Since $R^2 = z\overline{z}$, the last expression is
$$\frac{z}{z-a} + \frac{\overline{a}}{\overline{z}-\overline{a}} = \frac{R^2 - |a|^2}{|z-a|^2},$$
or alternatively,
$$\frac{1}{2}\Big(\frac{z+a}{z-a} + \frac{\overline{z}+\overline{a}}{\overline{z}-\overline{a}}\Big) = \operatorname{Re}\frac{z+a}{z-a}.$$
We thus have
$$u(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z)\, d\theta = \frac{1}{2\pi}\int_{|z|=R} \operatorname{Re}\frac{z+a}{z-a} u(z)\, d\theta.$$

If u is only continuous on $|\zeta| \le R$, $u(rz)$ is harmonic in a neighborhood of $|\zeta| \le R$ for $r < 1$, so we have
$$u(ra) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(rz)\, d\theta.$$
We now let r increase to 1 and use the continuity of y. □

Using $d\zeta/\zeta = i\, d\theta$, the Poisson formula can be rewritten
$$u(z) = \operatorname{Re}\Big[\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{\zeta+z}{\zeta-z} u(\zeta)\frac{d\zeta}{\zeta}\Big],$$
so u is the real part of
$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{\zeta+z}{\zeta-z} u(\zeta)\frac{d\zeta}{\zeta} + iC.$$
The conjugate harmonic function will be the imaginary part of this analytic function.

Note that when u is identically 1, we have
$$\int_{|z|=R} \frac{R^2 - |z|^2}{|z-a|^2}\, d\theta = 2\pi. \tag{24.1}$$

Write
$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\frac{e^{i\theta}+z}{e^{i\theta}-z} U(\theta)\, d\theta.$$
Here we are taking $R = 1$ and U is a piecewise continuous function on the boundary of the disk.

Theorem 24.2. *(Schwarz' theorem) If U is piecewise continuous, then $P_U(z)$ is harmonic for $|z| < 1$ and*

$$\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0)$$

provided U is continuous at θ_0.

Proof. By looking at $U - U(\theta_0)$, we may suppose $U(\theta) = 0$. Let $\varepsilon > 0$ and pick δ such that $|U(\theta) - U(\theta_0)| < \varepsilon$ if $|\theta - \theta_0| < \delta$. Let $\varphi(\theta) = 1$ if $|\theta - \theta_0| < \delta$ and 0 otherwise. We have

$$P_{(1-\varphi)U}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} ((1 - \varphi)U)(e^{i\theta})\, d\theta.$$

For z close to $e^{i\theta_0}$, $|e^{i\theta} - z| > \delta/2$ for θ such that $(1 - \varphi)(e^{i\theta}) \neq 0$, and therefore for such z

$$|P_{(1-\varphi)U}(z)| \leq \frac{1}{2\pi} (\sup_\theta |U(e^{i\theta})|)(1 - |z|^2)(2\pi) \to 0$$

as $z \to e^{i\theta_0}$. On the other hand, $|U\varphi|$ is bounded by ε on the boundary, so $P_{\varphi U}$ is bounded in absolute value by ε by the maximum principle. Then

$$\limsup_{z \to e^{i\theta_0}} |P_U(z)| \leq \limsup_{z \to e^{i\theta_0}} |P_{\varphi U}(z)| + \limsup_{z \to e^{i\theta_0}} |P_{(1-\varphi)U}(z)| \leq \varepsilon.$$

Since ε is arbitrary, $\lim_{z \to e^{i\theta_0}} P_U(z) = 0$ as required. □

25. Schwarz reflection principle.

Suppose Ω is a region that is symmetric with respect to the x axis, $\Omega^+ = \Omega \cap \{z : \operatorname{Im} z > 0\}$, $\Omega^- = \Omega \cap \{z : \operatorname{Im} z < 0\}$, and $\sigma = \Omega \cap \{z : \operatorname{Im} z = 0\}$.

Theorem 25.1. *Suppose v is continuous on $\Omega^+ \cup \sigma$, harmonic in Ω^+, and 0 on σ. Then v has a harmonic extension to Ω and $v(\overline{z}) = -v(z)$. If f is analytic on Ω^+ and its imaginary part v satisfies the conditions just given, then f has an analytic extension to Ω and $f(z) = \overline{f(\overline{z})}$.*

Proof. Let $V = v$ on Ω^+, 0 on σ, and $-v(\overline{z})$ on Ω^-. What we need to show is that V is harmonic on σ. Pick a point $x_0 \in \sigma$, consider a disk about x_0 contained in Ω, and let P_V be the Poisson integral of V in this disk. By symmetry, P_V is 0 on σ. So $v - P_V$ is harmonic in the upper half disk, 0 on the portion of the boundary of the disk that is in the upper half plane, and 0 on the portion of the disk that intersects the x axis. By the maximum priniciple, $v - P_V$ is identically 0 in the upper half disk. The same argument holds in the lower half disk, and $v - P_V$ is 0 on the intersection of the disk with the x axis. Therefore $v = P_V$ on the disk, so v is harmonic in the disk.

If $-u$ is the conjugate harmonic function for v in the disk, let us normalize so that $u = \operatorname{Re} f$. Define $U(z) = u(z) - u(\overline{z})$. Clearly U_x is 0 on the diameter of the disk, and on the diameter $U_y = 2u_y = -2v_x = 0$ since $v = 0$ on the diameter. So $U_x - iU_y$ vanishes on the diameter. Since it is analytic, it is identically 0, so U is constant. The constant must be zero, so $u(z) = u(\overline{z})$. □

26. More on harmonic functions.

A real-valued continuous function u satisfies the mean value property in a region Ω if

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

whenever $|z - z_0| \le r$ is contained in Ω.

