Functional analysis

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Chapter 1

Linear spaces

Functional analysis can best be characterized as infinite dimensional linear algebra. We will use some real analysis, complex analysis, and algebra, but functional analysis is not really an extension of any one of these.

1.1 Definitions

We start with a field F, which for us will always be the reals or the complex numbers. Elements of F will be called scalars.

A *linear space* is a set X together with two operations, addition (denoted "$x + y$") mapping $X \times X$ into X and scalar multiplication (denoted "ax") mapping $F \times X$ into X, having the following properties.
(1) $x + y = y + x$ for all $x, y \in X$;
(2) $x + (y + z) = (x + y) + z$ for all $x, y, z \in X$;
(3) there is an element of X denoted 0 such that $x + 0 = 0 + x = x$ for all $x \in X$;
(4) for each $x \in X$ there is an element $-x$ in X such that $x + (-x) = 0$;
(5) $a(bx) = (ab)X$ whenever $a, b \in F$ and $x \in F$;
(6) $a(x+y) = ax+ay$ and $(a+b)x = ax+bx$ whenever $x, y \in X$ and $a, b \in F$;
(7) $1x = x$ for all $x \in X$ where 1 is the identity for F.

A *vector space* is the same thing as a linear space.

Under the operation of addition we see that (1)–(4) says that X is an

Abelian group.

We use $x - y$ for $x + (-y)$.

By the same proofs as in the finite dimensional case, we have the following.

Lemma 1.1 *(1)* $0x = 0$ *and*
(2) $(-1)x = -x$.

Proof. First write $0x = (0+0)x = 0x + 0x$ and subtract $0x$ from both sides to get (1). Then write

$$0 = 0x = (1)x + (-1)x = x + (-1)x$$

and subtract x from both sides to get (2). □

We give a number of examples of linear spaces. We leave to the reader the verification that these satisfy the definition of linear spaces.

Example 1.2 Let $X = \mathbb{R}^n$ be the set of n-tuples of real numbers. This is a linear space over the reals. We have

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

and

$$a(x_1, \ldots, x_n) = (ax_1, \ldots, ax_n).$$

Example 1.3 Let $X = \mathbb{C}^n$ be the set of n-tuples of complex numbers. This is a linear space over the complex numbers. We define addition and scalar multiplication as in Example 1.2.

Example 1.4 Let X be the collection of all infinite sequences $(x_1, x_2, \ldots)$ of real numbers with addition being coordinate-wise and scalar multiplication also being coordinate-wise, that is,

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots)$$

and similarly for scalar multiplication.

Example 1.5 Let S be a set and let X be the collection of real-valued bounded functions on S. We define $f + g$ by

$$(f + g)(s) = f(s) + g(s) \tag{1.1}$$

for each $s \in S$ and af by

$$(af)(s) = af(s) \tag{1.2}$$

for each $s \in S$. A closely related example is to let X be the collection of complex-valued bounded functions on S.

Example 1.6 Let S be a topological space (so that the notion of continuous functions from S to $\mathbb{R}$ or to $\mathbb{C}$ makes sense) and let $X = C(S)$, the collection of real-valued continuous functions on S. with addition and scalar multiplication being defined by (1.1) and (1.2).

Example 1.7 Let $X = C^k(\mathbb{R})$, the set of k times continuously differentiable functions on $\mathbb{R}$, where addition and scalar multiplication being defined by (1.1) and (1.2).

Example 1.8 Let μ be a σ-finite measure and let $X = L^p(X, \mu)$, the set of functions f such that $|f|^p$ is integrable with respect to the measure μ. Addition and scalar multiplication are again given by (1.1) and (1.2).

Example 1.9 We can let X be the set of complex-valued functions that are analytic on the unit disk.

Example 1.10 Let X be the set of finite signed measures on a measurable space.

If X is a linear space and $Y \subset X$, then we say Y is a *linear subspace* of X if $ay \in Y$ and $x + y \in Y$ whenever $x, y \in Y$ and $a \in F$. This definition is the obvious generalization of the one given in linear algebra courses.

Let Y be a subset of X, not necessarily a linear subspace. Consider the collection

$$\{Z_\alpha : Z_\alpha \text{ is a linear subspace of } X, S \subset Z_\alpha\}.$$

It is easy to check that $\cap_\alpha Z_\alpha$ is a subspace of X, and it is called the *linear span* of S.

Proposition 1.11 *The linear span of S is equal to*

$$W = \Big\{ \sum_{i=1}^{n} a_i x_i : a_i \in F, x_i \in S, n \geq 1 \Big\}.$$

Proof. W is clearly a linear subspace of X containing Y, therefore the span of Y is contained in W. If Z_α is any linear subspace containing Y, then Z_α must contain W, therefore $\cap_\alpha Z_\alpha$ contains W. $\square$

1.2 Normed linear spaces

A *norm* is a map from $X \to \mathbb{R}$, denoted $\|x\|$, such that
(1) $\|0\| = 0$;
(2) $\|x\| > 0$ if $x \neq 0$;
(3) $\|x + y\| \leq \|x\| + \|y\|$ whenever $x, y \in X$; and
(4) $\|ax\| = |a| \, \|x\|$ whenever $x \in X$ and $a \in F$.
A linear space together with its norm is called a *normed linear space.*

If we define $d(x, y) = \|x - y\|$, then d is easily seen to be a metric, and we can use all the terminology of topology. Here are a few terms we will need right away. We define the open ball of radius r about x by

$$B(x, r) = \{y \in X : \|y - x\| < r\}.$$

The topology generated by the metric d is the smallest collection of subsets of X that contains all the open balls, has the property that the intersection of two elements in the topology is again in the topology, and has the property that the arbitrary union of elements of the topology is again in the topology. We write $x_n \to x$ and say x_n converges to x if $\|x_n - x\| \to 0$. A subset Y of X is closed if $y \in Y$ whenever $y_n \in Y$ for $n = 1, 2 \ldots$ and $y_n \to y$. A sequence $\{y_n\}$ of elements of X is a *Cauchy sequence* if given $\varepsilon > 0$ there exists N such that $d(y_n, y_m) < \varepsilon$ whenever $n, m \geq N$. A metric space$X$ is complete if every Cauchy sequence converges to a point in X. The space X is separable if there exists a countable subset of X that is *dense* in X, that is, such that the smallest closed set containing this countable subset is X itself.

Two norms $\|x\|_1$ and $\|x\|_2$ are equivalent if there exist constants c_1 and c_2 such that

$$c_1\|x\|_1 \le \|x\|_2 \le c_2\|x\|_1, \qquad x \in X.$$

Equivalent norms give rise to the same topology.

A subspace of a normed linear space is again a normed linear space.

For many purposes it is important to know whether a subspace is closed or not, closed meaning that the subspace is closed in the topological sense given above. Here is an example of a subspace that is not closed. Let $X = \ell^2$, the set of all sequences $\{x = (x_1, x_2, \ldots)\}$ with $\|x\| = (\sum_{j=1}^\infty |x_j|^2)^{1/2} < \infty$. Let Y be the collection of points in X such that all but finitely many coordinates are zero. Clearly Y is a linear subspace. Let $y_1 = (1, 0, \ldots)$, $y_2 = (1, \frac{1}{2}, 0, \ldots)$, $y_3 = (1, \frac{1}{2}, \frac{1}{4}, 0, \ldots)$ and so on. Each $y_k \in Y$. But it is easy to see that $|y_k - y| \to 0$ as $k \to \infty$, where $y = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$ and $y \notin Y$. Thus Y is not closed.

For another example, let $X = C(\mathbb{R})$ and $Y = C^1(\mathbb{R})$, and define $\|f\| = \sup_{r\in\mathbb{R}} |f(r)|$, the supremum norm. Clearly Y is a subspace of X, but we can find a sequence of continuously differentiable functions converging in the supremum norm to a function that is continuous but not everywhere differentiable.

1.3 Examples

We give some examples of normed linear spaces. A Banach space is a normed linear space that is complete.

Example 1.12 Let X be the collection of infinite sequences $x = \{a_1, a_2, \ldots\}$ with each $a_i \in \mathbb{C}$ and $\sup_i |a_i| < \infty$. Another name for such a space X is ℓ^∞. We define $\|x\|_\infty = \sup_j |a_j|$. This is a normed linear space from a result in real analysis, because we can identify ℓ^∞ with $L^\infty(\mathbb{N}, \mu)$, where $\mathbb{N}$ is the set of natural numbers and μ is counting measure, that is, $\mu(A)$ is equal to the number of elements of A. In fact ℓ^∞ is a Banach space.

Example 1.13 If $1 \le p < \infty$, ℓ^p is the collection of infinite sequences

$x = (a_1, a_2, \ldots)$ for which

$$\|x\|_p = \Big(\sum_j |a_j|^p\Big)^{1/p}$$

is finite. This is a complete normed linear space, hence a Banach space, because we can identify ℓ^p with $L^p(\mathbb{N}, \mu)$, where $\mathbb{N}$ and μ are as in Example 1.12.

Example 1.14 If S is a set, the collection of bounded functions on S with $\|f\|_\infty = \sup_s |f(s)|$ is a complete normed linear space. This is a well known result from undergraduate analysis. Most of the examples above are separable metric spaces. However the collection of bounded functions on S is separable if and only if S is countable - look at the collection $\{\chi_{\{y\}}\}$, $y \in Y$, where $\chi_{\{y\}}(x)$ equals 1 if $y = x$ and 0 otherwise.

Example 1.15 If S is a topological space, then the collection of continuous bounded functions with $\|f\| = \sup_s |f(s)|$ is also a Banach space.

Example 1.16 The L^p spaces are complete normed linear spaces.

Example 1.17 (Sobolev spaces) First consider one dimension. For $f \in C^\infty(\mathbb{R})$ we can define

$$\|f\|_{k,p} = \Big(\int_{\mathbb{R}} |f|^p + \int_{\mathbb{R}} |f'|^p + \cdots + \int_{\mathbb{R}} |f^{(k)}|^p\Big)^{1/p},$$

where $f^{(k)}$ is the k^{th} derivative of f. The set of C^∞ functions with compact support is not complete under this norm. We will discuss this in detail later.

In higher dimensions, let E be a domain in $\mathbb{R}^n$ and consider the C^∞ functions on E with

$$\int_E |D^j f(x)|^p \, dx$$

finite for all $|j| \leq k$. Here $j = (j_1, \ldots, j_n)$,

$$D^j = \frac{D^{j_1}}{\partial x_1^{j_1}} \cdots \frac{\partial^{j_n}}{\partial x_n^{j_n}},$$

and $|j| = j_1 + \cdots + j_n$. For a norm, we take

$$|f|_{k,p} = \Big(\sum_{|j| \le k} \int |D^j f(x)|^p \, dx \Big)^{1/p}.$$

This is not a complete space, but its completion is denoted $W^{k,p}$ and is called a Sobolev space.

Example 1.18 If X is the set of finite signed measures μ on a measurable space, setting $\|\mu\|$ equal to the total variation of μ makes this into a normed linear space.

We will discuss Banach spaces in more detail in Chapter 3.

1.4 Direct sums

If Y and Z are subspaces of X, we write $X = Y \oplus Z$ if for each $x \in X$, there exist a unique $y \in Y$ and $z \in Z$ such that $x = y + z$. The decomposition must be unique, i.e., there is only one y and one z that works for any particular x. Of course, y and z depend on x. In this case we say that X is the *direct sum* of Y and Z.

As an example, let $X = \mathbb{R}^3$, $Y = \{(x, 0, 0) : x \in \mathbb{R}\}$. There are lots of possibilities for Z, in fact, any plane in $\mathbb{R}^3$ that passes through the origin and does not contain the x axis. Given any choice of Z, though, there is only one way to write a given x as $y + z$.

We will frequently use *Zorn's lemma*, which is equivalent to the axiom of choice.

Suppose we have a *partially ordered* set S, which means that there is an order relation such that
(1) $a \le a$ for all $a \in S$,
(2) if $a \le b$ and $b \le a$, then $a = b$, and
(3)if $a \le b$ and $b \le c$, then $a \le c$.
A subset is *totally ordered* if for every pair x, y in the subset, either $x \le y$ or $y \le x$. An element u of a partially ordered set is an *upper bound* for a subset of S if $x \le u$ for every x in the subset. An element x of a partially ordered set is *maximal* if $y \ge x$ implies $y = x$.

Lemma 1.19 *(Zorn's lemma) Let X be a partially ordered set. If every totally ordered subset of X has an upper bound in X, then X has a maximal element.*

Note that it is not required that the upper bound for a totally ordered subset be in the subset.

Lemma 1.20 *Suppose Y is a subspace of a linear space X. Then there exists a linear subspace Z such that $X = Y \oplus Z$.*

Proof. Look at $\{Z : Z \text{ a subspace of } X, Z \cap Y = \{0\}\}$. We partially order this collection by inclusion: $Z_\alpha \leq Z_\beta$ if $Z_\alpha \subset Z_\beta$. If $\{Z_\alpha\}$ is a totally ordered subcollection, then $\cup_\alpha Z_\alpha$ is an upper bound in the collection. Let Z_0 be the maximal element guaranteed by Zorn's lemma.

Suppose there is a point $x \in X$ that is not in $Y \oplus Z_0$. We adjoin x to Z_0 to form Z_1 as follows: $Z_1 = \{ax + z : z \in Z_0, a \in \mathbb{R}\}$. Z_1 is a subspace of X that is strictly bigger than Z_0. We argue that $Z_1 \cap Y = \{0\}$, a contradiction to the fact that Z_0 is maximal.

x is not in the direct sum of Y and Z_0, so $x \notin Y$, or else we could write $x = x + 0$. If $w \neq 0$ and $w \in Z_1 \cap Y$, then there exist $a \in \mathbb{R}$ and $z \in Z_0$ such that $w = ax + z$. One possibility is that $a = 0$; but then $w = z \in Z_0 \cap Y$, which isn't possible since w is nonzero. The other possibility is that $a \neq 0$. But $w \in Y$, so

$$x = \frac{w}{a} + \frac{-z}{a} \in Y \oplus Z_0,$$

also a contradiction. □

If Z and U are normed linear spaces, we can make $Z \oplus U$ into a normed linear space by defining

$$|(z, u)| = |z| + |u|.$$

1.5 The unit ball in infinite dimensions

In finite dimensions, the closed unit ball is always compact, but this is not the case in infinite dimensions. As an example, consider ℓ^2. If e_i is the sequence which has a one in the ith place and 0 everywhere else, then $\|e_i - e_j\| = \sqrt{2}$ if $i \neq j$. But then $\{e_i\}$ is a sequence contained in the unit ball that has no convergent subsequence, hence the unit ball is not compact.

In fact, the closed unit ball $B = \{x : |x| \leq 1\}$ is never compact in infinite dimensions.

First we define what infinite dimensional means. Elements $x_1, x_2, \ldots, x_n$ of X are said to be *linearly dependent* if there exist $a_1, \ldots, a_n$ in F, not all equal to 0, such that $a_1x_1 + \cdots + a_nx_n = 0$. If $x_1, \ldots, x_n$ are not linearly dependent, they are *linearly independent.* A linear space X is finite dimensional if there are finitely many nonzero elements whose linear span is all of X. If X is not finite dimensional, it is infinite dimensional. We write $\dim X = n$ if there exist n linearly independent nonzero elements of X whose linear span is equal to X.

The key to proving that the unit ball in an infinite dimensional space is not compact is the following proposition.

Proposition 1.21 *Suppose Y is a finite dimensional subspace of X that is not all of X. There exists v such that $\|v\| = 1$ and $\inf_{z \in Y} \|v - z\| \geq 1/2$.*

Proof. Since Y is finite dimensional, it is closed. It is not all of X, so there exists $x \in X \setminus Y$. Let $d = \inf_{y \in Y} \|y - x\|$. We claim that $d > 0$. If not, then there exists a sequence $y_n \in Y$ such that $\|y_n - x\| \to 0$. This means that y_n converges to x. But Y is closed, so $x \in Y$, a contradiction.

Choose $w \in Y$ such that $\|x - w\| < 2d$. Let $z = x - w$ so $\|z\| < 2d$. If $y \in Y$, then $y + w \in Y$ and

$$\|z - y\| = \|x - (y + w)\| \geq d.$$

If we let $v = z/\|z\|$, then for all $y \in Y$

$$\|v - y\| = \Big\|\frac{z}{\|z\|} - y\Big\| = \frac{1}{\|z\|}\big\|z - \|z\|y\big\| \geq \frac{d}{2d} = \frac{1}{2}$$

since $\|z\|y \in Y$. □

Theorem 1.22 *Let X be an infinite dimensional normed linear space. Then the closed unit ball is not compact.*

Proof. Choose y_1 such that $\|y_1\| = 1$. Given $y_1, \ldots, y_{n-1}$, let Y_n be the linear span. Since Y_n is finite dimensional, it is closed. By Proposition 1.21 there exists y_n such that $\|y_n\| = 1$ and $\inf_{y \in Y_n} \|y - y_n\| \geq 1/2$. We continue by induction and find a sequence $\{y_n\}$ contained in the closed unit ball such that $\|y_j - y_n\| \geq 1/2$ if $j < n$, hence which has no convergent subsequence. $\square$

Chapter 2

Linear maps

2.1 Basic notions

Let X and Y be linear spaces over F. A map $M : X \to U$ is linear or is a *linear map* or is a *linear operator* if $M(x+y) = M(x) + M(y)$ and $M(ax) = aM(x)$ for all $x, y \in X$ and all $a \in F$.

Here are some examples of linear operators.

(1) Let $X = L^1(\mu)$, g be bounded and measurable, and define

$$Mf = \int f(x)g(x)\,\mu(dx).$$

Here M maps X into $\mathbb{R}$.

(2) Let X be the space of bounded functions on a set S, fix points $x_1, \ldots, x_n \in S$, and let $Mf = (f(x_1), \ldots, f(x_n))$. Here M maps X to $\mathbb{R}^n$.

(3) Let X be $\mathbb{R}^m$, let Y be $\mathbb{R}^n$, let $a_{ij} \in \mathbb{R}$ for $1 \le i \le n$, $1 \le j \le m$, and define the ith coordinate of Mx to be $\sum_{j=1}^n a_{ij}x_j$. This is just matrix multiplication.

In fact all linear maps in finite dimensions can be viewed in this way. To see this, let $e_1, \ldots, e_m$ be linearly independent nonzero elements of X and $f_1, \ldots, f_n$ linearly independent nonzero elements of Y. Since $\{f_1, \ldots, f_n\}$ spans Y, there exist elements $a_{i1}, \ldots, a_{in}$ of F such that $Me_i = \sum_{j=1}^n a_{ij} f_j$ for $i = 1, \ldots, m$.

(4) Let X be the set of bounded sequences, suppose that

$$\sup_i \sum_{j=1}^\infty |a_{ij}| < \infty,$$

and define the ith coordinate of Mx to be $\sum_{j=1}^\infty a_{ij}x_j$.

(5) Let X be the set of bounded measurable functions on some measure space with finite measure μ, and suppose $K(x,y)$ is jointly measurable and bounded. Define Mf by

$$Mf(x) = \int K(x,y)\,\mu(dy).$$

If M and N are linear maps from X into Y and a is a scalar, we define

$$(M+N)(x) = M(x) + N(x), \qquad (aM)(x) = aM(x).$$

Thus the set of linear maps from X into Y is a linear space, and we denote it by $\mathcal{L}(X,Y)$.

If $M : X \to Y$ and $N : Y \to Z$, we define $(NM)(x) = N(M(x))$.

An exercise is to show this is associative but not necessarily commutative. (Multiplication by matrices is an example to show commutativity need not hold.) It is distributive:

$$M(N+K) = MN + MK, \qquad (M+K)N = MN + KN.$$

We usually write Mx for $M(x)$.

Define the *identity* $I : X \to X$ by $Ix = x$. We will also write I_X when we want to emphasize the space.

We say $M : X \to Y$ is invertible if there exists $M^{-1} : Y \to X$ such that $M^{-1}M = I_X$, $MM^{-1} = I_Y$.

If M is linear and invertible, then M^{-1} is also linear. To see this, suppose $y_1 = Mx_1$ and $y_2 = Mx_2$ are elements of Y with $x_1, x_2 \in X$. Then $y_1 + y_2 = Mx_1 + Mx_2 = M(x_1 + x_2)$. Hence $M^{-1}(y_1 + y_2) = x_1 + x_2 = M^{-1}y_1 + M^{-1}y_2$. Similarly $M^{-1}(ay) = aM^{-1}y$.

Two linear spaces are said to be *isomorphic* if there exists a one-to-one linear mapping from one space onto the other.

Given two linear spaces Y and Z, we can define a new space $X = \{(y, z) : y \in Y, z \in Z\}$ and define $(y_1, z_1) + (y_2, z_2) = (y_1 + y_2, z_1, z_2)$ and define scalar multiplication similarly. Clearly Y is isomorphic to $Y' = \{(y, 0) : y \in Y\}$ and Z is isomorphic to $Z' = \{(0, z) : z \in Z\}$. Moreover we see that $X = Y' \oplus Z'$. If Y and Z are normed linear spaces, then X is also if we define $\|(y, z)\| = \|y\| + \|z\|$.

The *null space* or *kernel* of M is $N_M = \{x \in X : Mx = 0\}$ and the *range* of M is $R_M = \{Mx : x \in X\}$.

Observe that $N_M \subset X$ and $R_M \subset Y$.

Some easily checked facts: N_M and R_M are linear subspaces, and if L, M are invertible, then $(LM)^{-1} = M^{-1}L^{-1}$.

2.2 Boundedness and continuity

A linear map M from a normed linear space X into a normed linear space Y is a *bounded linear map* if

$$\|M\| = \sup\{\|Mx\| : \|x\| = 1\} \tag{2.1}$$

is finite.

A linear map M from a normed linear space X to a Banach space Y is continuous if $x_n \to x$ implies $Mx_n \to Mx$.

Proposition 2.1 *M is continuous if and only if it is bounded.*

Proof. If M is bounded,

$$\|M(x_n) - M(x)\| = \|M(x_n - x)\| \leq c\|x_n - x\| \to 0$$

for some $c < \infty$, and so it is continuous.

Suppose M is continuous but not bounded. Then there exist x_n such that $\|M(x_n)\| > n\|x_n\|$. If

$$y_n = \frac{1}{\sqrt{n}} \frac{x_n}{\|x_n\|},$$

then $\|y_n - 0\| = \|y_n\| \to 0$, but

$$\|M(y_n)\| > \frac{1}{\sqrt{n}} n \frac{\|x_n\|}{\|x_n\|} = \sqrt{n},$$

which does not tend to $0 = M(0)$, a contradiction. □

Define

$$\|M\| = \sup_{x \neq 0} \frac{\|Mx\|}{\|x\|},$$

or what is the same,

$$\|M\| = \sup_{\|x\|=1} \|Mx\|.$$

Proposition 2.2 *Suppose M and N are linear maps from a normed linear space X into a normedlinear space Y. Then $\|aM\| = |a|\,\|M\|$, $\|M\| \geq 0$ and equals 0 if and only if $M = 0$, and $\|M + N\| \leq \|M\| + \|N\|$.*

The proofs are easy.

Proposition 2.3 *N_M is closed.*

Proof. $\{0\}$ is closed, M is a continuous function from one metric space into another, so $N_M = M^{-1}(\{0\})$ is closed. □

Proposition 2.4 *Suppose X, Y, and Z are normed linear spaces, M is a linear map from X to Y, and N is a linear map from Y to Z. Then $\|NM\| \leq \|N\|\,\|M\|$.*

Proof. For $\|x\| = 1$,

$$\|NMx\| \leq \|N\|\,\|Mx\| \leq \|N\|\,\|M\|\,\|x\|.$$

□

2.3 Quotient spaces

Let X be a linear space and Y a subspace. We write $x_1 \equiv x_2$ and say that x_1 is equivalent to x_2 if $x_1 - x_2 \in Y$. This is an equivalence relation. Let $\overline{x}$ denote the equivalence class containing x. The collection of all such equivalence classes is denoted X/Y and called the *quotient space* of X with respect to Y,

Let's make X/Y into a linear space. If $\overline{x}_1, \overline{x}_2$ are in X/Y, define $\overline{x}_1 + \overline{x}_2$ to be $\overline{x_1 + x_2}$. To see that this is well defined, if z_1, z_2 are any two elements of $\overline{x}_1, \overline{x}_2$, resp., then $(x_1 + x_2) - (z_1 + z_2) = (x_1 - z_1) + (x_2 - z_2)$, the sum of two elements of Y, hence an element of Y. Hence $\overline{x_1 + x_2} = \overline{z_1 + z_2}$, and it doesn't matter in the definition which elements of $\overline{x}_1$ and $\overline{x}_2$we choose. We similarly define $a\overline{x} = \overline{ax}$. It is now routine to verify that X/Y is a linear space.

We define the *codimension* of Y by

$$\text{codim } Y = \dim X/Y.$$

Let's look at an example. Let $X = \mathbb{R}^5$ and suppose $Y = \{(x, y, 0, 0, 0) : x, y \in \mathbb{R}\}$. $x_1 \equiv x_2$ if and only if the 3^{rd} through 5^{th} coordinates of x_1 and x_2 agree. Therefore X/Y is (essentially - at least it is isomorphic to) the 3^{rd} through 5^{th} coordinates of points in $\mathbb{R}^5$, hence isomorphic to $\mathbb{R}^3$. We see codim $Y = \dim X/Y = 3$, while $\dim Y = 2$.

Proposition 2.5 *If $X = Y \oplus Z$, then X/Y is isomorphic to Z.*

Proof. If $\overline{x} \in X/Y$, then $x \in X$ and we can write $x = z + y$, where $z \in Z$ and $y \in Y$. Define $M\overline{x} = z$. We will show that M is an isomorphism.

First we need to show M is well defined. If x' is another element of $\overline{x}$, we can write $x' = z' + y'$ with $z' \in Z$ and $y' \in Y$. Then $x - x' = (z - z') + (y - y')$. Since we can also write $x - x' = 0 + (x - x')$ and we can write each element of X as a sum of elements of Z and Y in only one way, we must have $z - z' = 0$, or $z = z'$.

Next we show M is linear. If $x_1 + x_2 \in \overline{x_1 + x_2}$, then $x_1 = z_1 + y_1$, $x_2 = z_2 + y_2$, and then $x_1 + x_2 = (z_1 + z_2) + (y_1 + y_2)$. So $M(\overline{x_1 + x_2}) =$

$z_1 + z_2 = M\overline{x}_1 + M\overline{x}_2$. The linearity with respect to scalar multiplication is similar.

We show M is one-to-one. If $M\overline{x} = M\overline{x}'$, and we write $x = z + y$, $x' = z' + y'$, then $z = M\overline{x} = M\overline{x}' = z'$. Hence

$$x - x' = (z - z') + (y - y') = y - y' \in Y,$$

so $\overline{x} = \overline{x}'$.

Finally, M is onto, because if $z \in Z$, then $z = z + 0 \in X$ and $M\overline{z} = z$. □

Let $M : X \to Y$. We will need the fact that

Proposition 2.6 *X/N_M is isomorphic to R_M.*

Proof. If $\overline{x} \in X/N_M$, we define $\widetilde{M}\overline{x}$ to be Mx for any $x \in \overline{x}$. If x' is any other element of $\overline{x}$, then $x - x' \in N_M$, or $M(x - x') = 0$, or $Mx = Mx'$. So the map $\widetilde{M}$ is well defined. It is routine to check that $\widetilde{M}$ is linear.

To show $\widetilde{M}$ is one-to-one, if $\widetilde{M}\overline{x} = \widetilde{M}\overline{y}$, then $Mx = My$, or $M(x-y) = 0$, or $x - y \in N_M$, so $\overline{x} = \overline{y}$. To show $\widetilde{M}$ is onto, if $y \in R_M$, then $y = Mx$ for some $x \in X$. Then $\widetilde{M}\overline{x} = Mx = y$. □

2.4 Convex sets

A set $K \subset X$ is *convex* if $\lambda x + (1-\lambda)y \in K$ whenever $x, y \in K$ and $\lambda \in [0, 1]$.

A *convex combination* of $x_1, \ldots, x_m$ is a sum of the form

$$\sum_{i=1}^n a_i x_i,$$

where $\sum_{i=1}^n a_i = 1$, n is a positive integer, and all the a_i are non-negative.

If K is convex and $x_1, \ldots, x_n \in K$, each $a_i \geq 0$, and $\sum_{i=1}^n a_i = 1$, then $\sum_{i=1}^n a_i x_i \in K$. This can be proved easily using induction on n.

Lemma 2.7 *Linear subspaces are convex. Intersections of convex sets are convex. If $M : X \to Y$ is linear and $K \subset X$ is convex, then $\{M(x) : x \in K\}$ is convex.*

The proof is left as an exercise.

If $S \subset X$, the *convex hull* of S is the intersection of all the convex sets containing S.

Proposition 2.8 *The convex hull of S is equal to the set of all convex combinations of points of S.*

Proof. Let H be the convex hull of S and C the set of all convex combinations of points of S. If y is in C, then $y = \sum_{i=1}^n a_i x_i$ where each $x_i \in S$, each $a_i \geq 0$, and $\sum_{i=1}^n a_i = 1$. Therefore y is in each convex set containing S, hence $y \in H$. Thus $C \subset H$.

Suppose $y = \sum_{i=1}^n a_i x_i$, where $x_i \in S$, each $a_i \geq 0$, and $\sum_{i=1}^n a_i = 1$ and similarly $z = \sum_{j=1}^m b_j x'_j$. If $m < n$, we can set $b_{m+1}, \ldots, b_n = 0$ and $x_{m+1}, \ldots, x_n$ any point in S, and can thus assume $m \geq n$. Similarly we may without loss of generality assume $n \geq m$, hence $m = n$. If $\lambda \in [0, 1]$, we have

$$\lambda y + (1 - \lambda) z = \sum_{k=1}^{2n} c_k w_k,$$

where $c_k = \lambda a_k$ and $w_k = x_k$ if $k \leq n$ and $c_k = (1 - \lambda) b_{k-n}$ and $w_k = x'_{k-n}$ if $k > n$. Then each $c_k \geq 0$ and

$$\sum_{k=1}^{n} c_k + \sum_{k=n+1}^{2n} c_k = \lambda \sum_{k=1}^{n} a_k + (1 - \lambda) \sum_{k=1}^{n} b_k = \lambda + (1 - \lambda) = 1.$$

This proves that C is convex, and since C contains S we have $H \subset C$. □

If K is convex, and $E \subset K$, then E is an *extreme subset* of K if
(1) E is convex and non-empty, and
(2) if $x \in E$ and $x = \frac{y+z}{2}$ with $y, z \in K$, then $y, z \in E$.

If E is a single point, then the point is called an *extreme point* of K.

For an example, consider the case where K is a polygon (plus the interior) in $\mathbb{R}^2$. Each edge of K is an extreme subset. Each vertex is an extreme point.

Proposition 2.9 *If E is an extreme subset of F and F is an extreme subset of G, then E is an extreme subset of G.*

Proof. Suppose $x \in E$ and $x = (y+z)/2$ for $y, z \in G$. Since $x \in E \subset F$ and F is an extreme subset of G, then $y, z \in F$. But then since E is an extreme subset of F, we must have $y, z \in E$. □

2.5 Hahn-Banach theorem

A *linear functional* ℓ is a linear map from X to F. In this section we will take F to be the reals. In a later section we will consider the case when F is the complex numbers.

The *Hahn-Banach theorem* is a tool that lets us assert that there is a plentiful supply of linear functionals.

We will be working with a *sublinear functional* $p(x)$ in the statement of the theorem. Suppose $p : X \to \mathbb{R}$ is a sublinear functional if
(1) $p(ax) = ap(x)$ whenever $a > 0$ and $x \in X$ and
(2) $p(x + y) \le p(x) + p(y)$ if $x, y \in X$.

One example is to let $p(x) = c\|x\|$, where $c > 0$. This example requires X to be a normed linear space.

Here is another example that applies to linear spaces, whether or not they are normed linear spaces.

A point $x_0 \in S \subset X$ is *interior* to S if for all $y \in X$, there exists ε (depending on y) such that $x_0 + ty \in S$ if $-\varepsilon < t < \varepsilon$.

Let K be a convex set with 0 as an interior point. Define

$$p_K(x) = \inf \Big\{ b > 0 : \frac{x}{b} \in K \Big\}. \tag{2.2}$$

p_K is sometimes called the *gauge* of K.

Proposition 2.10 *p_K is a sublinear functional.*

Proof. It is clear that $p_K(ax) = ap_K(x)$ if $a > 0$. Let $x, y \in X$. If $p_K(x)$ or $p_K(y)$ is infinite, there is nothing to prove. So suppose both are finite and let $\varepsilon > 0$. Choose $p_K(x) < a < p_K(x) + \varepsilon$ and $p_K(y) < b < p_K(y) + \varepsilon$. Then $\frac{x}{a}$ and $\frac{y}{b}$ are in K. Letting $\lambda = a/(a+b)$, then by the convexity of K

$$\lambda \frac{x}{a} + (1-\lambda)\frac{y}{b} = \frac{x+y}{a+b}$$

is in K. So

$$p_K(x+y) \le a + b \le p_K(x) + p_K(y) + 2\varepsilon.$$

Since ε is arbitrary, we are done. □

Here is the Hahn-Banach theorem for real-valued linear functionals.

Theorem 2.11 *Suppose $p : X \to \mathbb{R}$ satisfies $p(ax) = ap(x)$ if $a > 0$ and $p(x+y) \le p(x) + p(y)$ if $x, y \in X$. Suppose Y is a linear subspace, ℓ is a linear functional on Y, and $\ell(y) \le p(y)$ for all $y \in Y$. Then ℓ can be extended to a linear functional on X satisfying $\ell(x) \le p(x)$ for all $x \in X$.*

Proof. If Y is not all of X, pick $z \in X \setminus Y$. Look at $Y_1 = \{y + az : y \in Y, a \in \mathbb{R}\}$. We want to define $\ell(z)$ to be some real number with the property that if we set

$$\ell(y + az) = \ell(y) + a\ell(z),$$

we would have $\ell(y) + a\ell(z) \le p(y + az)$ for all $y \in Y$ and $a \in \mathbb{R}$. This would give us an extension of ℓ from Y to Y_1.

For all $y, y' \in Y$,

$$\begin{aligned}\ell(y') + \ell(y) = \ell(y' + y) \le p(y' + y) &= p((y + z) + (y' - z)) \\ &\le p(y + z) + p(y' - z).\end{aligned}$$

So

$$\ell(y') - p(y' - z) \le p(y + z) - \ell(y).$$

This is true for all $y, y' \in Y$. So choose $\ell(z)$ to be a number between $\sup_{y'}[\ell(y') - p(y' - z)]$ and $\inf_y[p(y + z) - \ell(y)]$. Therefore

$$\ell(y') - p(y' - z) \le \ell(z) \le p(y + z) - \ell(y),$$

or

$$\ell(y) + \ell(z) \le p(y+z), \qquad \ell(y') - \ell(z) \le p(y'-z).$$

If $a > 0$,

$$\ell(y+az) = a\ell(\tfrac{y}{a} + z) \le ap(\tfrac{y}{a} + z) = p(y+az).$$

Similarly $\ell(y' - az) \le p(y' - az)$ if $a > 0$.

So we have extended ℓ from Y to Y_1, a larger space. Let $\{(Y_\alpha, \ell_\alpha)\}$ be the collection of all extensions of (Y, ℓ). We partially ordered this collection by saying $(Y_\alpha, \ell_\alpha) \le (Y_\beta, \ell_\beta)$ if $Y_\alpha \subset Y_\beta$ and ℓ_β is an extension of ℓ_α. If $\{(Y_\beta, \ell_\beta)\}$ is a totally ordered subset, define ℓ on $\cup_\beta Y_\beta$ by setting $\ell(z) = \ell_\beta(z)$ if $z \in Y_\beta$. By Zorn's lemma, there is a maximal extension. This maximal extension must be all of X, or else by the above we could extend it. □

Corollary 2.12 *Suppose that X is a normed linear space X, Y is a linear subspace of X, and ℓ is a bounded linear functional on Y. Then ℓ can be extended to a bounded linear functional on X with the same norm.*

Proof. Let M be the norm of ℓ as a bounded linear map on Y and set $p(x) = M\|x\|$ for $x \in X$. Our assumption tells us that $\ell(y) \le p(y)$ for $y \in Y$. Take the extension guaranteed by Theorem 2.11, and then $\ell(x) \le M\|x\|$ for all $x \in X$. Applying this also to $-x$, we have $-\ell(x) = \ell(-x) \le M\|x\|$, hence $|\ell(x)| \le M\|x\|$. □

This corollary is the version of the Hahn-Banach theorem one learns in real analysis courses.

2.6 Complex linear functionals

Now we turn to linear spaces (not necessarily normed) over the complex numbers.

Theorem 2.13 *Let X be a linear space over $\mathbb{C}$. Suppose $p \ge 0$ satisfies $p(ax) = |a|p(x)$ for all $x \in X, a \in \mathbb{C}$, and $p(x+y) \le p(x) + p(y)$ for all $x, y \in X$. If Y is a subspace of X, ℓ is a linear functional on Y, and $|\ell(y)| \le p(y)$ for all $y \in Y$, then ℓ can be extended to a linear functional on X with $|\ell(x)| \le p(x)$ for all x.*

For normed linear spaces an example would be $p(x) = M\|x\|$.

Proof. Write ℓ as $\ell(y) = \ell_1(y) + i\ell_2(y)$, the real and imaginary parts of ℓ. Since ℓ is linear,

$$i\ell(y) = \ell_1(iy) + i\ell_2(iy).$$

On the other hand

$$i\ell(y) = i\ell_1(y) - \ell_2(y)$$

by substituting in for $\ell(y)$ and multiplying by i. Equating the real parts, $\ell_1(iy) = -\ell_2(y)$.

One can work this in reverse to see that if ℓ_1 is a linear functional over the reals, and we define $\ell(x) = \ell_1(x) - i\ell_1(ix)$, we get a linear functional over the complexes.

To extend ℓ, we have

$$\ell_1(y) \le |\ell(y)| \le p(y).$$

Use Hahn-Banach to extend ℓ_1 to all of X and set $\ell(x) = \ell_1(x) - i\ell_1(ix)$.

We need to show that $|\ell(x)| \le p(x)$ for all x. Fix x and write $\ell(x) = ar$, where r is real and $|a| = 1$. Then

$$|\ell(x)| = r = a^{-1}\ell(x) = \ell(a^{-1}x).$$

Since $\ell(a^{-1}x) = |\ell(x)|$, it is real with no imaginary part, and therefore equals

$$\ell_1(a^{-1}x) \le p(a^{-1}x) = |a^{-1}|p(x) = p(x).$$

□

2.7 Positive linear functionals

Let S be an arbitrary set and let X be the collection of real-valued bounded functions on S. We say $x \le y$ if $x(s) \le y(s)$ for all $s \in S$. (We'll use $x \ge y$ if $y \le x$.) A function x is non-negative if $0 \le x$. Let Y be a linear subspace of X. ℓ is a *positive linear functional* on Y if $\ell(y) \ge 0$ whenever $y \ge 0$. Note that if $x \le y$, then $0 \le \ell(y - x) = \ell(y) - \ell(x)$, so $\ell(x) \le \ell(y)$.

One example is to take $\ell(y) = y(s_0)$ for some point s_0 in S. Or we could take a linear combination $\sum c_i y(s_i)$ provided all the $c_i \geq 0$. Another example is to take (S, μ) to be a measure space and let $\ell(y) = \int y(s)\, \mu(dx)$.

Proposition 2.14 *Let Y be a linear subspace and suppose there exists $y_0 \in Y$ such that $y_0(s) \geq 1$ for all s. Let ℓ be a positive linear functional on Y. Then ℓ can be extended to a positive linear functional on X.*

Proof. Define

$$p(x) = \inf\{\ell(y) : y \in Y, y \geq 0, y \geq x\}.$$

Since $-cy_0 \leq x \leq cy_0$ if x is bounded by c, we are not taking the infimum of an empty set. Since $x \leq cy_0$, then $p(x) \leq c\ell(y_0) < \infty$.

It is clear that $p(ax) = ap(x)$ if $x \in X$ and $a > 0$. To show that p is a sublinear functional, suppose $x_1, x_2 \in X$, let $\varepsilon > 0$, and choose $y_1, y_2 \in Y$ with $x_1 \leq y_1, x_2 \leq y_2$, $0 \leq y_1$, $0 \leq y_2$, $\ell(y_1) \leq p(x_1)+\varepsilon$, and $\ell(y_2) \leq p(x_2)+\varepsilon$. Then $y_1 + y_2 \in Y$, $y_1 + y_2 \geq 0$, $y_1 + y_2 \geq x_1 + x_2$, and so

$$p(x_1 + x_2) \leq \ell(y_1 + y_2) = \ell(y_1) + \ell(y_2) \leq p(x_1) + p(x_2) + 2\varepsilon.$$

Since ε is arbitrary, this proves sublinearity.

If $y \in Y$, $y' \geq 0$, and $y' \geq y$ is any other element in Y, then $\ell(y) \leq \ell(y')$, so $p(y) \geq \ell(y)$.

We now use Theorem 2.11 to extend ℓ to all of B. If $x \geq 0$, then $-x \leq 0$, so

$$\ell(-x) \leq p(-x) \leq l(0) = 0$$

since $0 \in Y$, and then $\ell(x) = -\ell(-x) \geq 0$. □

The additional assumption here is that there is a function in Y that is bounded below by a positive number. (If y_0 is bounded below by $\delta > 0$, look at y_0/δ.)

2.8 Separating hyperplanes

If ℓ is a real-valued linear functional on a linear space over the reals, then $\{x : \ell(x) = c\}$ is a *hyperplane*. This splits X into two parts, those x for which $\ell(x) > c$ and those for which $\ell(x) < c$.

Recall the definition of the gauge p_K of a convex set from (2.2).

Proposition 2.15 *(1) If K is convex, 0 is interior to K, and $x \in K$, then $p_K(x) \le 1$. If K is convex and x is interior to K, then $p_K(x) < 1$.*
(2) Let p be a positive sublinear functional. Then $\{x : p(x) < 1\}$ is convex and 0 is an interior point. Also $\{x : p(x) \le 1\}$ is convex.

We leave the proof to the reader. □

We now prove the *hyperplane separation theorem.*

Theorem 2.16 *Suppose K is a nonempty convex subset of a linear space X over the reals and all points of K are interior. If $y \notin K$, then there exist ℓ and c such that $\ell(x) < c$ for all $x \in K$ and $\ell(y) = c$.*

Proof. Without loss of generality, assume $0 \in K$. Note $p_K(x) < 1$ for all $x \in K$. Set $\ell(y) = 1$ and $\ell(ay) = a$. If $a \le 0$, $\ell(ay) \le 0 \le p_K(ay)$. If $a > 0$, then since $y \notin K$, $p_K(y) \ge 1$, and so $p_K(ay) \ge a = \ell(ay)$.

We let $Y = \{ay\}$ and use Hahn-Banach to extend ℓ to all of X. We have $\ell(x) \le p_K(x) < 1$ if $x \in K$ and $l(y) = 1$. We take $c = 1$. □

Corollary 2.17 *If K is convex with at least one interior point and $y \notin K$, there exists $\ell \ne 0$ such that $\ell(x) \le \ell(y)$ for all $x \in K$.*

$A + B$ is defined to be $\{a + b : a \in A, b \in B\}$.

Corollary 2.18 *Let H and M be disjoint convex sets, with at least one having an interior point. Then there exist ℓ and c such that*

$$\ell(u) \le c \le \ell(v), \qquad u \in H, v \in M.$$

Proof. $-M$ is convex, so $K = H + (-M)$ is convex. K must have an interior point. $H \cap M = \emptyset$, so $0 \notin K$. Let $y = 0$. There exists ℓ such that

$$\ell(x) \le \ell(0) = 0 \qquad x \in K.$$

If $u, v \in H, M$, resp., then $x = u - v \in K$, so $\ell(x) \le 0$, and hence $\ell(u) \le \ell(v)$. □

Chapter 3

Banach spaces

3.1 Preliminaries

A *Banach space* is a complete normed linear space.

If X and Y are metric spaces, a map $\varphi : X \to Y$ is an *isometry* if $d_Y(\varphi(x), \varphi(y)) = d_X(x, y)$ for all $x, y \in X$, where d_X is the metric for X and d_Y the one for Y. A metric space X^* is the *completion* of a metric space X is there is an isometry φ of X into X^* such that $\varphi(X)$ is dense in X^* and X^* is complete.

Recall that any metric space can be embedded in a complete metric space. See [1] for a proof of the following theorem.

Theorem 3.1 *If X is a metric space, then it has a completion X^*.*

Of course, if X is already complete, its completion is X itself and φ is the identity map.

We have already seen some examples of Banach space. We looked at

(1) ℓ^∞, the collection of infinite sequences $\{a_1, a_2, \ldots\}$ with each $a_i \in \mathbb{C}$ and $\sup_i |a_i| < \infty$. We define $\|x\|_\infty = \sup_j |a_j|$.

(2) If $1 \le p < \infty$, ℓ^p is the collection of infinite sequences for which

$$\|x\|_p = \Big(\sum_j |a_j|^p \Big)^{1/p}$$

is finite.

(3) If S is a set, the collection of bounded functions on S with $\|f\|_\infty = \sup_s |f(s)|$.

(4) If S is a topological space, then the collection of continuous bounded functions with $\|f\| = \sup_s |f(s)|$.

(5) The L^p spaces.

(6) We defined for $f \in C^\infty(\mathbb{R})$

$$\|f\|_{k,p} = \Big(\int_{\mathbb{R}} |f|^p + \int_{\mathbb{R}} |f'|^p + \cdots + \int_{\mathbb{R}} |f^{(k)}|^p \Big)^{1/p},$$

where $f^{(k)}$ is the k^{th} derivative of f. The set of C^∞ functions with compact support is not complete under this norm, but we can take its completion and that will be a Banach space.

In higher dimensions, let D be a domain in $\mathbb{R}^n$ and consider the C^∞ functions on D with

$$\int_D |\partial^\alpha f(x)|^p \, dx$$

finite for all $|\alpha| \le k$. Here $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For a norm, we take

$$\|f\|_{k,p} = \Big(\sum_{|\alpha| \le k} \int |\partial^\alpha f(x)|^p \, dx \Big)^{1/p}.$$

This is not a complete space, but its completion is denoted $W^{k,p}$ and is called a Sobolev space.

We wrote $\mathcal{L}(X, Y)$ for the set of linear maps from a Banach space X to a Banach space Y.

Proposition 3.2 *$\mathcal{L}$ is itself a Banach space.*

Proof. Let M_n be a Cauchy sequence. For each $x \in X$, $M_n x$ is a Cauchy sequence in Y. Since Y is complete, $M_n x$ converges, say to a point Nx. Let $\varepsilon > 0$. Note

$$|Nx - M_n x| \le \limsup_{m \to \infty} |M_m x - M_n x|,$$

which will be less than ε if n is large enough, independently of x. Thus M_n converges to N uniformly. Showing that N is a linear map is easy. □

3.2 Baire's theorem

We turn now to the Baire category theorem and some of its consequences. Recall that if A is a set, we use $\overline{A}$ for the closure of A and A^o for the interior of A. A set A is dense in X if $\overline{A} = X$ and A is nowhere dense if $(\overline{A})^o = \emptyset$.

The *Baire category theorem* is the following. Completeness of the metric space is crucial to the proof.

Theorem 3.3 *Let X be a complete metric space.*
(1) If G_n are open sets dense in X, then $\cap_n G_n$ is dense in X.
(2) X cannot be written as the countable union of nowhere dense sets.

Proof. We first show that (1) implies (2). Suppose we can write X as a countable union of nowhere dense sets, that is, $X = \cup_n E_n$ where $(\overline{E_n})^o = \emptyset$. We let $F_n = \overline{E_n}$, which is a closed set, and then $F_n^o = \emptyset$ and $X = \cup_n F_n$. Let $G_n = F_n^c$, which is open. Since $F_n^o = \emptyset$, then $\overline{G_n} = X$. Starting with $X = \cup_n F_n$ and taking complements, we see that $\emptyset = \cap_n G_n$, a contradiction to (1).

We must prove (1). Suppose $G_1, G_2, \ldots$ are open and dense in X. Let H be any non-empty open set in X. We need to show there exists a point in $H \cap (\cap_n G_n)$. We will construct a certain Cauchy sequence $\{x_n\}$ and the limit point, x, will be the point we seek.

Let $B(z, r) = \{y \in X : d(z, y) < r\}$, where d is the metric. Since G_1 is dense in X, $H \cap G_1$ is non-empty and open, and we can find x_1 and r_1 such that $\overline{B(x_1, r_1)} \subset H \cap G_1$ and $0 < r_1 < 1$. Suppose we have chosen x_{n-1} and r_{n-1} for some $n \geq 2$. Since G_n is dense, then $G_n \cap B(x_{n-1}, r_{n-1})$ is open and non-empty, so there exists x_n and r_n such that $\overline{B(x_n, r_n)} \subset G_n \cap B(x_{n-1}, r_{n-1})$ and $0 < r_n < 2^{-n}$. We continue and get a sequence x_n in X. If $m, n > N$, then x_m and x_m both lie on $B(x_N, r_N)$, and so $d(x_m, x_n) < 2r_N < 2^{-N+1}$. Therefore x_n is a Cauchy sequence, and since X is complete, x_n converges to a point $x \in X$.

It remains to show that $x \in H \cap (\cap_n G_n)$. Since x_n lies in $\overline{B(x_N, r_N)}$ if $n > N$, then x lies in each $\overline{B(x_N, r_N)}$, and hence in each G_N. Therefore $x \in \cap_n G_n$. Also,

$$x \in \overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1}) \subset \cdots \subset B(x_1, r_1) \subset H.$$

Thus we have found a point x in $H \cap (\cap_n G_n)$. □

A set $A \subset X$ is called *meager* or of the *first category* if it is the countable union of nowhere dense sets; otherwise it is of the *second category*.

3.3 Uniform boundedness theorem

An important application of the Baire category theorem is the *Banach-Steinhaus theorem*, also called the *uniform boundedness theorem*.

Theorem 3.4 *Suppose X is a Banach space and Y is a normed linear space. Let A be an index set and let $\{M_\alpha : \alpha \in A\}$ be a collection of bounded linear maps from X into Y. Then either there exists a positive real number $N < \infty$ such that $\|M_\alpha\| \le N$ for all $\alpha \in A$ or else $\sup_\alpha \|M_\alpha x\| = \infty$ for some x.*

Proof. Let $\ell(x) = \sup_{\alpha \in A} \|M_\alpha x\|$. Let $G_n = \{x : \ell(x) > n\}$. We argue that G_n is open. The map $x \to \|M_\alpha x\|$ is a continuous function for each α since M_α is a bounded linear functional. This implies that for each α, the set $\{x : \|M_\alpha x\| > n\}$ is open. Since $x \in G_n$ if and only if for some $\alpha \in A$ we have $\|M_\alpha x\| > n$, we conclude G_n is the union of open sets, hence is open.

Suppose there exists N such that G_N is not dense in X. Then there exists x_0 and r such that $\overline{B(x_0, r)} \cap G_N = \emptyset$. This can be rephrased as saying that if $\|x - x_0\| \le r$, then $\|M_\alpha(x)\| \le N$ for all $\alpha \in A$. If $\|y\| \le r$, we have $y = (x_0 + y) - x_0$. Then $\|(x_0 + y) - x_0\| = \|y\| \le r$, and hence $\|M_\alpha(x_0 + y)\| \le N$ for all α. Also, of course, $\|x_0 - x_0\| = 0 \le r$, and thus $\|M_\alpha(x_0)\| \le N$ for all α. We conclude that if $\|y\| \le r$ and $\alpha \in A$,

$$\|M_\alpha y\| = \|M_\alpha((x_0 + y) - x_0)\| \le \|M_\alpha(x_0 + y)\| + \|M_\alpha x_0\| \le 2N.$$

Consequently, $\sup_\alpha \|M_\alpha\| \le N$ with $N = 2N/r$.

The other possibility, by the Baire category theorem, is that every G_n is dense in X, and in this case $\cap_n G_n$ is dense in X. But $\ell(x) = \infty$ for every $x \in \cap_n G_n$. □

3.4 Open mapping theorem

The following theorem is called the *open mapping theorem*. It is important that M be onto. A mapping $M : X \to Y$ is *open* if $M(U)$ is open in Y whenever U is open in X. For a measurable set A, we let $M(A) = \{Mx : x \in A\}$.

Theorem 3.5 *Let X and Y be Banach spaces. A bounded linear map M from X onto Y is open.*

Proof. We need to show that if $B(x, r) \subset X$, then $M(B(x, r))$ contains a ball in Y. We will show $M(B(0, r))$ contains a ball centered at 0 in Y. Then using the linearity of M, $M(B(x, r))$ will contain a ball centered at Mx in Y. By linearity, to show that $M(B(0, r))$ contains a ball centered at 0, it suffices to show that $M(B(0, 1))$ contains a ball centered at 0 in Y.

Step 1. We show that there exists r such that $B(0, r2^{-n}) \subset \overline{M(B(0, 2^{-n}))}$ for each n. Since M is onto, $Y = \cup_{n=1}^{\infty} M(B(0, n))$. The Baire category theorem tells us that at least one of the sets $M(B(0, n))$ cannot be nowhere dense. Since M is linear, $M(B(0, 1))$ cannot be nowhere dense. Thus there exist y_0 and r such that $B(y_0, 4r) \subset \overline{M(B(0, 1))}$.

Pick $y_1 \in M(B(0, 1))$ such that $\|y_1 - y_0\| < 2r$ and let $z_1 \in B(0, 1)$ be such that $y_1 = Mz_1$. Then $B(y_1, 2r) \subset B(y_0, 4r) \subset \overline{M(B(0, 1))}$. Thus if $\|y\| < 2r$, then $y + y_1 \in B(y_1, 2r)$, and so

$$y = -Mz_1 + (y + y_1) \in \overline{M(-z_1 + B(0, 1))}.$$

Since $z_1 \in B(0, 1)$, then $-z_1 + B(0, 1) \subset B(0, 2)$, hence

$$y \in \overline{M(-z_1 + B(0, 1))} \subset \overline{M(B(0, 2))}.$$

By the linearity of M, if $\|y\| < r$, then $y \in \overline{M(B(0, 1))}$. It follows by linearity that if $\|y\| < r2^{-n}$, then $y \in \overline{M(B(0, 2^{-n}))}$. This can be rephrased as saying that if $\|y\| < r2^{-n}$ and $\varepsilon > 0$, then there exists x such that $\|x\| < 2^{-n}$ and $\|y - Mx\| < \varepsilon$.

Step 2. Suppose $\|y\| < r/2$. We will construct a sequence $\{x_j\}$ by induction such that $y = M(\sum_{j=1}^{\infty} x_j)$. By Step 1 with $\varepsilon = r/4$, we can find $x_1 \in$

$B(0, 1/2)$ such that $\|y - Mx_1\| < r/4$. Suppose we have chosen $x_1, \ldots, x_{n-1}$ such that

$$\Big\| y - \sum_{j=1}^{n-1} Mx_j \Big\| < r2^{-n}.$$

Let $\varepsilon = r2^{-(n+1)}$. By Step 1, we can find x_n such that $\|x_n\| < 2^{-n}$ and

$$\Big\| y - \sum_{j=1}^{n} Mx_j \Big\| = \Big\| \Big(y - \sum_{j=1}^{n-1} Mx_j \Big) - Mx_n \Big\| < r2^{-(n+1)}.$$

We continue by induction to construct the sequence $\{x_j\}$. Let $w_n = \sum_{j=1}^n x_j$. Since $\|x_j\| < 2^{-j}$, then w_n is a Cauchy sequence. Since X is complete, w_n converges, say, to x. But then $\|x\| < \sum_{j=1}^\infty 2^{-j} = 1$, and since M is continuous, $y = Mx$. That is, if $y \in B(0, r/2)$, then $y \in M(B(0, 1))$. □

Corollary 3.6 *M maps open sets onto open sets.*

Corollary 3.7 *If M is one-to-one, onto, and bounded, then M^{-1} is bounded.*

Proof. By the open mapping theorem, there exists d such that $B(0, d) \subset M(B(), 1))$. If $u \in U$ with $|u| = d/2$, there exists x with $|x| < 1$ and $Mx = u$. By homogeneity, if $u \in B_U(0, 1)$, there exists $x \in X$ with $Mx = u$ and $|x| < 2|u|/d$. So $x = M^{-1}u$ and

$$\|M^{-1}u\| = \|x\| < 2\|u\|/d,$$

so $|M^{-1}| < 2/d$. □

3.5 Closed graph theorem

A map $M : X \to U$ is *closed* if whenever $x_n \to x$ and $MX_n \to u$, then $Mx = u$. This is equivalent to the *graph* $\{x, Mx\}$ being a closed set.

If M is continuous, it is closed. If D is the differentiation operator on the set of differentiable functions on $[0, 1]$, then D is closed, but not continuous.

Theorem 3.8 *(Closed graph theorem) If X and U are Banach spaces and M a closed linear map, then M is continuous.*

Proof. Let $G = \{g = (x, Mx)\}$ with norm $\|g\| = \|x\| + \|Mx\|$. It is easy to see that $\|g\|$ is a norm, and we use the closedness of M to show that G is complete: If $g_n = (x_n, Mx_n)$ is a Cauchy sequence in G, then $\|x_n - x_m\| \leq \|g_n - g_m\|$ is a Cauchy sequence, so x_n converges, say to x. $\|Mx_n - Mx_m\| \leq \|g_n - g_m\|$, so Mx_n is a Cauchy sequence in U, and hence converges, say to y. Since G is closed, then $y = Mx$. Therefore g_n converges to (x, Mx).

Define $P : G \to X$ by $P(x, Mx) = x$, so that P is a projection onto the first coordinate.

$\|Pg\| = \|x\| \leq \|x\| + \|Mx\| = \|g\|$, so P is bounded with norm less than or equal to 1. P is linear and one-to-one, and onto, so P^{-1} is bounded, i.e., there exists c such that $c\|Pg\| \geq \|g\|$. So $(c-1)\|x\| \geq \|Mx\|$, which proves M is bounded. □

Corollary 3.9 *Suppose X has two norms such that if $\|x_n - x\|_1 \to 0$ and $\|x_n - y\|_2 \to 0$, then $x = y$. Suppose X is complete with respect to both norms. Then the norms are equivalent.*

Proof. Let $X_1 = (X, \|\cdot\|_1)$ and similarly X_2. Let $I : X_1 \to X_2$. The hypothesis is equivalent to I being closed. Therefore I and I^{-1} are bounded. □

Chapter 4

Hilbert spaces

Hilbert spaces are complete normed linear spaces that have an inner product. This added structure allows one to talk about orthonormal sets. We will give the definitions and basic properties. As an application we briefly discuss Fourier series.

4.1 Inner products

Recall that if a is a complex number, then $\overline{a}$ represents the complex conjugate. When a is real, $\overline{a}$ is just a itself.

Definition 4.1 Let H be a vector space where the set of scalars F is either the real numbers or the complex numbers. H is an *inner product space* if there is a map $\langle \cdot, \cdot \rangle$ from $H \times H$ to F such that
(1) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for all $x, y \in H$;
(2) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in H$;
(3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$, for $x, y \in H$ and $\alpha \in F$;
(4) $\langle x, x \rangle \geq 0$ for all $x \in H$;
(5) $\langle x, x \rangle = 0$ if and only if $x = 0$.

We define $\|x\| = \langle x, x \rangle^{1/2}$, so that $\langle x, x \rangle = \|x\|^2$. From the definitions it follows easily that $\langle 0, y \rangle = 0$ and $\langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$.

The following is the *Cauchy-Schwarz inequality*. The proof is the same as the one usually taught in undergraduate linear algebra classes, except for some complications due to the fact that we allow the set of scalars to be the complex numbers.

Theorem 4.2 *For all $x, y \in H$, we have*

$$|\langle x, y\rangle| \le \|x\|\,\|y\|.$$

Proof. Let $A = \|x\|^2$, $B = |\langle x, y\rangle|$, and $C = \|y\|^2$. If $C = 0$, then $y = 0$, hence $\langle x, y\rangle = 0$, and the inequality holds. If $B = 0$, the inequality is obvious. Therefore we will suppose that $C > 0$ and $B \neq 0$.

If $\langle x, y\rangle = Re^{i\theta}$, let $\alpha = e^{i\theta}$, and then $|\alpha| = 1$ and $\alpha\langle y, x\rangle = |\langle x, y\rangle| = B$. Since B is real, we have that $\overline{\alpha}\langle x, y\rangle$ also equals $|\langle x, y\rangle|$.

We have for real r

$$\begin{aligned}
0 &\le \|x - r\alpha y\|^2 \\
&= \langle x - r\alpha y, x - r\alpha y\rangle \\
&= \langle x, x\rangle - r\alpha\langle y, x\rangle - r\overline{\alpha}\langle x, y\rangle + r^2\langle y, y\rangle \\
&= \|x\|^2 - 2r|\langle x, y\rangle| + r^2\|y\|^2.
\end{aligned}$$

Therefore

$$A - 2Br + Cr^2 \ge 0$$

for all real numbers r. Since we are supposing that $C > 0$, we may take $r = B/C$, and we obtain $B^2 \le AC$. Taking square roots of both sides gives the inequality we wanted. □

From the Cauchy-Schwarz inequality we get the *triangle inequality*:

Proposition 4.3 *For all $x, y \in H$ we have*

$$\|x + y\| \le \|x\| + \|y\|.$$

Proof. We write

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&\le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2,
\end{aligned}$$

as desired. □

The triangle inequality implies

$$\|x - z\| \leq \|x - y\| + \|y - z\|.$$

Therefore $\|\cdot\|$ is a norm on H, and so if we define the distance between x and y by $\|x - y\|$, we have a metric space.

Definition 4.4 A *Hilbert space* H is an inner product space that is complete with respect to the metric $d(x, y) = \|x - y\|$.

Example 4.5 Let μ be a positive measure on a set X, let $H = L^2(\mu)$, and define

$$\langle f, g\rangle = \int f\overline{g}\, d\mu.$$

As is usual, we identify functions that are equal a.e. H is easily seen to be a Hilbert space. The completeness is a result of real analysis.

If we let μ be counting measure on the natural numbers, we get what is known as the space ℓ^2. An element of ℓ^2 is a sequence $a = (a_1, a_2, \ldots)$ such that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ and if $b = (b_1, b_2, \ldots)$, then

$$\langle a, b\rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}.$$

We get another common Hilbert space, n-dimensional Euclidean space, by letting μ be counting measure on $\{1, 2, \ldots, n\}$.

Proposition 4.6 *Let $y \in H$ be fixed. Then the functions $x \to \langle x, y\rangle$ and $x \to \|x\|$ are continuous.*

Proof. By the Cauchy-Schwarz inequality,

$$|\langle x, y\rangle - \langle x', y\rangle| = |\langle x - x', y\rangle| \leq \|x - x'\|\,\|y\|,$$

which proves that the function $x \to \langle x, y\rangle$ is continuous. By the triangle inequality, $\|x\| \leq \|x - x'\| + \|x'\|$, or

$$\|x\| - \|x'\| \leq \|x - x'\|.$$

The same holds with x and x' reversed, so

$$|\,\|x\| - \|x'\|\,| \le \|x - x'\|,$$

and thus the function $x \to \|x\|$ is continuous. □

4.2 Subspaces

Definition 4.7 A subset M of a vector space is a *subspace* if M is itself a vector space with respect to the same operations of addition and scalar multiplication. A *closed subspace* is a subspace that is closed relative to the metric given by $\langle \cdot, \cdot \rangle$.

For an example of a subspace that is not closed, consider ℓ^2 and let M be the collection of sequences for which all but finitely many elements are zero. M is clearly a subspace. Let $x_n = (1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots)$ and $x = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$. Then each $x_n \in M$, $x \notin M$, and we conclude M is not closed because

$$\|x_n - x\|^2 = \sum_{j=n+1}^{\infty} \frac{1}{j^2} \to 0$$

as $n \to \infty$.

Since $\|x+y\|^2 = \langle x+y, x+y \rangle$ and similarly for $\|x-y\|^2$, $\|x\|^2$, and $\|y\|^2$, a simple calculation yields the *parallelogram law*:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{4.1}$$

A set $E \subset H$ is *convex* if $\lambda x + (1 - \lambda x) \in E$ whenever $0 \le \lambda \le 1$ and $x, y \in E$.

Proposition 4.8 *Each non-empty closed convex subset E of H has a unique element of smallest norm.*

Proof. Let $\delta = \inf\{\|x\| : x \in E\}$. Dividing (4.1) by 4, if $x, y \in E$, then

$$\tfrac{1}{4}\|x-y\|^2 = \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2 - \left\|\frac{x+y}{2}\right\|^2.$$

Since E is convex, if $x, y \in E$, then $(x+y)/2 \in E$, and we have

$$\|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - 4\delta^2. \tag{4.2}$$

Choose $y_n \in E$ such that $\|y_n\| \to \delta$. Applying (4.2) with x replaced by y_n and y replaced by y_m, we see that

$$\|y_n - y_m\|^2 \leq 2\|y_n\|^2 + 2\|y_m\|^2 - 4\delta^2,$$

and the right hand side tends to 0 as m and n tend to infinity. Hence y_n is a Cauchy sequence, and since H is complete, it converges to some $y \in H$. Since $y_n \in E$ and E is closed, $y \in E$. Since the norm is a continuous function, $\|y\| = \lim \|y_n\| = \delta$.

If y' is another point with $\|y'\| = \delta$, then by (4.2) with x replaced by y' we have $\|y - y'\| = 0$, and hence $y = y'$. □

We say $x \perp y$, or x is *orthogonal* to y, if $\langle x, y\rangle = 0$. Let $x^\perp$, read "x perp," be the set of all y in X that are orthogonal to x. If M is a subspace, let $M^\perp$ be the set of all y that are orthogonal to all points in M. The subspace $M^\perp$ is called the *orthogonal complement* of M. It is clear from the linearity of the inner product that $x^\perp$ is a subspace of H. The subspace $x^\perp$ is closed because it is the same as the set $f^{-1}(\{0\})$, where $f(x) = \langle x, y\rangle$, which is continuous. Also, it is easy to see that $M^\perp$ is a subspace, and since

$$M^\perp = \cap_{x\in M} x^\perp,$$

$M^\perp$ is closed. We make the observation that if $z \in M \cap M^\perp$, then

$$\|z\|^2 = \langle z, z\rangle = 0,$$

so $z = 0$.

Lemma 4.9 *Let M be a closed subspace of H with $M \neq H$. Then $M^\perp$ contains a non-zero element.*

Proof. Choose $x \in H$ with $x \notin M$. Let $E = \{w - x : w \in M\}$. It is routine to check that E is a closed and convex subset of H. By Lemma 4.8, there exists an element $y \in E$ of smallest norm.

Note $y + x \in M$ and we conclude $y \neq 0$ because $x \notin M$.

We show $y \in M^\perp$ by showing that if $w \in M$, then $\langle w, y\rangle = 0$. This is obvious if $w = 0$, so assume $w \neq 0$. We know $y + x \in M$, so for any real number t we have $tw + (y + x) \in M$, and therefore $tw + y \in E$. Since y is the element of E of smallest norm,

$$\begin{aligned}\langle y, y\rangle &= \|y\|^2 \leq \|tw + y\|^2 \\ &= \langle tw + y, tw + y\rangle \\ &= t^2\langle w, w\rangle + 2t\mathrm{Re}\,\langle w, y\rangle + \langle y, y\rangle,\end{aligned}$$

which implies

$$t^2\langle w, w\rangle + 2t\mathrm{Re}\,\langle w, y\rangle \geq 0$$

for each real number t. Choosing $t = -\mathrm{Re}\,\langle w, y\rangle/\langle w, w\rangle$, we obtain

$$-\frac{|\mathrm{Re}\,\langle w, y\rangle|^2}{\langle w, w\rangle} \geq 0,$$

from which we conclude $\mathrm{Re}\,\langle w, y\rangle = 0$.

Since $w \in M$, then $iw \in M$, and if we repeat the argument with w replaced by iw, then we get $\mathrm{Re}\,\langle iw, y\rangle = 0$, and so

$$\mathrm{Im}\,\langle w, y\rangle = -\mathrm{Re}\,(i\langle w, y\rangle) = -\mathrm{Re}\,\langle iw, y\rangle = 0.$$

Therefore $\langle w, y\rangle = 0$ as desired. □

If in the proof above we set $Px = y + x$ and $Qx = -y$, then $Px \in M$ and $Qx \in M^\perp$, and we can write $x = Px+Qx$. We call Px and Qx the orthogonal projections of x onto M and $M^\perp$, resp. If we write $x = w + z = w' + z'$, with $w, w' \in M$ and $z, z' \in M^\perp$, then $w - w' = z' - z$ is in M and $M^\perp$, hence 0. Therefore each element of H can be written as the sum of an element of M and an element of $M^\perp$ in exactly one way.

Proposition 4.10 *P and Q are linear operators.*

Proof. Since

$$x + y = (Px + Qx) + (Py + Qy)$$

and also $x + y = P(x + y) + Q(x + y)$, we have

$$Px + Py - P(x + y) = Q(x + y) - Qx - Qy.$$

The left hand side is in M, the right hand side in $M^\perp$, so $Px+Py = P(x+y)$ and similarly for Q. To show $P(kx) = kPx$ and $Q(kx) = kQx$ is similar, so P and Q are linear. □

Proposition 4.11 *Suppose M is a closed subspace of H. Then*

1) $M^\perp$ is a closed linear subspace of H.

2) $H = M \oplus M^\perp$.

3) $(M^\perp)^\perp = M$.

Proof. 1) (We don't need M closed for this first part.) That $M^\perp$ is a linear subspace is easy. We already showed $M^\perp$ is closed.

2) We proved this: write $x = Px + Qx$.

3) If $y \in M$, then for any $v \in M^\perp$ we have $(y, v) = 0$, and hence $y \in (M^\perp)^\perp$. We thus need to show $(M^\perp)^\perp \subset M$.

By 2), $H = M \oplus M^\perp$. If $y \in (M^\perp)^\perp$, we can write $y = v + z$ with $z \in M \subset (M^\perp)^\perp$ and $v \in M^\perp$. Then $v = y - z \in (M^\perp)^\perp$. Since v is also in $M^\perp$, we see $v = 0$, or $y = z \in M$. □

4.3 Riesz representation theorem

The following is sometimes called the *Riesz representation theorem*, although usually that name is reserved for the theorem of real analysis about linear functionals on the set of continuous functions. To motivate the theorem, consider the case where H is n-dimensional Euclidean space. Elements of $\mathbb{R}^n$ can be identified with $n \times 1$ matrices and linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by multiplication on the left by a $m \times n$ matrix A. For bounded linear functionals on H, $m = 1$, so A is $1 \times n$, and the y of the next theorem is the vector associated with the transpose of A.

Theorem 4.12 *If L is a bounded linear functional on H, then there exists a unique $y \in H$ such that $Lx = \langle x, y \rangle$.*

Proof. The uniqueness is easy. If $Lx = \langle x, y \rangle = \langle x, y' \rangle$, then $\langle x, y - y' \rangle = 0$ for all x, and in particular, when $x = y - y'$.

We now prove existence. If $Lx = 0$ for all x, we take $y = 0$. Otherwise, let $M = \{x : Lx = 0\}$, take $z \neq 0$ in $M^\perp$, and let $y = \alpha z$ where $\alpha = \overline{Lz}/\langle z, z \rangle$. Notice $y \in M^\perp$,

$$Ly = \frac{\overline{Lz}}{\langle z, z \rangle} Lz = |Lz|^2/\langle z, z \rangle = \langle y, y \rangle,$$

and $y \neq 0$.

If $x \in H$ and

$$w = x - \frac{Lx}{\langle y, y \rangle} y,$$

then $Lw = 0$, so $w \in M$, and hence $\langle w, y \rangle = 0$. Then

$$\langle x, y \rangle = \langle x - w, y \rangle = Lx$$

as desired. □

4.4 Lax-Milgram lemma

Theorem 4.13 *(Lax-Milgram lemma) Let H be a Hilbert space and suppose*
(1) for each y, $B(x, y)$ is linear in x;
(2) for each x, $B(x, y_1 + y_2) = B(x, y_1) + B(x, y_2)$ and $B(x, cy) = \bar{c}B(x, y)$;
(3) there exists c such that $|B(x, y| \leq c\|x\| \, \|y\|$;
(4) there exists b such that $|B(y, y)| \geq b\|y\|^2$ for all y.
(We do not assume $B(x, y) = \overline{B(y, x)}$.) Then every bounded linear functional ℓ is of the form $\ell(x) = B(x, y)$ for some unique y.

Proof. For each y, $B(x, y)$ is a bounded linear functional of x, so there exists $z = z(y)$ such that $B(x, y) = (x, z)$ for all x, and z is unique.

If $Z = \{z : z = z(y) \text{ for some } y \in H\}$, then Z is a linear space.

Z is closed: setting $x = y$, and letting $z = z(y)$,

$$b\|y\|^2 \le B(y,y) = (y,z) \le c\|y\|\,\|z\|,$$

or

$$b\|y\| \le \|z\|.$$

If $z_n \in Z$ and $z_n \to z$, let y_n be a point such that $z_n = z(y_n)$. Then $B(x, y_n) = (x, z_n)$. So $B(x, y_n - y_m) = (x, z_n - z_m)$, hence $b\|y_n - y_m\| \le \|z_n - z_m\|$, and therefore y_n is a Cauchy sequence. H is complete; let y be the limit. Since $B(x, y_n) \to B(x, y)$ and $(x, z_n) \to (x, z)$, we have $B(x, y) = (x, z)$, and hence $z \in Z$.

$Z = H$: For each y, there exists $z(y)$ such that $B(x, y) = (x, z)$ for all y. If $Z \ne H$, there exists $x \in Z^\perp$. Applying the above with $y = x$, there exists $z(x)$ such that $B(x, x) = (x, z(x))$. Since $x \in Z^\perp$ and $z(x) \in Z$, $b\|x\|^2 \le B(x, x) = (x, z(x)) = 0$. So $x = 0$.

Existence: given ℓ, there exists y such that $\ell(x) = (x, y)$ for all x. Then $\ell(x) = B(x, z(y))$.

Uniqueness: if there are two such z, then $B(x, z-z') = B(x, z) - B(x, z') = \ell(x) - \ell(x) = 0$. Now set $x = z - z'$. □

4.5 Orthonormal sets

A subset $\{u_\alpha\}_{\alpha \in A}$ of H is *orthonormal* if $\|u_\alpha\| = 1$ for all α and $\langle u_\alpha, u_\beta \rangle = 0$ whenever $\alpha, \beta \in A$ and $\alpha \ne \beta$.

The *Gram-Schmidt procedure* from linear algebra also works in infinitely many dimensions. Suppose $\{x_n\}_{n=1}^\infty$ is a linearly independent sequence, i.e., no finite linear combination of the x_n is 0. Let $u_1 = x_1/\|x_1\|$ and define inductively

$$v_N = x_N - \sum_{i=1}^{N-1} \langle x_N, u_i \rangle u_i,$$
$$u_N = v_N/\|v_N\|.$$

We have $\langle v_N, u_i \rangle = 0$ if $i < N$, so $u_1, \ldots, u_N$ are orthonormal.

Proposition 4.14 *If* $\{u_\alpha\}_{\alpha\in A}$ *is an orthonormal set, then for each* $x \in H$*,*

$$\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 \le \|x\|^2. \tag{4.3}$$

This is called *Bessel's inequality*. This inequality implies that only finitely many of the summands on the left hand side of (4.3) can be larger than $1/n$ for each n, hence only countably many of the summands can be non-zero.

Proof. Let F be a finite subset of A. Let

$$y = \sum_{\alpha\in F} \langle x, u_\alpha\rangle u_\alpha.$$

Then

$$0 \le \|x-y\|^2 = \|x\|^2 - \langle x, y\rangle - \langle y, x\rangle + \|y\|^2.$$

Now

$$\langle y, x\rangle = \Big\langle \sum_{\alpha\in F} \langle x, u_\alpha\rangle u_\alpha, x\Big\rangle = \sum_{\alpha\in F} \langle x, u_\alpha\rangle \langle u_\alpha, x\rangle = \sum_{\alpha\in F} |\langle x, u_\alpha\rangle|^2.$$

Since this is real, then $\langle x, y\rangle = \langle y, x\rangle$. Also

$$\begin{aligned}\|y\|^2 = \langle y, y\rangle &= \Big\langle \sum_{\alpha\in F} \langle x, u_\alpha\rangle u_\alpha, \sum_{\beta\in F} \langle x, u_\beta\rangle u_\beta\Big\rangle\\ &= \sum_{\alpha,\beta\in F} \langle x, u_\alpha\rangle \overline{\langle x, u_\beta\rangle} \langle u_\alpha, u_\beta\rangle\\ &= \sum_{\alpha\in F} |\langle x, u_\alpha\rangle|^2,\end{aligned}$$

where we used the fact that $\{u_\alpha\}$ is an orthonormal set. Therefore

$$0 \le \|y-x\|^2 = \|x\|^2 - \sum_{\alpha\in F} |\langle x, u_\alpha\rangle|^2.$$

Rearranging,

$$\sum_{\alpha\in F} |\langle x, u_\alpha\rangle|^2 \le \|x\|^2$$

when F is a finite subset of A. If N is an integer larger than $n\|x\|^2$, it is not possible that $|\langle x, u_\alpha\rangle|^2 > 1/n$ for more than N of the α. Hence $|\langle x, u_\alpha\rangle|^2 \neq 0$ for only countably many α. Label those α's as $\alpha_1, \alpha_2, \ldots$. Then

$$\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 = \sum_{j=1}^{\infty} |\langle x, u_{\alpha_j}\rangle|^2 = \lim_{J\to\infty} \sum_{j=1}^{J} |\langle x, u_{\alpha_j}\rangle|^2 \leq \|x\|^2,$$

which is what we wanted. □

Proposition 4.15 *Suppose $\{u_\alpha\}_{\alpha\in A}$ is orthonormal. Then the following are equivalent.*
(1) If $\langle x, u_\alpha\rangle = 0$ for each $\alpha \in A$, then $x = 0$.
(2) $\|x\|^2 = \sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2$ for all x.
(3) For each $x \in H$, $x = \sum_{\alpha\in A} \langle x, u_\alpha\rangle u_\alpha$.