Proposition 26.1. *A continuous function which satisfies the mean value property in a region Ω is harmonic.*

Proof. Suppose $|z-z_0| \le r$ is contained in Ω. Let v be the harmonic function (constructed using the Poisson integral) that has the same boundary values as u on $|z - z_0| = r$. Then $u-v$ also has the mean value property inside $|z-z_0| < r$. A look at the maximum principle for harmonic functions shows that what we really used was the mean value property. So $u - v$ is less than or equal to the maximum on the boundary, which is 0, and similarly for $v - u$. Therefore $u = v$, and hence u is harmonic. □

Theorem 26.2. *(Harnack's inequality) Suppose u is harmonic and nonnegative in $|z - z_0| < \rho$, Then*

$$\frac{\rho - r}{\rho + r} u(z_0) \le u(z) \le \frac{\rho + r}{\rho - r} u(z_0), \qquad |z - z_0| < r < \rho.$$

Proof. Without loss of generality, let us take $z_0 = 0$. Check that

$$\frac{\rho^2 - |z|^2}{|\rho e^{i\theta} - z|^2} \le \frac{\rho^2 - |z|^2}{(\rho - |z|)^2} = \frac{\rho + |z|}{\rho - |z|} \le \frac{\rho + r}{\rho - r},$$

and similarly

$$\frac{\rho - r}{\rho + r} \le \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2}$$

for all $\theta \in [0, 2\pi)$ and all $|z| < r$. Since $u \ge 0$, by the Poisson formula,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{|\rho e^{i\theta} - z|^2} u(\rho e^{i\theta})\, d\theta \le \frac{\rho + r}{\rho - r}\frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\theta})\, d\theta,$$

which gives the upper bound, and the lower bound is proved the same way. □

Theorem 26.3. *(Harnack's principle) Suppose we have a sequence of domains Ω_n and harmonic functions u_n defined on Ω_n. Let Ω be a region such that every point of Ω has a neighborhood contained in all but finitely many Ω_n and in this neighborhood, $u_n(z) \le u_{n+1}(z)$ for n sufficiently large. Then either u_n increases uniformly to ∞ on every compact subset of Ω or else u_n tends to a harmonic limit function in Ω uniformly on compact sets.*

Proof. Suppose $\lim u_n(z_0) = \infty$ for some z_0. Then there exists r and m such that $u_n(z)$ are harmonic and nondecreasing for $n \ge m$ and $|z - z_0| < r$. Applying the lower bound of the Harnack inequality to $u_n - u_m$, we see u_n tends to ∞ uniformly in $|z - z_0| \le r/2$. If the limit is finite, using the upper bound shows that the u_n are uniformly bounded in this ball. So the sets where the limit are finite and infinite are both open. Since Ω is connected, one of them is empty.

If the limit is finite, $u_{n+p}(z) - u_n(z) \le 3(u_{n+p}(z_0) - u_n(z_0))$ for $|z - z_0| \le r/2$ and $n + p \ge n \ge m$. So convergence at z_0 implies uniform convergence in a neighborhood of z_0. By compactness, we have uniform convergence on every compact set. The limit is harmonic by using Poisson's formula. The case where the limit is infinite is similar. □

27. Convergence of analytic functions.

Often one has analytic functions f_n converging in some sense to f and one want to assert that f is analytic. But the domains of f_n may not all be the same. For example, if

$$f_n(z) = \frac{z}{2z^n + 1},$$

then f_n is not defined on all of $|z| < 1$. But for each z with $|z| < 1$, we have $f_n(z) \to z$.

The appropiate condition turns out to be that f_n converges uniformly to f on compact subsets of Ω. To be more precise, f_n converges uniformly to f on compact subsets of Ω if for each E that is compact and a subset of Ω, f_n converges uniformly to f on E. Saying f_n converges uniformly to f on E means that given $\varepsilon > 0$, there exists n_0 depending on ε and E such that if $n \ge n_0$, then $|f_n(z) - f(z)| > \varepsilon$ for all $z \in E$. (In the above example, $f_n(z) - z = -2z^{n+1}/(2z^n + 1)$, and this can be made small on each disk $|z| \le r$ for $r < 1$.)

Theorem 27.1. *(Weierstrass' theorem) Suppose f_n is analytic in a region Ω_n and f_n converges uniformly to a function f on compact subsets of a region Ω. Then f is analytic on Ω and moreover f_n' converges uniformly to f' on compact subsets of Ω.*

Proof. We have

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\zeta)\, d\zeta}{\zeta - z},$$

where C is the circle $|\zeta - a| = r$ and $|z-a| < r$. Taking the limit, we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z},$$

which proves that f is analytic in the disk.

Similarly,

$$f_n'(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\zeta)\,d\zeta}{(\zeta - z)^2}.$$

The limit on the right is

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2},$$

which is equal to $f'(z)$. Therefore $f_n'(z)$ converges to $f'(z)$. It is straightforward to show the convergence is uniform in the disk. Any compact subset of Ω can be covered by a finite number of disks, and therefore the convergence is uniform on every compact subset. □

If we have a power series $\sum a_n z^n$, then $\sum_{n=1}^N a_n z^n$ converges uniformly on $|z| \le r$ if $r < R$ and R is the radius of convergence. We conclude that the limit is analytic and taking the derivative term by term leads to a series that converges to the derivative of the limit function.

Suppose that a sequence of analytic functions f_n converges uniformly on the boundary of a disk $|z| \le a$. Then it converges unifornly in the closed disk as well. To see that, by the maximum principle,

$$\sup_{|z|\le a} |f_n(z) - f_m(z)| \le \sup_{|z|=a} |f_n(z) - f_m(z)| \to 0.$$

Related to Weierstrass' theorem is the following

Theorem 27.2. *(Hurwitz' theorem) If the functions f_n are analytic in Ω and never zero there and they converge uniformly on compacts to f, then f is never zero on Ω or else is identically zero.*

Proof. Suppose f is not identically zero in Ω. Since the zeros of f are isolated, given z_0 there exists r such that f is nonzero in $0 < |z - z_0| < 2r$. In particular, f has a positive minimum on $|z - z_0| = r$. Then $1/f_n(z)$ converges uniformly to $1/f(z)$ on $|z - z_0| = r$. Since f_n' converges uniformly to f' on $|z - z_0| = r$, we have

$$\frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f_n'(z)}{f_n(z)}dz \to \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f'(z)}{f(z)}dz.$$

The left hand side is the number of zeros of f_n in $|z - z_0| < r$ and is 0. So the right hand side is also 0, which says that f has no zeros inside $|z - z_0| < r$. □

28. Laurent series.

Consider the series

$$\sum_{n=-\infty}^{\infty} a_n z^n.$$

The terms with $n \geq 0$ will converge inside some disk of radius R_2 and the terms with $n < 0$ will converge outside some disk of radius R_1. If $R_1 < R_2$, then the sum will be an analytic function inside the annulus $R_1 < |z| < R_2$. Such a series is called a Laurent series.

Conversely, we have

Theorem 28.1. *Suppose f is analytic in the annulus $R_1 < |z-a| < R_2$. Then we have*

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-a)^n$$

with

$$A_n = \frac{1}{2\pi i} \int_{|z-a|=r} f(\zeta)(\zeta - a)^{-n-1}\, d\zeta \tag{28.1}$$

for any r with $R_1 < r < R_2$.