We make a few remarks. When (1) holds, we say the orthonormal set is *complete*. (2) is called *Parseval's identity*. In (3) the convergence is with respect to the norm of H and implies that only countably many of the terms on the right hand side are non-zero.

Proof. First we show (1) implies (3). Let $x \in H$. By Bessel's inequality, there can be at most countably many α such that $|\langle x, u_\alpha\rangle|^2 \neq 0$. Let $\alpha_1, \alpha_2, \ldots$ be an enumeration of those α. By Bessel's inequality, the series $\sum_i |\langle x, u_{\alpha_i}\rangle|^2$ converges. Using that $\{u_\alpha\}$ is an orthonormal set,

$$\Big\| \sum_{j=m}^{n} \langle x, u_{\alpha_j}\rangle u_{\alpha_j} \Big\|^2 = \sum_{j,k=m}^{n} \langle x, u_{\alpha_j}\rangle \overline{\langle x, u_{\alpha_k}\rangle} \langle u_{\alpha_j}, u_{\alpha_k}\rangle$$
$$= \sum_{j=m}^{n} |\langle x, u_{\alpha_j}\rangle|^2 \to 0$$

as $m, n \to \infty$. Thus $\sum_{j=1}^{n} \langle x, u_{\alpha_j}\rangle u_{\alpha_j}$ is a Cauchy sequence, and hence converges. Let $z = \sum_{j=1}^{\infty} \langle x, u_{\alpha_j}\rangle u_{\alpha_j}$. Then $\langle z - x, u_{\alpha_j}\rangle = 0$ for each α_j. By (1), this implies $z - x = 0$.

We see that (3) implies (2) because

$$\|x\|^2 - \sum_{j=1}^{n} |\langle x, u_{\alpha_j}\rangle|^2 = \Big\| x - \sum_{j=1}^{n} \langle x, u_{\alpha_j}\rangle u_{\alpha_j} \Big\|^2 \to 0.$$

That (2) implies (1) is clear. □

Example 4.16 Take $H = \ell^2 = \{x = (x_1, x_2, \ldots) : \sum |x_i|^2 < \infty\}$ with $\langle x, y\rangle = \sum_i x_i \overline{y_i}$. Then $\{e_i\}$ is a complete orthonormal system, where $e_i = (0, 0, \ldots, 0, 1, 0, \ldots)$, i.e., the only non-zero coordinate of e_i is the i^{th} one.

If K is a subset of a Hilbert space H, the set of finite linear combinations of elements of K is called the *span* of K.

A collection of elements $\{e_\alpha\}$ is a *basis* for H if the set of finite linear combinations of the e_α is dense in H. A basis, then, is an orthonormal subset of H such that the closure of its span is all of H.

Proposition 4.17 *Every Hilbert space has an orthonormal basis.*

This means that (3) in Proposition 4.15 holds.

Proof. If $B = \{u_\alpha\}$ is orthonormal, but not a basis, let V be the closure of the linear span of B, that is, the closure with respect to the norm in H of the set of finite linear combinations of elements of B. Choose $x \in V^\perp$, and if we let $B' = B \cup \{x/\|x\|\}$, then B' is a basis that is strictly bigger than B.

It is easy to see that the union of an increasing sequence of orthonormal sets is an orthonormal set, and so there is a maximal one by Zorn's lemma. By the preceding paragraph, this maximal orthonormal set must be a basis, for otherwise we could find a larger basis. □

4.6 Fourier series

An interesting application of Hilbert space techniques is to *Fourier series*, or equivalently, to *trigonometric series*. For our Hilbert space we take $H = L^2([0, 2\pi))$ and let

$$u_n = \frac{1}{\sqrt{2\pi}} e^{inx}$$

for n an integer. (n can be negative.) Recall that

$$\langle f, g\rangle = \int_0^{2\pi} f(x)\overline{g(x)}\,dx$$

and $\|f\|^2 = \int_0^{2\pi} |f(x)|^2\,dx$.

It is easy to see that $\{u_n\}$ is an orthonormal set:

$$\int_0^{2\pi} e^{inx}e^{-imx}\,dx = \int_0^{2\pi} e^{i(n-m)x}\,dx = 0$$

if $n \neq m$ and equals 2π if $n = m$.

Let $\mathcal{F}$ be the set of finite linear combinations of the u_n, i.e., the span of $\{u_n\}$. We want to show that $\mathcal{F}$ is a dense subset of $L^2([0, 2\pi))$. The first step is to show that the closure of $\mathcal{F}$ with respect to the supremum norm is equal to the set of continuous functions f on $[0, 2\pi)$ with $f(0) = f(2\pi)$. We will accomplish this by using the Stone-Weierstrass theorem.

We identify the set of continuous functions on $[0, 2\pi)$ that take the same value at 0 and 2π with the continuous functions on the circle. To do this, let $S = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$ be the unit circle in $\mathbb{C}$. If f is continuous on $[0, 2\pi)$ with $f(0) = f(2\pi)$, define $\widetilde{f} : S \to \mathbb{C}$ by $\widetilde{f}(e^{i\theta}) = f(\theta)$. Note $\widetilde{u}_n(e^{i\theta}) = e^{in\theta}$.

Let $\widetilde{\mathcal{F}}$ be the set of finite linear combinations of the $\widetilde{u}_n$. S is a compact metric space. Since the complex conjugate of $\widetilde{u}_n$ is $\widetilde{u}_{-n}$, then $\widetilde{\mathcal{F}}$ is closed under the operation of taking complex conjugates. Since $\widetilde{u}_n \cdot \widetilde{u}_m = \widetilde{u}_{n+m}$, it follows that $\mathcal{F}$ is closed under the operation of multiplication. That it is closed under scalar multiplication and addition is obvious. $\widetilde{u}_0$ is identically equal to 1, so $\widetilde{\mathcal{F}}$ vanishes at no point. If $\theta_1, \theta_2 \in S$ and $\theta_1 \neq \theta_2$, then $\theta_1 - \theta_2$ is not an integer multiple of 2π, so

$$\frac{\widetilde{u}_1(\theta_1)}{\widetilde{u}_1(\theta_2)} = e^{i(\theta_1-\theta_2)} \neq 1,$$

or $\widetilde{u}_1(\theta_1) \neq \widetilde{u}_1(\theta_2)$. Therefore $\mathcal{F}$ separates points. By the Stone-Weierstrass theorem, the closure of $\mathcal{F}$ with respect to the supremum norm is equal to the set of continuous complex-valued functions on S.

If $f \in L^2([0, 2\pi))$, then

$$\int |f - f\chi_{[1/m, 2\pi-1/m]}|^2 \to 0$$

by the dominated convergence theorem as $m \to \infty$. Recall that any function in $L^2([1/m, 2\pi - 1/m])$ can be approximated in L^2 by continuous functions which have support in the interval $[1/m, 2\pi - 1/m]$. By what we showed above, a continuous function with support in $[1/m, 2\pi - 1/m]$ can be approximated uniformly on $[0, 2\pi)$ by elements of $\mathcal{F}$. Finally, if g is continuous on $[0, 2\pi)$ and $g_m \to g$ uniformly on $[0, 2\pi)$, then $g_m \to g$ in $L^2([0, 2\pi))$ by the dominated convergence theorem. Putting all this together proves that $\mathcal{F}$ is dense in $L^2([0, 2\pi))$.

It remains to show the completeness of the u_n. If f is orthogonal to each u_n, then it is orthogonal to every finite linear combination, that is, to every element of $\mathcal{F}$. Since $\mathcal{F}$ is dense in $L^2([0, 2\pi))$, we can find $f_n \in \mathcal{F}$ tending to f in L^2. Then

$$\|f\|^2 = |\langle f, \overline{f}\rangle| \le |\langle f - f_n, \overline{f}\rangle| + |\langle f_n, \overline{f}\rangle|.$$

The second term on the right of the inequality sign is 0. The first term on the right of the inequality sign is bounded by $\|f - f_n\|\, \|f\|$ by the Cauchy-Schwarz inequality, and this tends to 0 as $n \to \infty$. Therefore $\|f\|^2 = 0$, or $f = 0$, hence the $\{u_n\}$ are complete. Therefore $\{u_n\}$ is a complete orthonormal system.

Given f in $L^2([0, 2\pi))$, write

$$c_n = \langle f, u_n\rangle = \int_0^{2\pi} f\overline{u_n}\, dx = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-inx}\, dx,$$

the Fourier coefficients of f. Parseval's identity says that

$$\|f\|^2 = \sum_n |c_n|^2.$$

For any f in L^2 we also have

$$\sum_{|n| \le N} c_n u_n \to f$$

as $N \to \infty$ in the sense that

$$\Big\| f - \sum_{|n| \le N} c_n u_n \Big\|_2 \to 0$$

as $N \to \infty$.

Using $e^{inx} = \cos nx + i \sin nx$, we have

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = A_0 + \sum_{n=1}^{\infty} B_n \cos nx + \sum_{n=1}^{\infty} C_n \sin nx,$$

where $A_0 = c_0$, $B_n = c_n + c_{-n}$, and $C_n = i(c_n - c_{-n})$. Conversely, using $\cos nx = (e^{inx} + e^{-inx})/2$ and $\sin nx = (e^{inx} - e^{-inx})/2i$,

$$A_0 + \sum_{n=1}^{\infty} B_n \cos nx + \sum_{n=1}^{\infty} C_n \sin nx = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

if we let $c_0 = A_0$, $c_n = B_n/2 + C_n/2i$ for $n > 0$ and $c_n = B_n/2 - C_n/2i$ for $n < 0$. Thus results involving the u_n can be transferred to results for series of sines and cosines and vice versa.

If c_n are the Fourier coefficients of f, then

$$\sum_{-N}^{N} c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) k_N(\theta)\, d\theta,$$

where

$$\begin{aligned} k_N(\theta) = \sum_{n=-N}^{N} e^{-in\theta} &= e^{iN\theta} \Big(\frac{e^{-(2N+1)i\theta} - 1}{e^{-i\theta} - 1} \Big) \\ &= \frac{e^{-(N+1)i\theta} - e^{iN\theta}}{e^{-i\theta} - 1} \\ &= \frac{e^{-(N+\frac{1}{2})i\theta} - e^{i(N+\frac{1}{2})\theta}}{e^{-i\theta/2} - e^{i\theta/2}} \\ &= \frac{\sin(N + \frac{1}{2})\theta}{\sin \theta/2}. \end{aligned}$$

4.7 The Radon-Nikodym theorem

We can use Hilbert space techniques to give an alternate proof of the *Radon-Nikodym theorem.*

Suppose μ and ν are finite measures on a space S and we have the condition $\nu(A) \le \mu(A)$ for all measurable A. For $f \in L^2(\mu)$, define

$$\ell(f) = \int f \, d\nu.$$

Our condition implies $\int h \, d\nu \le \int h \, d\mu$ if $h \ge 0$. We use this with $h = |f|$, use Cauchy-Schwarz, and obtain

$$\begin{aligned} |\ell(f)| = \Big| \int f \, d\nu \Big| &\le \int |f| \, d\nu \le \int |f| \, d\mu \\ &\le \Big(\mu(S)\Big)^{1/2} \Big(\int f^2 \, d\mu \Big)^{1/2} \le c \|f\|_{L^2(\mu)}. \end{aligned}$$

There exists g such that $\ell(f) = \langle f, g \rangle$, which translates to

$$\int f \, d\nu = \int f g \, d\mu.$$

Letting $f = \chi_A$, we get $\nu(A) = \int_A g \, d\mu$.

If ν is absolutely continuous with respect to μ, we let $\rho = \mu + \nu$ and apply the above to ν and ρ and also to μ and ρ. The absolute continuity implies that $d\mu/d\rho > 0$ a.e., and we use

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\rho} \Big/ \frac{d\mu}{d\rho}.$$

4.8 The Dirichlet problem

Let D be a bounded domain in $\mathbb{R}^n$, contained in $B(0, K)$, say, where this is the ball of radius K about 0. Let $\langle f, g \rangle$ be the usual L^2 scalar product for real valued functions. It is easy to see that if $C_0^\infty(D)$ is the set of C^∞ functions that vanish on the boundary of D, then the completion of $C^\infty(D)$ with respect to the L^2 norm is simply $L^2(D)$. Define

$$\mathcal{E}(f, g) = \int_D \langle \nabla f(x), \nabla g(x) \rangle \, dx.$$

Clearly $\mathcal{E}$ is bilinear and symmetric.

If we start with

$$f(x_1, \dots, x_n) = \int_{-K}^{x_1} \frac{\partial f}{\partial x_1}(y, x_2, \dots, x_n)\, dy$$

and apply Cauchy-Schwarz, we have

$$|f(x_1, \dots, x_n)|^2 \le \int_{-K}^{K} 1\, dy \int_{-K}^{K} |\nabla f(y, x_2, \dots, x_n)|^2\, dy.$$

Integrating over $(x_2, \dots, x_n) \in [-K, K]^{n-1}$ we obtain

$$\int_D |f(x)|^2\, dx \le c \int_D |\nabla f(x)|^2\, dx,$$

or in other words,

$$\langle f, f \rangle \le c\mathcal{E}(f, f).$$

If $\mathcal{E}(f, f) = 0$, then $\langle f, f \rangle = 0$, and so $f = 0$ (a.e., of course). This proves that $\mathcal{E}$ is positive. We let H_0^1 be the completion of $C_0^\infty(D)$ with respect to the norm induced by $\mathcal{E}$. The superscript 1 refers to the fact we are working with first derivatives, the subscript 0 to the fact that our functions vanish on the boundary. $\mathcal{E}$ is an example of a *Dirichlet form*.

Recall the *divergence theorem*:

$$\int_{\partial D} (F, n)\, d\sigma = \int_D \operatorname{div} F\, dx,$$

where D is a reasonably smooth domain, ∂D is the boundary of D, n is the outward pointing unit normal, and σ is surface measure. In three dimensions, this is also known as Gauss' theorem, and along with Green's theorem and Stokes' theorem are consequences of the fundamental theorem of calculus.

If we apply the divergence theorem to $F = u\, \nabla v$, then

$$\frac{\partial}{\partial x_1} F_1 = \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} + u \frac{\partial^2 v}{\partial x_1^2},$$

and so

$$\operatorname{div} F = \langle \nabla u, \nabla v \rangle + u \Delta v,$$

where Δv is the Laplacian. Also,

$$\langle \operatorname{div} F, n\rangle = u\frac{\partial v}{\partial n},$$

where $\frac{\partial v}{\partial n}$ is the normal derivative of v. We then get Green's first identity:

$$\int_D u\Delta v + \int_D \langle \nabla u, \nabla v\rangle = \int_{\partial D} u\frac{\partial v}{\partial n}.$$

Our goal is to solve the equation $\Delta v = g$ in D with $v = 0$ on the boundary of D. This is *Poisson's equation*, while the *Dirichlet problem* more properly refers to the equation $\Delta v = 0$ in D with v equal to some pre-specified function f on the boundary of D.

If we have a solution v and $u \in C_0^\infty(D)$, then by Green's identity we get

$$\int_D u(x)g(x)\,dx = -\int_D \langle \nabla u(x), \nabla v(x)\rangle\,dx.$$

So one way of formulating a (weak) solution to Poisson's equation is: given $g \in L^2(D)$, find $v \in H_0^1$ such that

$$\mathcal{E}(u, v) = -\int ug$$

for all $u \in C_0^\infty(D)$.

After all this, it is easy to find a weak solution to the Poisson equation. Suppose $g \in H_0^1$. Define $\ell(u) = -(u, g)$. Then

$$|\ell(u)| \le \|g\|\,\|u\| \le c\|g\|\mathcal{E}(u, u)^{1/2}.$$

By the Riesz representation theorem for Hilbert spaces, there exists $v \in H_0^1$ such that $\ell(u) = \mathcal{E}(u, v)$ for all u. So

$$\mathcal{E}(u, v) = \ell(u) = -\langle u, g\rangle,$$

and v is the desired solution.

Chapter 5

Duals of normed linear spaces

5.1 Bounded linear functionals

If X is a normed linear space, a *linear functional* ℓ is a linear map from X to F, the field of scalars. ℓ is continuous if $\|x_n - x\| \to 0$ implies $\ell(x_n) \to \ell(x)$. ℓ is bounded if there exists c such that $|\ell(x)| \le c\|x\|$ for all x.

Theorem 5.1 *A linear functional ℓ is continuous if and only if it is bounded.*

This is a special case of Proposition 2.1.

The collection of all continuous linear functionals of X is called the dual of X, written X' or X^*.

Note $N_\ell = \ell^{-1}(\{0\})$ is closed, since ℓ is continuous.

Define

$$\|\ell\| = \sup_{\|x\| \neq 0} \frac{|\ell(x)|}{\|x\|}.$$

By linearity, this is the same as $\sup_{\|x\|=1} |\ell(x)|$.

Proposition 5.2 *X^* is a Banach space.*

This is Proposition 3.2.

5.2 Extensions of bounded linear functionals

Proposition 5.3 *Let X be a normed linear space, Y a subspace, ℓ a linear functional on Y with $|\ell(y)| \le c\|y\|$ for all $y \in Y$. Then ℓ can be extended to a bounded linear functional on X with the same bound on X as on Y.*

Proof. This is the Hahn-Banach theorem with $p(x) = c\|x\|$. □

$y_1, \ldots, y_N$ are said to be linearly independent if $\sum_{i=1}^N c_i y_i = 0$ implies all the c_i are zero.

Theorem 5.4 *Suppose $y_1, \ldots, y_N$ are linearly independent and $a_1, \ldots, a_N$ are scalars. Then there exists a bounded linear functional ℓ such that $\ell(y_j) = a_j$.*

Proof. Let Y be the span of $y_1, \ldots, y_N$. If $y \in Y$, then y can be written as $\sum b_j y_j$ in only one way, for if $\sum b'_j y_j$ is another way, then

$$\sum (b_j - b'_j) y_j = y - y = 0,$$

and so $b_j = b'_j$ for all j. Define

$$\ell\Big(\sum b_j y_j\Big) = \sum a_j b_j.$$

Now use the preceding theorem to extend ℓ to all of X. □

Theorem 5.5 *If X is a normed linear space, then*

$$\|y\| = \max_{\|\ell\|=1} |\ell(y)|.$$

Proof. $|\ell(y)| \le \|\ell\|\,\|y\|$, so the maximum on the right hand side is less than or equal to y.

If $y \in X$, let $Y = \{ay\}$ and define $\ell(ay) = a\|y\|$. Then the norm of ℓ on Y is 1. Now extend ℓ to all of X so as to have norm 1. □

Theorem 5.6 *(Spanning criterion) Let Y be the closed linear span of $\{y_j\}$. Suppose that whenever ℓ is a bounded linear functional such that $\ell(y_j) = 0$ for all j, then $\ell(z) = 0$. We conclude that $z \in Y$.*

Proof. If $\ell(y_j) = 0$ for all j, then $\ell(y)$ for all y of the form $\sum a_j y_j$, and by continuity of ℓ, for all $y \in Y$.

Suppose $z \notin Y$. Then

$$\inf_{y \in Y} \|z - y\| = d > 0.$$

Let $Z = \{y + az : y \in Y\}$. Define ℓ_0 on Z by $\ell_0(y + az) = a$. We note that ℓ is well defined, since if $y + az = y' + a'z$, then $(a' - a)z = y - y'$. Since $z \notin Y$ and $y - y' \in Y$, then $a' - a = 0$, and it follows that $y = y'$. Then

$$\|y + az\| = |a| \left\| -\frac{-y}{a} + z \right\| \geq d\|a\|.$$

Therefore on Z, ℓ_0 is bounded by d^{-1}. Extend ℓ_0 to all of X. But then $\ell_0(y_j) = 0$ while $\ell_0(z) = 1$. □

5.3 Uniform boundedness

Theorem 5.7 *Let X be a Banach space and $\{\ell_\nu\}$ a collection of bounded functionals such that $|\ell_\nu(x)| \leq M(x)$ for all ν and each x. Then there exists c such that $\|\ell_\nu\| \leq c$.*

In other words, if the ℓ_ν are bounded pointwise, they are bounded uniformly.

This is just a special case of the uniform boundedness principle (Banach-Steinhaus theorem).

5.4 Reflexive spaces

If $x \in X$, define the linear functional L_x on X^* by

$$L_x(\ell) = \ell(x).$$

It is clear that L_x is linear. Since

$$|L_x(\ell)| = |\ell(x)| \le \|\ell\| \, \|x\|,$$

we see that $\|L_x\| \le \|x\|$. Define ℓ' on $Y = \{ax\}$ by $\ell'(ax) = a\|x\|$. Note the norm of ℓ' on Y is 1. Use Hahn-Banach to extend this to a linear functional on X. Then

$$|L_x(\ell')| = |\ell'(x)| = \|x\|,$$

and since $\|\ell'\| = 1$, we conclude $\|L_x\| = \|x\|$. So we can isomorphically embed X into X^{**}.

Corollary 5.8 *Let X be a normed linear space, $\{x_\nu\}$ a subset such that for all $\ell \in X^*$ we have*

$$|\ell(x_\nu)| \le M(\ell) \qquad \text{for all } x_\nu.$$

Then there exists c such that $|x_\nu| \le c$ for all x_ν.

Proof. Write $L_\nu(\ell) = \ell(x_\nu)$. So each x_ν acts as a bounded linear functional on X^*. □

A Banach space is *reflexive* if $X^{**} = X$.

If $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the dual of L^p is isomorphic to L^q. Hence the L^p spaces are reflexive.

Theorem 5.9 *Hilbert spaces are reflexive.*

Proof. Recall $X^* = X$, and the result follows from this. To see $X^* = X$, if ℓ is a linear functional, there exists y such that $\ell(x) = \langle x, y\rangle$ for all x. If we show $|\ell| = \|y\|$, this gives an isometry between X and X^*. By Cauchy-Schwarz, $|\ell(x)| = |\langle x, y\rangle| \le \|x\| \, \|y\|$, so $\|\ell\| \le \|y\|$. Taking $x = y$, $\ell(y) = \|y\|^2$, hence $\|\ell\| \ge \|y\|$. □

Proposition 5.10 *If X is a normed linear space over $\mathbb{C}$ and X^* is separable, then X is separable.*

Proof. Since X^* is separable, there is a countable dense subset $\{\ell_n\}$. Recall $\|\ell_n\| = \sup_{\|x\|=1} |\ell_n(x)|$. So for each n there exists $x_n \in X$ such that $\|x_n\| = 1$ and $\ell_n(x_n) > \frac{1}{2}\|\ell_n\|$.

We claim the linear span of $\{x_n\}$ is dense in X. To prove this, we start by showing that if ℓ is a linear functional on X that vanishes on $\{x_n\}$, then ℓ vanishes identically.

Suppose not and that there exists ℓ such that $\ell(x_n) = 0$ for all x_n but $\ell \neq 0$. We can normalize so that $\|\ell\| = 1$. Since the ℓ_n are dense in X^*, there exists ℓ_n such that $\|\ell - \ell_n\| < 1/3$. Therefore $\|\ell_n\| > 2/3$. Then

$$\tfrac{1}{3} > |(\ell - \ell_n)(x_n)| = |\ell_n(x_n)| > \tfrac{1}{2}\|\ell_n\| > \tfrac{1}{2} \cdot \tfrac{2}{3},$$

a contradiction.

If $z \in X$, any linear functional that vanishes on $\{x_n\}$ is identically zero. By the spanning criterion, z is in the closed linear span of the x_n, and thus the closed linear span is all of X. Then the set of finite linear combinations of the x_n where all the coefficients have rational coordinates, is also dense in X, and is countable. □

By the Riesz representation theorem from real analysis, the dual of $X = C([0,1])$ is the set of finite signed measures on $[0,1]$. X is separable, but X^* is not, since $\|\delta_x - \delta_y\| = 2$ whenever $x \neq y$. It follows that X is not reflexive: if it were, we would have X^{**} separable, but X^* not, contradicting the previous proposition.

5.5 Weak convergence

Let X be a normed linear space. We say x_n converges to x *weakly*, written $w - \lim x_n = x$ or $x_n \xrightarrow{w} x$ if $\ell(x_n) \to \ell(x)$ for all $\ell \in X^*$.

x_n converges to x *strongly*, written $s - \lim x_n = x$ or $x_n \xrightarrow{s} x$ if $\|x_n - x\| \to 0$.

Strong convergence implies weak convergence because

$$|\ell(x_n) - \ell(x)| = |\ell(x_n - x)| \le \|\ell\|\,\|x_n - x\| \to 0.$$

As an example where we have weak convergence but not strong convergence, let $X = \ell^2$ and let e_n be the element whose n^{th} coordinate is 1 and all other coordinate coordinates are 0. Since $\|e_n\| = 1$, then e_n does not converge strongly to 0. But it does converge weakly to 0. To see this, if ℓ is any bounded linear functional on X, then ℓ is of the form $\ell(x) = \langle x, y\rangle$ for some $y \in X$, which means $y = (b_1, b_2, \ldots)$ with $\sum_j |b_j|^2 < \infty$. In particular, $b_j \to 0$. Then $\ell(e_n) = \overline{b}_n \to 0 = \ell(0)$.

This example stretches to any Hilbert space. If $\{x_n\}$ is an orthonormal sequence in the space, $\ell(x_n) = \langle x_n, y\rangle$ for some y. By Bessel's inequality, $\sum |\langle x_n, y\rangle|^2 \le \|y\|^2$, so $\langle x_n, y\rangle \to 0$,

Proposition 5.11 *Let X be a normed linear space and suppose x_n converges weakly to x. Then $\|x\| \le \liminf \|x_n\|$.*

Proof. There exists ℓ such that $\|\ell\| = 1$ and $|\ell(x)| = \|x\|$. Then $|\ell(x)| = \lim |\ell(x_n)|$ and $|\ell(x_n)| \le \|\ell\|\,\|x_n\| = \|x_n\|$. □

5.6 Weak* convergence

We say $u_n \in X^*$ is *weak* convergent* to u if $\lim u_n(x) = u(x)$ for all $x \in X$.

If X is reflexive, then weak* convergence is the same as weak convergence.

Weak convergence in probability theory can be identified as weak* convergence in functional analysis.

As an example, if S is a compact Hausdorff space and $X = C(S)$, then X^* is the collection of finite signed measures. Saying a sequence of measures ν_n converges in the weak* sense means that $\int f\, d\nu_n$ converges for each continuous function f.

A set is *weak* sequentially compact* if every sequence in the set has a subsequence which converges in the weak* sense to an element of the set.

5.7 Approximating the δ function

Let k_n be a sequence of integrable functions on the interval $[-1,1]$. They *approximate the δ function* (or are an *approximation to the identity*) if

$$\int_{-1}^{1} f(t)k_n(t)\,dt \to f(0) \tag{5.1}$$

as $n \to \infty$ for all f continuous on $[-1,1]$.

As an example, we could take $k_n(t) = n\chi_{[0,1/n]}(t)$.

Theorem 5.12 *k_n approximates the δ function on $[-1,1]$ if and only if the following three properties hold.*

(1) $\int_{-1}^{1} k_n(t)\,dt \to 1$.

(2) If g is C^∞ and 0 in a neighborhood of 0, then

$$\int_{-1}^{1} g(t)k_n(t)\,dt \to 0$$

as $n \to \infty$.

(3) There exists c such that $\int_{-1}^{1} |k_n(t)|\,dt \le c$ for all n.

Proof. If (1)–(3) hold, write $f = (f - f(0)) + f(0)$, and we may suppose without loss of generality that $f(0) = 0$. Choose $g \in C^\infty$ such that g is 0 in a neighborhood of 0 and $\|g - f\| < \varepsilon$. We have

$$\Big|\int_{-1}^{1}(f-g)k_n\Big| \le \varepsilon \int |k_n| \le c\varepsilon$$

and

$$\int gk_n \to 0.$$

So

$$\limsup \Big|\int fk_n\Big| \le c\varepsilon.$$

Since ε is arbitrary, this shows (5.1).

If (5.1) holds, then (1) holds by taking f identically 1 and (2) holds by taking f equal to g. So we must show (3). If X is the set C of continuous functions on $[-1,1]$, then X^* is the collection of finite signed measures (by the Riesz representation theorem of real analysis). Let $m_n(dt) = k_n(t)\,dt$ and $m_0(dt) = \delta_0(dt)$. Then (5.1) says that $m_n(f) \to m_0(f)$ for all $f \in C$, or m_n converges to m_0 in the sense of weak-* convergence. $\limsup |m_n(f)| < \infty$, so $|m_n(f)| \le M(f)$ for all f, and by the uniform boundedness principle, $\|m_n\| \le c$. Note by the proof of the Riesz representation theorem, $\|m_n\|$ is the total mass of m_n, which is $\int_{-1}^1 |k_n(t)|\,dt$. □

From the approximation of the δ-function, we can show that there exists a continuous function f whose Fourier series diverges at 0.

We look at the set of continuous functions on S^1, the unit circle. We say $f(\theta)$ has Fourier series $\sum_{-\infty}^{\infty} a_n e^{in\theta}$ with

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta.$$

The Fourier series converges at 0 if

$$\lim_{N\to\infty}\sum_{-N}^{N} a_n = f(0).$$

Recall from Section 4.6 that

$$\sum_{-N}^{N} a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)k_N(\theta)\,d\theta,$$

where

$$k_N(\theta) == \frac{\sin(N+\frac{1}{2})\theta}{\sin\theta/2}.$$

So the convergence of the Fourier series at 0 is equivalent to k_N being an approximation to the δ function. And if (3) fails, then $\sum_{-N}^{N} a_n$ does not converge for some f.

Since $|\sin x| \le |x|$, then

$$\Big|\frac{1}{\sin x/2}\Big| \ge \frac{2}{|x|},$$

and therefore

$$\begin{aligned}\int_{-\pi}^{\pi} |k_N(\theta)|\,d\theta &\geq 2\int_{-\pi}^{\pi} |\sin(N+\tfrac{1}{2})\theta| \frac{d\theta}{|\theta|}\\ &= 4\int_0^{(N+\frac{1}{2})\pi} |\sin x| \frac{dx}{x}\\ &\geq c\log N\end{aligned}$$

5.8 Weak and weak* topologies

The *weak topology* is the coarsest topology (i.e., fewest sets) in which all bounded linear functionals are continuous.

Bounded linear functionals are continuous in the usual norm topology (also called the strong topology), so the weak topology is coarser than the strong topology.

Recall that a topology is a collection of sets that contain $\emptyset$ and X and which is closed under arbitrary unions and finite intersections. Elements of the topology are called open sets. A subcollection of a topology is a basis if every open set can be written as the union of elements of the subcollection. A subcollection of the topology is a subbasis is the set of finite intersections of elements of the subcollection is a basis.

Let $\mathcal{S}$ be the collection of sets of the form

$$\{x : a < \ell(x) < b\}$$

for reals $a < b$ and $\ell \in X^*$.

Proposition 5.13 *If we are considering real-valued bounded linear functionals, then $\mathcal{S}$ is a subbasis for the weak topology.*

Proof. Since ℓ is continuous in the weak topology and $\{x : a < \ell(x) < b\} = \ell^{-1}((a,b))$, then $\mathcal{S}$ is a subcollection of the weak topology.

Any topology with respect to which all bounded linear functionals are continuous must contain $\mathcal{S}$, and the smallest such is the topology generated by $\mathcal{S}$. □

Finite intersections of sets in $\mathcal{S}$ are unbounded if X is infinite dimensional, so every open set in the weak topology when X is infinite dimensional is unbounded. The open unit ball ($\{x : \|x\| < 1\}$) is a set that is open in the strong topology but not the weak topology.

5.9 The Alaoglu theorem

Consider X^*, where X is a Banach space. For $x \in X$, define $L_x : X^* \to \mathbb{R}$ by $L_x(\ell) = \ell(x)$. The weak* topology is the coarsest topology on X^* with respect to which all the L_x with $x \in X$ are continuous. As above, a subbasis for the weak* topology is the collection of sets $\{\ell : a < \ell(x) < b\}$, where $a < b \in \mathbb{R}$ and $x \in X$.

Lemma 5.14 *Suppose f is a function from a topological space $(X, \mathcal{T})$ to a topological space $(Y, \mathcal{U})$. Let $\mathcal{S}$ be a subbase for Y. If $f^{-1}(G) \in \mathcal{T}$ whenever $G \in \mathcal{S}$, then f is continuous.*

Proof. Let $\mathcal{B}$ be the collection of finite intersections of elements of $\mathcal{S}$. By the definition of subbase, $\mathcal{B}$ is a base for Y. Suppose $H = G_1 \cap G_2 \cap \cdots \cap G_n$ with each $G_i \in \mathcal{S}$. Since $f^{-1}(H) = f^{-1}(G_1) \cap \cdots \cap f^{-1}(G_n)$ and $\mathcal{T}$ is closed under the operation of finite intersections, then $f^{-1}(H) \in \mathcal{T}$. If J is an open subset of Y, then $J = \cup_{\alpha \in I} H_\alpha$, where I is a non-empty index set and each $H_\alpha \in \mathcal{B}$. Then $f^{-1}(J) = \cup_{\alpha \in I} f^{-1}(H_\alpha)$, which proves $f^{-1}(J) \in \mathcal{T}$. That is what we needed to show. □

Recall that the product topology on $\prod_{\alpha \in I} X_\alpha$ is the one generated by the sets $\{\pi_\alpha^{-1}(G) : \alpha \in I, G \text{ open in } X_\alpha\}$. Here π_β is the projection of $\prod_{\alpha \in I} X_\alpha$ onto X_β, that is, if $x = \{x_\alpha\}$ so that x_α is the α^{th} coordinate of x, then $\pi_\beta(x) = x_\beta$.

Theorem 5.15 *(*Alaoglu theorem*) The closed unit ball B in X^* is compact in the weak* topology.*

There is a connection with the Prohorov theorem of probability theory. Let $X = C(S)$ where S is a compact Hausdorff space. If μ_n is a sequence of probability measures, then the μ_n are elements of the closed unit ball in X^*. The Alaoglu theorem implies there must be a subsequence which converges in the weak* sense.

Proof. If $\ell \in B$, then $|\ell(x)| \le \|x\|$. Let

$$P = \prod_{x \in X} I_x,$$

where $I_x = [-\|x\|, \|x\|]$. Map B into P by setting $\varphi(\ell) = \{\ell(x)\}$, the function whose x^{th} coordinate is $\ell(x)$.

We will show that φ is one-to-one, continuous, and onto $\varphi(B)$. Hence to show B is compact, it suffices to show $\varphi(B)$ is compact. By Tychonov's theorem, P is compact. So it suffices to show that $\varphi(B)$ is closed.

That φ is one-to-one is obvious. To show φ is continuous, we use the lemma and show that $\varphi^{-1}(G)$ is open when G is a subbasic open set in the product topology. The collection of sets $\{\{f \in P : a < f(x) < b\}, x \in X, a < b \in \mathbb{R}\}$ is a subbasis for the product topology. If G is such a set, then

$$\varphi^{-1}(G) = \{\ell \in B : a < \ell(x) < b\} = \{\ell \in B : a < L_x(\ell) < b\}$$

is open in the weak* topology. To show φ is open, we need to show, using the lemma, that $\varphi(G)$ is relatively open in P if G is of the form $\{\ell : a < \ell(x) < b\}$. But then $\varphi(G) = \{f \in \varphi(B) : a < f(x) < b\}$ is relatively open in P.

Let p be in the closure of $\varphi(B)$. We will show $p = \varphi(\ell)$ for some $\ell \in B$. Fix $x, y \in X$ and $c \in \mathbb{R}$. If we show that $p(x+y) = p(x) + p(y)$ and $p(cx) = cp(x)$, then p will be equal to $\varphi(\ell)$ for a linear functional on X. Since $p \in \varphi(B) \subset P$, then $|p(x)| \le \|x\|$ for each x, and p will be equal to $\varphi(\ell)$ for $\ell \in B$, which finishes the proof.

We prove that $p(x+y) = p(x) + p(y)$. For each m, the set

$$\{f \in P : p(x) - 2^{-m} < f(x) < p(x) + 2^{-m}, p(y) - 2^{-m} < f(y) < p(y) + 2^{-m}, \\ p(x+y) - 2^{-m} < f(x+y) < p(x+y) + 2^{-m}\}$$

is the intersection of three subbasic sets in the product topology, and hence is open. Since p is a limit point of $\varphi(B)$, there exists q_m in B such that $\varphi(q_m)$ is in this set. We conclude $\varphi(q_m)(x) \to p(x)$, and similarly with x replaced by y and by $x + y$. Since q_m is a bounded linear functional,

$q_m(x+y) = q_m(x) + q_m(y)$. Passing to the limit, $p(x+y) = p(x) + p(y)$ as required. □

5.10 Transpose of a bounded linear map

Suppose $M : X \to U$. We define the *transpose* M' (or M^*) as follows. $M' : U^* \to X^*$. If $\ell \in U^*$, $\ell(Mx)$ is a linear functional on X, and we call this linear functional $M'\ell$.

Sometimes the notation $\ell(u) = \langle u, \ell \rangle$. With this notation,

$$\langle Mx, \ell \rangle = \ell(Mx) = M'\ell(x) = \langle x, M'\ell \rangle,$$

which justifies the name adjoint or transpose.

Proposition 5.16 *(1) M' is bounded and $\|M'\| = \|M\|$.*

(2) $(M+N)' = M' + N'$.

(3) If $M : X \to U$ and $N : U \to W$ are linear, then $(NM)' = M'N'$.

Proof. (1) $\|M'\| = \sup_{\|\ell\|=1} \|M'\ell\|$ (recall $M'\ell \in X^*$) and

$$\|M'\ell\| = \sup_{\|x\|=1} |M'\ell(x)| = \sup_{\|x\|=1} |\ell(Mx)|.$$

So

$$\|M'\| = \sup_{\|\ell\|=1, \|x\|=1} |\ell(Mx)| = \sup_{\|x\|=1} \|Mx\| = \|M\|.$$

(2) is easy.

To prove (3), we write

$$m(NM)x = (N'm)(Mx) = (M'N'm)(x),$$

so $m(NM) = M'N'm)$, and our result follows. □

Chapter 6

Convexity

6.1 Locally convex topological spaces

We look at topologies other than those defined in terms of linear functionals.

A *topological linear space* is a linear space over the reals with a Hausdorff topology satisfying
(1) $(x, y) \to x + y$ is a continuous mapping from $X \times X \to \mathbb{R}$.
(2) $(k, x) \to kx$ is a continuous mapping from $F \times X \to X$.
If in addition,
3) Every open set containing the origin contains a convex open set containing the origin
then we have a *locally convex topological linear space* (LCT).

It is an exercise to show that the weak and weak* topologies are locally convex (i.e., satisfy (3)).

Topological linear spaces just satisfying (1) and (2) are not satisfactory, because there may not be enough linear functionals. If we have (3), then we can use the hyperplane separation theorem to produce linear functionals. Thus convexity is important.

Proposition 6.1 *In a LCT linear space,*

(1) if T is open, so are $T - x_0, kT$, and $-T$.

(2) Every point of an open set T is interior to T.

Proof. (1) The map $\varphi : y \to x_0 + y$ is the composition of the maps $y \to (x_0, y)$ and $(x_0, y) \to x_0 + y$. If A and B are open in X, the inverse image of $A \times B$ under the first map is B if $x_0 \in A$ and $\emptyset$ if $x_0 \notin A$, which is open in either case. By Lemma 5.14, since the inverse image of subbasic sets is open, the first map is continuous. Therefore φ is continuous. $T - x$ is the inverse image of T under φ. kT is similar.

(2) Suppose $0 \in T$. Fix $x \in X$. $k \to kx$ is continuous, so $\{k : kx \in T\}$ is open. Since $0 \in T$, then 0 is in this set, and therefore there exists an interval about 0 such that $kx \in T$ if k is in this interval. This is true for all x, and therefore 0 is an interior point. Use translation if the point we are interested in is other than 0. □

6.2 Separation of points

In a LCT space, we can talk about continuous linear functionals, but not bounded linear functionals.

Proposition 6.2 *Continuous linear functionals in a LCT linear space X separate points: if $y \neq z$, there exists ℓ such that $\ell(y) \neq \ell(z)$.*

Proof. Without loss of generality assume $y = 0$. There exists an open set T that contains 0 but not z, since the topology is Hausdorff. We can take T to be convex. By looking at $T \cap (-T)$, we may assume that T is symmetric, that is, $T = -T$. $0 \in T$ is interior, so the gauge function p_T is finite. Recall $p_T(u) < 1$ if $u \in T$.

By the hyperplane separation theorem, there exists ℓ such that $\ell(z) = 1$ and $\ell(x) \leq p_T(x)$ for all x. Since $\ell(y) = \ell(0)$, then ℓ separates.

It remains to prove that ℓ is continuous.

We first show $H = \{w : \ell(w) < c\}$ is open. If $w \in H$ and $u \in T$, let $r = c - \ell(w)$. Then

$$\ell(w + ru) = \ell(w) + r\ell(u) \leq \ell(w) + rp_T(u) < \ell(w) + c - \ell(w) = c,$$

so $w + ru \in H$. Therefore the inverse image under ℓ of $(-\infty, c)$ contains $w + rT$, an open neighborhood of w. Hence H is open.

A similar argument shows $J = \{w : \ell(w) > d\}$ is open. Let $w \in J$, $u \in T$, and $r = d - \ell(w)$. Since r is negative,

$$\ell(w - ru) = \ell(w) + r\ell(-u) \geq \ell(w) + rp_T(-u) > \ell(w) + r = d,$$

so $w - ru \in J$. As above, J is open.

Since the inverse images of $(-\infty, c)$ and (d, ∞) are open and the collection of such sets is a subbasis for the topology of the real line, ℓ is continuous. □

Using the extended hyperplane separation theorem, we have

Corollary 6.3 *Let K be a closed convex set in a LCT space, $z \notin K$. There exists a continuous linear functional ℓ such that $\ell(y) \leq c$ for $y \in K$ and $\ell(z) > c$.*

6.3 Krein-Milman theorem

We will use the easy fact that if E is an extreme subset of a convex set K and p is an extreme point for E, then p is an extreme point for K.

Theorem 6.4 *(Krein-Milman) Let K be a nonempty, compact, convex subset of a LCT linear space X. Then*
(1) K has at least one extreme point.
(2) K is the closure of the convex hull of its extreme points.

Proof. (1) Let $\{E_j\}$ be the collection of all nonempty closed extreme subsets of K. It is nonempty because it contains K. We partially order by reverse inclusion: $E \leq F$ if $E \supset F$. We show that if we have a totally ordered subcollection, $\cap_j E_j$ is an upper bound with respect to "$\leq$," and hence by Zorn's lemma a maximal element, which means that it contains no strictly smaller extreme subset.

The intersection of any finite totally ordered subcollection $\{E_j\}$ is just the smallest one. Since K is compact, by the finite intersection property, the intersection of any totally ordered subcollection is nonempty. (If $\cap E_j = \emptyset$, then $\{E_j^c\}$ forms an open cover of K, so there is a finite subcover, and then the intersection of those finitely many E_j is empty, a contradiction.) The intersection of closed sets is closed, and it is easy to check that the intersection of extreme sets is extreme.

By Zorn's lemma, there is a maximal element E, an extreme subset that contains no strictly smaller extreme subset. We claim E is a single point. If not, there exists a continuous linear functional ℓ that separates 2 of the points of E. Let μ be the maximum value of ℓ on E. Since E is compact, this maximum value is attained. Let $M = \{x \in E : \ell(x) = \mu\}$. $M \neq E$ since ℓ is not constant. ℓ is continuous and E is closed, so M is closed. $\ell^{-1}(\{\mu\})$ is the inverse image of an extreme set and we can check that it therefore is itself extreme, so M is extreme in E, and since E is extreme in K, M is extreme in K. But this contradicts the fact that E was a minimal extreme subset.

(2) Let K_e be the extreme points of K. We'll show that if z is not in the closure of the convex hull, then $z \notin K$. There exists a continuous linear functional ℓ such that $\ell(y) \leq c$ for $y \in \overline{K}_e$ and $\ell(z) > c$. K is compact and ℓ is continuous, so ℓ achieves it maximum on a closed subset E of K. E is extreme, and E must contain an extreme point p. Since $p \in E \subset K_e$, then $\ell(p) \leq c$. Since $\ell(p) = \max_K \ell(x)$, then $\ell(x) \leq \ell(p) \leq c$ for all $x \in K$. Since $\ell(z) > c$, then $z \notin K$. □

6.4 Choquet's theorem

Here is a theorem of Choquet.

Theorem 6.5 *Suppose K is a nonempty compact convex subset of a LCT linear space X. Let K_e be the set of extreme points. If $u \in K$, there exists a measure m_u of total mass 1 on $\overline{K}_e$ such that*

$$u = \int_{\overline{K}_e} e\, m_u(de)$$

in the weak sense.

A measure with total mass one is a probability measure, but this theorem has nothing to do with probability.

The equation holding in the weak sense means

$$\ell(u) = \int_{\overline{K}_e} \ell(e)\, m_u(de)$$

for all continuous linear functionals ℓ.

Proof. Let m, M be the minimum and maximum of ℓ on K. K is compact, so these values are achieved. Then $\{x \in K : \ell(x) = m\}$ is an extreme subset of K and similarly with m replaced by M. They each contain extreme points. So if $u \in K$,

$$\min_{p \in K_e} \ell(p) \le \ell(u) \le \max_{p \in K_e} \ell(p). \tag{6.1}$$

If ℓ_1 and ℓ_2 are equal on K_e, then applying the above to $\ell_1 - \ell_2$ shows they are equal on K.

Let L be the class of continuous functions on $\overline{K}_e$ that are the restriction of a continuous linear functional. Fix u. Define r on L by setting

$$r(\ell) = \ell(u).$$

If L contains the constant function 1, then by (6.1) we have $r(\ell) = \ell(u) = 1$. If L does not contain the constant functions, adjoin the constant function $f_0 = 1$ to L and set $r(f_0) = 1$. The set L is a linear subspace of $C(\overline{K}_e)$. Check that r is a positive linear functional on L.

Now use Hahn-Banach to extend r from L to $C(\overline{K}_e)$.

$\overline{K}_e$ is a closed subset of K, hence compact. r is a positive linear functional on $C(\overline{K}_e)$. By the Riesz representation theorem from measure theory, there exists a measure m such that

$$r(f) = \int_{\overline{K}_e} f\, dm.$$

Since $r(f_0) = 1$, then $m(\overline{K}_e) = 1$. □

An example: in $\mathbb{R}^3$, let K be the unit circle in the (x, y) plane together with $\{(1, 0, z) : |z| \le 1\}$. Then $(1, 0, 0) \notin K_e$, so the collection of extreme points is not closed.

Choquet proved an important extension of his theorem in that we can take the integral to be over K_e rather than its closure, provided K is metrizable.

We write $b(\mu)$ for $\int e\, \mu(de)$, the barycenter of μ.

Theorem 6.6 (Choquet) *If K is convex, compact, and metrizable, and $x \in K$, there exists a probability measure μ supported on the extreme points of K such that $x = b(\mu)$.*

A function f is concave if $-f$ is convex. Let $\mathcal{S}$ be the set of continuous concave functions on K. $\mathcal{S}$ is closed under the operation $\wedge$ (the operation of taking the greatest lower bound) and is closed under addition. It is an exercise to show that $\mathcal{S}-\mathcal{S}$ is closed under $\wedge$ and $\vee$. $\mathcal{S}-\mathcal{S}$ contains constants, and since it contains linear functions, it separates points. By the Stone-Weierstrass theorem, $\mathcal{S}-\mathcal{S}$ is dense in $C(K)$.

Let us write $\lambda \prec \mu$ if $\int f\,d\lambda \ge \int f\,d\mu$ for all $f \in \mathcal{S}$, where μ and λ are probability measures on K. The idea of the existence of an extremal measure is the following. If x is not extremal, it can be written as $\sum_i a_i x_i$. Any x_i that is not an extreme point has a similar representation. The measure $\sum_i a_i \delta_{x_i}$ is "closer" to the boundary than δ_x and we will see this means $\sum_i a_i \delta_{x_i} \succ \delta_x$. The desired representation will come if we find the measure that is maximal with respect to $\prec$.

Proposition 6.7 *There exists μ such that $\delta_x \prec \mu$ and $\mu \succ \lambda$ whenever $\lambda \succ \delta_x$.*

Proof. We use Zorn's lemma. Suppose I is a totally ordered set and μ_i is a probability measure on K for each $i \in I$ with $\mu_i \prec \mu_j$ if $i < j$. If $f \in \mathcal{S}$, then $\int f\,d\mu_i$ decreases as i increases. Let $\ell(f)$ denote the limit. In this context, this means that given $\varepsilon > 0$, there exists $i_0 \in I$ such that $|\ell(f) - \int f\,d\mu_i| < \varepsilon$ if $i > i_0$. Because $\int f\,d\mu_i$ decreases in i, it is easy to see that $\ell(f) = \inf_{i\in I} \int f\,d\mu_i$. Define ℓ on $\mathcal{S}-\mathcal{S}$ by $\ell(f) = \lim_i \int f\,d\mu_i$. Because all the μ_i have total mass 1, ℓ is a bounded linear operator, and we can then extend ℓ to $C(K)$, the continuous functions on K. By the Riesz representation theorem, there exists a measure μ such that $\ell(f) = \int f\,d\mu$ for all $f \in C(K)$. If $f \in \mathcal{S}-\mathcal{S}$ is nonnegative, $\ell(f) = \lim_i \int f\,d\mu_i \ge 0$, or μ is a positive measure. Since $\ell(1) = 1$, μ is a probability measure. Because $\int f\,d\mu = \ell(f) = \inf_i \int f\,d\mu_i$ if $f \in \mathcal{S}$, $\mu \succ \mu_i$ for all $i \in I$. Thus μ is an upper bound for the μ_i. By Zorn's lemma, then, $\{\lambda : \lambda \succ \delta_x\}$ has a maximal element. □

We must now show that μ has barycenter x and that μ is supported on the extreme points of K.

Proposition 6.8 *If $\mu \succ \delta_x$, then $b(\mu) = x$.*

Proof. Suppose ℓ is any linear functional on K. Then $\ell \in \mathcal{S}$ and $\ell \in -\mathcal{S}$. Thus $\int \ell \, d\mu \leq \int \ell \, d\delta_x$ and $\int (-\ell) \, d\mu \leq \int (-\ell) \, d\delta_x$, or $\int \ell \, d\mu = \int \ell \, d\delta_x$ for all ℓ linear. That implies $b(\mu) = x$. □

If $f \in C(K)$, the continuous functions on K, let

$$\widetilde{f} = \inf\{g \in \mathcal{S}, g \geq f\}. \tag{6.2}$$

$\widetilde{f}$ is the least concave majorant of f. Note $f \leq \widetilde{f}$ and that $\widetilde{f}$ is bounded, since the constant function $g \equiv \sup_K f$ is continuous and dominates f.

The existence part is completed by the following.

Proposition 6.9 *If μ is maximal with respect to $\succ$, then μ is supported on the extreme points of K.*

Proof. Suppose μ is maximal. We first show

$$\int f \, d\mu = \int \widetilde{f} \, d\mu \tag{6.3}$$

for all $f \in C(K)$. Let $E = C(K)$ and define

$$P(f) = \int \widetilde{f} \, d\mu. \tag{6.4}$$

Since $\widetilde{f + g} \leq \widetilde{f} + \widetilde{g}$, P is clearly sublinear. Suppose $\int f \, d\mu \neq \int \widetilde{f} \, d\mu$ for some $f \in C(K)$. Since $\mathcal{S} - \mathcal{S}$ is dense in $C(K)$, there exists $f \in \mathcal{S}$ such that $\int f \, d\mu \neq \int \widetilde{f} \, d\mu$. Let $F = \{cf : c \in \mathbb{R}\}$ and let $\ell(f) = P(f) = \int \widetilde{f} \, d\mu$.

$0 \leq P(f) + P(-f)$ by sublinearity, so $\ell(-f) = -P(f) \leq P(-f)$, or ℓ is dominated by P on F. Use the Hahn-Banach theorem to extend ℓ to E. Since ℓ is a linear functional on $C(K)$, there exists a measure ν such that $\ell g = \int g \, d\nu$ for all $g \in C(K)$ by the Riesz representation theorem. We claim ν is a probability. $\ell(1) \leq P(1) = 1$, and $-\ell(1) = \ell(-1) \leq P(-1) = -1$, or $\ell(1) = 1$. If $g \geq 0$, $-\ell(g) = \ell(-g) \leq P(-g) = \int (\widetilde{-g}) \, d\mu \leq 0$, or $\ell(g) \geq 0$. Thus ν is a positive measure with total mass 1, which proves the claim.

If $h \in \mathcal{S}$, then $\widetilde{h} = h$ and $\int h\,d\nu = \ell(h) \le P(h) = \int \widetilde{h}\,d\mu = \int h\,d\mu$, or $\nu \succ \mu$. Moreover, $\int f\,d\nu = \ell(f) = P(f) = \int \widetilde{f}\,d\mu > \int f\,d\mu$, or $\nu \ne \mu$. This contradicts the maximality of μ. Therefore $\int f\,d\mu = \int \widetilde{f}\,d\mu$ for all $f \in C(K)$.

Since (6.3) holds and $f \le \widetilde{f}$, μ must be concentrated on $B_f = \{x \in K : f(x) = \widetilde{f}(x)\}$ for all $f \in -\mathcal{S}$. Since K is metrizable, $C(K)$ has a countable dense subset. Since $\mathcal{S} - \mathcal{S}$ is dense in $C(K)$, we can find a countable sequence $f_n \in -\mathcal{S}$ that separate points. We may normalize so that $\|f_n\|_\infty \le 1$. Let $f = \sum_n (f_n)^2/2^n$. Then f is convex, continuous, and also strictly convex. Since $\widetilde{f}$ is concave, $\widetilde{f}(x) > f(x)$ for all x in K that are not extremal. So B_f is contained in the set of extreme points of K, which proves the proposition.
□

Chapter 7

Sobolev spaces

7.1 Weak derivatives

Let C_K^∞ be the set of C^∞ functions on $\mathbb{R}^n$ that have compact support and have partial derivatives of all orders. For $j = (j_1, \ldots, j_n)$, write

$$D^j f = \frac{\partial^{j_1+\cdots+j_n} f}{\partial_{x_1}^{j_1} \cdots \partial_{x_n}^{j_n}},$$

and set $|j| = j_1 + \cdots + j_n$. We use the convention that $\partial^0 f/\partial x_i^0$ is the same as f.

Let f, g be locally integrable. We say that $D^j f = g$ in the *weak sense* or g is the *weak j^{th} order partial derivative* of f if

$$\int f(x)\, D^j \varphi(x)\, dx = (-1)^{|j|} \int g(x)\varphi(x)\, dx$$

for all $\varphi \in C_K^\infty$. Note that if $g = D^j f$ in the usual sense, then integration by parts shows that g is also the weak derivative of f.

Let

$$W^{k,p}(\mathbb{R}^n) = \{f : f \in L^p, D^j f \in L^p \text{ for each } j \text{ such that } |j| \le k\}.$$

Set

$$\|f\|_{W^{k,p}} = \sum_{\{j: 0 \le |j| \le k\}} \|D^j f\|_p,$$

where we set $D^0 f = f$.

This is slightly different than

$$\|f\|_0 = \Big(\sum_{|j|\le k} \int |D^j f|^p\Big)^{1/p},$$

which is the definition we used for the $W^{k,p}$ norm earlier. However they are equivalent norms. To see this, we use the inequality

$$(a+b)^p \le c_p a^p + c_p b^p, \qquad a, b \ge 0,$$

where $c_p = 2^{p-1}$ when $p \ge 1$ and $c_p = 1$ when $p < 1$. An induction argument leads to

$$\Big(\sum_{i=1}^N a_i\Big)^p \le c(p,N) \sum_{i=1}^N a_i^p$$

if each $a_i \ge 0$.

We have

$$\|f\|_{W^{k,p}}^p = \Big(\sum_{|j|\le k} \|D^j f\|_p\Big)^p \le c(p,k) \sum_{j\le k} \|D^j f\|_p^p = c(p,k)\|f\|_0^p$$

for one direction. For the other,

$$\|f\|_0 \le c(1/p, k) \sum_{|j|\le k} \Big(\int |D^j f|^p\Big)^{1/p} = \|f\|_{W^{k,p}}.$$

Theorem 7.1 *The space $W^{k,p}$ is complete.*

Proof. Let f_m be a Cauchy sequence in $W^{k,p}$. For each j such that $|j| \le k$, we see that $D^j f_m$ is a Cauchy sequence in L^p. Let g_j be the L^p limit of $D^j f_m$. Let f be the L^p limit of f_m. Then

$$\int f_m\, D^j \varphi = (-1)^{|j|} \int (D^j f_m)\varphi \to (-1)^{|j|} \int g_j \varphi$$

for all $\varphi \in C_K^\infty$. On the other hand, $\int f_m\, D^j\varphi \to \int f\, D^j \varphi$. Therefore

$$(-1)^{|j|} \int g_j \varphi = \int f\, D^j \varphi$$

for all $\varphi \in C_K^\infty$. We conclude that $g_j = D^j f$ a.e. for each j such that $|j| \le k$. We have thus proved that $D^j f_m$ converges to $D^j f$ in L^p for each j such that $|j| \le k$, and that suffices to prove the theorem. □

7.2 Sobolev inequalities

Lemma 7.2 *If $k \geq 1$ and $f_1, \ldots, f_k \geq 0$, then*

$$\int f_1^{1/k} \cdots f_k^{1/k} \leq \Big(\int f_1 \Big)^{1/k} \cdots \Big(\int f_k \Big)^{1/k}.$$

Proof. We will prove

$$\Big(\int f_1^{1/k} \cdots f_k^{1/k} \Big)^k \leq \Big(\int f_1 \Big) \cdots \Big(\int f_k \Big). \tag{7.1}$$

We will use induction. The case $k = 1$ is obvious. Suppose (7.1) holds when k is replaced by $k-1$ so that

$$\Big(\int f_1^{1/(k-1)} \cdots f_{k-1}^{1/(k-1)} \Big)^{k-1} \leq \Big(\int f_1 \Big) \cdots \Big(\int f_{k-1} \Big). \tag{7.2}$$

Let $p = k/(k-1)$ and $q = k$ so that $p^{-1} + q^{-1} = 1$. Using Hölder's inequality,

$$\int (f_1^{1/k} \cdots f_{k-1}^{1/k}) f_k^{1/k} \\ \leq \Big(\int f_1^{1/(k-1)} \cdots f_{k-1}^{1/(k-1)} \Big)^{(k-1)/k} \Big(\int f_k \Big)^{1/k}.$$

Taking both sides to the k^{th} power, we obtain

$$\Big(\int (f_1^{1/k} \cdots f_{k-1}^{1/k}) f_k^{1/k} \Big)^k \\ \leq \Big(\int f_1^{1/(k-1)} \cdots f_{k-1}^{1/(k-1)} \Big)^{(k-1)} \Big(\int f_k \Big).$$

Using (7.2), we obtain (7.1). Therefore our result follows by induction. □

Let C_K^1 be the continuously differentiable functions with compact support. The following theorem is sometimes known as the *Gagliardo-Nirenberg inequality.*

Theorem 7.3 *There exists a constant c_1 depending only on n such that if $u \in C_K^1$, then*

$$\|u\|_{n/(n-1)} \leq c_1 \| \, |\nabla u| \, \|_1.$$

We observe that u having compact support is essential; otherwise we could just let u be identically equal to one and get a contradiction. On the other hand, the constant c_1 does not depend on the support of u.

Proof. For simplicity of notation, set $s = 1/(n-1)$. Let $K_{j_1\cdots j_m}$ be the integral of $|\nabla u(x_1,\ldots,x_n)|$ with respect to the variables $x_{j_1},\ldots,x_{j_m}$. Thus

$$K_1 = \int |\nabla u(x_1,\ldots,x_n)|\,dx_1$$

and

$$K_{23} = \int\int |\nabla u(x_1,\ldots,x_n)|\,dx_2\,dx_3.$$

Note K_1 is a function of $(x_2,\ldots,x_n)$ and K_{23} is a function of $(x_1,x_4,\ldots,x_n)$.

If $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, then since u has compact support,

$$\begin{aligned}|u(x)| &= \Big|\int_{-\infty}^{x_1} \frac{\partial u}{\partial x_1}(y_1,x_2,\ldots,x_n)\,dy_1\Big|\\ &\le \int_{\mathbb{R}} |\nabla u(y_1,x_2,\ldots,x_n)|\,dy_1\\ &= K_1.\end{aligned}$$

The same argument shows that $|u(x)|\le K_i$ for each i, so that

$$|u(x)|^{n/(n-1)} = |u(x)|^{ns} \le K_1^s K_2^s\cdots K_n^s.$$

Since K_1 does not depend on x_1, Lemma 7.2 shows that

$$\begin{aligned}\int |u(x)|^{ns}\,dx_1 &\le K_1^s \int K_2^s\cdots K_n^s\,dx_1\\ &\le K_1^s\Big(\int K_2\,dx_1\Big)^s\cdots\Big(\int K_n\,dx_1\Big)^s.\end{aligned}$$

Note that

$$\int K_2\,dx_1 = \int\Big(\int |\nabla u(x_1,\ldots,x_n)|\,dx_2\Big)\,dx_1 = K_{12},$$

and similarly for the other integrals. Hence

$$\int |u|^{ns}\,dx_1 \le K_1^s K_{12}^s\cdots K_{1n}^s.$$

Next, since K_{12} does not depend on x_2,

$$\begin{aligned}\int |u(x)|^{ns}\,dx_1\,dx_2 &\le K_{12}^s \int K_1^s K_{13}^s \cdots K_{1n}^s\,dx_2\\ &\le K_{12}^s\Big(\int K_1\,dx_2\Big)^s\Big(\int K_{13}\,dx_2\Big)^s\cdots\Big(\int K_{1n}\,dx_2\Big)^s\\ &= K_{12}^s K_{12}^s K_{123}^s \cdots K_{12n}^s.\end{aligned}$$

We continue, and get

$$\int |u(x)|^{ns}\,dx_1\,dx_2\,dx_3 \le K_{123}^s K_{123}^s K_{123}^s K_{1234}^s \cdots K_{123n}^s$$

and so on, until finally we arrive at

$$\int |u(x)|^{ns}\,dx_1\,\cdots\,dx_n \le \Big(K_{12\cdots n}^s\Big)^n = K_{12\cdots n}^{ns}.$$

If we then take the $ns = n/(n-1)$ roots of both sides, we get the inequality we wanted. □

From this we can get the *Sobolev inequalities*.

Theorem 7.4 *Suppose $1 \le p < n$ and $u \in C_K^1$. Then there exists a constant c_1 depending only on n such that*

$$\|u\|_{np/(n-p)} \le c_1 \|\,|\nabla u|\,\|_p.$$

Proof. The case $p = 1$ is the case above, so we assume $p > 1$. The case when u is identically equal to 0 is obvious, so we rule that case out. Let

$$r = \frac{p(n-1)}{n-p},$$

and note that $r > 1$ and

$$r - 1 = \frac{np-n}{n-p}.$$

Let $w = |u|^r$. Since $r > 1$, then $x \to |x|^r$ is continuously differentiable, and by the chain rule, $w \in C^1$. We observe that

$$|\nabla w| \le c_2 |u|^{r-1} |\nabla u|.$$

Applying Theorem 7.3 to w and using Hölder's inequality with $q = \frac{p}{p-1}$, we obtain

$$\begin{aligned} \Big(\int |w|^{n/(n-1)}\Big)^{\frac{n-1}{n}} &\le c_3 \int |\nabla w| \\ &\le c_4 \int |u|^{(np-n)/(n-p)} |\nabla u| \\ &\le c_5 \Big(\int |u|^{np/(n-p)}\Big)^{\frac{p-1}{p}} \Big(\int |\nabla u|^p\Big)^{1/p}. \end{aligned}$$

The left hand side is equal to

$$\Big(\int |u|^{np/(n-p)}\Big)^{\frac{n-1}{n}}.$$

Divide both sides by

$$\Big(\int |u|^{np/(n-p)}\Big)^{\frac{p-1}{p}}.$$

Since

$$\frac{n-1}{n} - \frac{p-1}{p} = \frac{1}{p} - \frac{1}{n} = \frac{n-p}{pn},$$

we get our result. □

We can iterate to get results on the L^p norm of f in terms of the L^q norm of $D^k f$ when $k > 1$.

Theorem 7.5 *Suppose $k \ge 1$. Suppose $p < n/k$ and we define q by $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. Then there exists c_1 such that*

$$\|f\|_q \le c \Big\| \sum_{\{j:|j|=k\}} |D^k f| \Big\|_p.$$

7.3 Morrey's inequality

Morrey's inequality shows that if $f \in L^p$ for large enough p, then f is Hölder continuous.

Let

$$\|f\|_{C^\gamma} = \|f\|_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\gamma}.$$

Theorem 7.6 *Suppose $p > n$ and $u \in C^1(\mathbb{R}^n)$ with compact support. Let $\gamma = 1 - \frac{n}{p}$. Then there exists a constant c depending only on p and n such that*

$$\|u\|_{C^\gamma} \le c\|u\|_{W^{1,p}}.$$

Proof. We will prove the Hölder estimate first and then do the L^∞ estimate. Let's take $x = 0$.

If $v \in \partial B(0,1)$ and $0 < s < r$, then

$$\begin{aligned}|u(x+sv) - u(x)| &= \Big| \int_0^s \frac{d}{dt} u(x+tv)\,dt \Big| = \Big| \int_0^s \langle \nabla u(x+tv), v\rangle\,dt \Big| \\ &\le \int_0^s |\nabla u(x+tv)|\,dt.\end{aligned}$$

Integrating over $v \in \partial B(0,1)$, if σ is surface measure,

$$\int_{\partial B(0,1)} |u(x+sv) - u(x)|\,d\sigma(v) \le \int_0^s \int_{\partial B(0,1)} |\nabla u(x+tv)|\,d\sigma(v)\,dt.$$

We change to rectangular coordinates, with $x + tv = y$, so that $t = |y - x|$ and $d\sigma(v) = t^{-n+1}\,d\sigma(y)$:

$$\int_{\partial B(0,1)} |u(x+sv) - u(x)|\,d\sigma(v) \le \int_{B(0,s)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\,d\sigma(y).$$

If we change the integral on the right to being over $B(0,r)$, this just makes the integral larger.

Now multiply by s^{n-1} and integrate over s from 0 to r to get

$$\int_{B(0,r)} |u(y) - u(x)|\,dy \le \frac{r^n}{n} \int_{B(0,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\,dy.$$

If $x, y \in \mathbb{R}^n$, set $r = |y - x|$ and let $W = B(x,r) \cap B(y,r)$. We see that if $z \in W$, then

$$|u(x) - u(y)| \le |u(x) - u(z)| + |u(z) - u(y)|,$$

and then integrating over such z,

$$|u(x) - u(y)| \le \frac{1}{|W|}\int_W |u(x) - u(z)|\,dz + \frac{1}{|W|}\int_W |u(y) - u(z)|\,dz,$$

where $|A|$ is the Lebesgue measure of A.

We estimate the first integral, the second being almost identical. Note $|B(x,r)|/|W|$ does not depend on r. By Hölder's inequality,

$$\begin{aligned}\frac{1}{|W|}\int_W &|u(x){-}u(z)|\,dz\\ &\le c\Big(\int_{B(x,r)} |\nabla u(z)|^p\,dz\Big)^{1/p}\Big(\int_{B(x,r)} \frac{1}{|x-z|^{\frac{p(n-1)}{p-1}}}\Big)^{\frac{p-1}{p}}\\ &\le c\Big(r^{n-\frac{p(n-1)}{p-1}}\Big)^{\frac{p-1}{p}}\|\nabla u\|_p\\ &\le cr^{1-\frac{n}{p}}\|\nabla u\|_p\\ &= c_1|x-y|^{1-\frac{n}{p}}\|\nabla u\|_p.\end{aligned}$$

This argument works no matter what x is.

Now we turn to the L^∞ estimate. Suppose $\|u\|_{W^{1,p}} = 1$. If there exists x such that $|u(x)| \ge M$ (where M will be chosen in a moment), then from

$$|u(x) - u(y)| \le c_1|x-y|^{1-\frac{n}{p}}\|\nabla u\|_p \le c_1|x-y|^{1-\frac{n}{p}},$$

we see that $|u(y)| \ge M/2$ in $B(x,1)$ as long as $M \ge 2c_1$. But

$$1 = \|u\|_{W^{1,p}} \ge \|u\|_p \ge \Big(\int_{B(x,1)} |u(y)|^p\,dy\Big)^{1/p} \ge c_2 M.$$

Take $M = \max(2c_1, 2/c_2)$. We then get a contradiction to the assumption that there exists x with $|u(x)| \ge M$. □

If $r \ge 0$ is an integer and $\alpha \in (0,1)$, define

$$\|f\|_{C^{r,\alpha}} = \sum_{|j|\le r}\sup_x |D^j f(x)| + \sum_{|j|=r}\sup_{x\ne y}\frac{|D^j f(x) - D^j f(y)|}{|x-y|^\alpha},$$

and let $C^{r,\alpha}$ be the set of functions whose norm is finite.

Also part of the Sobolev embedding theorem is the following.

Theorem 7.7 *If*

$$\frac{k-r-\alpha}{n} = \frac{1}{p},$$

then $W^{k,p}(\mathbb{R}^n) \subset C^{r,\alpha}$ *and for* $f \in C^{k+\alpha}$ *with compact support,*

$$\|f\|_{C^{r,\alpha}} \leq c\|f\|_{W^{k,p}}.$$

This follows easily from the Morrey inequality.

Chapter 8

Distributions

For simplicity of notation, in this chapter we restrict ourselves to dimension one, but everything we do can be extended to $\mathbb{R}^n$, $n > 1$, although in some cases a more complicated proof is necessary.

8.1 Definitions and examples

We use C_K^∞ for the set of C^∞ functions on $\mathbb{R}$ with compact support. Let $Df = f'$, the derivative of f, $D^2 f = f''$, the second derivative, and so on, and we make the convention that $D^0 f = f$.

If f is a continuous function on $\mathbb{R}$, let $\operatorname{supp}(f)$ be the *support* of f, the closure of the set $\{x : f(x) \neq 0\}$. If $f_j, f \in C_K^\infty$, we say $f_j \to f$ in the C_K^∞ sense if there exists a compact subset K such that $\operatorname{supp}(f_j) \subset K$ for all j, f_j converges uniformly to f, and $D^m f_j$ converges uniformly to $D^m f$ for all m.

We have not claimed that C_K^∞ with this notion of convergence is a Banach space, so it doesn't make sense to talk about bounded linear functionals. But it does make sense to consider continuous linear functionals. A map $F : C_K^\infty \to \mathbb{C}$ is a *continuous linear functional on* C_K^∞ if $F(f+g) = F(f) + F(g)$ whenever $f, g \in C_K^\infty$, $F(cf) = cF(f)$ whenever $f \in C_K^\infty$ and $c \in \mathbb{C}$, and $F(f_j) \to F(f)$ whenever $f_j \to f$ in the C_K^∞ sense.

A *distribution* is defined to be a complex-valued continuous linear functional on C_K^∞.

Here are some examples of distributions.

Example 8.1 If g is a continuous function, define

$$G_g(f) = \int_{\mathbb{R}} f(x)g(x)\,dx, \qquad f \in C_K^\infty. \tag{8.1}$$

It is routine to check that G_g is a distribution.

Note that knowing the values of $G_g(f)$ for all $f \in C_K^\infty$ determines g uniquely up to almost everywhere equivalence. Since g is continuous, g is uniquely determined at every point by the values of $G_g(f)$.

Example 8.2 Set $\delta(f) = f(0)$ for $f \in C_K^\infty$. This distribution is the *Dirac delta function*.

Example 8.3 If g is integrable and $k \geq 1$, define

$$F(f) = \int_{\mathbb{R}} D^k f(x)g(x)\,dx, \qquad f \in C_K^\infty.$$

Example 8.4 If $k \geq 1$, define $F(f) = D^k f(0)$ for $f \in C_K^\infty$.

There are a number of operations that one can perform on distributions to get other distributions. Here are some examples.

Example 8.5 Let h be a C^∞ function, not necessarily with compact support. If F is a distribution, define $M_h(F)$ by

$$M_h(F)(f) = F(fh), \qquad f \in C_K^\infty.$$

It is routine to check that $M_h(F)$ is a distribution.

Example 8.1 shows how to consider a continuous function g as a distribution. Defining G_g by (8.1),

$$M_h(G_g)(f) = G_g(fh) = \int (fh)g = \int f(hg) = G_{hg}(f).$$

Therefore we can consider the operator M_h we just defined as an extension of the operation of multiplying continuous functions by a C^∞ function h.

Example 8.6 If F is a distribution, define $D(F)$ by

$$D(F)(f) = F(-Df), \qquad f \in C_K^\infty.$$

Again it is routine to check that $D(F)$ is a distribution.

If g is a continuously differentiable function and we use (8.1) to identify the function g with the distribution G_g, then

$$\begin{aligned} D(G_g)(f) = G_g(-Df) &= \int (-Df)(x)g(x)\,dx \\ &= \int f(x)(Dg)(x)\,dx = G_{Dg}(f), \qquad f \in C_K^\infty, \end{aligned}$$

by integration by parts. Therefore $D(G_g)$ is the distribution that corresponds to the function that is the derivative of g. However, $D(F)$ is defined for any distribution F. Hence the operator D on distributions gives an interpretation to the idea of taking the derivative of any continuous function.

Example 8.7 Let $a \in \mathbb{R}$ and define $T_a(F)$ by

$$T_a(F)(f) = F(f_{-a}), \qquad f \in C_K^\infty,$$

where $f_{-a}(x) = f(x+a)$. If G_g is given by (8.1), then

$$\begin{aligned} T_a(G_g)(f) = G_g(f_{-a}) &= \int f_{-a}(x)g(x)\,dx \\ &= \int f(x)g(x-a)\,dx = G_{g_a}(f), \qquad f \in C_K^\infty, \end{aligned}$$

by a change of variables, and we can consider T_a as the operator that translates a distribution by a.

Example 8.8 Define R by

$$R(F)(f) = F(Rf), \qquad f \in C_K^\infty,$$

where $Rf(x) = f(-x)$. Similarly to the previous examples, we can see that R reflects a distribution through the origin.

Example 8.9 Finally, we give a definition of the convolution of a distribution with a continuous function h with compact support. Define $C_h(F)$ by

$$C_h(F)(f) = F(f * Rh), \qquad f \in C_K^\infty,$$

where $Rh(x) = h(-x)$. To justify that this extends the notion of convolution, note that

$$\begin{aligned} C_h(G_g)(f) &= G_g(f * Rh) = \int g(x)(f * Rh)(x)\,dx \\ &= \int\int g(x)f(y)h(y-x)\,dy\,dx = \int f(y)(g*h)(y)\,dy \\ &= G_{g*h}(f), \end{aligned}$$

or C_h takes the distribution corresponding to the continuous function g to the distribution corresponding to the function $g * h$.

One cannot, in general, define the product of two distributions or quantities like $\delta(x^2)$.

8.2 Distributions supported at a point

We first define the support of a distribution. We then show that a distribution supported at a point is a linear combination of derivatives of the delta function.

Let G be open. A distribution F is zero on G if $F(f) = 0$ for all C_K^∞ functions f for which $\operatorname{supp}(f) \subset G$.

Lemma 8.10 *If F is zero on G_1 and G_2, then F is zero on $G_1 \cup G_2$.*

Proof. This is just the usual partition of unity proof. Suppose f has support in $G_1 \cup G_2$. We will write $f = f_1 + f_2$ with $\operatorname{supp}(f_1) \subset G_1$ and $\operatorname{supp}(f_2) \subset G_2$. Then $F(f) = F(f_1) + F(f_2) = 0$, which will achieve the proof.

Fix $x \in \operatorname{supp}(f)$. Since G_1, G_2 are open, we can find h_x such that h_x is non-negative, $h_x(x) > 0$, h_x is in C_K^∞, and the support of h_x is contained either in G_1 or in G_2. The set $B_x = \{y : h_x(y) > 0\}$ is open and contains x.

By compactness we can cover supp f by finitely many sets $\{B_{x_1}, \ldots, B_{x_m}\}$. Let h_1 be the sum of those h_{x_i} whose support is contained in G_1 and let h_2 be the sum of those h_{x_i} whose support is contained in G_2. Then let

$$f_1 = \frac{h_1}{h_1 + h_2} f, \qquad f_2 = \frac{h_2}{h_1 + h_2} f.$$

Clearly $\operatorname{supp}(f_1) \subset G_1$, $\operatorname{supp}(f_2) \subset G_2$, $f_1 + f_2 > 0$ on $G_1 \cup G_2$, and $f = f_1 + f_2$. □

If we have an arbitrary collection of open sets $\{G_\alpha\}$, F is zero on each G_α, and $\operatorname{supp}(f)$ is contained in $\cup_\alpha G_\alpha$, then by compactness there exist finitely many of the G_α that cover $\operatorname{supp}(f)$. By Lemma 8.10, $F(f) = 0$.

The union of all open sets on which F is zero is an open set on which F is zero. The complement of this open set is called the *support* of F.

Example 8.11 The support of the Dirac delta function is $\{0\}$. Note that the support of $D^k\delta$ is also $\{0\}$.