Proof. For simplicity, let us assume $a = 0$. If $|z| < R_2$, choose r such that $|z| < r < R_2$ and set

$$f_1(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)\, d\zeta}{\zeta - z}.$$

We know f_1 is analytic in $|z| < r$. By Cauchy's integral theorem, the value of $f(z)$ does not depend on r. We can expand f_1 in a Taylor series as $\sum_{n=0}^{\infty} A_n z^n$, and the coefficients satisfy (28.1). If $|z| > R_1$, choose r such that $R_1 < r < |z|$ and define

$$f_2(z) = -\frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)\, d\zeta}{\zeta - z}.$$

Again, the value of $f_2(z)$ does not depend on r by Cauchy's theorem. If $R_1 < |z| < R_2$, choose $R_1 < r_1 < |z| < r_2 < R_2$. Then

$$f_1(z) + f_2(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{\zeta - z},$$

where γ is the cycle consisting of the circle $|z| = r_2$ traversed in the counterclockwise direction and the circle $|z| = r_1$ traversed in the clockwise direction. By Cauchy's integral formula, this is equal to $f(z)$. It remains to expand $f_2(z)$ in negative powers of z and to determine the coefficients. Let $w = 1/z$ and $\omega = 1/\zeta$. $f_2(w)$ is analytic inside $|w| < 1/R_1$ and can be expanded as a Taylor series $\sum B_n w^n$ with

$$B_n = \frac{1}{2\pi i} \int_{|\omega|=1/r} \frac{f(1/\omega)\, d\omega}{\omega^{n+1}} = \frac{1}{2\pi i} \int_{|\zeta|=r} f(\zeta)(\zeta)^{n-1}\, d\zeta.$$

□

29. Equicontinuity.

Sometimes we will want to know that given a sequence of functions $f_n : \Omega \to \mathbb{C}$ that there is subsequence which converges uniformly. The key idea is equicontinuity.

Given a function f, it is continuous at z_0 if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. f may be continuous at every point in a domain, and it isn't always true that we can choose δ independent of z_0. If we can, we have uniform continuity: given $\varepsilon > 0$ there exists δ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$ for all z, z_0.

We can carry this one step further: given a collection $\mathcal{F}$ of functions, each function in $\mathcal{F}$ might be continuous. If we can choose δ independent of which function f in $\mathcal{F}$, then the collection is equicontinuous. The way the definition reads is that a collection $\mathcal{F}$ is equicontinuous if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$ for all z, z_0 and all f in the collection $\mathcal{F}$.

The main result is the Ascoli-Arzelà theorem.

Theorem 29.1. *Suppose $\mathcal{F}$ is an equicontinuous collection of functions mapping a compact set E to $\mathbb{C}$. Suppose also that $\{|f(z)| : f \in \mathcal{F}\}$ is bounded for each $z \in E$. Then there exists a subsequence of functions f_n in $\mathcal{F}$ such that f_n converges uniformly on E.*

Proof. Let $\{z_i\}$ be a dense subset of E. The collection of numbers $\{|f(z_1)| : f \in \mathcal{F}\}$ is a bounded collection of complex numbers, so there exists a sequence $f_{11}, f_{12}, \ldots$ such that $f_{1n}(z_1)$ converges. Now $\{|f_{1n}(z_2)| : n = 1, 2, \ldots\}$ is a bounded collection of complex numbers, so there is a subsequence $f_{21}, f_{22}, \ldots$ of $\{f_{1n}\}$ such that $f_{2n}(z_2)$ converges. Then take a subsequence $\{f_{3n}\}$ of $\{f_{2n}\}$ such that $f_{3n}(z_3)$ converges and so on. Finally, look at $\{f_{nn}\}$. For each n, except for a finite number of terms this sequence is a subsequence of $\{f_{in}\}$, and so $f_{nn}(z_i)$ converges for each i. (This is called Cantor's diagonalization argument.)

Let $\varepsilon > 0$. There exists a δ such that if $|z - z_i| < \delta$, then $|f(z) - f(z_i)| < \varepsilon/3$ for each $f \in \mathcal{F}$. The collection of open balls $\{B(z_i, \delta/2)\}$ covers E, so there is a finite subcollection $B_1, \ldots, B_k$ that covers E; this is because E is compact. Let $w_1, \ldots, w_k$ be the centers; each w_j is some z_i. There exists n_0 such that if $n, m \geq n_0$, then $|f_{nn}(w_j) - f_{mm}(w_j)| < \varepsilon/3$ for each $j = 1, 2, \ldots, k$. Now pick $z \in E$. It is in some B_j, and if $m, n \geq n_0$ we have

$$\begin{aligned} |f_{nn}(z) - f_{mm}(z)| &\leq |f_{nn}(z) - f_{nn}(z_j)| + |f_{nn}(z_j) - f_{mm}(z_j)| \\ &\quad + |f_{mm}(z_j) - f_{mm}(z)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore the sequence f_{nn} converges uniformly. $\square$

Here is a converse.

Theorem 29.2. *Suppose $\mathcal{F}$ is a collection of continuous functions defined on a compact set with the property that every sequence in $\mathcal{F}$ has a subsequence which converges uniformly. Then $\mathcal{F}$ is equicontinuous.*

Proof. If $\mathcal{F}$ is not equicontinuous, there exists $\varepsilon > 0$, points x_n and y_n in the compact set, and functions f_n in $\mathcal{F}$ such that $|x_n - y_n| < 1/n$ and $|f_n(x_n) - f_n(y_n)| > 3\varepsilon$. Take a subsequence $\{n_j\}$ such that f_{n_j} converges uniformly, say, to f. The uniform limit of continuous functions is continuous, so f is continuous. Our set is compact, so f is uniformly continuous. Hence there exists δ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. For n_j large, $|x_{n_j} - y_{n_j}| \le 1/n_j < \delta$ and $\sup_x |f_{n_j}(x) - f(x)| < \varepsilon$. But then

$$\begin{aligned} |f_{n_j}(x_{n_j}) - f_{n_j}(y_{n_j})| &\le |f_{n_j}(x_{n_j}) - f(x_{n_j})| + |f(x_{n_j}) - f(y_{n_j})| \\ &\quad + |f(y_{n_j}) - f_{n_j}(y_{n_j})| \\ &< 3\varepsilon, \end{aligned}$$

a contradiction. $\square$

A collection of analytic functions $\mathcal{F}$ is called normal if every sequence in $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets.

Theorem 29.3. *Let Ω be a region and suppose $\mathcal{F}$ is a collection of analytic functions which is uniformly bounded on each compact subset of Ω. Then $\mathcal{F}$ is a normal family.*

Saying $\mathcal{F}$ is uniformly bounded on a compact set E means that there exists M such that $|f(z)| \le M$ for all $z \in E$ and all $f \in \mathcal{F}$.