Define

$$\|f\|_{C^N(K)} = \max_{0 \le k \le N} \sup_{x \in K} |D^k f(x)|.$$

Proposition 8.12 *Let F be a distribution and K a fixed compact set. There exist N and c depending on F and K such that if $f \in C_K^\infty$ has support in K, then*

$$|F(f)| \le c\|f\|_{C^N(K)}.$$

Proof. Suppose not. Then for each m there exists $f_m \in C_K^\infty$ with support contained in K such that $F(f_m) = 1$ and $\|f\|_{C^m(K)} \le 1/m$. Therefore $f_m \to 0$ in the sense of C_K^∞. However $F(f_m) = 1$ while $F(0) = 0$, a contradiction. □

Proposition 8.13 *Suppose F is a distribution and* $\operatorname{supp}(F) = \{0\}$. *There exists N such that if $f \in C_K^\infty$ and $D^j f(0) = 0$ for $j \le N$, then $F(f) = 0$.*

Proof. Let $\varphi \in C^\infty$ be 0 on $[-1,1]$ and 1 on $|x|>2$. Let $g=(1-\varphi)f$. Note $\varphi f=0$ on $[-1,1]$, so $F(\varphi f)=0$ because F is supported on $\{0\}$. Then

$$F(g)=F(f)-F(\varphi f)=F(f).$$

Thus is suffices to show that $F(g)=0$ whenever $g\in C_K^\infty$, $\operatorname{supp}(g)\subset[-3,3]$, and $D^jg(0)=0$ for $0\le j\le N$.

Let $K=[-3,3]$. By Proposition 8.12 there exist N and c depending only on F such that $|F(g)|\le c\|g\|_{C^N(K)}$. Define $g_m(x)=\varphi(mx)g(x)$. Note that $g_m(x)=g(x)$ if $|x|>2/m$.

Suppose $|x|<2/m$ and $g\in C_K^\infty$ with support in $[-3,3]$ and $D^jg(0)=0$ for $j\le N$. By Taylor's theorem, if $j\le N$,

$$\begin{aligned} D^jg(x)&=D^jg(0)+D^{j+1}g(0)x+\cdots+D^Ng(0)\frac{x^{N-j}}{(N-j)!}+R\\ &=R, \end{aligned}$$

where the remainder R satisfies

$$|R|\le\sup_{y\in\mathbb{R}}|D^{N+1}g(y)|\frac{|x|^{N+1-j}}{(N+1-j)!}.$$

Since $|x|<2/m$, then

$$|D^jg(x)|=|R|\le c_1m^{j-1-N} \tag{8.2}$$

for some constant c_1.

By the definition of g_m and (8.2),

$$|g_m(x)|\le c_2|g(x)|\le c_3m^{-N-1},$$

where c_2 and c_3 are constants. Again using (8.2),

$$|Dg_m(x)|\le|\varphi(mx)|\,|Dg(x)|+m|g(x)|\,|D\varphi(mx)|\le c_4m^{-N}.$$

Continuing, repeated applications of the product rule show that if $k\le N$, then

$$|D^kg_m(x)|\le c_5m^{k-1-N}$$

for $k\le N$ and $|x|\le 2/m$, where c_5 is a constant.

Recalling that $g_m(x) = g(x)$ if $|x| > 2/m$, we see that $D^j g_m(x) \to D^j g(x)$ uniformly over $x \in [-3, 3]$ if $j \le N$. We conclude

$$F(g_m - g) = F(g_m) - F(g) \to 0.$$

However, each g_m is 0 in a neighborhood of 0, so by the hypothesis, $F(g_m) = 0$; thus $F(g) = 0$. □

By Example 8.6, $D^j\delta$ is the distribution such that $D^j\delta(f) = (-1)^j D^j f(0)$.

Theorem 8.14 *Suppose F is a distribution supported on $\{0\}$. Then there exist N and constants c_i such that*

$$F = \sum_{i=0}^{N} c_i D^i \delta.$$

Proof. Let $P_i(x)$ be a C_K^∞ function which agrees with the polynomial x^i in a neighborhood of 0. Taking derivatives shows that $D^j P_i(0) = 0$ if $i \ne j$ and equals $i!$ if $i = j$. Then $D^j\delta(P_i) = (-1)^i i!$ if $i = j$ and 0 otherwise.

Use Proposition 8.13 to determine the integer N. Suppose $f \in C_K^\infty$. By Taylor's theorem, f and the function

$$g(x) = \sum_{i=0}^{N} D^i f(0) P_i(x)/i!$$

agree at 0 and all the derivatives up to order N agree at 0. By the conclusion of Proposition 8.13 applied to $f - g$,

$$F\Big(f - \sum_{i=0}^{N} \frac{D^i f(0)}{i!} P_i\Big) = 0.$$

Therefore

$$\begin{aligned} F(f) &= \sum_{i=0}^{N} \frac{D^i f(0)}{i!} F(P_i) = \sum_{i=0}^{N} (-1)^i \frac{D^i\delta(f)}{i!} F(P_i) \\ &= \sum_{i=0}^{N} c_i D^i \delta(f) \end{aligned}$$

if we set $c_i = (-1)^i F(P_i)/i!$ Since f was arbitrary and the c_i do not depend on f, this proves the theorem. □

8.3 Distributions with compact support

In this section we consider distributions whose supports are compact sets.

Theorem 8.15 *If F has compact support, there exist a non-negative integer L and continuous functions g_j such that*

$$F = \sum_{j \le L} D^j G_{g_j}, \tag{8.3}$$

where G_{g_j} is defined by Example 8.1.

Example 8.16 The delta function is the derivative of h, where h is 0 for $x < 0$ and 1 for $x \ge 0$. In turn h is the derivative of g, where g is 0 for $x < 0$ and $g(x) = x$ for $x \ge 0$. Therefore $\delta = D^2 G_g$.

Proof. Let $h \in C_K^\infty$ and suppose h is equal to 1 on the support of F. Then $F((1-h)f) = 0$, or $F(f) = F(hf)$. Therefore there exist N and c_1 such that

$$|F(hf)| \le c_1 \|hf\|_{C^N(K)}.$$

By the product rule,

$$|D(hf)| \le |h(Df)| + |(Dh)f| \le c_2 \|f\|_{C^N(K)},$$

and by repeated applications of the product rule,

$$\|hf\|_{C^N(K)} \le c_3 \|f\|_{C^N(K)}.$$

Hence

$$|F(f)| = |F(hf)| \le c_4 \|f\|_{C^N(K)}.$$

Let $K = [-x_0, x_0]$ be a closed interval containing the support of F. Let $C^N(K)$ be the N times continuously differentiable functions whose support is contained in K. We will use the fact that $C^N(K)$ is a complete metric space with respect to the metric $\|f - g\|_{C^N(K)}$.

Define
$$\|f\|_{H^M} = \Big(\sum_{k\le M}\int |D^k f|^2\,dx\Big)^{1/2}, \qquad f\in C_K^\infty,$$
and let H^M be the completion of $\{f\in C_K^\infty : \mathrm{supp}\,(f)\subset K\}$ with respect to this norm. It is routine to check that H^M is a Hilbert space.

Suppose $M=N+1$ and $x\in K$. Then using the Cauchy-Schwarz inequality and the fact that $K=[-x_0,x_0]$,
$$\begin{aligned}|D^j f(x)| &= |D^j f(x) - D^j f(-x_0)| = \Big|\int_{-x_0}^x D^{j+1} f(y)\,dy\Big|\\ &\le |2x_0|^{1/2}\Big(\int_{\mathbb{R}} |D^{j+1} f(y)|^2\,dy\Big)^{1/2}\\ &\le c_5\Big(\int_{\mathbb{R}} |D^{j+1} f(y)|^2\,dy\Big)^{1/2}.\end{aligned}$$
This holds for all $j\le N$, hence
$$\|u\|_{C^N(K)} \le c_6\|u\|_{H^M}. \tag{8.4}$$

Recall the definition of completion. If $g\in H^M$, there exists $g_m\in C^N(K)$ such that $\|g_m-g\|_{H^M}\to 0$. In view of (8.4), we see that $\{g_m\}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{C^N(K)}$. Since $C^N(K)$ is complete, then g_m converges with respect to this norm. The only possible limit is equal to g a.e. We may therefore conclude $g\in C^N(K)$ whenever $g\in H^M$.

Since $|F(f)|\le c_4\|f\|_{C^N(K)}\le c_4c_6\|f\|_{H^M}$, then F can be viewed as a bounded linear functional on H^M. By the Riesz representation theorem for Hilbert spaces (Theorem 4.12), there exists $g\in H^M$ such that
$$F(f) = \langle f,g\rangle_{H^M} = \sum_{k\le M}\langle D^k f, D^k g\rangle, \qquad f\in H^M.$$

Now if $g_m\to g$ with respect to the H^M norm and each $g_m\in C^N(K)$, then
$$\begin{aligned}\langle D^k f, D^k g\rangle &= \lim_{m\to\infty}\langle D^k f, D^k g_m\rangle = \lim_{m\to\infty}(-1)^k\langle D^{2k} f, g_m\rangle\\ &= (-1)^k\langle D^{2k} f, g\rangle = (-1)^k G_g(D^{2k} f)\\ &= (-1)^k D^{2k} G_g(f)\end{aligned}$$

if $f \in C_K^\infty$, using integration by parts and the definition of the derivative of a distribution. Therefore

$$F = \sum_{k \le M} (-1)^k D^{2k} G_{g_k},$$

which gives our result if we let $L = 2M$, set $g_j = 0$ if j is odd, and set $g_{2k} = (-1)^k g$. □

8.4 Tempered distributions

Let $\mathcal{S}$ be the class of complex-valued C^∞ functions u such that $|x^j D^k u(x)| \to 0$ as $|x| \to \infty$ for all $k \ge 0$ and all $j \ge 1$. $\mathcal{S}$ is called the *Schwartz class*. An example of an element in the Schwartz class that is not in C_K^∞ is e^{-x^2}.

Define

$$\|u\|_{j,k} = \sup_{x \in \mathbb{R}} |x|^j |D^k u(x)|.$$

We say $u_n \in \mathcal{S}$ converges to $u \in \mathcal{S}$ in the sense of the Schwartz class if $\|u_n - u\|_{j,k} \to 0$ for all j, k.

A continuous linear functional on $\mathcal{S}$ is a function $F : \mathcal{S} \to \mathbb{C}$ such that $F(f + g) = F(f) + F(g)$ if $f, g \in \mathcal{S}$, $F(cf) = cF(f)$ if $f \in \mathcal{S}$ and $c \in \mathbb{C}$, and $F(f_m) \to F(f)$ whenever $f_m \to f$ in the sense of the Schwartz class. A *tempered distribution* is a continuous linear functional on $\mathcal{S}$.

Since $C_K^\infty \subset \mathcal{S}$ and $f_n \to f$ in the sense of the Schwartz class whenever $f_n \to f$ in the sense of C_K^∞, then any continuous linear functional on $\mathcal{S}$ is also a continuous linear functional on C_K^∞. Therefore every tempered distribution is a distribution.

Any distribution with compact support is a tempered distribution. If g grows slower than some power of $|x|$ as $|x| \to \infty$, then G_g is a tempered distribution, where $G_g(f) = \int f(x)g(x)\,dx$.

For $f \in \mathcal{S}$, recall that we defined the Fourier transform $\mathcal{F}f = \widehat{f}$ by

$$\widehat{f}(u) = \int f(x) e^{ixu}\,dx.$$

Theorem 8.17 *$\mathcal{F}$ is a continuous map from $\mathcal{S}$ into $\mathcal{S}$.*

Proof. For elements of $\mathcal{S}$, $D^k(\mathcal{F}f) = \mathcal{F}((ix)^k)f)$. If $f \in \mathcal{S}$, $|x^k f(x)|$ tends to zero faster than any power of $|x|^{-1}$, so $x^k f(x) \in L^1$. This implies $D^k\mathcal{F}f$ is a continuous function, and hence $\mathcal{F}f \in C^\infty$.

By an exercise,

$$u^j D^k(\mathcal{F}f)(u) = i^{k+j}\mathcal{F}(D^j(x^k f))(u). \tag{8.5}$$

Using the product rule, $D^j(x^k f)$ is in L^1. Hence $u^j D^k \mathcal{F}f(u)$ is continuous and bounded. This implies that every derivative of $\mathcal{F}f(u)$ goes to zero faster than any power of $|u|^{-1}$. Therefore $\mathcal{F}f \in \mathcal{S}$.

Finally, if $f_m \to f$ in the sense of the Schwartz class, it follows by the dominated convergence theorem that $\mathcal{F}(f_m)(u) \to \mathcal{F}(f)(u)$ uniformly over $u \in \mathbb{R}$ and moreover $|u|^k D^j(\mathcal{F}(f_m)) \to |u|^k D^j(\mathcal{F}(f))$ uniformly over $\mathbb{R}$ for each j and k. □

If F is a tempered distribution, define $\mathcal{F}F$ by

$$\mathcal{F}F(f) = F(\widehat{f})$$

for all $f \in \mathcal{S}$. We verify that $\mathcal{F}G_g = G_{\widehat{g}}$ if $g \in \mathcal{S}$ as follows:

$$\begin{aligned}\mathcal{F}(G_g)(f) = G_g(\widehat{f}) &= \int \widehat{f}(x)g(x)\,dx \\ &= \int\int e^{iyx} f(y)g(x)\,dy\,dx = \int f(y)\widehat{g}(y)\,dy \\ &= G_{\widehat{g}}(f)\end{aligned}$$

if $f \in \mathcal{S}$.

Note that for the above equations to work, we used the fact that $\mathcal{F}$ maps $\mathcal{S}$ into $\mathcal{S}$. Of course, $\mathcal{F}$ does not map C_K^∞ into C_K^∞. That is why we define the Fourier transform only for tempered distributions rather than all distributions.

Theorem 8.18 *$\mathcal{F}$ is an invertible map on the class of tempered distributions and $\mathcal{F}^{-1} = (2\pi)^{1/2}\mathcal{F}R$. Moreover $\mathcal{F}$ and R commute.*

Proof. We know

$$f(x) = (2\pi)^{-1/2} \int \widehat{f}(-u) e^{ixu}\, du, \qquad f \in \mathcal{S},$$

so $f = (2\pi)^{-1/2}\mathcal{F}R\mathcal{F}f$, and hence $\mathcal{F}R\mathcal{F} = (2\pi)^{1/2}I$, where I is the identity. Then if H is a tempered distribution,

$$\begin{aligned}(2\pi)^{-1/2}\mathcal{F}R\mathcal{F}H(f) &= R\mathcal{F}H((2\pi)^{-1/2}\mathcal{F}f) = \mathcal{F}H((2\pi)^{-1/2}R\mathcal{F}f)\\ &= H((2\pi)^{-1/2}\mathcal{F}R\mathcal{F}f) = H(f).\end{aligned}$$

Thus

$$(2\pi)^{-1/2}\mathcal{F}R\mathcal{F}H = H,$$

or

$$(2\pi)^{-1/2}\mathcal{F}R\mathcal{F} = I.$$

We conclude $A = (2\pi)^{-1/2}\mathcal{F}R$ is a left inverse of $\mathcal{F}$ and $B = (2\pi)^{-1/2}R\mathcal{F}$ is a right inverse of $\mathcal{F}$. Hence $B = (A\mathcal{F})B = A(\mathcal{F}B) = A$, or $\mathcal{F}$ has an inverse, namely, $(2\pi)^{-1/2}\mathcal{F}R$, and moreover $R\mathcal{F} = \mathcal{F}R$. □

Chapter 9

Banach algebras

9.1 Normed algebras

An *algebra* is a linear space over $+$ and a ring over $\cdot$. We assume there is an identity for the multiplication, which we call I. Our algebras will be over the scalar field $\mathbb{C}$; the reasons will be very apparent shortly.

An algebra is a *normed algebra* if the linear space is normed and $\|NM\| \le \|N\|\,\|M\|$ and $\|I\| = 1$. If the normed algebra is complete, it is called a Banach algebra.

One example is to let $\mathcal{L} = \mathcal{L}(X, X)$, the set of linear maps from X into X. Another is to let $\mathcal{L}$ be the collection of bounded continuous functions on some set. A third example is to let $\mathcal{L}$ be the collection of bounded functions that are analytic in the unit disk.

An element M of $\mathcal{L}$ is *invertible* if there exists $N \in \mathcal{L}$ such that $NM = MN = I$.

M has a left inverse A if $AM = I$ and a right inverse B if $MB = I$. If it has both, then $B = AMB = A$, and so M is invertible.

Proposition 9.1 *(1) If M and K are invertible, then*

$$(MK)^{-1} = K^{-1}M^{-1}.$$

(2) If M and K commute and MK is invertible, then M and K are

invertible.

Proof. (1) is easy. For (2), let $N = (MK)^{-1}$. Then $MKN = I$, so KN is a right inverse for M. Also, $I = NMK = NKM$, so NK is a left inverse for M. Since M has a left and right inverse, it is invertible. The argument for K is similar. □

Proposition 9.2 *If K is invertible, then so is $L = K - A$ provided $\|A\| < 1/\|K^{-1}\|$.*

Proof. First we suppose $K = I$. If $\|B\| < 1$, then

$$\Big\| \sum_m^n B^i \Big\| \le \sum_m^n \|B^i\| \le \sum_m^n \|B\|^i$$

is a Cauchy sequence, so $S = \sum_{i=0}^\infty B^i$ converges.We see $BS = \sum_{i=1}^\infty B^i = S - I$, so $(I - B)S = I$. Similarly $S(I - B) = I$.

For the general case, write $K - A = K(I - K^{-1}A)$, and let $B = K^{-1}A$. Then $\|B\| \le \|K^{-1}\| \, \|A\| < 1$, and

$$(K - A)^{-1} = (I - K^{-1}A)^{-1}K^{-1}.$$

□

We note for future reference the equation

$$(I - B)^{-1} = \sum_{i=0}^\infty B^i. \tag{9.1}$$

The *resolvent* set of M, $\rho(M)$, is the set of $\lambda \in \mathbb{C}$ such that $\lambda I - M$ is invertible. The *spectrum* of M, $\sigma(M)$, is the set of λ for which $\lambda I - M$ is not invertible. We frequently write $\lambda - M$ for $\lambda I - M$. We will also use R_λ for $(\lambda I - M)^{-1}$.

Let $F : G \to X$, where $G \subset \mathbb{C}$, and write F_z for $F(z)$. F_z is *strongly analytic* if

$$\lim_{h \to 0} \frac{F_{z+h} - F_z}{h}$$

exists in the norm topology for all $z \in G$, that is, there exists an element F'_z such that the norm of $(F_{z+h} - F_z)/h - F'_z$ tends to 0 as $h \to 0$.

One can check that much of complex analysis can be extended to strongly analytic functions. There are two approaches one could follow to show this. One is to recall that much of complex analysis is derived from Cauchy's theorem, and that in turn is based on the fact that $\int_C cz\,dz = 0$ and $\int_C c\,dz = 0$ when C is the boundary of a rectangle. If we replace c by $M \in \mathcal{L}$, the same argument goes through.

The other argument is that if ℓ is a bounded linear functional on $\mathcal{L}$, then $f(z) = \ell(F_z)$ is analytic in the usual sense, and by Riemann sum approximations one can show that

$$\ell\Big(\int_C F_z\,dz\Big) = \int_C \ell(F_z)\,dz = 0$$

for suitable closed curves C. This is true for every ℓ, and so $\int_C F_z\,dz = 0$. Many of the other theorems of complex analysis can be proved in a similar way.

Proposition 9.3 *(1) $\rho(M)$ is open in $\mathbb{C}$.*

(2) $(z - M)^{-1}$ is an analytic function of z in $\rho(M)$.

Proof. If $\lambda \in \rho(M)$, letting $K = \lambda I - M$ and $A = -hI$, $K - A = (\lambda + h) - M$ is invertible if $|h| = \|A\| < 1/\|K^{-1}\|$, which happens if h is small. So $\lambda + h \in \rho(M)$.

For (2),

$$\lambda - M + hI = (\lambda - M)(I + hR_\lambda),$$

and so

$$(\lambda - M + hI)^{-1} = \Big(\sum_{i=0}^\infty (-1)^i (\lambda - M)^i h^i\Big)(\lambda - M)^{-1}.$$

Therefore

$$((\lambda + h) - M)^{-1} = \sum_{n=0}^\infty (-1)^n (\lambda - M)^{n-1} h^n = \Big(\sum_{i=0}^\infty (-hR_\lambda)^i\Big) R_\lambda$$

for h small. So the resolvent can be expanded in a power series in h which is valid if $|h| < \|(\lambda - M)^{-1}\|^{-1}$. We then have

$$\left\| \frac{R_{\lambda+h} - R_\lambda}{h} - (-R_\lambda^2) \right\| \to 0$$

as $h \to 0$. □

We write

$$r(M) = \sup_{\lambda \in \sigma(M)} |\lambda|,$$

and call this the *spectral radius* of M.

Theorem 9.4 *$\sigma(M)$ is closed, bounded, and nonempty.*

Proof. $\rho(M)$ is open, so $\sigma(M)$ is closed.

$$(zI - M)^{-1} = z^{-1}(I - Mz^{-1})^{-1} = \sum_{n=0}^{\infty} M^n z^{-n-1}$$

converges if $\|z^{-1}M\| < 1$, or equivalently, $|z| > \|M\|$. Therefore, if $|z| > \|M\|$, then $z \in \rho(M)$. Hence the spectrum is contained in $B_{\|M\|}(0)$.

Suppose $\sigma(M)$ is empty. For $z > \|M\|$ we have

$$R_z = (z - M)^{-1} = z^{-1}(I - z^{-1}M)^{-1}.$$

Let ℓ be any bounded linear functional on $\mathcal{L}$. We conclude that $f(z) = \ell(R_z)$ is analytic and $f(z) = \ell(R_z) \to 0$ as $|z| \to \infty$.

We thus know that f is analytic on $\mathbb{C}$, i.e., it is an entire function, and that $f(z)$ tends to 0 as $|z| \to \infty$. Therefore f is a bounded entire function. By Liouville's theorem from complex analysis, f must be constant. Since f tends to 0 as $|z|$ tends to infinity, that constant must be 0. This holds for all ℓ, so R_z must be equal to 0 for all z. But then we have $I = (z - M)R_z = 0$, a contradiction. □

A key result is the *spectral radius formula*. First we need a consequence of the uniform boundedness principle.

Lemma 9.5 *If $\mathcal{B}$ is a Banach space and $\{x_n\}$ a subset of $\mathcal{B}$ such that $\sup_n |f(x_n)|$ is finite for each bounded linear functional f, then $\sup_n \|x_n\|$ is finite.*

Proof. For each $x \in \mathcal{B}$, define a linear functional L_x on $\mathcal{B}^*$, the dual space of $\mathcal{B}$, by

$$L_x(f) = f(x), \qquad f \in \mathcal{B}^*.$$

We already have shown that $\|L_x\| = \|f\|$.

Since $\sup_n |L_{x_n}(f)| = \sup_n |f(x_n)|$ is finite for each $f \in \mathcal{B}^*$, by the uniform boundedness principle,

$$\sup_n \|L_{x_n}\| < \infty.$$

Since $\|L_{x_n}\| = \|x_n\|$, we obtain our result. □

Theorem 9.6 *(Spectral radius formula)*

$$r(M) = \lim_{k\to\infty} \|M^k\|^{1/k}.$$

Proof. Fix k for the moment. If we write $n = kq + r$,

$$\Big\| \sum_{n=0}^{\infty} \frac{M^n}{z^{n+1}} \Big\| \le \sum \frac{\|M^n\|}{|z|^{n+1}} \le \sum_{n=0}^{k-1} \frac{\|M\|^r}{|z|^{r+1}} \sum_q \Big(\frac{\|M^k\|}{|z|^k} \Big)^q.$$

So $\sum M^n |z|^{-n-1}$ converges absolutely if $\|M^k\|/|z|^k < 1$, or if $|z| > \|M^k\|^{1/k}$.

If $|z| > \|M^k\|^{1/k}$, then $z \in \rho(M)$. Hence if $\lambda \in \sigma(M)$, then $|\lambda| \le \|M^k\|^{1/k}$. This is true for all k, so $r(M) \le \liminf_{k\to\infty} \|M^k\|^{1/k}$.

For the other direction, if $z \in \mathbb{C}$ with $|z| < 1/r(M)$, then $|1/z| > r(M)$, and thus $1/z \notin \sigma(M)$ by the definition of $r(M)$. Hence $I - zM = z(z^{-1}I - M)$ if invertible if $z \neq 0$. Clearly $I - zM$ is invertible when $z = 0$ as well.

Suppose ℓ a linear functional on $\mathcal{L}$. The function $F(z) = \ell((I - zM)^{-1})$ is analytic in $B(0, 1/r(M)) \subset \mathbb{C}$. We know from complex analysis that a function has a Taylor series that converges absolutely in any disk on which the function is analytic. Therefore F has a Taylor series which converges absolutely at each point of $B(0, 1/r(M))$.

Let us identify the coefficients of the Taylor series. If $|z| < 1/\|M\|$, then we see that

$$F(z) = \ell\Big(\sum_{n=0}^{\infty} z^n M^n\Big) = \sum_{n=0}^{\infty} \ell(M^n) z^n. \tag{9.2}$$

Therefore $F^{(n)}(0) = n!\ell(M^n)$, where $F^{(n)}$ is the n^{th} derivative of F. We conclude that the Taylor series for F in $B(0, 1/r(M))$ is

$$F(z) = \sum_{n=0}^{\infty} \ell(M^n) z^n. \tag{9.3}$$

The difference between (9.2) and (9.3) is that the former is valid in the ball $B(0, 1/\|M\|)$ while the latter is valid in $B(0, 1/r(M))$.

We conclude from this that

$$\sum_{n=0}^{\infty} \ell(z^n M^n)$$

converges absolutely for z in the ball $B(0, 1/r(M))$, and consequently

$$\lim_{n\to\infty} |\ell(z^n M^n)| = 0$$

if $|z| < 1/r(M)$. By Lemma 9.5 there exists a real number K such that

$$\sup_n \|z^n M^n\| \leq K$$

for all $n \geq 1$ and all $z \in B(0, 1/r(M))$. This implies that

$$|z| \, \|M^n\|^{1/n} \leq K^{1/n},$$

and hence

$$|z| \limsup_{n\to\infty} \|M^n\|^{1/n} \leq 1$$

if $|z| < 1/r(M)$. Thus

$$\limsup_{n\to\infty} \|M^n\|^{1/n} \leq r(M),$$

which completes the proof. □

We say λ is an *eigenvalue* for M with associated *eigenvector* $x \neq 0$ if $Mx = \lambda x$. Note that not every element of $\sigma(M)$ is an eigenvalue of M. For example, if $M : \ell^2 \to \ell^2$ is defined by

$$M(x_1, x_2, \ldots) = (x_1, x_2/2, x_3/4, \ldots),$$

then $1, 1/2, 1/4, \ldots$ are eigenvalues. Since the spectrum is closed, then $0 \in \sigma(M)$, but 0 is not an eigenvalue for M.

9.2 Functional calculus

We can define $p(M) = \sum_{i=1}^n a_i M^i$ for any polynomial p. Suppose $f(z) = \sum_{j=0}^\infty a_j z^j$ is an analytic function with radius of convergence R. If $r(M) < R$, then there exists $r(M) < S < R$, and for j sufficiently large, $\|M^j\| \leq S^j$. Since

$$\Big\| \sum_{j=m+1}^n a_j M^j \Big\| \leq \sum_{j=m+1}^n |a_j| \, \|M^j\| \leq \sum_{j=m+1}^n |a_j| S^j$$

for m and n large enough and $\sum |a_j| \, S^j$ converges (since S is less than the radius of convergence for f), then we can define $f(M)$ for for any analytic function whose power series' radius of convergence is larger than the spectral radius of M as the limit of the polynomials $\sum_{j=0}^n a_j M^j$.

Let G be a domain containing $\sigma(M)$, f analytic in G, C a closed curve in $G \cap \rho(M)$ whose winding number is 1 about each point in $\sigma(M)$ and 0 about each point of G^c. Define

$$f(M) = \frac{1}{2\pi i} \int_C (z - M)^{-1} f(z) \, dz.$$

By Cauchy's theorem, this is independent of the contour chosen.

Theorem 9.7 *(1) If f is a polynomial, the two definitions agree.*

(2) Suppose f and g are analytic functions defined on a ball containing $\overline{B(0, r(M))}$. Then $f(M)g(M) = (fg)(M)$.

Proof. (1) Take a c larger than $r(M)$ and let $C = \{|z| = c\}$. Expanding $(z-M)^{-1}$ in a power series, which is valid if $|z| = c$, we have

$$\frac{1}{2\pi i}\int_C (z-M)^{-1} z^n \, dz = \frac{1}{2\pi i}\sum_{j=0}^{\infty} M^j \int_C z^{n-j-1}\, dz = M^n.$$

(2) The radius of convergence of fg is at least as large as the smaller of the radii of convergence of f and g, and hence is larger than $r(M)$. (2) follows easily by approximating f and g by polynomials. □

Here is the *spectral mapping theorem* for polynomials.

Theorem 9.8 *Suppose A is a bounded linear operator and P is a polynomial. Then $\sigma(P(A)) = P(\sigma(A))$.*

By $P(\sigma(A))$ we mean the set $\{P(\lambda) : \lambda \in \sigma(A)\}$.

Proof. We first suppose $\lambda \in \sigma(P(A))$ and prove that $\lambda \in P(\sigma(A))$. Factor

$$\lambda - P(x) = c(x-a_1)\cdots(x-a_n).$$

Since $\lambda \in \sigma(P(A))$, then $\lambda - P(A)$ is not invertible, and therefore for at least one i we must have that $A - a_i$ is not invertible. That means that $a_i \in \sigma(A)$. Since a_i is a root of the equation $\lambda - P(x) = 0$, then $\lambda = P(a_i)$, which means that $\lambda \in P(\sigma(A))$.

Now suppose $\lambda \in P(\sigma(A))$. Then $\lambda = P(a)$ for some $a \in \sigma(A)$. We can write

$$P(x) = \sum_{i=0}^{n} b_i x^i$$

for some coefficients b_i, and then

$$P(x) - P(a) = \sum_{i=1}^{n} b_i(x^i - a^i) = (x-a)Q(x)$$

for some polynomial Q, since $x - a$ divides $x^i - a^i$ for each $i \geq 1$. We then have

$$P(A) - \lambda = P(A) - P(a) = (A-a)Q(A).$$

If $P(A) - \lambda$ were invertible, then by an earlier lemma we would have that $A - a$ is invertible, a contradiction. Therefore $P(A) - \lambda$ is not invertible, i.e., $\lambda \in \sigma(P(A))$. □

9.3 Commutative Banach algebras

We look at *commutative Banach algebras* with a unit. Commutative means $MN = NM$ for all $M, N \in \mathcal{L}$.

p is a *multiplicative functional* on $\mathcal{L}$ if p is a homomorphism from $\mathcal{L}$ into $\mathbb{C}$.

Proposition 9.9 *Every homomorphism is a contraction.*

Proof. $M = IM$, so $p(M) = p(IM) = p(I)p(M)$, or $p(I) = 1$. If K is invertible,

$$p(K)p(K^{-1}) = p(KK^{-1}) = p(I) = 1,$$

so $p(K) \neq 0$. Suppose $|p(M)| > \|M\|$ for some M. Then if $B = M/p(M)$, we have $\|B\| < 1$, so $K = I - B$ is invertible. But

$$p(K) = p(I) - p(M/p(M)) = 1 - 1 = 0,$$

a contradiction.

Our goal in this section is to show that if $p(K) \neq 0$ for all homomorphisms, then K is invertible.

$\mathcal{I} \subset \mathcal{L}$ is an *ideal* if $\mathcal{I}$ is a linear subspace, $\mathcal{I} \neq \{0\}$, and if $M \in \mathcal{L}$ and $J \in \mathcal{I}$, then $MJ \in \mathcal{I}$. $\mathcal{I}$ is a *proper ideal* if $\mathcal{I} \neq \mathcal{L}$.

As an example, let $\mathcal{L} = C(S)$, let $r \in S$, and let $\mathcal{I} = \{f : f(r) = 0\}$.

If $I \in \mathcal{I}$, then $\mathcal{I} = \mathcal{L}$. If $\mathcal{I}$ contains an invertible element, then $\mathcal{I}$ contains the identity, and hence equals $\mathcal{L}$.

Lemma 9.10 *Let q be a homomorphism from $\mathcal{L}$ onto $\mathcal{A}$, but where q is not an isomorphism and $q(\mathcal{L}) \neq 0$. Then*

(1) $\{K \in \mathcal{L} : q(K) = 0\}$ *is a proper ideal. (This set is called the kernel of* q*.)*

(2) If $\mathcal{I}$ *is a proper ideal, then* $\mathcal{I}$ *is the kernel of some non-trivial homomorphism.*

Proof. (1) is easy. For (2), let $\mathcal{A} = \mathcal{L}/\mathcal{I}$. Let q map M into the equivalence class containing M. Then the kernel of q is $\mathcal{I}$. □

Proposition 9.11 *If* $K \in \mathcal{L}$, $K \neq 0$, *and* K *not invertible, then* K *lies in some proper ideal.*

Proof. Look at $K\mathcal{L} = \{KM : M \in \mathcal{L}\}$. Note $K\mathcal{L}$ does not contain the identity. □

Lemma 9.12 *Every proper ideal is contained in a maximal proper ideal.*

Proof. Let $\mathcal{J}$ be a proper ideal. Order the set of proper ideals that contain $\mathcal{J}$ by inclusion. The union of a totally ordered subcollection will be an upper bound. (Note that if $I \notin \mathcal{I}_\alpha$ for all α, then $I \notin \cup_\alpha \mathcal{I}_\alpha$.) Then use Zorn's lemma to find a maximal element of this collection. This element will be in the collection, and hence will be a proper ideal containing $\mathcal{J}$. □

A *division algebra* is one where every nonzero element is invertible.

Proposition 9.13 *If* $\mathcal{M}$ *is a maximal proper ideal of* $\mathcal{L}$, *then* $\mathcal{A} = \mathcal{L}/\mathcal{M}$ *is a division algebra.*

Proof. Suppose $C \in \mathcal{A}$, $C \neq 0$, and C is not invertible. Then $\mathcal{J} = C\mathcal{A} = \{CM : M \in \mathcal{A}\}$ is a proper ideal contained in $\mathcal{A}$. Let $q : \mathcal{L} \to \mathcal{L}/\mathcal{M} = \mathcal{A}$ be the usual map. $\mathcal{R} = q^{-1}(\mathcal{J})$ is easily checked to be a proper ideal in $\mathcal{L}$. If $M \in \mathcal{M}$, then $q(M) = 0$. So $\mathcal{M} = q^{-1}(\{0\})$ is contained in $\mathcal{R} = q^{-1}(\mathcal{J})$. $\mathcal{M}$ is a proper subset of $\mathcal{R}$ because $\mathcal{J} \neq \{0\}$. This contradicts that $\mathcal{M}$ is maximal. □

Lemma 9.14 *The closure of a proper ideal is a proper ideal.*

Proof. The only thing to prove is that $I \notin \overline{\mathcal{I}}$. We know $I \notin \mathcal{I}$, and so if $N \in B(I, 1)$, the ball of radius 1 about I, then N is invertible, and hence not in $\mathcal{I}$. So $B(I, 1)$ is an open set about I that is disjoint from $\mathcal{I}$. Therefore $I \notin \overline{\mathcal{I}}$. □

Lemma 9.15 *If $\mathcal{M}$ is a maximal proper ideal, then $\mathcal{M}$ is closed.*

Proof. If not, $\overline{\mathcal{M}}$ is a proper ideal strictly larger than $\mathcal{M}$. □

Lemma 9.16 *If $\mathcal{I}$ is a closed ideal in $\mathcal{L}$, then $\mathcal{L}/\mathcal{I}$ is a Banach algebra.*

Proposition 9.17 *If $\mathcal{A}$ is a Banach algebra with unit that is a division algebra, then $\mathcal{A}$ is isomorphic to $\mathbb{C}$.*

Proof. If $K \in \mathcal{A}$, there exists $\kappa \in \sigma(K)$. So $\kappa I - K$ is not invertible. Therefore $\kappa I - K = 0$, or $K = \kappa I$. The map $K \to \kappa$ is the desired isomorphism. □

Theorem 9.18 *$K \in \mathcal{L}$ is invertible if and only if $p(K) \neq 0$ for all homomorphisms p of $\mathcal{L}$ into $\mathbb{C}$.*

Proof. Suppose K is not invertible. K is in some maximal proper ideal $\mathcal{M}$. Then $\mathcal{M}$ is closed, $\mathcal{L}/\mathcal{M}$ is a division algebra, and is isomorphic to $\mathbb{C}$.

$$p : \mathcal{L} \to \mathcal{L}/\mathcal{M} \to \mathbb{C}$$

is a homomorphism onto $\mathbb{C}$, and its null space is $\mathcal{M}$. Since $K \in \mathcal{M}$, then $p(K) = 0$. □

9.4 Absolutely convergent Fourier series

Let $\mathcal{L}$ be the set of continuous functions from the unit circle S^1 to the complex functions such that $f(\theta) = \sum c_n e^{in\theta}$ with $\sum |c_n| < \infty$. We let the norm of f be $\sum |c_n|$.

We check that $\mathcal{L}$ is a Banach algebra. To do that, we use the fact that the Fourier coefficients for fg are the convolution of those for f and those for g, and that the convolution of two ℓ^1 functions is in ℓ^1, so $fg \in \mathcal{L}$. Here is the verification. If f has Fourier coefficients a_n and g has Fourier coefficients b_n, let

$$c_n = \sum_j a_j b_{n-j}.$$

Then to see that c_n are the Fourier coefficients of fg, write

$$\begin{aligned}\sum_n c_n e^{inx} &= \sum_n \sum_j a_j b_{n-j} e^{i(n-j)x} e^{ijx} \\ &= \sum_k \sum_j a_j b_k e^{ikx} e^{ijx} = f(x)g(x).\end{aligned}$$

To see that the norm of fg is less than or equal to the norm of f times the norm of g,

$$\sum_n |c_n| \leq \sum_n \sum_j |a_j|\,|b_{n-j}| = \sum_k \sum_j |a_j|\,|b_k|.$$

If $w \in S^1$, set $p_w(f) = f(w)$. p_w is a homomorphism from $\mathcal{L}$ to $\mathbb{C}$.

Proposition 9.19 *If p is a homomorphism from $\mathcal{L}$ to $\mathbb{C}$, then there exists w such that $p(f) = f(w)$ for all $f \in \mathcal{L}$.*

Proof. $p(I) = 1$ and $|p(M)| \leq \|M\|$, so p has norm 1. Then

$$|p(e^{i\theta})| \leq 1, \qquad |p(e^{-i\theta})| \leq 1,$$

and

$$1 = p(1) = p(e^{i\theta})p(e^{-i\theta}).$$

We must have $|p(e^{i\theta})| = 1$, or we would have inequality in the above.

Therefore there exists w such that $p(e^{i\theta}) = e^{iw}$. Since p is a homomorphism, by induction $p(e^{in\theta}) = e^{inw}$. By linearity,

$$p\Big(\sum_{n=-N}^{N} c_n e^{in\theta}\Big) = \sum_{n=-N}^{N} c_n e^{inw}.$$

If $f \in \mathcal{L}$, since p is continuous and $\sum |c_n| < \infty$, we have $p(f) = f(w)$. □

Theorem 9.20 *Suppose f has an absolutely convergent Fourier series and f is never 0 on S^1. Then $1/f$ also has an absolutely convergent Fourier series.*

Proof. If p is a homomorphism on $\mathcal{L}$, then $p(f) = f(w)$ for some w. Since f is never 0, $p(f) \neq 0$ for all non-trivial homomorphisms p. This implies f is invertible in $\mathcal{L}$. □

Chapter 10

Compact maps

10.1 Basic properties

A subset S is *precompact* if $\overline{S}$ is compact. Recall that if A is a subset of a metric space, A is precompact if and only if every sequence in A has a subsequence which converges in $\overline{A}$. Also, A is compact if and only if A is complete and totally bounded. Write B_1 for the unit ball in X.

A map K from a Banach space X to a Banach space U is *compact* if $K(B_1)$ is precompact in U.

One example is if K is degenerate, so that R_K is finite dimensional. The identity on ℓ^2 is not compact.

The following facts are easy:

(1) If C_1, C_2 are precompact subsets of a Banach space, then $C_1 + C_2$ is precompact.

(2) If C is precompact, so is the convex hull of C.

(3) If $M : X \to U$ and C is precompact in X, then $M(C)$ is precompact in U.

Proposition 10.1 *(1) If K_1 and K_2 are compact maps, so is $kK_1 + K_2$.*

(2) If $X \xrightarrow{L} U \xrightarrow{M}$, where M is bounded and L is compact, then ML is compact.

(3) In the same situation as (2), if L is bounded and M is compact, then ML is compact.

(4) If K_n are compact maps and $\lim \|K_n - K\| = 0$, *then K is compact.*

Proof. (1) For the sum, $(K_1 + K_2)(B_1) \subset K_1(B_1) + K_2(B_1)$, and the multiplication by k is similar.

(2) $ML(B_1)$ will be compact because $L(B_1)$ is compact and M is continuous.

(3) $L(B_1)$ will be contained in some ball, so $ML(B_1)$ is precompact.

(4) Let $\varepsilon > 0$. Choose n such that $\|K_n - K\| < \varepsilon$. $K_n(B_1)$ can be covered by finitely many balls of radius ε, so $K(B_1)$ is covered by the set of balls with the same centers and radius 2ε. Therefore $K(B_1)$ is totally bounded. □

We can use (4) to give a more complicated example of a compact operator. Let $X = U = \ell^2$ and define

$$K(a_1, a_2, \ldots,) = (a_1/2, a_2/2^2, a_3/2^3, \ldots).$$

It is the limit in norm of K_n, where

$$K_n(a_1, a_2, \ldots) = (a_1/2, a_2/2^2, \ldots, a_n/2^n, 0, \ldots).$$

Note that any bounded operator K on ℓ^2 maps B_1 into a set of the form $[-M, M]^{\mathbb{N}}$. By Tychonoff, this is compact in the product topology. However it is not necessarily compact in the topology of the space ℓ^2.

Proposition 10.2 *If X and Y are Banach spaces and $K : X \to Y$ is compact and Z is a closed subspace of X, then the map $K|_Z$ is compact.*

Let A be a bounded linear operator on a Banach space. If z is a complex number and I is the identity operator, then $zI - A$ is a bounded linear operator on H which might or might not be invertible. We define the *spectrum* of A by

$$\sigma(A) = \{z \in \mathbb{C} : zI - A \text{ is not invertible}\}.$$

We sometimes write $z - A$ for $zI - A$. The *resolvent set* for A is the set of complex numbers z such that $z - A$ is invertible. A non-zero element z is an *eigenvector* for A with corresponding *eigenvalue* λ if $Az = \lambda z$.

10.2 Compact symmetric operators

If A is a bounded operator on H, a Hilbert space over the complex numbers, the *adjoint* of A, denoted A^*, is the operator on H such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all x and y.

It follows from the definition that the adjoint of cA is $\overline{c}A^*$ and the adjoint of A^n is $(A^*)^n$. If $P(x) = \sum_{j=0}^n a_j x^j$ is a polynomial, the adjoint of $P(A) = \sum_{j=0}^n a_j A^j$ will be

$$\overline{P}(A^*) = \sum_{j=0}^n \overline{a}_j P(A^*).$$

The adjoint operator always exists.

Proposition 10.3 *If A is a bounded operator on H, there exists a unique operator A^* such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all x and y.*

Proof. Fix y for the moment. The function $f(x) = \langle Ax, y\rangle$ is a linear functional on H. By the Riesz representation theorem for Hilbert spaces, there exists z_y such that $\langle Ax, y\rangle = \langle x, z_y\rangle$ for all x. Since

$$\langle x, z_{y_1+y_2}\rangle = \langle Ax, y_1 + y_2\rangle = \langle Ax, y_1\rangle + \langle Ax, y_2\rangle = \langle x, z_{y_1}\rangle + \langle x, z_{y_2}\rangle$$

for all x, then $z_{y_1+y_2} = z_{y_1} + z_{y_2}$ and similarly $z_{cy} = cz_y$. If we define $A^*y = z_y$, this will be the operator we seek.

If A_1 and A_2 are two operators such that $\langle x, A_1y\rangle = \langle Ax, y\rangle = \langle x, A_2y\rangle$ for all x and y, then $A_1y = A_2y$ for all y, so $A_1 = A_2$. Thus the uniqueness assertion is proved. □

A bounded linear operator A mapping H into H is called *symmetric* if

$$\langle Ax, y\rangle = \langle x, Ay\rangle \tag{10.1}$$

for all x and y in H. Other names for symmetric are *Hermitian* or *self-adjoint*. When A is symmetric, then $A^* = A$, which explains the name "self-adjoint."

Example 10.4 For an example of a symmetric bounded linear operator, let $(X, \mathcal{A}, \mu)$ be a measure space with μ a σ-finite measure, let $H = L^2(X)$, and let $F(x, y)$ be a jointly measurable function from $X \times X$ into $\mathbb{C}$ such that $F(y, x) = \overline{F(x, y)}$ and

$$\int\int F(x,y)^2 \, \mu(dx) \, \mu(dy) < \infty. \tag{10.2}$$

Define $A : H \to H$ by

$$Af(x) = \int F(x, y) f(y) \, \mu(dy). \tag{10.3}$$

You can check that A is a bounded symmetric operator.

Here is an example of a compact symmetric operator.

Example 10.5 Let $H = L^2([0, 1])$ and let $F : [0, 1]^2 \to \mathbb{R}$ be a continuous function with $F(x, y) = \overline{F(y, x)}$ for all x and y. Define $K : H \to H$ by

$$Kf(x) = \int_0^1 F(x, y) f(y) \, dy.$$

We discussed in Example 10.4 the fact that K is a bounded symmetric operator. Let us show that it is compact.

If $f \in L^2([0, 1])$ with $\|f\| \le 1$, then

$$\begin{aligned} |Kf(x) - Kf(x')| &= \Big| \int_0^1 [F(x, y) - F(x', y)] f(y) \, dy \Big| \\ &\le \Big(\int_0^1 |F(x, y) - F(x', y)|^2 \, dy \Big)^{1/2} \|f\|, \end{aligned}$$

using the Cauchy-Schwarz inequality. Since F is continuous on $[0, 1]^2$, which is a compact set, then it is uniformly continuous there. Let $\varepsilon > 0$. There exists δ such that

$$\sup_{|x-x'|<\delta} \sup_y |F(x, y) - F(x', y)| < \varepsilon.$$

Hence if $|x - x'| < \delta$, then $|Kf(x) - Kf(x')| < \varepsilon$ for every f with $\|f\| \le 1$. In other words, $\{Kf : \|f\| \le 1\}$ is an equicontinuous family.

Since F is continuous, it is bounded, say by N, and therefore

$$|Kf(x)| \le \int_0^1 N|f(y)|\,dy \le N\|f\|,$$

again using the Cauchy-Schwarz inequality. If Kf_n is a sequence in $K(B_1)$, then $\{Kf_n\}$ is a bounded equicontinuous family of functions on $[0,1]$, and by the Ascoli-Arzelà theorem, there is a subsequence which converges uniformly on $[0,1]$. It follows that this subsequence also converges with respect to the L^2 norm. Since every sequence in $K(B_1)$ has a subsequence which converges, the closure of $K(B_1)$ is compact. Thus K is a compact operator.

We have the following proposition.

Proposition 10.6 *Suppose A is a bounded symmetric operator.*
(1) (Ax, x) is real for all $x \in H$.
(2) The function $x \to \langle Ax, x\rangle$ is not identically 0 unless $A = 0$.
(3) $\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|$.

Proof. (1) This one is easy since

$$\langle Ax, x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle},$$

where we use $\overline{z}$ for the complex conjugate of z.

(2) If $\langle Ax, x\rangle = 0$ for all x, then

$$\begin{aligned} 0 &= \langle A(x+y), x+y\rangle = \langle Ax, x\rangle + \langle Ay, y\rangle + \langle Ax, y\rangle + \langle Ay, x\rangle \\ &= \langle Ax, y\rangle + \langle y, Ax\rangle = \langle Ax, y\rangle + \overline{\langle Ax, y\rangle}. \end{aligned}$$

Hence $\mathrm{Re}\,\langle Ax, y\rangle = 0$. Replacing x by ix and using linearity,

$$\mathrm{Im}\,(\langle Ax, y\rangle) = -\mathrm{Re}\,(i\langle Ax, y\rangle) = -\mathrm{Re}\,(\langle A(ix), y\rangle) = 0.$$

Therefore $\langle Ax, y\rangle = 0$ for all x and y. We conclude $Ax = 0$ for all x, and thus $A = 0$.

(3) Let $\beta = \sup_{\|x\|=1} |\langle Ax, x\rangle|$. By the Cauchy-Schwarz inequality,

$$|\langle Ax, x\rangle| \le \|Ax\|\,\|x\| \le \|A\|\,\|x\|^2,$$

so $\beta \le \|A\|$.

To get the other direction, let $\|x\| = 1$ and let $y \in H$ such that $\|y\| = 1$ and $\langle y, Ax\rangle$ is real. Then

$$\langle y, Ax\rangle = \tfrac{1}{4}(\langle x+y, A(x+y)\rangle - \langle x-y, A(x-y)\rangle).$$

We used that $\langle y, Ax\rangle = \langle Ay, x\rangle = \langle Ax, y\rangle = \langle x, Ay\rangle$ since $\langle y, Ax\rangle$ is real and A is symmetric. Then

$$\begin{aligned} 16|\langle y, Ax\rangle|^2 &\le \beta^2(\|x+y\|^2 + \|x-y\|^2)^2 \\ &= 4\beta^2(\|x\|^2 + \|y\|^2)^2 \\ &= 16\beta^2. \end{aligned}$$

We used the parallelogram law (equation (4.1)) in the first equality. We conclude $|\langle y, Ax\rangle| \le \beta$.

If $\|y\| = 1$ but $\langle y, Ax\rangle = re^{i\theta}$ is not real, let $y' = e^{-i\theta}y$ and apply the above with y' instead of y. We then have

$$|\langle y, Ax\rangle| = |\langle y', Ax\rangle| \le \beta.$$

Setting $y = Ax/\|Ax\|$, we have $\|Ax\| \le \beta$. Taking the supremum over x with $\|x\| = 1$, we conclude $\|A\| \le \beta$. □

If $(Ax, x) \ge 0$ for all x, we say A is positive, and write $A \ge 0$. Writing $A \le B$ means $B - A \ge 0$. For matrices, one uses the words "positive definite."

Now suppose A is compact.

Proposition 10.7 *If $x_n \xrightarrow{w}$, then $Ax_n \xrightarrow{s}$.*

Proof. If $x_n \xrightarrow{w} x$, then $Ax_n \xrightarrow{w} Ax$, since $\langle Ax_n, y\rangle = \langle x_n, Ay\rangle \to \langle x, Ay\rangle = \langle Ax, y\rangle$. If x_n converges weakly, then $\|x_n\|$ is bounded so Ax_n lies in a precompact set.

Any subsequence of Ax_n has a further subsequence which converges strongly. The limit must be Ax. □

We will use the following easy lemma repeatedly.

Lemma 10.8 *If K is a compact operator and $\{x_n\}$ is a sequence with $\|x_n\| \le 1$ for each n, then $\{Kx_n\}$ has a convergent subsequence.*

Proof. Since $\|x_n\| \le 1$, then $\{\frac{1}{2}x_n\} \subset B_1$. Hence $\{\frac{1}{2}Kx_n\} = \{K(\frac{1}{2}x_n)\}$ is a sequence contained in $\overline{K(B_1)}$, a compact set and therefore has a convergent subsequence. □

We now prove the *spectral theorem* for compact symmetric operators.

Theorem 10.9 *Suppose H is a separable Hilbert space over the complex numbers and K is a compact symmetric linear operator. There exist a sequence $\{z_n\}$ in H and a sequence $\{\lambda_n\}$ in $\mathbb{R}$ such that*
(1) $\{z_n\}$ is an orthonormal basis for H,
(2) each z_n is an eigenvector with eigenvalue λ_n, that is, $Kz_n = \lambda_n z_n$,
(3) for each $\lambda_n \neq 0$, the dimension of the linear space $\{x \in H : Kx = \lambda_n x\}$ is finite,
(4) the only limit point, if any, of $\{\lambda_n\}$ is 0; if there are infinitely many distinct eigenvalues, then 0 is a limit point of $\{\lambda_n\}$.

Note that part of the assertion of the theorem is that the eigenvalues are real. (3) is usually phrased as saying the non-zero eigenvalues have finite *multiplicity.*

Proof. If $K = 0$, any orthonormal basis will do for $\{z_n\}$ and all the λ_n are zero, so we suppose $K \neq 0$. We first show that the eigenvalues are real, that eigenvectors corresponding to distinct eigenvalues are orthogonal, the multiplicity of non-zero eigenvalues is finite, and that 0 is the only limit point of the set of eigenvalues. We then show how to sequentially construct a set of eigenvectors and that this construction yields a basis.

If λ_n is an eigenvalue corresponding to a eigenvector $z_n \neq 0$, we see that

$$\begin{aligned}\lambda_n\langle z_n, z_n\rangle &= \langle \lambda_n z_n, z_n\rangle = \langle Kz_n, z_n\rangle = \langle z_n, Kz_n\rangle \\ &= \langle z_n, \lambda_n z_n\rangle = \overline{\lambda}_n\langle z_n, z_n\rangle,\end{aligned}$$

which proves that λ_n is real.

If $\lambda_n \neq \lambda_m$ are two distinct eigenvalues corresponding to the eigenvectors z_n and z_m, we observe that

$$\begin{aligned}\lambda_n\langle z_n, z_m\rangle &= \langle \lambda_n z_n, z_m\rangle = \langle Kz_n, z_m\rangle = \langle z_n, Kz_m\rangle \\ &= \langle z_n, \lambda_m z_m\rangle = \lambda_m\langle z_n, z_m\rangle,\end{aligned}$$

using that λ_m is real. Since $\lambda_n \neq \lambda_m$, we conclude $\langle z_n, z_m\rangle = 0$.

Suppose $\lambda_n \neq 0$ and that there are infinitely many orthonormal vectors x_k such that $Kx_k = \lambda_n x_k$. Then

$$\|x_k - x_j\|^2 = \langle x_k - x_j, x_k - x_j\rangle = \|x_k\|^2 - 2\langle x_k, x_j\rangle + \|x_j\|^2 = 2$$

if $j \neq k$. But then no subsequence of $\lambda_n x_k = Kx_k$ can converge, a contradiction to Lemma 10.8. Therefore the multiplicity of λ_n is finite.

Suppose we have a sequence of distinct non-zero eigenvalues converging to a real number $\lambda \neq 0$ and a corresponding sequence of eigenvectors each with norm one. Since K is compact, there is a subsequence $\{n_j\}$ such that Kz_{n_j} converges to a point in H, say w. Then

$$z_{n_j} = \frac{1}{\lambda_{n_j}} Kz_{n_j} \to \frac{1}{\lambda} w,$$

or $\{z_{n_j}\}$ is an orthonormal sequence of vectors converging to $\lambda^{-1}w$. But as in the preceding paragraph, we cannot have such a sequence.

Since $\{\lambda_n\} \subset \overline{B(0, r(K))}$, a bounded subset of the complex plane, if the set $\{\lambda_n\}$ is infinite, there will be a subsequence which converges. By the preceding paragraph, 0 must be a limit point of the subsequence.

We now turn to constructing eigenvectors. By Lemma 10.6(3), we have

$$\|K\| = \sup_{\|x\|=1} |\langle Kx, x\rangle|.$$

We claim the maximum is attained. If $\sup_{\|x\|=1} \langle Kx, x\rangle = \|K\|$, let $\lambda = \|K\|$; otherwise let $\lambda = -\|K\|$. Choose x_n with $\|x_n\| = 1$ such that $\langle Kx_n, x_n\rangle$ converges to λ. There exists a subsequence $\{n_j\}$ such that Kx_{n_j} converges, say to z. Since $\lambda \neq 0$, then $z \neq 0$, for otherwise $\lambda = \lim_{j\to\infty} \langle Kx_{n_j}, x_{n_j}\rangle = 0$. Now

$$\begin{aligned}\|(K - \lambda I)z\|^2 &= \lim_{j\to\infty} \|(K - \lambda I)Kx_{n_j}\|^2 \\ &\leq \|K\|^2 \lim_{j\to\infty} \|(K - \lambda I)x_{n_j}\|^2\end{aligned}$$

and

$$\begin{aligned}\|(K-\lambda I)x_{n_j}\|^2 &= \|Kx_{n_j}\|^2 + \lambda^2\|x_{n_j}\|^2 - 2\lambda\langle x_{n_j}, Kx_{n_j}\rangle \\ &\le \|K\|^2 + \lambda^2 - 2\lambda\langle x_{n_j}, Kx_{n_j}\rangle \\ &\to \lambda^2 + \lambda^2 - 2\lambda^2 = 0.\end{aligned}$$

Therefore $(K-\lambda I)z = 0$, or z is an eigenvector for K with corresponding eigenvalue λ.

Suppose we have found eigenvalues $z_1, z_2, \ldots, z_n$. Let X_n be the linear subspace spanned by $\{z_1, \ldots, z_n\}$ and let $Y = X_n^\perp$ be the orthogonal complement of X_n, that is, the set of all vectors orthogonal to every vector in X_n. If $x \in Y$ and $k \le n$, then

$$\langle Kx, z_k\rangle = \langle x, Kz_k\rangle = \overline{\lambda}_k\langle x, z_k\rangle = 0,$$

or $Kx \in Y$. Hence K maps Y into Y. It is an exercise to show that $K|_Y$ is a compact symmetric operator. If Y is non-zero, we can then look at $K|_Y$, and find a new eigenvector z_{n+1}.

It remains to prove that the set of eigenvectors forms a basis. Suppose y is orthogonal to every eigenvector. Then

$$\langle Ky, z_k\rangle = \langle y, Kz_k\rangle = \langle y, \lambda_k z_k\rangle = 0$$

if z_k is an eigenvector with eigenvalue λ_k, so Ky is also orthogonal to every eigenvector. Suppose X is the closure of the linear subspace spanned by $\{z_k\}$, $Y = X^\perp$, and $Y \ne \{0\}$. If $y \in Y$, then $\langle Ky, z_k\rangle = 0$ for each eigenvector z_k, hence $\langle Ky, z\rangle = 0$ for every $z \in X$, or $K : Y \to Y$. Thus $K|_Y$ is a compact symmetric operator, and by the argument already given, there exists an eigenvector for $K|_Y$. This is a contradiction since Y is orthogonal to every eigenvector. □

Remark 10.10 If $\{z_n\}$ is an orthonormal basis of eigenvectors for K with corresponding eigenvalues λ_n, let E_n be the projection onto the subspace spanned by z_n, that is, $E_n x = \langle x, z_n\rangle z_n$. A vector x can be written as $\sum_n \langle x, z_n\rangle z_n$, thus $Kx = \sum_n \lambda_n\langle x, z_n\rangle z_n$. We can then write

$$K = \sum_n \lambda_n E_n.$$

For general bounded symmetric operators there is a related expansion where the sum gets replaced by an integral, which we'll do later on.

Remark 10.11 If z_n is an eigenvector for K with corresponding eigenvalue λ_n, then $Kz_n = \lambda_n z_n$, so

$$K^2 z_n = K(Kz_n) = K(\lambda_n z_n) = \lambda_n K z_n = (\lambda_n)^2 z_n.$$

More generally, $K^j z_n = (\lambda_n)^j z_n$. Using the notation of Remark 10.10, we can write

$$K^j = \sum_n (\lambda_n)^j E_n.$$

If Q is any polynomial, we can then use linearity to write

$$Q(K) = \sum_n Q(\lambda_n) E_n.$$

It is a small step from here to make the definition

$$f(K) = \sum_n f(\lambda_n) E_n$$

for any bounded and Borel measurable function f.

If $\alpha_1 \geq \alpha_2 \geq \cdots > 0$ and $Az_n = \alpha_n z_n$, then our construction shows that

$$\alpha_N = \max_{x \perp z_1, \cdots, z_{N-1}} \frac{\langle Ax, x\rangle}{\|x\|^2}.$$

This is known as the *Rayleigh principle.*

Let

$$R_A(x) = \frac{\langle Ax, x\rangle}{\|x\|^2}.$$

Proposition 10.12 *Let A be compact and symmetric and let α_k be the non-negative eigenvalues with $\alpha_1 \geq \alpha_2 \geq \cdots$. Then*

(1) (Fisher's principle)

$$\alpha_N = \max_{S_N} \min_{x \in S_N} R_A(x),$$

where the maximum is over all linear subspaces S_N of dimension N.

(2) (Courant's principle)

$$\alpha_N = \min_{S_{N-1}} \max_{x \perp S_{N-1}} R_A(x),$$

where the minimum is over all linear subspaces of dimension $N-1$.

Proof. Let $z_1, \ldots, z_N$ be eigenvectors with corresponding eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N$. Let T_N be the linear subspace spanned by $\{z_1, \ldots, z_N\}$. If $y \in T_N$, we have $y = \sum_{j=1}^N c_j z_j$ for some complex numbers c_j and then

$$\begin{aligned}\langle Ay, y\rangle &= \sum_{i=1}^N \sum_{j=1}^N c_i \overline{c_j} \langle Az_i, z_j\rangle = \sum_i \sum_j c_i \overline{c_j} \alpha_i \langle z_i, z_j\rangle \\ &= \sum_i |c_i|^2 \alpha_i \geq \sum_i |c_i|^2 \alpha_N \\ &= \langle y, y\rangle\end{aligned}$$

using the fact that the z_i are orthogonal by our construction.

(1) Let z_k be the eigenvectors. Let S_N be a subspace of dimension N. There exists $y \in S_N$ such that $\langle y, z_k\rangle = 0$ for $k = 1, \ldots, N-1$. Since

$$\alpha_N = \max_{x \perp z_1, \ldots, z_{N-1}} R_A(x),$$

then y is one of the vectors over which the max is being taken, so $R_A(y) \leq \alpha_N$ for this y. So $\min_{x \in S_N} R_A(x) \leq \alpha_N$. This is true for all spaces of dimension N. So the right hand side is less than or equal to α_N.

Now we show the right hand side is greater than or equal to α_N. Let S_N be the linear span of $\{z_1, \ldots, z_N\}$. By the first paragraph of the proof, $R_A(x) \geq \alpha_N$ for every $x \in S_N$, and $R_A(x) = \alpha_N$ when $x = z_N$. So $\min_{x \in S_N} R_A(x) = \alpha_N$. The maximum over all subspaces of dimension N will be larger than the value for this particular subspace, so the right hand side is at least as large as α_N.

(2) Let S_{N-1} be a subspace of dimension $N-1$ and let T_N be the span of $\{z_1, \ldots, z_N\}$. Since the dimension of T_N is larger than that of S_{N-1},

there must be a vector $y \in T_N$ perpendicular to S_{N-1}. Since $y \in T_N$, then $R_A(y) \geq \alpha_N$ by the first paragraph of this proof, so

$$\max_{x \perp S_{N-1}} R_A(x) \geq R_A(y) \geq \alpha_N.$$

Taking the minimum over all spaces S_{N-1} shows that right hand side is greater than or equal to α_N.

If $x \perp T_{N-1}$, then $x = \sum_{j=N+1}^{\infty} c_j z_j$, and then

$$\begin{aligned} \langle Ax, x \rangle &= \sum_{j=N}^{\infty} \sum_{k=N}^{\infty} c_j \overline{c_k} \alpha_j \langle z_j, z_k \rangle \\ &= \sum_{j=N}^{\infty} \alpha_j |c_j|^2 \leq \alpha_N \sum_{j=N}^{\infty} |c_j|^2 \\ &= \alpha_N \langle x, x \rangle. \end{aligned}$$

Therefore $R_A(x) \leq \alpha_N$. This leads to

$$\min_{S_{N-1}} \max_{x \perp S_{N-1}} R_A(x) \leq \max_{x \perp T_{N-1}} R_A(x) \leq \alpha_N,$$

since T_{N-1} is a particular subspace of dimension $N-1$. □

Proposition 10.13 *Suppose $A \leq B$ with eigenvalues α_k, β_k, resp., ordered to be decreasing. Then $\alpha_k \leq \beta_k$ for all k.*

Proof. $A \leq B$ implies $\langle Ax, x \rangle \leq \langle Bx, x \rangle$, so $R_A(x) \leq R_B(x)$. Now use either Fisher's or Courant's principle. □

10.3 Mercer's theorem

We will need to use *Dini's theorem* from analysis.

Proposition 10.14 *Suppose g_n are continuous functions on $[0,1]$ with $g_n(x) \leq g_{n+1}(x)$ for each n and x and $g_\infty(x) = \lim_{n\to\infty} g_n(x)$ is continuous. Then g_n converges to g uniformly.*

Proof. Let $f_n = g_\infty - g_n$, so the f_n are continuous and decrease to 0. Let $\varepsilon > 0$. If $G_n(x) = \{x \in [0,1] : f_n(x) < \varepsilon\}$, then G_n is an open set (with respect to the relative topology on $[0,1]$), since f_n is continuous. Since $f_n(x) \to 0$, each x will be in some G_n. Thus $\{G_n\}$ is an open cover for $[0,1]$. Let $G_{n_1}, \dots, G_{n_m}$ be a finite subcover. If $n \ge \max(n_1, \dots, n_m)$ and $x \in [0,1]$, then x is in some G_{n_j} and $f_n(x) \le f_{n_j}(x) < \varepsilon$. Thus the convergence is uniform. □

Define $K : L^2[0,1] \to L^2[0,1]$ by

$$Ku(x) = \int_0^1 K(x,y)u(y)\,dy.$$

K^* has kernel $\overline{K(y,x)}$.

Suppose K is continuous, symmetric, and real-valued. Then K is compact, as we showed before. Therefore there exists a complete orthonormal system $\{e_j\}$ of eigenvectors. Let κ_j be the eigenvalue corresponding to e_j. $K : L^2 \to C[0,1]$, so $e_j = \kappa_j^{-1} K e_j$ is continuous if $\kappa_j \ne 0$.

Theorem 10.15 *(Mercer) Suppose K is real-valued, symmetric, and continuous. Suppose K is positive: $\langle Ku, u\rangle \ge 0$ for all $u \in H$. Then*

$$K(x,y) = \sum_j \kappa_j e_j(x)\overline{e_j(y)},$$

and the series converges uniformly and absolutely.

An example is to let $K = P_t$, the transition density of absorbing or reflecting Brownian motion.

Proof. First we observe that the κ_j are non-negative. To see this, let $u = e_j$, and we have

$$0 \le \langle e_j, Ke_j\rangle = \kappa_j \langle e_j, e_j\rangle.$$

$K \ge 0$ on the diagonal: Suppose $K(r,r) < 0$ for some r. Then $K(x,y) < 0$ if $|x - r|, |y - r| < \delta$ for some δ. Take $u = \chi_{[r-\delta/2, r+\delta/2]}$. Then

$$\langle Ku, u\rangle = \int\int K(x,y)u(y)x(s)\,ds\,dt < 0,$$

a contradiction.

Let $K_N(x,y) = \sum_{j=1}^N \kappa_j e_j(x)\overline{e_j(y)}$. If $f = \sum_{k=1}^\infty \langle f, e_k\rangle e_k$, we have

$$\begin{aligned} K_N f(x) &= \int_0^1 \sum_{j=1}^N \kappa_j e_j(x)\overline{e_j(y)} \sum_{k=1}^\infty \langle f, e_k\rangle e_k(y)\, dy \\ &= \sum_{j=1}^N \kappa_j \langle f, e_j\rangle e_j(x). \end{aligned}$$

We have

$$Kf(x) = \sum_{j=1}^\infty \langle f, e_j\rangle K e_j(x) = \sum_{j=1}^\infty \langle f, e_j\rangle \kappa_j e_j(x).$$

We conclude that $K - K_N$ is a positive operator, since

$$\langle f, (K-K_N)f\rangle = \sum_{k=1}^\infty \sum_{j=1}^N \kappa_j |\langle f, e_j\rangle|^2 \langle e_k, e_j\rangle = \sum_{j=1}^N \kappa_j |\langle f, e_j\rangle|^2 \geq 0.$$

As above, $K - K_N$ is non-negative on the diagonal, which implies that

$$\sum_{j=1}^N \kappa_j |e_j(x)|^2 \leq K(x,x).$$

Each term is non-negative, so the sum converges for each x. Let $J(x)$ be the limit.

Let $M = \sup_{x,y\in[0,1]} |K(x,y)|$. By Cauchy-Schwarz,

$$\begin{aligned} |K_N(x,y)| &\leq \Big(\sum_{j=1}^N \kappa_j |e_j(x)|^2\Big)^{1/2} \Big(\sum_{j=1}^N \kappa_j |e_j(y)|^2\Big)^{1/2} \\ &= (K_N(x,x))^{1/2} (K_N(y,y))^{1/2}. \end{aligned}$$

Fix x. By the same argument,

$$\begin{aligned}\Big|\sum_{j=m}^{n}\kappa_j e_j(x)\overline{e_j(y)}\Big|\\ &\le\Big(\sum_{j=m}^{n}\kappa_j|e_j(x)|^2\Big)^{1/2}\Big(\sum_{j=m}^{n}\kappa_j|e_j(y)|^2\Big)^{1/2}\\ &\le\Big(\sum_{j=m}^{n}\kappa_j|e_j(x)|^2\Big)^{1/2}M^{1/2}.\end{aligned}$$

The last line goes to 0 as $m,n\to\infty$ since $K_N(x,x)\to J(x)\le M$. Therefore, for each x, the functions $K_N(x,\cdot)$ converge uniformly. Let's call the limit $L(x,y)$. Then $L(x,y)$ will be continuous in y for each x.

Given f, let

$$f_N(x)=\sum_{j=1}^{N}\langle f,e_j\rangle e_j(x).$$

Note

$$\begin{aligned}Kf_N(x)&=\sum_{j=1}^{N}\langle f,e_j\rangle Ke_j(x)\\ &=\sum_{j=1}^{N}\langle f,e_j\rangle\kappa_j e_j(x)\\ &=K_Nf(x).\end{aligned}$$

We have

$$\|f-f_N\|^2=\sum_{j=N+1}^{\infty}|\langle f,e_j\rangle|^2\to 0$$

as $N\to\infty$ by Bessel's inequality, so

$$|Kf(x)-Kf_N(x)|\le\int_0^1|K(x,y)|\,|f(y)-f_N(y)|\,dy\le M\|f-f_N\|$$

by Cauchy-Schwarz. Therefore $K_Nf(x)\to Kf(x)$ as $N\to\infty$.

By dominated convergence,

$$K_Nf(x)=\int_0^1 K_N(x,y)f(y)\,dy\to\int_0^1 L(x,y)f(y)\,dy.$$

We therefore have

$$\int_0^1 L(x,y)f(y)\,dy = Kf(x)$$

for all $f \in L^2[0,1]$. This implies that (x is still fixed) $K(x,y) = L(x,y)$ for almost every y. With x fixed, both sides are continuous functions of y, hence they are equal for every y.

This is true for each x, and $K(x,y)$ is continuous, hence L is continuous. We now can apply Dini's theorem to conclude that $K_N(x,x)$ converges to $L(x,x) = J(x)$ uniformly. Finally, again by Cauchy-Schwarz,

$$\sum_{j=m}^{n} \kappa_j |e_j(x)|\,|\overline{e_j(y)}| \\ \le \Big(\sum_{j=m}^{n} \kappa_j |e_j(x)|^2\Big)^{1/2} \Big(\sum_{j=m}^{n} \kappa_j |e_j(y)|^2\Big)^{1/2},$$

and this proves that $K_N(x,y)$ converges to K uniformly and absolutely. □

10.4 Positive compact operators

We'll do the *Krein-Rutman theorem*, which is a generalization of the Perron-Frobenius theorem for matrices.

Theorem 10.16 *Suppose Q is compact and Hausdorff and $X = C(Q)$, the complex-valued continuous functions on Q. Suppose $K : C(Q) \to C(Q)$ and K is compact. Suppose further than K maps real-valued functions to real-valued functions. Finally, suppose that whenever $f \ge 0$ and f is not identically zero, then Kf is strictly positive. Then K has a positive eigenvalue σ of multiplicity one, the associated eigenfunction is positive, and all the other eigenvalues of K are strictly smaller in absolute value than σ.*

Examples include matrices with all positive entries, the semigroup P_t when $t = 1$ for reflecting Brownian motion on a bounded interval, and

$$Kf(x) = \int K(x,y)f(y)\,\mu(dy),$$

where K is jointly continuous, positive, and μ is a finite measure. We have seen that the operator K is compact.

Proof. If $f \le g$ and $f \not\equiv g$, then $g - f \ge 0$, so $K(g-f) > 0$, or $Kf < Kg$.

Step 1. We show there exists a non-zero eigenvalue. Let f be the identically one function. Since Kf is continuous and everywhere positive, there exists a positive number b such that $Kf \ge b = bf$.

If f and b are any pair such that $f \ge 0$, and $Kf \ge bf$, then

$$b^2 f \le bKf = K(bf) \le K(Kf) = K^2 f,$$

and continuing,

$$b^n f \le K^n f.$$

Since $f \ge 0$,

$$b^n \|f\| \le \|K^n f\| \le \|K^n\| \, \|f\|,$$

so

$$r(K) = \lim \|K^n\|^{1/n} \ge b.$$

Therefore $r(K)$ is strictly positive. Since K is compact, the set of eigenvalues of K is nonempty. We have shown that there exists a non-zero eigenvalue for K. Moreover, any b that satisfies $Kf \ge bf$ for some $f \ge 0$ is less than or equal to $r(K)$.

Step 2. K is compact, so there exists an eigenvalue λ and an eigenfunction g such that $Kg = \lambda g$, $|\lambda| = r(K)$. Let λ and g be any pair with $|\lambda| = r(K)$.

(a) We claim: if $f = |g|$ and $\sigma = |\lambda|$, then $\sigma f \le Kf$.

Proof: Let $x \in Q$. Multiply g by $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $\alpha\lambda g(x)$ is real and non-negative. Of course α depends on x. Write $g = u + iv$. Then

$$Ku(x) + iKv(x) = Kg(x) = \lambda g(x).$$

Looking at the real part,

$$\lambda g(x) = (Ku)(x).$$

Next, $u \le |g| = f$, and

$$|\lambda| f(x) = |\lambda g(x)| = Ku(x) \le (Kf)(x). \tag{10.4}$$

Then

$$\sigma f(x) \le Kf(x). \tag{10.5}$$

Although g depends on α, which depends on x, neither σ nor f depend on x. Since x was arbitrary, the inequality (10.5) holds for all x.

(b) We claim

$$\sigma f = Kf.$$

Proof: If not, there exists x such that $\sigma f(x) < Kf(x)$. By continuity, there exists a neighborhood N about x such that

$$\sigma f(s) + \delta \le Kf(s), \qquad s \in N.$$

Let $h > 0$ in N, 0 outside of N, and so $Kh > 0$.

We will find $c, \varepsilon > 0$ and set $F = f + \varepsilon h$, $\kappa = \sigma + c\varepsilon$, and get $\kappa F \le KF$. This will be a contradiction to Step 1: if $bf \le Kf$, then we know $b \le r(K)$; use this with b replaced by κ and f replaced by F.

(i) Now $Kh > 0$, so there exists $c \le 1$ such that $cf \le Kh$. If $s \in N$,

$$\begin{aligned} KF(s) &= Kf(s) + \varepsilon Kh(s) \ge Kf(s) + \varepsilon cf(s) \\ &\ge \sigma f(s) + \delta + \varepsilon cf(s). \end{aligned}$$

Then

$$\begin{aligned} \kappa F(s) &= (\sigma + c\varepsilon)(f + \varepsilon h)(s) = \sigma f(s) + \varepsilon cf(s) + \sigma\varepsilon h(s) \\ &\quad + c\varepsilon^2 h(s) \\ &\le KF(s) - \delta + \varepsilon cf(s) + \sigma\varepsilon h(s) + c\varepsilon^2 h(s). \end{aligned}$$

Since h is bounded above, we can take ε small enough so that the last line is less than or equal to $KF(s)$.

(ii) If $s \notin N$, then $h(s) = 0$ and

$$\begin{aligned} \kappa F(s) &= \kappa f(s) = (\sigma + c\varepsilon) f(s) = \sigma f(s) + \varepsilon cf(s) \\ &\le Kf(s) + \varepsilon Kh(s) = KF(s), \end{aligned}$$

using that $cf \le Kh$.

Step 3. We next show that any other eigenvalue that has absolute value σ is in fact equal to σ. Let G be any eigenfunction corresponding to λ with

$|\lambda| = \sigma$. Fix $x \in Q$. As before, we may assume $\lambda G(x) \geq 0$. As before, write $G = u + iv$ and then $\lambda G(x) = Ku(x)$. We have $u \leq |G| = f$.

Suppose $u < f$ at some point $y \in Q$. Then $u \leq f$ and $u < f$ at one point means that we have $Ku < Kf$ at every point, and so

$$|\lambda| f(x) = |\lambda G(x)| = \lambda G(x) = Ku(x) < Kf(x).$$

So $\sigma f(x)) < Kf(x)$. But we showed $\sigma f = Kf$. Therefore u is identically equal to f. This implies that G is real and positive, and then it follows that λ is real and positive. Since $G = \sigma^{-1} KG$, G is strictly positive.

Step 4. Finally, we show σ has multiplicity 1. If not, there exist distinct real eigenfunctions f_1, f_2. But some linear combination H of f_1, f_2 will be real, take the value 0, but not be identically zero. As before $|H|$ will be an eigenfunction that is non-negative, and must also take the value 0. Moreover the corresponding eigenvalue is σ. But then $0 < K|H| = \sigma|H|$, a contradiction to $|H|$ taking the value 0. □

Chapter 11

Spectral theory

11.1 Preliminaries

Suppose A is a bounded linear operator over a complex-valued Hilbert space. If $y \in H$ is fixed, then $\ell(x) = \langle Ax, y \rangle$ is a bounded linear functional on H. Therefore there exists $z = z_y \in H$ such that $\ell(x) = \langle x, z \rangle$ for all x. We define $A^* y = z_y$, so we have

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

for all x and y.