Proof. Let E_n be the set of points in $\Omega \cap \{|z| \le n\}$ whose distance from the boundary of Ω is greater than or equal to $1/n$. Then each E_n is closed and bounded, hence compact. If we show that $\mathcal{F}$ has a convergent subsequence on each E_n, we can use Cantor's diagonalization argument to find a subsequence which converges uniformly on each E_n. Since any compact subset of Ω is contained in E_n for n sufficiently large, that will prove the theorem.

So let E_n be one of these compact sets. Let $\varepsilon > 0$. Let $z_0 \in E_n$ and let C be the boundary of a circle of radius r, where r is less than $1/n$. So $|z - z_0| < r$ is contained in Ω. By Cauchy's integral formula

$$\begin{aligned} f(z) - f(z_0) &= \frac{1}{2\pi i} \int_C \Big(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \Big) f(\zeta)\, d\zeta \\ &= \frac{z - z_0}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)(\zeta - z_0)}. \end{aligned}$$

Therefore if $|z - z_0| < r/2$ and $|f|$ is bounded by M on E_n, we have

$$|f(z) - f(z_0)| \le \frac{4M|z - z_0|}{r}.$$

So if we choose $\delta < r/2$ and $\delta < \varepsilon r/(4M)$, we have $|f(z) - f(z_0)| < \varepsilon$ for every $f \in \mathcal{F}$ and z in $|z - z_0| < r/2$. We can cover E_n by finitely many balls of radius $r/2$, so this proves equicontinuity of $\mathcal{F}$ on E_n. We complete the proof by applying Theorem 29.1. □

Traditionally, one says that $\mathcal{F}$ is a normal family if there is a subsequence which converges uniformly on compact subsets or if there is a subsequence which converges uniformly to ∞ on compact subsets of Ω. A sequence f_n converges uniformly to ∞ on a compact set E if $1/f_n$ converges uniformly to 0.

As an example, consider $\mathcal{F} = \{f : f \text{ is analytic on } \Omega, |f(z)| > 1 \text{ for all } z \in \Omega\}$. We claim $\mathcal{F}$ is a normal family. To see this, let $g(z) = 1/z$. Then $\{g \circ f : f \in \mathcal{F}\}$ is uniformly bounded by 1, so is a normal family. Therefore there exists a subsequence f_n such that $g \circ f_n$ converges uniformly on compact subsets. If the limit F is everywhere nonzero, then f_n converges uniformly on compacts to $g \circ F$, since $g \circ g$ is the identity. If F is identically 0, then f_n converges uniformly on compacts to ∞. Each f_n is complex-valued, so $g \circ f_n$ is never 0, and by Hurwitz' theorem, either F is identically 0 or never 0.

30. Partial fractions.

The goal in this section is to represent meromorphic functions in the form

$$\sum_i P_i(1/(z - b_i)) + g(z),$$

where b_i are the poles and P_i are polynomials. Such an expansion doesn't always converge, so we have to include convergence enhancing terms. The theorem, due to Mittage-Leffler, is

Theorem 30.1. *Suppose $b_i \to \infty$ and let P_i be polynomials without constant term. Then there are functions which are meromorphic in the whole plane with poles at b_i and corresponding singular parts $P_i(1/(z - b_i))$. Moreover, the most general meromorphic function of this kind can be written*

$$f(z) = \sum_i \Big[P_i\Big(\frac{1}{z - b_i}\Big) - p_i(z)\Big] + g(z),$$

where the p_i are polynomials and $g(z)$ is analytic in the whole plane.

Proof. By a change of coordinate systems we may suppose no b_i equals 0. The function $P_i(1/(z - b_i))$ is analytic for $|z| < |b_i|$. Let $p_i(z)$ be a partial sum of degree n_i of the Taylor series. Recall that the remainder term for Taylor series is $f_n(z)(z-a)^n$, where

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta-a)^n(\zeta-z)},$$

and C is a circle centered at a. If the maximum of $|P_i|$ on $|z| \le |b_i|/2$ if M_i, then

$$\left|P_i\Big(\frac{1}{z-b_i}\Big) - p_i(z)\right| \le 2M_i\Big(\frac{2|z|}{|b_i|}\Big)^{n_i+1}$$

for $|z| \le |b_i|/4$. If we choose n_i such that $2^{n_i} \ge M_i 2^i$, then the sum of the remainder terms will converge.

In any disk $|z| \le R$, provided we omit the terms with $|b_i| \le 4R$, we get uniform convergence. By Weierstrass' theorem, the remaining series represents an analytic function in $|z| \le R$. Hence the full series is meromorphic in the whole plane with the desired singular parts. □

Consider the example

$$f(z) = \frac{\pi^2}{\sin^2(\pi z)}.$$

The function $\frac{\pi}{\sin \pi z} - \frac{1}{z}$ is bounded near 0, so has a removable singularity there. The singular part at the origin of f is thus $1/z^2$ and since f is periodic, its singular part at $z = n$ is $1/(z-n)^2$. The series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \tag{30.1}$$

is convergent for $z \ne n$ (compare to $\sum 1/n^2$) and is uniformly convergent on any compact set after omission of the terms which become infinite on this set. So

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + g(z),$$

where g is analytic in the whole plane. Since both f and the series have period 1, so does g. If $z = x + iy$, then

$$\sin \pi z = \frac{1}{2i}(e^{i\pi x - \pi y} - e^{-i\pi x + \pi y}),$$

which tends to ∞ as $y \to \infty$ or $y \to -\infty$. The convergence of the series (30.1) is uniform for $|y| \ge 1$ and by taking the limit term by term, we see that (30.1) tends to 0 as $|y| \to \infty$. This implies that $|g(z)|$ is bounded in the strip $0 \le x \le 1$, and by periodicity, $|g(z)|$ is bounded in the whole plane. So g is constant. Since the limit as $|y| \to \infty$ is 0, the constant must be 0, or g is identically zero.

Here is another example. Let

$$g(z) = \frac{1}{z} + \sum_{n\neq 0}\Big(\frac{1}{z-n} + \frac{1}{n}\Big).$$

We introduce the $1/n$ to get convergence: the n^{th} term is $z/(n(z-n))$, and we compare to $\sum 1/n^2$. The convergence is uniform on every compact set if we omit the terms which become infinite, so we can differentiate term by term and obtain

$$g'(z) = -\sum_n \frac{1}{(z-n)^2} = -\frac{\pi^2}{\sin^2 \pi z}.$$

Therefore $g(z) = \pi \cot \pi z + c$. If the terms corresponding to n and $-n$ are bracketed together, we have

$$\pi \cot \pi z = c + \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{1}{z-n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}.$$

The left hand side is odd, and the right hand side is equal to c plus an odd function, hence $c = 0$.