Note

$$\begin{aligned} \langle x, A^*(y_1 + y_2) \rangle &= \langle Ax, y_1 + y_2 \rangle = \langle Ax, y_1 \rangle + \langle Ax, y_2 \rangle \\ &= \langle x, A^* y_1 \rangle + \langle x, A^* y_2 \rangle = \langle x, A^* y_1 + A^* y_2 \rangle. \end{aligned}$$

We conclude $A^*(y_1 + y_2) = A^* y_1 + A^* y_2$. Similarly $A^*(cy) = cA^* y$, so A^* is a linear operator.

If $x, y \in H$, we have

$$|\langle x, A^* y \rangle| = |\langle Ax, y \rangle| \le \|A\| \, \|x\| \, \|y\|.$$

Taking the supremum over $\|x\| = 1$, we have $\|A^* y\| \le \|A\| \, \|y\|$, and taking the supremum over $\|y\| = 1$, we get $\|A^*\| \le \|A\|$. Replacing A by A^* we obtain $\|A^{**}\| \le \|A^*\|$ and noticing that $A^{**} = A$, we have

$$\|A^*\| = \|A\|.$$

It is easy to check that $(A+B)^* = A^* + B^*$, and since

$$\langle cAx, y\rangle = c\langle Ax, y\rangle = c\langle x, A^*y\rangle = \langle x, \overline{c}A^*y\rangle$$

for all x and y, we have $(cA)^* = \overline{c}A^*$. We note that

$$\langle A^2x, y\rangle = \langle Ax, A^*y\rangle = \langle x, (A^*)^2y\rangle,$$

so$(A^2)^* = (A^*)^2$. This holds for all positive powers n by an induction argument. If $P(z) = \sum_{j=0}^n c_j z^j$, let $\overline{P}(z) = \sum_{j=0}^n \overline{c_j} z^n$. We then have that $P(A)^* = \overline{P}(A^*)$.

In the case that $H = \mathbb{C}^n$, we can identify vectors in $\mathbb{C}^n$ with $n\times 1$ matrices and an operator A is identified with a $n \times n$ matrix. Then $\langle X, Y\rangle = X^T\overline{Y}$, where B^T is the transpose of a matrix B. Saying $\langle AX, Y\rangle = \langle X, A^*Y\rangle$ is the same as saying that $X^TA^T\overline{Y} = (AX)^T\overline{Y}$ is equal to $X^T\overline{A}^*\overline{Y}$ for all X and Y. Hence $A^T = \overline{A}^*$, or $A^* = \overline{A}^T$, the conjugate transpose of A.

We say A is a symmetric operator over a complex-valued Hilbert space if $A = A^*$, or equivalently, if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all x and $y \in H$.

If A is compact, we can write $x = \sum a_n e_n$ and $Ax = \sum \lambda_n a_n e_n$. Let E_n be the projection onto the eigenspace with eigenvector λ_n, so $x = \sum E_n x$ and $Ax = \sum \lambda_n E_n(x)$.

If we define a projection-valued measure $E(S)$ by

$$E(S) = \sum_{\lambda_n \in S} E_n$$

for S a Borel subset of $\mathbb{R}$, then $x = \int E(d\lambda)x$ and $Ax = \int \lambda E(d\lambda)x$.

Here E is a pure point measure. In general, we get the same result, but E might not be pure point.

When we move away from compact operators, the spectrum can become much more complicated. Let us look at an instructive example.

Example 11.1 Let $H = L^2([0,1])$ and define $A : H \to H$ by $Af(x) = xf(x)$. There is no difficulty seeing that A is bounded and symmetric.

We first show that no point in $[0,1]^c$ is in the spectrum of A. If z is a fixed complex number and either has a non-zero imaginary part or has a real part that is not in $[0,1]$, then $z-A$ has the inverse $Bf(x)=\frac{1}{z-x}f(x)$. It is obvious that B is in fact the inverse of $z-A$ and it is a bounded operator because $1/|z-x|$ is bounded on $x\in[0,1]$.

If $z\in[0,1]$, we claim $z-A$ does not have a bounded inverse. The function that is identically equal to 1 is in $L^2([0,1])$. The only function g that satisfies $(z-A)g=1$ is $g=1/(z-x)$, but g is not in $L^2([0,1])$, hence the range of $z-A$ is not all of H.

We conclude that $\sigma(A)=[0,1]$. We show now, however, that no point in $[0,1]$ is an eigenvalue for A. If $z\in[0,1]$ were an eigenvalue, then there would exist a non-zero f such that $(z-A)f=0$. Since our Hilbert space is L^2, saying f is non-zero means that the set of x where $f(x)\neq 0$ has positive Lebesgue measure. But $(z-A)f=0$ implies that $(z-x)f(x)=0$ a.e., which forces $f=0$ a.e. Thus A has no eigenvalues.

We have shown that the spectrum of a bounded symmetric operator is closed and bounded and never empty because the collection of bounded symmetric operators is a Banach algebra, although not a commutative one.

We proved the spectral radius formula when we studied Banach algebras:

$$r(A)=\lim_{n\to\infty}\|A^n\|^{1/n}.$$

We have the following important corollary.

Proposition 11.2 *If A is a symmetric operator, then*

$$\|A\|=r(A).$$

Proof. It suffices to show that $\|A^n\|=\|A\|^n$ when n is a power of 2. We show this for $n=2$ and the general case follows by induction.

On the one hand, $\|A^2\|\le\|A\|^2$. On the other hand,

$$\begin{aligned}\|A\|^2&=(\sup_{\|x\|=1}\|Ax\|)^2=\sup_{\|x\|=1}\|Ax\|^2\\&=\sup_{\|x\|=1}\langle Ax,Ax\rangle=\sup_{\|x\|=1}\langle A^2x,x\rangle\\&\le\|A^2\|\end{aligned}$$

by the Cauchy-Schwarz inequality. □

The following corollary will be important in the proof of the spectral theorem.

Corollary 11.3 *Let A be a symmetric bounded linear operator.*
(1) If P is a polynomial with real coefficients, then

$$\|P(A)\| = \sup_{z\in\sigma(A)} |P(z)|.$$

(2) If P is a polynomial with complex coefficients, then

$$\|P(A)\| \le 2 \sup_{z\in\sigma(A)} |P(z)|.$$

A later proposition will provide an improvement of assertion (2).

Proof. (1) Since P has real coefficients, then $P(A)$ is symmetric and

$$\begin{aligned}\|P(A)\| = r(P(A)) &= \sup_{z\in\sigma(P(A))} |z| \\ &= \sup_{z\in P(\sigma(A))} |z| = \sup_{w\in\sigma(A)} |P(w)|,\end{aligned}$$

where we used Corollary 11.2 for the first equality and the spectral mapping theorem for the third.

(2) If $P(z) = \sum_{j=0}^n (a_j + ib_j)z^j$, let $Q(z) = \sum_{j=0}^n a_j z^n$ and $R(z) = \sum_{j=0}^m b_j z^n$. By (1),

$$\|P(A)\| \le \|Q(A)\| + \|R(A)\| \le \sup_{z\in\sigma(A)} |Q(z)| + \sup_{z\in\sigma(A)} |R(z)|,$$

and (2) follows. □

We will also need the fact that the spectrum of a bounded symmetric operator is real. We know that each eigenvalue of a bounded symmetric operator is real, but as we have seen, not every element of the spectrum is an eigenvalue.

Proposition 11.4 *If A is bounded and symmetric, then $\sigma(A) \subset \mathbb{R}$.*

Proof. Suppose $\lambda = a + ib$, $b \neq 0$. We want to show that λ is not in the spectrum.

If r and s are real numbers, rewriting the inequality $(r - s)^2 \geq 0$ yields the inequality $2rs \leq r^2 + s^2$. By the Cauchy-Schwarz inequality

$$2a\langle x, Ax\rangle \leq 2|a|\, \|x\|\, \|Ax\| \leq a^2\|x\|^2 + \|Ax\|^2.$$

We then obtain the inequality

$$\begin{aligned} \|(\lambda - A)x\|^2 &= \langle (a + bi - A)x, (a + bi - A)x\rangle \\ &= (a^2 + b^2)\|x\|^2 + \|Ax\|^2 - (a + bi)\langle Ax, x\rangle \\ &\quad - (a - bi)\langle x, Ax\rangle \\ &= (a^2 + b^2)\|x\|^2 + \|Ax\|^2 - 2a\langle Ax, x\rangle \\ &\geq b^2\|x\|^2. \end{aligned} \tag{11.1}$$

This inequality shows that $\lambda - A$ is one-to-one, for if $(\lambda - A)x_1 = (\lambda - A)x_2$, then

$$0 = \|(\lambda - A)(x_1 - x_2)\| \geq b^2\|x_1 - x_2\|^2.$$

Suppose λ is in the spectrum of A. Since $\lambda - A$ is one-to-one but not invertible, it cannot be onto. Let R be the range of $\lambda - A$. We next argue that R is closed.

If $y_k = (\lambda - A)x_k$ and $y_k \to y$, then (11.1) shows that

$$b^2\|x_k - x_m\|^2 \leq \|y_k - y_m\|^2,$$

or x_k is a Cauchy sequence. If x is the limit of this sequence, then

$$(\lambda - A)x = \lim_{n\to\infty} (\lambda - A)x_k = \lim_{n\to\infty} y_k = y.$$

Therefore R is a closed subspace of H but is not equal to H. Choose $z \in R^\perp$. For all $x \in H$,

$$0 = \langle (\lambda - A)x, z\rangle = \langle x, (\overline{\lambda} - A)z\rangle.$$

This implies that $(\overline{\lambda} - A)z = 0$, or $\overline{\lambda}$ is an eigenvalue for A with corresponding eigenvector z. However we know that all the eigenvalues of a bounded symmetric operator are real, hence $\overline{\lambda}$ is real. This shows λ is real, a contradiction. □

11.2 Functional calculus

Let f be a continuous function on $\mathbb{C}$ and let A be a bounded symmetric operator on a separable Hilbert space over the complex numbers. We describe how to define $f(A)$.

We have shown that the spectrum of A is a closed and bounded subset of $\mathbb{C}$, hence a compact set. By the Stone-Weierstrass theorem we can find polynomials P_n (with complex coefficients) such that P_n converges to f uniformly on $\sigma(A)$. Then

$$\sup_{z\in\sigma(A)} |(P_n - P_m)(z)| \to 0$$

as $n, m \to \infty$. By Corollary 11.3

$$\|(P_n - P_m)(A)\| \to 0$$

as $n, m \to \infty$, or in other words, $P_n(A)$ is a Cauchy sequence in the space $\mathcal{L}$ of bounded symmetric linear operators on H. We call the limit $f(A)$.

The limit is independent of the sequence of polynomials we choose. If Q_n is another sequence of polynomials converging to f uniformly on $\sigma(A)$, then

$$\lim_{n\to\infty} \|P_n(A) - Q_n(A)\| \le 2 \sup_{z\in\sigma(A)} |(P_n - Q_n)(z)| \to 0,$$

so $Q_n(A)$ has the same limit $P_n(A)$ does.

We record the following facts about the operators $f(A)$ when f is continuous.

Proposition 11.5 *Let f be continuous on $\sigma(A)$.*
(1) $\langle f(A)x, y\rangle = \langle x, \overline{f}(A)y\rangle$ for all $x, y \in H$.

(2) If f is equal to 1 on $\sigma(A)$, then $f(A) = I$, the identity.
(3) If $f(z) = z$ on $\sigma(A)$, then $f(A) = A$.
(4) $f(A)$ and A commute.
(5) If f and g are two continuous functions, then $(f+g)(A) = f(A) + g(A)$ and $f(A)g(A) = (fg)(A)$.
(6) $\|f(A)\| \le \sup_{z\in\sigma(A)} |f(z)|$.

Proof. The proofs of (1)-(4) are routine and follow from the corresponding properties of $P_n(A)$ when P_n is a polynomial. Let us prove (5) and (6) and leave the proofs of the others to the reader.

(5) Let P_n and Q_n be polynomials converging uniformly on $\sigma(A)$ to f and g, respectively. Then P_nQ_n will be polynomials converging uniformly to fg. The second assertion of (5) now follows from

$$(fg)(A) = \lim_{n\to\infty} (P_nQ_n)(A) = \lim_{n\to\infty} P_n(A)Q_n(A) = f(A)g(A).$$

The limits are with respect to the norm on bounded operators on H. The first assertion of (5) is similar.

(6) Since f is continuous on $\sigma(A)$, so is $g = |f|^2$. Let P_n be polynomials with real coefficients converging to g uniformly on $\sigma(A)$. By Corollary 11.3(1),

$$\|g(A)\| = \lim_{n\to\infty} \|P_n(A)\| \le \lim_{n\to\infty} \sup_{z\in\sigma(A)} |P_n(z)| = \sup_{z\in\sigma(A)} |g(z)|.$$

If $\|x\| = 1$, using (1) and (5),

$$\begin{aligned}\|f(A)x\|^2 &= \langle f(A)x, f(A)x\rangle = \langle x, \overline{f}(A)f(A)x\rangle = \langle x, g(A)x\rangle \\ &\le \|x\|\ \|g(A)x\| \le \|g(A)\| \le \sup_{z\in\sigma(A)} |g(z)| \\ &= \sup_{z\in\sigma(A)} |f(z)|^2.\end{aligned}$$

Taking the supremum over the set of x with $\|x\| = 1$ yields

$$\|f(A)\|^2 \le \sup_{z\in\sigma(A)} |f(z)|^2,$$

and (6) follows. □

We have the spectral mapping theorem for continuous functions.

Theorem 11.6 *Suppose A is symmetric and suppose f is continuous on $\sigma(A)$. Then*

$$\sigma(f(A)) = f(\sigma(A)).$$

Here $f(\sigma(A)) = \{f(\lambda) : \lambda \in \sigma(A)\}$.

Proof. *Step 1.* Suppose $\mu \notin f(\sigma(A))$; we show $\mu \notin \sigma(f(A))$, and then conclude $\sigma(f(A)) \subset f(\sigma(A))$. If $\mu \neq f(\lambda)$ for some $\lambda \in \sigma(A)$, then $f(z) - \mu$ does not vanish on $\sigma(A)$. So we can find a continuous function g that agrees with $(f(z) - \mu)^{-1}$ on $\sigma(A)$.

If we take polynomials P_n converging to $f - \mu$ and polynomials converging to g uniformly on $\sigma(A)$ and let $R_n(z) = (P_n(z) - \mu))(Q_n(z)) - 1$, then R_n converges uniformly to 0 on $\sigma(A)$. Therefore $R_n(A)$ converges to 0 in norm. So

$$[f(A) - \mu I]g(A) = I.$$

Therefore $g(A)$ is the inverse of $f(A) - \mu I$, and hence $\mu \notin \sigma(f(A))$.

Step 2. We show $f(\sigma(A)) \subset \sigma(f(A))$. Suppose $\lambda \in \sigma(A)$. We need to show $f(\lambda) \in \sigma(f(A))$. Suppose not, that is, $f(\lambda) - f(A)$ is invertible. Let P_n be polynomials converging to f uniformly on $\sigma(A)$. We use the fact that $K - B$ is invertible if K is invertible and the norm of B is less than $1/\|K^{-1}\|$. We set $K = f(\lambda) - f(A)$ and

$$B = f(\lambda) - P_n(z) - f(A) + P_n(A).$$

Then if n is sufficiently large and z is sufficiently close to λ, then $P_n(z) - P_n(A)$ will be invertible. Thus for such n and z, we see that $P_n(z) \notin \sigma(P_n(A)) = P_n(\sigma(A))$. Since P_n converges uniformly to f on $\sigma(A)$, we conclude $f(\lambda) \notin f(\sigma(A))$, a contradiction. □

A is a *positive* operator if $\langle Ax, x\rangle \geq 0$ for all x.

Proposition 11.7 *Let A be bounded and symmetric. A is positive if and only if $\sigma(A) \geq 0$.*

Proof. If $\sigma(A) \geq 0$, then $f(\lambda) = \sqrt{\lambda}$ is a continuous real-valued function for $\lambda \geq 0$, and so $N = f(A) = \sqrt{A}$ exists and is a symmetric operator because $N^* = \overline{f}(A) = f(A) = N$. Then

$$\langle Ax, x\rangle = \langle N^2 x, x\rangle = \langle Nx, Nx\rangle \geq 0.$$

Now suppose A is positive. If $\lambda \in \sigma(A)$ is strictly negative,

$$\|x\| \, \|(A-\lambda)x\| \geq \langle x, (A-\lambda)x \rangle = \langle x, Ax \rangle - \lambda \|x\|^2 \geq -\lambda \|x\|^2,$$

using Cauchy-Schwarz. Dividing both sides by $\|x\|$, we have

$$\|(\lambda - A)x\| \geq (-\lambda)\|x\|.$$

Similarly to the proof that the spectrum of a symmetric operator is contained in the reals, we see that $\lambda - A$ is one-to-one, its range R is closed, and $\overline{\lambda}$ is an eigenvalue for A. But then

$$\langle Az, z \rangle = \langle \lambda z, z \rangle = \lambda \|z\|^2 < 0,$$

a contradiction, where we used the fact that λ is real and hence $\overline{\lambda} = \lambda$. □

Corollary 11.8 *Every positive symmetric operator has a positive symmetric square root.*

11.3 Riesz representation theorem

The Riesz representation theorem for positive linear functionals on $\mathcal{C}(X)$ is proved in real analysis. We will need the version for complex-valued bounded linear functionals. See [1] for a proof.

Theorem 11.9 *If S is a compact metric space and I is a bounded complex-valued linear functional on $\mathcal{C}(X)$, there exists a unique finite complex-valued measure μ on the Borel σ-algebra such that*

$$I(f) = \int f \, d\mu$$

for each $f \in \mathcal{C}(X)$. Moreover the total variation of μ is

$$\sup \Big\{ \int f \, d\mu : \sup_{z \in S} |f(z)| \leq 1 \Big\}.$$

11.4 Spectral resolution

We now want to define $f(A)$ when f is a bounded Borel measurable function on $\mathbb{C}$. Fix $x, y \in H$. If f is a continuous function on $\mathbb{C}$, let

$$L_{x,y} f = \langle f(A)x, y \rangle. \tag{11.2}$$

It is easy to check that $L_{x,y}$ is a bounded linear functional on $\mathcal{C}(\sigma(A))$, the set of continuous functions on $\sigma(A)$. By the Riesz representation theorem for complex-valued linear functionals, there exists a complex measure $\mu_{x,y}$ such that

$$\langle f(A)x, y \rangle = L_{x,y} f = \int_{\sigma(A)} f(z)\, \mu_{x,y}(dz)$$

for all continuous functions f on $\sigma(A)$.

Define

$$L_{x,y} f = \int f(z)\, \mu_{x,y}(dz)$$

for all f that are bounded and Borel measurable on $\sigma(A)$.

We have the following properties of $\mu_{x,y}$.

Proposition 11.10 *(1)* $\mu_{x,y}$ *is linear in* x.
(2) $\mu_{y,x} = \overline{\mu_{x,y}}$.
(3) The total variation of $\mu_{x,y}$ *is less than or equal to* $\|x\|\,\|y\|$.

Proof. (1) The linear functional $L_{x,y}$ defined in (11.2) is linear in x and

$$\int f\, d(\mu_{x,y} + \mu_{x',y}) = L_{x,y} f + L_{x',y} f = L_{x+x',y} f = \int f\, d\mu_{x+x',y}.$$

By the uniqueness of the Riesz representation, $\mu_{x+x',y} = \mu_{x,y} + \mu_{x'+y}$. The proof that $\mu_{cx,y} = c\mu_{x,y}$ is similar.

(2) follows from the fact that if f is continuous on $\sigma(A)$, then

$$\begin{aligned}
\int f\, d\mu_{y,x} &= L_{y,x} f = \langle f(A)y, x \rangle = \langle y, \overline{f}(A)x \rangle \\
&= \overline{\langle \overline{f}(A)x, y \rangle} = \overline{L_{x,y} \overline{f}} = \overline{\int \overline{f}\, d\mu_{x,y}} \\
&= \int f\, d\overline{\mu_{x,y}}.
\end{aligned}$$

Now use the uniqueness of the Riesz representation.

(3) This follows from the Riesz representation theorem. □

If f is a bounded Borel measurable function on $\mathbb{C}$, then $L_{y,x}\overline{f}$ is linear in y. Note that

$$|L_{y,x}f| \le \sup_{z\in\sigma(A)} |f(z)| \|\mu_{x,y}\|_{TV} \le \sup_{z\in\sigma(A)} |f(z)|\, \|x\|\, \|y\|,$$

where TV stands for "total variation." Thus $L_{y,x}$ is a bounded linear functional on $\mathbb{C}$ with norm bounded by $\|x\|\,\|y\|$. By the Riesz representation theorem for Hilbert spaces, there exists $w_x \in H$ such that $L_{y,x}\overline{f} = \langle y, w_x\rangle$ for all $y \in H$. We then have that for all $y \in H$,

$$\begin{aligned} L_{x,y}f &= \int_{\sigma(A)} f(z)\, \mu_{x,y}(dz) = \int_{\sigma(A)} f(z)\, \overline{\mu_{y,x}(dz)} \\ &= \overline{\int_{\sigma(A)} \overline{f(z)}\, \mu_{y,x}(dz)} = \overline{L_{y,x}\overline{f}} \\ &= \overline{\langle y, w_x\rangle} = \langle w_x, y\rangle. \end{aligned}$$

Since

$$\langle y, w_{x_1+x_2}\rangle = L_{y,x_1+x_2}\overline{f} = L_{y,x_1}\overline{f} + L_{y,x_2}\overline{f} = \langle y, w_{x_1}\rangle + \langle y, w_{x_2}\rangle$$

for all y and

$$\langle y, w_{cx}\rangle = L_{y,cx}\overline{f} = \overline{c}L_{y,x}\overline{f} = \overline{c}\langle y, w_x\rangle = \langle y, cw_x\rangle$$

for all y, we see that w_x is linear in x. We define $f(A)$ to be the linear operator on H such that $f(A)x = w_x$.

In the particular case when A is also compact, if (λ_n, φ_n) are the eigenvalue/eigenvector pairs with $\{\varphi_n\}$ an orthonormal basis, we have

$$x = \sum_{n=1}^{\infty} \langle x, \varphi_n\rangle \varphi_n$$

and

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \varphi_n\rangle \varphi_n.$$

Then

$$A^2x = \sum_{n=1}^{\infty} \lambda_n \langle x, \varphi_n \rangle A^2 \varphi_n = \sum_{n=1}^{\infty} \lambda_n^2 \langle x, \varphi_n \rangle \varphi_n.$$

Generalizing this, we have

$$P(A)x = \sum_{n=1}^{\infty} P(\lambda_n) \langle x, \varphi_n \rangle \varphi_n,$$

and passing to the limit for continuous functions, and then for bounded and Borel measurable f,

$$f(A)x = \sum_{n=1}^{\infty} f(\lambda_n) \langle x, \varphi_n \rangle \varphi_n.$$

Specializing further to matrices, if A is a diagonal matrix with diagonal entries $A_{jj} = \lambda_j$, then $f(A)$ is the diagonal matrix with diagonal entries $f(\lambda_j)$.

If C is a Borel measurable subset of $\mathbb{C}$, we let

$$E(C) = \chi_C(A). \tag{11.3}$$

Remark 11.11 Later on we will write the equation

$$f(A) = \int_{\sigma(A)} f(z)\, E(dz). \tag{11.4}$$

Let us give the interpretation of this equation. If $x, y \in H$, then

$$\langle E(C)x, y \rangle = \langle \chi_C(A)x, y \rangle = \int_{\sigma(A)} \chi_C(z)\, \mu_{x,y}(dz).$$

Therefore we identify $\langle E(dz)x, y \rangle$ with $\mu_{x,y}(dz)$. With this in mind, (11.4) is to be interpreted to mean that for all x and y,

$$\langle f(A)x, y \rangle = \int_{\sigma(A)} f(z)\, \mu_{x,y}(dz).$$

Theorem 11.12 *(1) $E(C)$ is symmetric.*
(2) $\|E(C)\| \leq 1$.
(3) $E(\emptyset) = 0, E(\sigma(A)) = I$.
(4) If C, D are disjoint, $E(C \cup D) = E(C) + E(D)$.
(5) $E(C \cap D) = E(C)E(D)$.
(6) $E(C)$ and A commute.
(7) $E(C)^2 = E(C)$, so $E(C)$ is a projection. If C, D are disjoint, then $E(C)E(D) = 0$.
(8) $E(C)$ and $E(D)$ commute.

Proof. (1) This follows from

$$\begin{aligned}\langle x, E(C)y\rangle &= \overline{\langle E(C)y, x\rangle} = \overline{\int \chi_C(z)\,\mu_{y,x}(dz)}\\ &= \int \chi_C(z)\,\mu_{x,y}(dz) = \langle E(C)x, y\rangle.\end{aligned}$$

(2) Since the total variation of $\mu_{x,y}$ is bounded by $\|x\|\,\|y\|$, we obtain (2).

(3) $\mu_{x,y}(\emptyset) = 0$, so $E(\emptyset) = 0$. If f is identically equal to 1, then $f(A) = I$, and

$$\langle x, y\rangle = \int_{\sigma(A)} \mu_{x,y}(dz) = \langle E(\sigma(A))x, y\rangle.$$

This is true for all y, so $x = E(\sigma(A))x$ for all x.

(4) holds because $\mu_{x,y}$ is a measure, hence finitely additive.

(5) If we prove that

$$f(A)g(A) = (fg)(A) \tag{11.5}$$

for f and g bounded and Borel measurable on $\sigma(A)$, we can apply this with $f = \chi_C$ and $g = \chi_D$. Then $fg = \chi_{C\cap D}$, and we get (5).

Now

$$\langle f_n(A)g_m(A)x, y\rangle = \langle (f_ng_m)(A)x, y\rangle \tag{11.6}$$

when f_n and g_m are continuous. The right hand side equals

$$\int (f_ng_m)(z)\,\mu_{x,y}(dz),$$

which converges to

$$\int (f_n g)(z)\, \mu_{x,y}(dz) = \langle (f_n g)(A)x, y\rangle$$

when $g_m \to g$ boundedly and a.e. with respect to $\mu_{x,y}$. The left hand side of (11.6) equals

$$\langle g_m(A)x, \overline{f}_n(A)y\rangle = \int g_m(z)\, \mu_{\overline{f}_n(A)x,y}(dz),$$

which converges to

$$\int g(z)\, \mu_{\overline{f}_n(A)x,y}(dz) = \langle g(A)x, \overline{f}_n(A)y\rangle$$

as long as g_m also converges a.e. with respect to $\mu_{\overline{f}_n(A)x,y}$. So we have

$$\langle f_n(A)g(A)x, y\rangle = \langle (f_n g)(A)x, y\rangle. \tag{11.7}$$

If we let f_n converge to f boundedly and a.e. with respect to $\mu_{x,y}$, the right hand side converges as in the previous paragraph to $\langle (fg)(A)x, y\rangle$. The right hand side of (11.7) is equal to

$$\langle g(A)x, \overline{f}_n(A)y\rangle = \overline{\langle \overline{f}_n(A)y, g(A)x\rangle}. \tag{11.8}$$

If $\overline{f}_n$ converges to $\overline{f}$ a.e with respect to $\mu_{y,g(A)x}$, the right hand side of (11.8) converges by arguments similar to the above to

$$\overline{\langle \overline{f}(A)y, g(A)x\rangle} = \langle g(A)x, \overline{f}(A)y\rangle = \langle f(A)g(A)x, y\rangle.$$

(6) Let $h(z) = z$ and apply (11.5) with $f = \chi_C$ and $g = h$ to get $\chi_C(A)A = (\chi_C h)(A)$. Then apply (11.5) with $f = h$ and $g = \chi_C$ to get $A\chi_C(A) = (h\chi_C)(A)$.

(7) Setting $C = D$ in (5) shows $E(C) = E(C)^2$, so $E(C)$ is a projection. If $C \cap D = \emptyset$, then $E(C)E(D) = E(\emptyset) = 0$, as required.

(8) Writing

$$E(C)E(D) = E(C \cap D) = E(D \cap C) = E(D)E(C)$$

proves (8). □

The family $\{E(C)\}$, where C ranges over the Borel subsets of $\mathbb{C}$ is called the *spectral resolution of the identity.* We explain the name in just a moment.

Here is the *spectral theorem* for bounded symmetric operators.

Theorem 11.13 *Let H be a separable Hilbert space over the complex numbers and A a bounded symmetric operator. There exists a operator-valued measure E satisfying (1)–(8) of Theorem 11.12 such that*

$$f(A) = \int_{\sigma(A)} f(z)\, E(dz), \tag{11.9}$$

for bounded Borel measurable functions f. Moreover, the measure E is unique.

Remark 11.14 When we say that E is an operator-valued measure, here we mean that (1)–(8) of Theorem 11.12 hold. We use Remark 11.11 to give the interpretation of (11.9).

Remark 11.15 If f is identically one, then (11.9) becomes

$$I = \int_{\sigma(A)} E(d\lambda),$$

which shows that $\{E(C)\}$ is a decomposition of the identity. This is where the name "spectral resolution" comes from.

Proof of Theorem 11.13. Given Remark 11.14, the only part to prove is the uniqueness, and that follows from the uniqueness of the measure $\mu_{x,y}$. □

Proposition 11.16 *Suppose $A_1, \ldots, A_m$ are pairwise disjoint. Then*

$$\Big\| \sum_{i=1}^m c_i E(A_i) \Big\| = \max_{1 \le i \le m} |c_i|.$$

Proof. By letting $A_{m+1} = \sigma(A) \setminus \cup_{i=1}^m A_i$ and setting $c_{m+1} = 0$, we may suppose without loss of generality that the union of the A_i is $\sigma(A)$. Let $r = \max_i |c_i|$. Given x, let $x_i = E(A_i)x$. Then

$$\langle x_i, x_j \rangle = \langle E(A_i)x, E(A_j)x \rangle = \langle x, E(A_i)E(A_j)x \rangle = 0$$

if $i \neq j$. We have

$$\begin{aligned}\Big\| \sum_i c_i E(A_i)x \Big\|^2 &= \Big\langle \sum_i c_i E(A_i)x, \sum_j c_j E(A_j)x \Big\rangle = \Big\langle \sum_i c_i x_i, \sum_j c_j x_j \Big\rangle \\ &= \sum_i |c_i|^2 \|x_i\|^2 \leq r^2 \sum_i \|x_i\|^2 \leq r^2 \|x\|^2.\end{aligned}$$

Therefore the operator norm is less than or equal to r. If j is such that $|c_j| = r$, then take x in the range of $E(A_j)$, and then $\sum_i c_i E(A_i)x = c_j x$, which implies that the norm is equal to r. □

Suppose $\chi_B(A)$ is defined for every Borel measurable subset B of $\sigma(A)$. If f is simple, i.e., $f = \sum c_i \chi_{A_i}$, where the A_i are disjoint, we could define $f(A) = \sum c_i E(A_i)$. By the previous proposition, we know

$$\|f(A)\| = \sup_{\lambda \in \sigma(A)} |f(\lambda)|$$

if f is simple. If f is bounded and measurable, we can take f_n simple converging to f uniformly. Then

$$\|f_n(A) - f_m(A)\| = \sup_{\lambda \in \sigma(A)} |f_n - f_m|,$$

since $f_n - f_m$ is simple, and therefore $f_n(A)$ is a Cauchy sequence. We define $f(A)$ to be the limit of $f_n(A)$. This allows us to define $f(A)$ for f bounded and Borel measurable provided we know how to define $\chi_C(A)$.

Proposition 11.17

$$\|f(A)x\|^2 = \int |f(\lambda)|^2 \, m_{x,x}(d\lambda).$$

Proof. If f is bounded and measurable

$$\|f(A)x\|^2 = \langle f(A)x, f(A)x \rangle = \langle |f|^2(A)x, x \rangle = \int |f^2(\lambda)| \, m_{x,x}(d\lambda).$$

□

11.5 Normal operators

We need a simple lemma.

Lemma 11.18 *If U is a bounded linear operator on H, then $\|U^*U\| = \|U\|^2$.*

Proof. On the one hand

$$\|U^*U\| \le \|U^*\|\,\|U\|,$$

and we saw at the beginning of this chapter that $\|U^*\| = \|U\|$.

On the other hand,

$$\|Ux\|^2 = \langle Ux, Ux\rangle = \langle U^*Ux, x\rangle \le \|U^*U\|\,\|x\|^2.$$

Taking the sup over $\|x\| = 1$, we get our result. □

Let $\mathcal{F}$ be a Banach algebra with unit.

Proposition 11.19 *If $Q \in \mathcal{F}$, then*

$$\sigma(Q) = \{p(Q) : p \textit{ a homomorphism of } \mathcal{F} \textit{ into } \mathbb{C}\}.$$

Proof. $\lambda \in \sigma(Q)$ if and only if $\lambda I - Q$ is not invertible, which happens if and only if $p(\lambda I - Q) = 0$ for some homomorphism p. Since $p(I) = 1$, this happens if and only if $\lambda = p(Q)$ for some p. □

Proposition 11.20 *If p is a homomorphism, then $p(T^*) = \overline{p(T)}$.*

Proof. Let $A = (T + T^*)/2$ and $B = (T - T^*)/2$. Then $A^* = A$, $B^* = -B$, $T = A + B$, $T^* = A - B$, and so $p(T) = p(A) + p(B)$ and similarly with T replaced by T^*.

It will suffice to show $p(A)$ is real and $p(B)$ is imaginary, for then we have

$$p(T^*) = p(A) - p(B), \qquad p(T) = p(A) + p(B),$$

and these two are complex conjugates of each other.

Write $p(A) = a + ib$ and let $U = A + itI$, so that $U^* = A - itI$. Then

$$U^*U = A^2 + t^2 I.$$

We have $p(U) = a+i(b+t)$, so $|p(U)|^2 = a^2+(b+t)^2$. We know from Chapter 9 that homomorphisms are contractions, so $|p(U)| \le \|U\|$, and hence

$$a^2 + (b+t)^2 = |p(U)|^2 \le \|U\|^2 = \|U^*U\| \le \|A\|^2 + t^2$$

for all t, which can only happen if $b = 0$. (If $b > 0$, take t large positive, and t large negative if $b < 0$.) The operator iB is symmetric, so apply the above to iB. □

An alternate proof that $p(A)$ is real is that since A is symmetric, its spectrum is contained in the real line and we know $p(A) \in \sigma(A)$.

Proposition 11.21 *If T and T^* commute, then $\|T\| = r(T)$.*

Proof. We already know this if T is symmetric. Note T^*T is always symmetric.

For general T,

$$\begin{aligned}
\|T\|^2 = \|T^*T\| = r(T^*T) &= \sup_{\lambda \in \sigma(T^*T)} |\lambda| \\
&= \sup_{p \in \mathcal{F}} |p(T^*T)| = \sup_{p \in \mathcal{F}} |p(T)|^2 \\
&= \Big(\sup_{p \in \mathcal{F}} |p(T)| \Big)^2 = \Big(\sup_{\lambda \in \sigma(T)} |\lambda| \Big)^2 \\
&= (r(T))^2.
\end{aligned}$$

□

An operator N is *normal* if $N^*N = NN^*$.

We will see that there is a spectral resolution for the identity as for symmetric operators, but now the spectrum is not necessarily real. In the case of matrices, normal matrices are diagonalizable, but the eigenvalues can be complex.

Lemma 11.22 *Let $R(x, y)$ be a polynomial, N normal, and $Q = R(N, N^*)$. Then*
$$\sigma(Q) = \{R(\lambda, \overline{\lambda}) : \lambda \in \sigma(N)\}.$$

Proof. Operators of the form $R(N, N^*)$ are a commutative algebra with unit. Let $\mathcal{F}$ be the closure in the operator norm.

Since N and N^* commute, they each commute with Q, and so Q and Q^* commute. Now $p(Q) = R(p(N), p(N^*)) = R(p(N), \overline{p(N)})$. Then $\sigma(Q)$ is equal to the set of points $R(p(N), \overline{p(N)})$ where p is a homomorphism, which is the same as the set of $R(\lambda, \overline{\lambda})$ where $\lambda \in \sigma(N)$. □

Theorem 11.23 *Let N be normal. There exists an orthogonal projection valued measure E on $\sigma(N)$ such that $I = \int_{\sigma(N)} dE$ and $N = \int_{\sigma(N)} \lambda\, E(d\lambda)$.*

Proof. Let $q(x, y)$ be a polynomial in x and y. If we let $w = x + yi \in \mathbb{C}$, we can let $x = (w + \overline{w})/2$, $y = (w - \overline{w})/2$, and write $q(x, y) = R(w, \overline{w})$ for some polynomial R. Set $Q = R(N, N^*)$. By the above lemma we have $\sigma(Q) = R(\lambda, \overline{\lambda})$ for $\lambda \in \sigma(N)$. We have $\|Q\| = r(Q)$, since Q and Q^* commute. Therefore
$$\|Q\| = \sup_{\lambda \in \sigma(N)} |R(\lambda, \overline{\lambda})|.$$
Also, $R(\lambda, \overline{\lambda}) = q(\frac{1}{2}(\lambda + \overline{\lambda}), \frac{1}{2}(\lambda - \overline{\lambda}))$. Now we can define $f(N)$ as the limit of polynomials, and the rest of the proof is as before. □

11.6 Unitary operators

U is a *unitary* operator if it is linear, isometric, one-to-one, and onto. (Cf. rotations) So $\|Ux\| = \|x\|$, or $\langle Ux, Ux\rangle = \langle x, x\rangle$. By polarization, $\langle Ux, Uy\rangle = \langle x, y\rangle$, so $\langle x, U^*Uy\rangle = \langle x, y\rangle$, which implies $U^*U = I$. U is invertible, since it is one-to-one and onto, and thus $U^{-1} = U^*$.

$U^*U = I = UU^*$, so unitary operators are also normal operators.

Proposition 11.24 *If U is unitary, then $\sigma(U) \subset \{|z| = 1\}$.*

Proof. $(\lambda I - U) = \lambda(I - U/\lambda)$. Since U is an isometry, then $\|U\| = 1$. Then $I - \frac{1}{\lambda}U$ is invertible if $\frac{1}{|\lambda|}\|U\| < 1$, or if $|\lambda| > 1$.

Now suppose $|\lambda| < 1$. $(\lambda I - U) = U(\lambda U^{-1} - I) = U(\lambda U^* - I)$. Since $\|\lambda U^*\| = |\lambda| < 1$, then $I - \lambda U^*$ is invertible. □

Proposition 11.25 *Suppose T is a bounded normal operator.*
(1) If $\sigma(T) \subset \mathbb{R}$, then T is symmetric.
(2) If $\sigma(T) \subset \{|z| = 1\}$, then T is unitary.