Consider

$$\lim_{m\to\infty} \sum_{-m}^{m} \frac{(-1)^n}{z-n} = \frac{1}{z} + \sum_{1}^{\infty} (-1)^n \frac{2z}{z^2-n^2}.$$

By splitting into terms with even n and with odd n, we can write this as the limit of

$$\sum_{n=-k}^{k} \frac{1}{z-2n} - \sum_{n=-k-1}^{k} \frac{1}{z-1-2n},$$

which is

$$\frac{\pi}{2}\cot\frac{\pi z}{2} - \frac{\pi}{2}\cot\frac{\pi(z-1)}{2} = \frac{\pi}{\sin \pi z}.$$

31. Infinite products.

The infinite product $p_1 p_2 \cdots p_n \cdots = \prod_{n=1}^{\infty}$ is evaluated by looking at the limit $\lim_{N\to\infty} p_1 \cdots p_N$. The product is said to converge if all but a finite number of the the p_n's are nonzero and the partial products formed from the nonvanishing factors tends to a nonzero limit. If the product converges,

$$p_n = \prod_{i=n_0}^{n} p_i / \prod_{i=n_0}^{n-1} p_i \to 1.$$

So we write instead $p_i = 1 + a_i$.

Look at the sum

$$\sum_{n=1}^{\infty} \log(1+a_n). \tag{31.1}$$

We use the principal branch for the logarithm. i

Theorem 31.1. *The partial sums of (31.1) converge if and only if the infinite product* $\prod_{n=1}^{\infty}(1+a_n)$ *converges.*

Proof. If the series (31.1) converge to S, then the partial products are given by

$$\prod_{n=1}^{N}(1+a_n) = \exp\Big(\sum_{n=1}^{N}\log(1+a_n)\Big)$$

and they converge to e^S, which is nonzero. Therefore, if (31.1) does not converge, neither does the infinite product.

Suppose $P_N = \prod_{n=1}^{N}(1+a_n)$ converges to P. We choose the principal branch of $\log P$ and we determine $\arg P_n$ by the condition $\arg P - \pi < \arg P_n \le \arg P + \pi$. So $\log P_n = \log|P_n| + i\arg P_n$, and $S_n = \log P_n + h_n 2\pi i$, where h_n is well defined. For two consecutive terms,

$$(h_{n+1}-h_n)2\pi i = \log(1+a_{n+1}) + \log P_n - \log P_{n+1}.$$

For n sufficiently large, $|\arg(1+a_{n+1})| < 2\pi/3$, $|\arg P_n - \arg P| < 2\pi/3$, and $|\arg P_{n+1} - \arg P| < 2\pi/3$. By looking at the imaginary parts, we conclude $|h_{n+1}-h_n| < 1$ for n large, and hence h_n is constant from some point on. Therefore $S_n \to \log P + h2\pi i$. □

We say an infinite product converges absolutely if the series (31.1) converges absolutely.

Theorem 31.2. *The infinite product* $\prod_n(1+a_n)$ *converges absolutely if and only if the infinite sum* $\sum |a_n|$ *converges.*

Proof. If either $\sum |a_n|$ or (31.1) converges, then $a_n \to 0$. For $|a_n|$ sufficiently small we have

$$\tfrac{1}{2}|a_n| \le |\log(1+a_n)| \le 2|a_n|,$$

and the result follows. □

32. Canonical products.

An entire function is one that is analytic in the whole plane. If g is entire, then $f = e^g$ is entire, and is never zero. Conversely, if f is entire and never zero, then f'/f is entire and will be the derivative of an entire function g. Then fe^{-g} has derivative zero, hence is constant, so $f = ce^g$. Absorbing the constant into g, we can write $f = e^g$.

If f is entire and has zeros as 0 and at $a_1, \ldots, a_n$, then we can similarly write

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{N}\Big(1 - \frac{z}{a_n}\Big).$$

We would like a similar expression, but the corresponding infinite product may not converge. Instead we have the following:

Theorem 32.1. *Suppose either there is a finite collection $a_1, \dots, a_N$ or a sequence $a_n \to \infty$. There exists an entire function with these and no other zeros. Every entire function with these zeros can be written*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \Big(1 - \frac{z}{a_n}\Big) e^{(z/a_n)+(z/a_n)^2/2+\cdots+(z/a_n)^{m_n}/m_n},$$

where the product is taken over all $a_n \neq 0$, the m_n are certain integers, and g is an entire function.

Proof. The difficulty is only in the convergence of the infinite product. Once we have that, if f is any function with the given zeros, let F be an entire function of the desired form with exactly the same zeros. Then the zeros of f and F will cancel when we form the ratio f/F; this function is analytic except at the zeros, where it has removable singularities. Morever, f/F is never 0. So we can write $f/F = e^G$ for an entire function G, and consequently $f = Fe^G$, which shows that f has the desired form.

Let $s_n(z)$ be the n^{th} factor in the infinite product. We fix $R > 0$ and show that $\prod_{n=1}^{\infty} s_n(z)$ converges for $|z| < R$. In forming the product, we need only consider those terms with $|a_n| > 4R$. If we can show $\sum_{\{n:|a_n|>4R\}} \log s_n(z)$ converges absolutely and uniformly on $|z| < R$, then the sum will be an analytic function. Since the infinite product will be the exponential of this, the product will be analytic as well.

Expanding $\log(1 - z/a_n)$ in a Taylor series, we see that $\log s_n(z)$ is the remainder after n terms. So

$$\log s_n(z) = \frac{1}{m_n+1}\Big(\frac{z}{a_n}\Big)^{m_n+1} + \frac{1}{m_n+2}\Big(\frac{z}{a_n}\Big)^{m_n+2} + \cdots,$$

and

$$|\log s_n(z)| \leq \frac{1}{m_n+1}\Big(\frac{R}{|a_n|}\Big)^{m_n+1}\Big(1 - \frac{R}{|a_n|}\Big)^{-1}.$$

If the series

$$\sum_{n=1}^{\infty} \frac{1}{m_n+1}\Big(\frac{R}{|a_n|}\Big)^{m_n+1} \tag{32.1}$$

converges, then $\log s_n(z) \to 0$, and hence $\operatorname{Im} \log s_n(z) \in (-\pi, \pi]$ for n large. Then $\sum \log s_n(z)$ will be absolutely and uniformly convergent for $|z| < R$. (The uniform convergence will not be affected by those terms where $|a_n| \leq R$.) If we choose $m_n = n$, the series (32.1) converges, using, for example, the root test. □

Corollary 32.2. *Every function which is meromorphic in the plane is the ratio of two entire functions.*

Proof. If F is meromorphic, we can find an entire function g with the poles of F as zeros. So $f = Fg$ is entire, and $F = f/g$. □

If we can find a representation of the form in Theorem 32.1 with all the m_n equal to an integer h, then the product is called a canonical product and h is the genus of the product. What one needs is that $\sum (R/|a_n|)^{h+1}/(h+1)$ converges for all R, which happens when $\sum |a_n|^{-(h+1)} < \infty$.

As an example, let us find the product representation of $\sin \pi z$. $\sum 1/n$ diverges and $\sum 1/n^2$ converges, so we take $h = 1$, and look for a representation of the form

$$z e^{g(z)} \prod_{n \neq 0} \Big(1 - \frac{z}{n}\Big) e^{z/n}.$$

We take the logarithmic derivative, and obtain

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n \neq 0} \Big(\frac{1}{z-n} + \frac{1}{n}\Big).$$

This is justified because we have uniform convergence on any compact set not containing the points $\ldots, -n, \ldots, -1, 0, 1, \ldots, n, \ldots$. If we compare this with the partial fraction expansion for $\pi \cot \pi z$, $g' = 0$, or g is a constant. Since $\lim_{z \to 0} \sin \pi z / z = \pi$, then $e^{g(z)} = \pi$, and we obtain

$$\sin \pi z = \pi z \prod_{n \neq 0} \Big(1 - \frac{z}{n}\Big) e^{z/n}.$$

If we combine the n and $-n$ terms together, we have

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big).$$

33. Jensen's formula.

Theorem 33.1. *Suppose f is analytic on $|z| \le \rho$ and has no zero at 0, no zeros on $|z| = \rho$, and zeros at $a_1, \ldots, a_n$. Then*

$$\log |f(0)| = -\sum_{i=1}^{n} \log(\rho/|a_i|) + \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \, d\theta.$$

Proof. First suppose f has no zeros at all. $\log |f(z)|$ is harmonic except at the zeros of f so

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \, d\theta \tag{33.1}$$

by the mean value property for harmonic functions.

Now suppose f has zeros at $a_1, \ldots, a_n$. The function

$$F(z) = f(z) \prod_{i=1}^{n} \frac{\rho^2 - \overline{a}_i z}{\rho(z - a_i)}$$

has no zeros in the disk and $|F(z)| = |f(z)|$ on $|z| = \rho$. Applying (33.1) to F,

$$\log|F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(\rho e^{i\theta})|\, d\theta.$$

Our result follows by substituting for the value of $F(0)$. □

It turns out Jensen's formula is still valid is f has zeros on $|z| = \rho$ and there is a variation in the case f has a zero at 0. The formula relates the modulus of f on the boundary to the modulus of the zeros of f.

34. The Riemann mapping theorem.

The Riemann mapping theorem says that every simply connected region other than $\mathbb{C}$ itself is conformally equivalent to the unit disk.

Theorem 34.1. *Let Ω be simply connected and not equal to $\mathbb{C}$, and let $z_0 \in \Omega$. There exists a unique analytic function f on Ω such that $f(z_0) = 0$, $f'(z_0) > 0$, and f is a one-to-one mapping of Ω onto $|w| < 1$.*

A one-to-one function is called univalent.

Proof. First we look at uniqueness. If f_1, f_2 are two such functions, then $f_1 \circ f_2^{-1}$ maps the unit disk one-to-one onto itself and $f_1 \circ f_2^{-1}(0) = 0$. So $f_1 \circ f_2^{-1}$ is a linear fractional transformation. Since this function maps the unit disk one-to-one onto itself, it is equal to $e^{i\theta} z$ for some θ. Since the derivative is real and positive, then $\theta = 0$ and the function is the identity.

Now we begin the proof of existence. We first reduce to the case when Ω is bounded as follows. Suppose $a \notin \Omega$. Without loss of generality we may suppose $a = 0$. Let $h(z) = \sqrt{z}$; we can define a single-valued function since Ω is simply connected. Let $\Omega' = h(\Omega)$. Suppose $z_1 \in \Omega'$; then there exists $\varepsilon > 0$ such that $|z - z_1| < \varepsilon$ is contained in Ω'. We claim $|z + z_1| < \varepsilon$, the ball of radius ε about $-z_1$, is disjoint from Ω'. To see this, if w is a point in Ω' with $|w + z_1| < \varepsilon$, then $-w$ is in the ball of radius ε about z_1 and hence is also in Ω'. But then $(-w)^2 = w^2 \in \Omega$, a contradiction to h being one-to-one.

Now invert Ω' through the circle $|z + z_1| = \varepsilon$ to get Ω''. We have Ω'' is bounded. By a translation and dilation, we may suppose that Ω'' is contained in the unit disk. Now drop the primes.

Our strategy is the following. Let $\mathcal{F}$ be the collection of univalent analytic functions g on Ω that satisfy $|g(z)| \le 1$ on Ω, $g(z_0) = 0$, and $g'(z_0) > 0$. The function we want is the element of $\mathcal{F}$ that maximizes $f'(z_0)$.

We show that $\mathcal{F}$ is not empty. To see this, let g be a linear fractional transformation mapping the unit disk into itself with $g(z_0) = 0$. We can multiply g by a rotation to get $g'(z_0) > 0$. Such a g restricted to Ω will be in $\mathcal{F}$.

Next we define f. Let $B = \sup_{g \in \mathcal{F}} g'(z_0)$. B is finite by Cauchy's integral formula for $g'(z_0)$ together with the fact that g is bounded. Take a sequence $g_n \in \mathcal{F}$ such that $g_n'(z_0) \uparrow B$. The g_n's are bounded, so there exists a subsequence g_{n_j} which converges on compacts. Call the limit f and observe that $|f(z)| \le 1$ on Ω, $f(z_0) = 0$, and $f'(z_0) = B$.

We show f is one-to-one. f is not constant since $f'(z_0) = B > 0$. Choose $z_1 \in \Omega$. If $g \in \mathcal{F}$, then $g_1(z) = g(z) - g(z_1)$ is never 0 in $\Omega - \{z_1\}$ since functions in $\mathcal{F}$ are one-to-one. $f(z) - f(z_1)$ is the limit of such g_1's. By Hurwitz' theorem, $f(z) - f(z_1)$ is either identically zero or never zero. It is not constant, so it is never 0, or $f(z) \ne f(z_1)$ if $z \ne z_1$.

The hardest part is showing f is onto. The idea is this: suppose $f(z) \ne w_0$ for some w_0 in the unit disk. Do a linear fractional transformation to send w_0 to 0. Then $\sqrt{f}$ will have derivative at z_0 greater than the derivative of f.

So suppose $f(z) \ne w_0$ for some w_0 in the unit disk. We are in a simply connected domain; if φ is never 0, we can define $\log \varphi = \int \varphi'/\varphi$ and $\sqrt{\varphi} = \exp(\frac{1}{2} \log \varphi)$. So we can define

$$F(z) = \sqrt{\frac{f(z) - w_0}{1 - \overline{w}_0 f(z)}}$$

as a single valued function. F is one-to-one and bounded in modulus by 1. Let

$$G(z) = \frac{|F'(z_0)|}{F'(z_0)} \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)} F(z)}.$$

Note $G(z_0) = 0$ and G is bounded in modulus by 1. A calculation shows

$$\begin{aligned} G'(z_0) &= \frac{|F'(z_0)|}{F'(z_0)} \frac{(1 - \overline{F(z_0)} F(z_0)) F'(z_0) - (F(z_0) - F(z_0))(-\overline{F(z_0)} F'(z_0))}{(1 - \overline{F(z_0)} F(z_0))^2} \\ &= \frac{|F'(z_0)|}{F'(z_0)} \frac{F'(z_0)}{1 - |F(z_0)|^2} \\ &= \frac{|F'(z_0)|}{1 - |F(z_0)|^2}. \end{aligned}$$

Also,

$$F'(z) = \frac{1}{2F(z)} \frac{f'(z)(1 - \overline{w}_0 f(z)) - (f(z) - w_0)(-\overline{w}_0 f'(z))}{(1 - \overline{w}_0 f(z))^2}.$$

Observe $F(z_0) = \sqrt{-w_0}$ and

$$F'(z_0) = \frac{B - |w_0|^2 B}{2\sqrt{-w_0}}.$$

So

$$G'(z_0) = \frac{B(1 - |w_0|^2)}{2\sqrt{|w_0|}(1 - |w_0|)} = \frac{1 + |w_0|}{2\sqrt{|w_0|}} B. \tag{34.1}$$

Note that if $w_0 = \rho e^{i\theta}$, then

$$|\sqrt{-w_0}| = |\sqrt{\rho} i e^{i\theta/2}| = \sqrt{\rho} = \sqrt{|w_0|}.$$

Also, if $r < 1$, $0 < (1-r)^2 = 1 - 2r + r^2$, so $1 + r^2 > 2r$, or $(1+r^2)/(2r) > 1$. We apply this with $r = \sqrt{|w_0|}$ and see from (34.1) that $G'(z_0)$ is strictly larger than B, a contradiction. Therefore f must be onto. □

When can f be extended to be a map from the closure of Ω to the closed unit disk? It turns out that it can when the boundary of Ω is a Jordan curve. However, it is not always possible to extend f: the slit disk and Littlewood's crocodile are tow examples.

35. Picard's theorem.

If g is an entire function, then so is $f = e^g$ and f is never 0. But it turns out that is the only value f does not take. The same is true of any nonconstant entire function: it can omit at most one point. That is Picard's theorem.

We start with

Lemma 35.1. *Suppose f is analytic on $|z| < 1$, $f(0) = 0$, $f'(0) = 1$, and $|f(z)|$ is bounded by M for $|z| < 1$. Then $M \geq 1$ and the image of the unit disk under f contains $|z| < 1/(6M)$.*

Proof. We write

$$f(z) = z + a_2 z^2 + \cdots.$$

Let $0 < r < 1$. We know

$$|a_n| = n!|f^{(n)}(0)| \leq M/r^n.$$

Letting $r \uparrow 1$, we obtain $|a_n| \leq M$. In particular, $M \geq |a_1| = 1$.

Suppose $|z| = 1/(4M)$. Then

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n z^n| \geq \frac{1}{4M} - \sum_{n=2}^{\infty} M \frac{1}{(4M)^n} \\ &= \frac{1}{4M} - \frac{1/(16M)}{1 - 1/(4M)} \geq \frac{1}{6M}. \end{aligned}$$

Now suppose $|w| < 1/(6M)$. Let $g(z) = f(z) - w$ and let C be the circle $|z| = 1/(4M)$. On C,

$$|f(z) - g(z)| = |w| < \frac{1}{6M} \le |f(z)|.$$

By Rouché's theorem, f and g have the same number of zeros inside $|z| < 1/(4M)$. $f(0) = 0$, so g has at least one zero, which proves that f takes the value w. □

Lemma 35.2. *Suppose g is analytic on $|z| < R$, $g(0) = 0$, $g'(0) = \mu > 0$ and $|g|$ is bounded by M on this disk. Then the image of $|z| < R$ under g contains the disk about zero of radius $R^2\mu^2/(6M)$.*

Proof. Apply Lemma 35.1 to $f(z) = g(Rz)/(Rg'(0))$. □

Lemma 35.3. *If f is analytic on $|z - a| < r$ and*

$$|f'(z) - f'(a)| < |f'(a)|, \qquad |z - a| < r,$$

then f is one-to-one.

Proof. Let z_1, z_2 be distinct points in $|z - a| < r$ and let γ be the line segment connecting them. Then

$$\begin{aligned} |f(z_1) - f(z_2)| = \Big|\int_\gamma f'(z)\,dz\Big| &\ge \Big|\int_\gamma f'(a)\,dz\Big| - \Big|\int_\gamma [f'(z) - f'(a)]\,dz\Big| \\ &\ge |f'(a)|\,|z_2 - z_1| - \int_\gamma |f'(z) - f'(a)|\,|dz| > 0. \end{aligned}$$

Therefore $f(z_2) \ne f(z_1)$. □

We now prove Bloch's theorem. This is important in itself, as well as a key step in proving Picard's theorem.

Theorem 35.4. *Suppose f is analytic on $|z| \le 1$, $f(0) = 0$, and $f'(0) = 1$. Then there exists a disk S contained in $|z| < 1$ on which f is one-to-one and $f(S)$ contains a disk of radius $1/72$.*

Proof. First we show how to define S. Let

$$K(r) = \sup\{|f'(z)| : |z| = r\}, \qquad h(r) = (1 - r)K(r).$$

h is continuous on $[0, 1]$, $h(0) = 1$, and $h(1) = 0$. Let $r_0 = \sup\{r : h(r) = 1\}$. So $h(r_0) = 1$, $r_0 < 1$, and $h(r) < 1$ if $r > r_0$. Let a be such that $|a| = r_0$ and $|f'(a)| = K(r_0)$. So $|f'(a)| = 1/(1 - r_0)$. Let $\rho_0 = \frac{1}{2}(1 - r_0)$ and let S be the disk $|z - a| < \rho_0/3$.

Next we show f is one-to-one on S. The idea is that we picked a such that $(1-|a|)|f'(a)|=1$, and by bounds on $f'(z)$ we can get $f'(z)-f'(a)$ small if $|z-a|$ is small. Here are the details. If $|z-a|<\rho_0$, then $|z-a|<\frac{1}{2}(1-r_0)$, or $|z|<\frac{1}{2}(1-r_0)+r_0=\frac{1}{2}(1+r_0)$. then

$$\begin{aligned}|f'(z)|&<K(\tfrac{1}{2}(1+r_0))=h(\tfrac{1}{2}(1+r_0))\frac{1}{1-\frac{1}{2}(1+r_0)}\\&<\frac{1}{1-\frac{1}{2}(1+r_0)}=\frac{1}{\rho_0}.\end{aligned}$$

We use here that $\frac{1}{2}(1+r_0)>r_0$, which says that h evaluated at this value is strictly less than one. Consequently, if $|z-a|<\rho_0$ we have

$$|f'(z)-f'(a)|\le|f'(z)|+|f'(a)|<\frac{3}{2\rho_0}.$$

We now apply Schwarz' lemma to $(f'(z)-f'(a))/(\frac{3}{2\rho_0^2})$. This is 0 at a and bounded by ρ_0 if $|z-a|=\rho_0$. Therefore

$$\Big|\frac{f'(z)-f'(a)}{\frac{3}{2\rho_0^2}}\Big|\le|z-a|,$$

or

$$|f'(z)-f'(a)|\le\frac{|z-a|}{2\rho_0^2}.$$

So if z is in S, $|f'(z)-f'(a)|<1/(2\rho_0)=|f'(a)|$. Applying Lemma 35.3 shows f is one-to-one on S.

Finally we show that the image of S under f contains a disk of radius 1/72. For $|z|<\rho_0/3$, let $g(z)=f(z+a)-f(a)$. Then $g(0)=0$ and $|g'(0)|=|f'(a)|=1/(2\rho_0)$. For such z, the line segment γ connecting a to $z+a$ is contained in S, which in turn is contained in $|z-a|<\rho_0$, so

$$|g(z)|=\Big|\int_\gamma f'(w)\,dw\Big|\le\frac{1}{\rho_0}|z|<\frac{1}{3}.$$

We now apply Lemma 35.2: the image of $|z|<\rho_0/3$ under g contains the disk of radius s, where

$$s=\frac{(\rho_0/3)^2(1/(2\rho_0)^2}{6(\frac{1}{3})}=\frac{1}{72}.$$

□

If f is one-to-one on the unit disk, $f(0)=0$, and $|f'(0)|=1$, then Koebe's $\frac{1}{4}$ theorem, also called the distortion theorem, says that the image of the unit disk under f contains the ball of radius 1/4 centered at the origin, and

We now prove the (little) Picard theorem.

Theorem 35.5. *If f is entire and omits 2 values, then it is constant.*

Proof. If $f(z) \neq a, b$, then $(f(z)-a)/(b-a)$ omits 0 and 1. So without loss of generality, we may assume the two values it omits are 0 and 1.

Next we claim there exists g analytic such that

$$f(z) = -\exp(i\pi \cosh(2g(z))).$$

To see this, since f never vanishes, by Corollary 18.2, $\log f$ can be defined. Let $F(z) = \log f(z)/(2\pi i)$. If $F(a) = n$ for some integer n, then $f(n) = e^{2\pi i n} = 1$, a contradiction. So F is never equal to an integer. Since $F \neq 0, 1$, we can define $\sqrt{F(z)}$ and $\sqrt{F(z)-1}$. Let

$$H(z) = \sqrt{F(z)} - \sqrt{F(z)-1}.$$

H is never 0, so let $g = \log H$. We calculate

$$\begin{aligned} H + \frac{1}{H} &= \frac{H^2+1}{H} = \frac{F + (F-1) - 2\sqrt{F}\sqrt{F-1} + 1}{\sqrt{F}\sqrt{F-1}} \\ &= \frac{F + (F-1) - 2\sqrt{F}\sqrt{F-1} + 1}{\sqrt{F}\sqrt{F-1}} \frac{\sqrt{F} + \sqrt{F-1}}{\sqrt{F} + \sqrt{F-1}} = 2\sqrt{F}. \end{aligned}$$

Then

$$\begin{aligned} \cosh(2g) + 1 &= \tfrac{1}{2}(e^{2g} + e^{-2g}) + 1 = \tfrac{1}{2}(e^{g} + e^{-g})^2 \\ &= \tfrac{1}{2}(H + \tfrac{1}{H})^2 = 2F = \frac{1}{\pi i} \log f. \end{aligned}$$

So

$$f = e^{\log f} = e^{\pi i + \pi i \cosh 2g} = -e^{\pi i \cosh(2g)}.$$

Third, we show g never takes the values

$$\{\pm \log(\sqrt{n} + \sqrt{n-1}) + \tfrac{1}{2} i m \pi : n \geq 1, m = 0, \pm 1, \ldots\}.$$

We will do the case with the "+"; the case with "−" is similar. If $g(a) = +\log(\sqrt{n} + \sqrt{n-1} + \frac{1}{2} i m\pi$, then

$$\begin{aligned} 2\cosh(2g(a)) &= e^{2g(a)} + e^{-2g(a)} \\ &= e^{im\pi}(\sqrt{n} + \sqrt{n-1})^2 + e^{-im\pi}(\sqrt{n} + \sqrt{n-1})^{-2} \\ &= (-1)^m[(\sqrt{n} + \sqrt{n-1})^2 + (\sqrt{n} - \sqrt{n-1})^2] \\ &= (-1)^m 2[n + n - 1] = 2(-1)^m(2n-1). \end{aligned}$$

So

$$\cosh(2g(a)) = (-1)^m(2n-1).$$

But $2n-1$ is odd, so $e^{(-1)^m(2n-1)\pi i} = -1$, or $f(a) = 1$, a contradiction.

We now finish the proof. Which points are omitted by g? For a given n they differ by $\pi i/2$, which is bounded in modulus by 3. For a given m they differ by

$$\log(\sqrt{n+1}+\sqrt{n}) - \log(\sqrt{n}+\sqrt{n-1}) = \log\Big(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}+\sqrt{n-1}}\Big).$$

This is largest when $n = 1$, and the value is

$$\log(1+\sqrt{2}) < \log e = 1.$$

Therefore the omitted points form the vertices of rectangles, and the diagonal of any rectangle is at most $\sqrt{10}$. But by Bloch's theorem, the image of $|z-a| < R$ under g contains a disk of radius $R|g'(a)|/72$. Pick a such that $g'(a) \neq 0$ and then pick R large so that $R|g'(a)|/72 > 4$. This gives us a contradiction. $\square$