Proof. (1) Let $q(\lambda, \overline{\lambda}) = \lambda - \overline{\lambda}$. Then

$$T - T^* = q(T, T^*),$$

and

$$\|T - T^*\| = \sup_{\lambda \in \sigma(T)} q(\lambda, \overline{\lambda}) = 0$$

since $\lambda = \overline{\lambda}$ if λ is real.

(2) Let $q(\lambda, \overline{\lambda}) = \overline{\lambda}\lambda - 1$. Then

$$T^*T - I = q(T, T^*),$$

and

$$\|T^*T - I\| = \sup_{\lambda \in \sigma(T)} q(\lambda, \overline{\lambda}) \le \sup_{|\lambda|=1} (|\lambda|^2 - 1) = 0.$$

□

Chapter 12

Unbounded operators

12.1 Definitions

Let D be a subspace of a Hilbert space H. In this chapter D will almost never be closed. An *unbounded operator* T in H with domain D is a linear mapping from D into H. We will write $D(T)$ for the domain of T. T is *densely defined* if $D(T)$ is dense in H.

For an example, let $H = L^2[0,1]$, let $D = C^1[0,1]$, and let $Tf = f'$. Note T is not a bounded operator. For another example, let $D = \{f \in C^2 : f(0) = f(1) = 0\}$ and $Uf = f''$. Then one can show that $\{-n^2\pi^2\}$ are eigenvalues.

Recall that $\mathcal{G}(T)$, the graph of T, is the set $\{(x, Tx) : x \in D(T)\}$. If U is an extension of T, that means that $D(T) \subset D(U)$ and $Ux = Tx$ if $x \in D(T)$. Note U will be an extension of T if and only if $\mathcal{G}(T) \subset \mathcal{G}(U)$. One often writes $T \subset U$ to mean that U is an extension of T.

A *closed operator* in H is one whose graph is a closed subspace of $H \times H$. This is equivalent to saying that whenever $x_n \to x$ and $Tx_n \to y$, then $x \in D(T)$ and $y = Tx$.

Proposition 12.1 *If $D(T) = H$ and T is closed, then T is a bounded operator.*

Proof. Recall the closed graph theorem, which says that if M is a closed linear map from a Banach space to itself, then M is bounded. The proposition follows immediately from this. $\square$

Given a densely defined operator T, we want to define its adjoint T^*. First we define $D(T^*)$ to be the set of $y \in H$ such that the linear functional $\ell(x) = \langle Tx, y\rangle$ is continuous (i.e., bounded) on $D(T)$. If $y \in D(T^*)$, the Hahn-Banach theorem allows us to extend ℓ to a bounded linear functional on H. By the Riesz representation theorem for Hilbert spaces, there exists $z_y \in H$ such that

$$\ell(x) = \langle x, z_y\rangle, \qquad x \in D(T).$$

Of course z_y depends on y. We then define $T^*y = z_y$.

Since T is densely defined, it is routine to check that T^* is well defined and also that T^* is an operator in H, that is, $D(T^*)$ is a subspace of H and T^* is linear.

For an example, let $H = L^2[0,1]$, $D(T) = \{f \in C^1 : f(0) = f(1) = 0\}$, and $Tf = f'$. If $f \in D(T)$ and $g \in C^1$, then

$$\langle Tf, g\rangle = \int_0^1 f'(x)\overline{g}(x)\,dx = f(1)\overline{g}(1) - f(0)\overline{g}(0) - \int_0^1 f(x)\overline{g}'(x)\,dx = \langle f, -\overline{g}'\rangle$$

by integration by parts. Thus $|\langle Tf, g\rangle| \le \|f\|\,\|\overline{g}'\|$ is a bounded linear functional, and we see that $C^1 \subset D(T^*)$ and $T^*g = -g'$ if $g \in C^1$.

Some care is needed for the sum and composition of unbounded operators. We define

$$D(S+T) = D(S) \cap D(T)$$

and

$$D(ST) = \{x \in D(T) : Tx \in D(S)\}.$$

Proposition 12.2 *If S, T, and ST are densely defined operators in H, then*

$$T^*S^* \subset (ST)^*. \tag{12.1}$$

If in addition S is bounded, then

$$T^*S^* = (ST)^*.$$

Proof. Suppose $x \in D(ST)$ and $y \in D(T^*S^*)$. Since $x \in D(T)$ and $S^*y \in D(T^*)$, then

$$\langle Tx, S^*y\rangle = \langle x, T^*S^*y\rangle.$$

Since $Tx \in D(S)$ and $y \in D(S^*)$, then

$$\langle STx, y\rangle = \langle Tx, S^*y\rangle.$$

Therefore

$$\langle STx, y\rangle = \langle x, T^*S^*y\rangle.$$

Assume now that S is bounded and $y \in D((ST)^*)$. Then S^* is also bounded and $D(S^*)$ is therefore equal to H. Hence

$$\langle Tx, S^*y\rangle = \langle STx, y\rangle = \langle x, (ST)^*y\rangle$$

for every $x \in D(ST)$. Thus $S^*y \in D(T^*)$, and so $y \in D(T^*S^*)$. Now combine with (12.1). □

An operator T in H is symmetric if $\langle Tx, y\rangle = \langle x, Ty\rangle$ whenever x, y are both in $D(T)$. Thus a densely defined symmetric operator T is one such that $T \subset T^*$. If $T = T^*$, we say T is *self-adjoint*. Note that the domains of T and T^* are crucial here. This is not an issue with bounded operators because every symmetric bounded operator is self-adjoint.

Let us look at some examples. These will all be the same operator, but with different domains. Let $H = L^2[0, 1]$. Let $D(S)$ be the set of absolutely continuous functions f on $[0, 1]$ such that $f' \in L^2$. Let $D(T)$ be the set of $f \in D(S)$ such that in addition $f(0) = f(1)$, and let $D(U)$ be the set of functions in $D(S)$ such that $f(0) = f(1) = 0$. Note that if $f' \in L^2$, then

$$|f(t) - f(s)| = \Big| \int_s^t f'(x)\,dx \Big| \le \|f'\|_{L^2} |t - s|^{1/2}$$

by Cauchy-Schwarz, so functions in any of these domains can be well defined at points.

The operator will be the same in each case: $Sf = if'$, and the same for Tf and Uf provided f is in the appropriate domain. We see that $U \subset T \subset S$. We will show that T is self-adjoint, U is symmetric but not self-adjoint, and S is not symmetric.

By integration by parts,

$$\begin{aligned}\langle Tf, g\rangle &= \int_0^1 (if')\overline{g} \qquad (12.2)\\ &= if(1)\overline{g}(1) - if(0)\overline{g}(0) - \int_0^1 if(\overline{g})'\\ &= if(1)\overline{g}(1) - if(0)\overline{g}(0) + \int_0^1 f\overline{(ig')}.\end{aligned}$$

Thus if $f, g \in D(T)$, we have $\langle Tf, g\rangle = \langle f, Tg\rangle$, since $f(1) = f(0)$ and $g(1) = g(0)$ for $f, g \in D(T)$.

The same calculation with T replaced by S shows that S is not symmetric. The calculation with T replaced by U shows that U is symmetric. Moreover (12.2) shows that $U \subset S^*$.

Suppose $g \in D(T^*)$ and $\phi = T^*g$. Let $\Phi(x) = \int_0^x \phi(y)\,dy$. If $f \in D(T)$, then

$$\int_0^1 if'\overline{g} = \langle Tf, g\rangle = \langle f, \phi\rangle = f(1)\overline{\Phi}(1) - \int_0^1 f'\overline{\Phi},$$

the last equality by integration by parts. Since $D(T)$ contains non-zero constants, take f identically equal to 1 to conclude that $\Phi(1) = 0$. Therefore we have

$$\int_0^1 f'\overline{G} = 0$$

whenever $f \in D(T)$ and

$$G = ig - \Phi.$$

Taking the complex conjugate and replacing f by $\overline{f}$,

$$\int_0^1 f'G = 0$$

if $f \in D(T)$.

We claim G is constant (a.e.). Suppose $a < b$ is such that $[a, a+h], [b, b+h]$ are both subsets of $[0, 1]$ and take f such that

$$f' = \frac{1}{h}\chi_{[a,a+h]} - \frac{1}{h}\chi_{[b,b+h]}.$$

Then $f \in D(T)$ and so

$$\frac{1}{h}\int_a^{a+h} G(x)\,dx - \frac{1}{h}\int_b^{b+h} G(x)\,dx = 0.$$

There is a set N of Lebesgue measure 0 such that if $y \notin N$, then

$$\frac{1}{h}\int_y^{y+h} G(x)\,dx \to G(y).$$

So if $a, b \notin N$, taking the limit shows $G(a) = G(b)$. Since we are on L^2, we can modify G on a set of Lebesgue measure 0 and take G constant.

This implies that $g = -i\Phi + c$ is absolutely continuous and $g' = -i\phi \in L^2$. Also, $g(0) = -i\Phi(0) + c = -i\Phi(1) + c$, hence $g \in D(T)$. Thus $T^* \subset T$.

In the case of U: if $g \in D(U^*)$ and $f \in D(U)$, then $f(1) = 0$ and so

$$\int_0^1 if'\overline{g} = f(1)\overline{\Phi}(1) - \int_0^1 f'\overline{\Phi} = -\int_0^1 f'\overline{\Phi}.$$

If $G = ig - \Phi$, then $\int_0^1 f'G = 0$. As before G is constant, so $g = -i\Phi + c$, but now we no longer know that $\Phi(1) = 0$. So $g(1)$ might not equal $g(0)$. Therefore $U^* \subset S$.

If $g \in D(S)$ and $f \in D(U)$, we have

$$\langle Uf, g\rangle = if(1)\overline{g}(1) - if(0)\overline{g}(0) + \int_0^1 f\overline{(ig')} = \langle f, Ug\rangle.$$

Hence $g \in D(U^*)$. Thus $S \subset U^*$, and with the above $U^* = S$. Hence U is not self-adjoint.

Proposition 12.3 *Let H be a Hilbert space over $\mathbb{C}$, A self-adjoint. Then A is closed.*

Proof. A is closed: if $x_n \to x$ and $Ax_n \to u$, then

$$\langle Ax_n, y\rangle = \langle x_n, Ay\rangle \to \langle x, Ay\rangle = \langle Ax, y\rangle.$$

Also $\langle Ax_n, y\rangle \to \langle u, y\rangle$. This is true for all y, so $Ax = u$. □

If A is defined on all of H and is self-adjoint, we conclude that A is bounded.

We say z is in the resolvent set of A if $A-zI$ maps D one-to-one onto H.

Proposition 12.4 *If z is not real, then z is in the resolvent set. Equivalently, $\sigma(A)\subset\mathbb{R}$.*

Proof. (1) $R = \mathrm{Range}\,(A-zI)$ is a closed subspace.

R is equal to the set of all vectors u of the form $Av - zv = u$ for some $v \in D$. Then$\langle Av, v\rangle - z\langle v, v\rangle = \langle u, v\rangle$. A is self-adjoint, so $\langle Av, v\rangle = \langle v, Av\rangle = \overline{\langle Av, v\rangle}$ is real. Looking at the imaginary parts,

$$-\mathrm{Im}\,(z\|v\|^2) = \mathrm{Im}\,\langle u, v\rangle,$$

so $|\mathrm{Im}\,z|\|v\|^2 \le \|u\|\,\|v\|$, or

$$\|v\| \le \frac{1}{|\mathrm{Im}\,z|}\|u\|.$$

If $u_n \in R$ and $u_n \to u$, then $\|v_n - v_m\| \le (1/|\mathrm{Im}\,z|)\|u_n - u_m\|$, so v_n is a Cauchy sequence, and hence converges to some point v.

Since $Av_n - zv_n = u_n \to u$ and zv_n converges to zv, then Av_n converges to $u + zv$. Since A is self-adjoint, it is closed, and so $v \in D(A)$. Since $\langle Av_n, w\rangle = \langle v_n, Aw\rangle$ for $w \in D$, then $\langle u + zv, w\rangle = \langle v, Aw\rangle$, which implies and $Av = u + zv$, or $u = (A - z)v \in R$.

(2) $R = H$. If not, there exists $x \ne 0$ such that x is orthogonal to R, and then

$$\langle Av - zv, x\rangle = \langle Av, x\rangle - \langle v, \overline{z}x\rangle = 0$$

for all $v \in D$. Then $\langle Av, x\rangle = \langle v, \overline{z}x\rangle$, so $x \in D$ and $Ax = \overline{z}x$. But then $\langle x, Ax\rangle = z\langle x, x\rangle$ is not real, a contradiction.

(3) $A-zI$ is one-to-one. If not, there exists $x \in D$ such that $(A-zI)x = 0$. But then $\|x\| \le (1/|\mathrm{Im}\,z|)\|0\| = 0$, or $x = 0$. □

If we set $R(z) = (A - zI)^{-1}$ the resolvent, we have

$$\|R(z)\| \le \frac{1}{|\mathrm{Im}\,z|}.$$

If $u, w \in H$ and $v = R(z)u$, then $(A - z)v = u$, and

$$\begin{aligned}\langle u, R(\overline{z})w\rangle &= \langle (A-z)v, R(\overline{z})w\rangle = \langle v, ((A-\overline{z})R(\overline{z})w\rangle \\ &= \langle v, w\rangle = \langle R(z)u, w\rangle.\end{aligned}$$

So the adjoint of $R(z)$ is $R(\overline{z})$.

Theorem 12.5 *Let A be a symmetric operator. A is self-adjoint if and only if $\sigma(A) \subset \mathbb{R}$.*

Proof. That A self-adjoint implies that all non-real z are in the resolvent set has already been proved. We thus have to show that if A is symmetric and $\sigma(A) \subset \mathbb{R}$, then A is self-adjoint.

If $x, y \in D(A)$,

$$\langle (A-z)x, y\rangle = \langle x, (A-\overline{z})y\rangle.$$

If z is not real, then $z \notin \sigma(A)$, so $z - A$ is invertible and $A - z$ and $A - \overline{z}$ map $D(A)$ one-to-one and onto H. For $f, g \in H$, we can define $x = (A - z)^{-1}f$ and $y \in (A - \overline{z})g$, and we note that x and y are both in $D(A)$. We then have

$$\langle f, (A-\overline{z})^{-1}g\rangle = \langle (A-z)^{-1}f, g\rangle$$

for all $f, g \in H$.

Now we show A is self-adjoint. Take z non-real and suppose $v \in D(A^*)$. Set $w = A^*v \in H$. We have

$$\langle Ax, v\rangle = \langle x, A^*v\rangle$$

for all $x \in D(A)$. Subtract $z\langle x, v\rangle$ from both sides:

$$\langle (A-z)x, v\rangle = \langle x, (A^*-\overline{z})v\rangle.$$

Let $g = (A^* - \overline{z})v$ and $f = (A - z)x$. Then

$$\begin{aligned}\langle f, v\rangle &= \langle (A-z)x, v\rangle = \langle x, (A^*-\overline{z})v\rangle \\ &= \langle (A-z)^{-1}f, g\rangle = \langle f, (A-\overline{z})^{-1}g\rangle.\end{aligned}$$

The set of f of the form $(A - z)x$ for $x \in D(A)$ is all of H, hence $v = (A - \overline{z})^{-1}g$, which is in $D(A)$. In particular, $D(A^*) \subset D(A)$. We have $(A - \overline{z})v = g = (A^* - \overline{z})v$, so $A^*v = Av$. □

12.2 Cayley transform

Define

$$U = (A - i)(A + i)^{-1}.$$

This is the image of the operator A under the function

$$F(z) = \frac{z - i}{z + i}, \tag{12.3}$$

which maps the real line to $\partial B(0, 1) \setminus \{1\}$, and is called the *Cayley transform* of A.

Proposition 12.6 *U is a unitary operator.*

Proof. $A + i$ and $A - i$ each map $D(A)$ one-to-one onto H, so U maps H onto itself.

U is norm preserving: Let $u \in H$, $v = (A+i)^{-1}u$, $w = Uu$. So $(A+i)v = u$, $(A - i)v = w$. We need to show $\|u\| = \|w\|$.

We have

$$\begin{aligned} \|u\|^2 &= \langle (A + i)v, (A + i)v\rangle = \|Av\|^2 + \|v\|^2 + i\langle v, Av\rangle - i\langle Av, v\rangle \\ &= \|Av\|^2 + \|v\|^2, \end{aligned}$$

and similarly

$$\|w\| = \langle (A - i)v, (A - i)v\rangle = \|Av\|^2 + \|v\|^2.$$

□

Proposition 12.7 *Given A and U as above and E the spectral resolution for U, $E(\{1\}) = 0$.*

Proof. Write E_1 for $E(\{1\})$. If $E_1 \neq 0$, there exists $z \neq 0$ in the range of E_1, so $z = E_1 w$. Then

$$Uz = \int_{\sigma(U)} \lambda\, E(d\lambda)z = \int_{\sigma(U)} \lambda\, (E - E_1)(d\lambda)z + \int_{\{1\}} \lambda\, E_1(d\lambda)z.$$

The first integral is zero since $(E - E_1)(A)$ and E_1 are orthogonal for all A. The second integral is equal to

$$E_1 z = E_1 E_1 w = E_1 w = z$$

since E_1 is a projection.

We conclude z is an eigenvector for U with eigenvalue 1. So

$$(A - iI)(A + iI)^{-1} z = z.$$

Let $v = (A + iI)^{-1} z$, or $z = (A + iI)v$. Then

$$z = (A - iI)(A + iI)^{-1} z = (A - iI)v,$$

and then $iv = -iv$, so $v = 0$, and hence $z = 0$, a contradiction. □

12.3 Spectral theorem

When M is a bounded symmetric operator and f is bounded and measurable, we defined $f(M)$ in Chapter 11. We now want to define $f(M)$ for some unbounded functions f.

Proposition 12.8 *Let M be a bounded operator and f a measurable function. Let*

$$D_f = \Big\{x : \int_{\sigma(M)} |f(\lambda)|^2 \, \mu_{x,x}(d\lambda) < \infty\Big\}.$$

Then

(1) D_f is a dense subspace of H.

(2) If $x, y \in H$,

$$\int_{\sigma(M)} |f(\lambda)| \, |\mu_{x,y}|(d\lambda) \le \|y\| \Big(\int_{\sigma(M)} |f(\lambda)|^2 \, \mu_{x,x}(d\lambda) \Big)^{1/2}.$$

(3) If f is bounded and $v = f(M)z$, then

$$\mu_{x,v}(d\lambda) = \overline{f}(\lambda) \, \mu_{x,z}(d\lambda), \qquad x, z \in H.$$

Proof. (1) Let $S \subset \sigma(M)$ and $z = x + y$.

$$\|E(S)z\|^2 \leq (\|E(S)x\| + \|E(S)y\|)^2 \leq 2\|E(S)x\|^2 + 2\|E(S)y\|^2.$$

So

$$\mu_{z,z}(S) \leq 2\mu_{x,x}(S) + 2\mu_{y,y}(S).$$

This is true for all S, so

$$\mu_{z,z}(d\lambda) \leq 2\mu_{x,x}(d\lambda) + 2\mu_{y,y}(d\lambda).$$

This proves that D_f is a subspace.

Let $S_n = \{\lambda \in \sigma(M) : |f(\lambda)| < n\}$. Then if $x = E(S_n)z$,

$$E(S)x = E(S)E(S_n)E(S_n)z = E(S \cap S_n)E(S_n)z = E(S \cap S_n)x,$$

so $\mu_{x,x}(S) = \mu_{x,x}(S \cap S_n)$. Then

$$\int_{\sigma(M)} |f(\lambda)|^2 \, \mu_{x,x}(d\lambda) = \int_{S_n} |f(\lambda)|^2 \, \mu_{x,x}(d\lambda) \leq n^2 \|x\|^2 < \infty.$$

To see this last line, we know it holds when $|f|^2$ is replaced by g and g is the characteristic function of a set. It holds for g simple by linearity, and then it holds for $g = |f|^2$ by monotone convergence. Therefore the range of $E(S_n) \subset D(f)$. $\sigma(M) = \cup_n S_n$, so

$$\|E(S_n)y - y\|^2 = \|E(S_n)(y) - E(\sigma(M))(y)\|^2 = \int |\chi_{\sigma(M)\setminus S_n}(\lambda)|^2 \, \mu_{y,y}(d\lambda) \to 0$$

by dominated convergence. Hence y is in the closure of D_f.

(2) If $x, y \in H$, f bounded,

$$f(\lambda) \, \mu_{x,y}(d\lambda) \ll |f(\lambda)| \, |\mu_{x,y}|(d\lambda),$$

so there exists u with $|u| = 1$ such that

$$u(\lambda) f(\lambda) \, \mu_{x,y}(d\lambda) = |f(\lambda)| \, |\mu_{x,y}|(d\lambda).$$

Hence

$$\int_{\sigma(M)} |f(\lambda)| \, |\mu_{x,y}|(d\lambda) = (uf(M)x, y) \leq \|uf(M)x\| \, \|y\|.$$

But
$$\|uf(M)x\|^2 = \int |uf|^2\, d\mu_{x,x} = \int |f|^2\, d\mu_{x,x}.$$
So (2) holds for bounded f. Now take a limit and use monotone convergence.

(3) Let g be continuous.
$$\int_{\sigma(M)} g\, d\mu_{x,v} = (g(M)x, v) = (g(M)x, f(M)z)$$
$$= ((\overline{f}g)(M)x, z) = \int g\overline{f}\, d\mu_{x,z}.$$
this is true for all g continuous, so $d\mu_{x,x} = \overline{f}\, d\mu_{x,z}$. □

Theorem 12.9 *Let E be a resolution of the identity.*

(1) Suppose $f : \sigma(M) \to \mathbb{C}$ is measurable. There exists a densely defined operator $f(M)$ with domain D_f and
$$\langle f(M)x, y\rangle = \int_{\sigma(M)} f(\lambda)\, \mu_{x,y}(d\lambda) \tag{12.4}$$
$$\|f(M)x\| = \int_{\sigma(M)} |f(\lambda)|^2\, \mu_{x,x}(d\lambda). \tag{12.5}$$

(2) If $D_{fg} \subset D_g$, then $f(M)g(M) = (fg)(M)$.

(3) $f(M)^* = \overline{f}(M)$ and $f(M)f(M)^* = f(M)^*f(M) = |f|^2(M)$.

Proof. (1) If $x \in D_f$, then $\ell(y) = \int_{\sigma(M)} f\, d\mu_{x,y}$ is a bounded linear functional with norm at most $(\int |f|^2\, d\mu_{x,x})^{1/2}$ by (2) of the preceding proposition. Choose $f(M)x \in H$ to satisfy (1) for all y.

Let $f_n = f\chi_{(|f|\le n)}$. Then $D_{f-f_n} = D_f$ since $\int |f - f_n|^2\, d\mu_{x,x}$ is finite if and only if $\int |f|^2\, d\mu_{x,x}$ is finite, using that f_n is bounded. By the dominated convergence theorem,
$$\|f(M)x - f_n(M)x\|^2 \le \int_{\sigma(M)} |f - f_n|^2\, d\mu_{x,x} \to 0.$$

Since f_n is bounded, (12.5) holds with f_n. Now let $n \to \infty$.

(2): Define $g_m = g\chi_{(|g| \le m)}$. Since f_n and g_m are bounded, (2) follows for f_n, g_m. Now let $m \to \infty$ and then $n \to \infty$.

(3) We know this holds for f_n since f_n is bounded. Now let $n \to \infty$. □

Theorem 12.10 *(Change of measure principle) Let E be a resolution of the identity on A, $\Phi : A \to B$ one-to-one and bimeasurable. Let $E'(S') = E(\Phi^{-1}(S'))$. Then E' is a resolution of the identity on B, and*

$$\int_B f \, d\mu'_{x,y} = \int_B (f \circ \Phi) \, d\mu_{x,y}.$$

Bimeasurable means that Φ and Φ^{-1} are both measurable. Saying that E is a resolution of the identity on A means that $E(C)$ is symmetric for every measurable subset C of A, $\|E(C)\| \le 1$, $E(\emptyset) = 0$, $E(A) = I$, $E(C \cup D) = E(C) + E(D)$ if C and D are disjoint, and $E(C \cap D) = E(C)E(D)$. Finally, $\langle E(C)x, y\rangle = \int \chi_C(z) \, \mu_{x,y}(dz)$ characterizes the measure $\mu_{x,y}$ and similarly for $\mu'_{x,y}$.

Proof. Prove for f the indicator of a set, use linearity, and take limits. □

Theorem 12.11 *(Spectral theorem) Let A be a self-adjoint operator on a Hilbert space over the complex numbers. There exists a resolution of the identity E such that*

$$A = \int_{\sigma(A)} z \, E(dz).$$

Proof. Start with the unbounded operator A. Let $U = (A - iI)(A + iI)^{-1}$. Then U is unitary with a spectrum on $\partial B(0,1) \setminus \{1\}$. Let the resolution of the identity for U be given by $\widetilde{E}$.

Let F be defined as in (12.3) and define $\Phi = F^{-1}$, which is a map taking $\partial B(0,1) \setminus \{1\}$ to $\mathbb{R}$. Thus

$$\Phi(z) = \frac{i(1+z)}{1-z}.$$

We check that $A = \Phi(U)$. Since the range of Φ is $\mathbb{R}$, then $\Phi(U)$ is self-adjoint by Theorem 12.9(3). Since $\Phi(z)(1 - z) = i(1 + z)$, Theorem 12.9(2) implies that

$$\Phi(U)(I - U) = i(I + U).$$

In particular, the range of $I - U$ is contained in the domain of $\Phi(U)$. From the definition of the Cayley transform, we have

$$A(I - U) = i(I + U)$$

and the domain of A is equal to the range of $I - U$. Thus $A \subset \Phi(U)$. Since both A and $\Phi(U)$ are self-adjoint,

$$\Phi(U) = \Phi(U)^* \subset A^* = A \subset \Phi(U),$$

and hence $A = \Phi(U)$.

Let $E(S) = \widetilde{E}(\Phi^{-1}(S))$. We have

$$\langle Ax, y\rangle = \langle \Phi(U)x, y\rangle = \int_{\sigma(U)} \Phi(z)\, \langle \widetilde{E}(dz)x, y\rangle.$$

By the change of measure principle, this is equal to

$$\int_{\sigma(A)} z\, \langle E(dz)x, y\rangle.$$

□

Chapter 13

Semigroups

13.1 Strongly continuous semigroups

Let X be a Banach space over the complex numbers, $T(t) = T_t$ linear bounded operators for $t \geq 0$. T is a *semigroup* if $T_{t+s} = T_t T_s$, $T_0 = I$.

These come up in PDE and in probability. For example, if one wants to solve the equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x), \qquad u(0,x) = f(x),$$

where f is a given function (this is the heat equation on $\mathbb{R}$), the solution is given by $u(t,x) = T_t f(x)$ for a certain semigroup T_t.

If X_t is a Markov process, then $T_t f(x) = \mathbb{E}^x f(X_t)$ will be a semigroup, where $\mathbb{E}^x$ means expectation starting at x.

Here is an example: if X is a Hilbert space and $\{\varphi_n\}$ is an orthonormal basis and λ_j a sequence of real numbers increasing to infinity, let

$$T_t f = \sum_{j=1}^{\infty} e^{-\lambda_j t} \langle f, \varphi_j \rangle \varphi_j.$$

Another example is to let

$$T_t f(x) = \int f(y) \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}\, dy \tag{13.1}$$

where X is the set of continuous functions on $\mathbb{R}$ vanishing at infinity.

A third example is given by the next proposition.

Proposition 13.1 *Let $A : X \to X$ be bounded. Then $T_t = e^{tA}$ (defined as $e^{tA} = \sum t^n A^n/n!$) is a semigroup that is continuous in the norm topology.*

Proof. This follows easily from the functional calculus for operators. □

We say T_t is *strongly continuous* at $t = 0$ if $\|T_t x - x\| \to 0$ as $t \to 0$ for all $x \in X$.

Proposition 13.2 *Suppose T_t is a strongly continuous semigroup at 0.*

(1) There exists b and k such that $\|T_t\| \le be^{kt}$.

(2) $T_t x$ is strongly continuous in t for all $x \in X$.

Proof. We claim $\|T_t\|$ is bounded near 0. If not, there exists $t_j \to 0$ such that $\|T_{t_j}\| \to \infty$. By the uniform boundedness principle, $T_{t_j} x$ cannot converge to x for all x, a contradiction to strong continuity. So there exists a, b such that $\|T_t\| \le b$ for $t \le a$.

Write $t = na + r$. $T_t = T_a^n T_r$, so

$$\|T_t\| \le \|T_a\|^n \|T_r\| \le b^{n+1} \le be^{kt}$$

with $k = \frac{1}{a} \log b$.

(2) $T_t x - T_s x = T_s[T_{t-s} x - x]$, so

$$\|T_t x - T_s x\| \le \|T_s\| \, \|T_{t-s} x - x\| \to 0.$$

□

Suppose D is dense in X and $A : D \to X$ is closed. $z \in \rho(A)$, the resolvent set, if $z - A$ maps $D = D(A)$ one-to-one onto X. Thus $\rho(A) = \sigma(A)^c$. Write $R(z) = R_z = (zI - A)^{-1}$.

Since A is closed, then R_z is closed. To see this, suppose $x_n \to x$ and $y_n = R_z x_n \to y$. Then

$$Ay_n = zy_n - (z-A)y_n = zy_n - x_n \to zy - x.$$

Since A is closed, $y \in D(A)$ and $Ay = zy - x$, or $(z-A)y = x$. So $y = R_z x$, which proves R_z is closed.

R_z is defined on all of X, so by the closed graph theorem, R_z is a bounded operator.

Let T be a strongly continuous one parameter semigroup. The *infinitesimal generator* A is defined by

$$Ax = \lim_{h\to 0} \frac{T_h x - x}{h},$$

where we mean that the difference of the two sides goes to 0 in norm. The domain of A consists of those x for which the strong limit exists.

As an example, with T_t defined by (13.1), if $f \in C^2$ vanishes at infinity, then using Taylor's theorem,

$$\begin{aligned}\frac{T_h f(x) - f(x)}{h} &= \frac{1}{h}\int [f(y)-f(x)]\frac{1}{\sqrt{2\pi h}}e^{-(y-x)^2/2h}\,dy\\ &\qquad + f'(x)\int (y-x)\frac{1}{\sqrt{2\pi h}}e^{-(y-x)^2/2h}\,dy\\ &\qquad + \tfrac{1}{2}f''(x)\int (y-x)^2\frac{1}{\sqrt{2\pi h}}e^{-(y-x)^2/2h}\,dy\\ &\qquad + \int E(h)\frac{1}{\sqrt{2\pi h}}e^{-(y-x)^2/2h}\,dy\\ &= \tfrac{1}{2}f''(x) + E(h)/h \to \tfrac{1}{2}f''(x),\end{aligned}$$

where $E(h)$ is a remainder term that goes to 0 faster than h; we used standard facts about the Gaussian density. One can improve the above to show that the convergence is uniform, and we can then conclude that $C^2 \subset D(A)$ and $Af = \frac{1}{2}f''$.

Proposition 13.3 *(1) A commutes with T_t in the sense that if $x \in D(A)$, then $T_t x \in D(A)$ and $AT_t x = T_t Ax$.*

(2) $D(A)$ is dense in X.

(3) $D(A^n)$ is dense.

(4) A is closed.

(5) If $\|T_t\| \le be^{kt}$ and $\operatorname{Re} z > k$, then $z \in \rho(A)$. The resolvent of A is the Laplace transform of T_t.

Proof. (1)

$$\frac{T_{t+h} - T_t}{h} x = T_t \frac{T_h - I}{h} x = \frac{T_h - I}{h} T_t x.$$

If $x \in D(A)$, the middle term converges to $T_t Ax$. So the limit exists in the third term, and therefore $T_t x \in D(A)$. Moreover $\frac{d}{dt} T_t x = T_t Ax = AT_t x$.

(2) We claim

$$T_t x - x = A \int_0^t T_s x \, ds.$$

To see this, $T_s x$ is a continuous function of s. Using a Riemann sum approximation,

$$\begin{aligned} \frac{T_h - I}{h} \int_0^t T_s x \, ds &= \frac{1}{h} \int_0^t [T_{s+h} x - T_s x] \, ds \\ &= \frac{1}{h} \int_t^{t+h} T_s x \, ds - \frac{1}{h} \int_0^h T_s x \, ds \\ &\to T_t x - x. \end{aligned}$$

So $\int_0^t T_s x \, ds \in D(A)$. But $\frac{1}{t} \int_0^t T_s x \, ds \to x$.

(3) Let ϕ be C^∞ and supported in $(0, 1)$. Let

$$x_\phi = \int_0^1 \phi(s) T_s x \, ds.$$

Then

$$Ax_\phi = \int_0^1 \phi(s) AT_s x \, ds = \int_0^1 \phi(s) \frac{\partial}{\partial s} T_s x \, ds = -\int_0^1 \phi'(s) T_s x \, ds$$

by integration by parts. Repeating, $x_\phi \in D(A^n)$. Now take ϕ_j approximating the identity.

(4) $T_t x - x = \int_0^t T_s A x \, ds$: To see this, both are 0 at 0. The derivative on the left is $T_t A x$, which is the same as the derivative on the right. Let $x_n \in D(A)$, $x_n \to x$, $Ax_n \to y$. Then

$$T_t x_n - x_n = \int_0^t T_s A x_n \, ds \to \int_0^t T_s y \, ds.$$

The left hand term converges to $T_t x - x$. Divide by t and let $t \to 0$. The right hand side converges to y. Therefore $x \in D(A)$ and $Ax = y$.

(5) Let

$$L(z)x = \int_0^\infty e^{-zs} T_s x \, ds.$$

The Riemann integral converges when $\operatorname{Re} z > k$.

$$\|L(z)x\| \le \int_0^\infty b e^{(k - \operatorname{Re} z)s} \|x\| \, ds \le \frac{b}{\operatorname{Re} z - k} \|x\|.$$

We claim $L(z) = R_z$. Check that $e^{-zt} T_t$ is also a semigroup with infinitesimal generator $A - zI$.

Hence

$$e^{-zt} T_t - x = (A - zI) \int_0^t e^{-zs} T_s x \, ds.$$

As $t \to \infty$, the left hand side tends to $-x$ and the right hand side tends to $(A - zI)L(z)x$. Since A is closed, $x = (zI - A)L(z)x$. So $L(z)$ is the right inverse of $(zI - A)$. Similarly, we see that it is also the left inverse. □

13.2 Generation of semigroups

Proposition 13.4 *A strongly continuous semigroup of operators is uniquely defined by its infinitesimal generator.*

Proof. If S, T have the same generator, let $x \in D(A)$ and

$$\frac{d}{dt} S_t T_{s-t} x = S(t) A T_{s-t} x - S_t A T_{s-t} x = 0.$$

Therefore

$$0 = \int_0^s \frac{d}{dr} S_r T_{s-r} x \, dr = S_s T_0 x - S_0 T_s x,$$

or $S_s x = T_s x$. Now use the fact that $D(A)$ is dense. $\square$

T_t is a contraction if $\|T_t\| \le 1$ for all t.

Proposition 13.5 *The infinitesimal generator of a strongly continuous semigroup of contractions has* $(0, \infty) \subset \rho(A)$ *and*

$$\|R_\lambda\| = \|(\lambda I - A)^{-1}\| \le \frac{1}{\lambda}. \tag{13.2}$$

Proof. We already did this: this is the case $b = 1, k = 0$. We have

$$\|L(z)x\| \le \frac{1}{|\mathrm{Re}\, z - k|} \|x\|.$$

$\square$

Proposition 13.6 *Suppose* B *is an extension of* A *and there exists* $\lambda \in \rho(A) \cap \rho(B)$. *Then* $A = B$.

Proof. Suppose $x \in D(B) \setminus D(A)$. We know $(\lambda - B)x \in X$, so

$$(\lambda - A)^{-1}(\lambda - B)x \in D(A) \subset D(B).$$

Then

$$(\lambda - B)(\lambda - A)^{-1}(\lambda - B)x = (\lambda - A)(\lambda - A)^{-1}(\lambda - B)x = (\lambda - B)x.$$

Hit both sides with $(\lambda - B)^{-1}$ to obtain $(\lambda - A)^{-1}(\lambda - B)x = x$. So $x \in D(A)$, a contradiction. $\square$

Theorem 13.7 *(Hille-Yosida theorem) Let* A *be a densely defined unbounded operator such that* $(0, \infty) \subset \rho(A)$ *and*

$$\|R_\lambda\| = \|(\lambda I - A)^{-1}\| \le \frac{1}{\lambda}. \tag{13.3}$$

Then A *is the infinitesimal generator of a strongly continuous semigroup of contractions.*

Note that saying $(0,\infty) \subset \rho(A)$ implies that $\lambda - A$ is one-to-one and onto from the domain of A to the Banach space, which means the range of $\lambda - A$ is all of the Banach space.

Proof. Note $nR_n - I = R_n A$ since $R_n(nI - A) = I$. Let $A_n = nAR_n$. Then $A_n = n^2 R_n - nI$, so A_n is a bounded operator. Define $T_n(t) = e^{tA_n}$.

Step 1. We show $nR_n x \to x$ for all x.

To prove this,

$$\|nR_n x - x\| = \|R_n A(x)\| \leq \frac{1}{n}\|Ax\|,$$

so the claim is true for $x \in D(A)$. Since $\|nR_n\| \leq 1$ and $D(A)$ is dense in X, this proves the claim.

Step 2. We show that if $x \in D(A)$, then $A_n(x) \to A(x)$:

$$A_n x = nAR_n x = nR_n Ax \to Ax.$$

Step 3. We show that $T_n(s)x$ converges for all x.

We have

$$T_n(t) = e^{tA_n} = e^{-nt} e^{n^2 R_n t} = e^{-nt} \sum \frac{(n^2 t)^m}{m!} (R_n)^m,$$

so $\|T_n(t)\| \leq e^{nt} e^{nt} = 1$.

A_n and A_m commute with T_n and T_m.

$$\frac{d}{dt} T_n(s-t) T_m(t) x = T_n(s-t) T_m(t) [A_m - A_n] x.$$

The norm of the right hand side is bounded by $\|A_n x - A_m x\|$. So

$$\|T_n(s)x - T_m(s)x\| \leq s\|A_n x - A_m x\| \to 0$$

as $n, m \to \infty$. Therefore $T_n(s)x$ converges, say, to $T_s x$, uniformly in s. $D(A)$ is dense so this holds for all x.

$T_n(s)$ is a strongly continuous semigroup of contractions, so the same holds for T_s.

Step 4. It remains to show that A is the infinitesimal generator of T. We have

$$T_n(t)x - x = \int_0^t T_n(s)A_n x\, ds.$$

If $x \in D(A)$, we can let $n \to \infty$ to get

$$T_t x - x = \int_0^t T_s Ax\, ds.$$

If B is the generator of T, dividing by t and letting $t \to 0$, we get $D(A) \subset D(B)$ and $B = A$ on $D(A)$. So B is an extension of A. If $\lambda > 0$, then $\lambda \in \rho(A), \rho(B)$, which implies B cannot be a proper extension by Proposition 13.6. □

13.3 Perturbation of semigroups

Lemma 13.8 *(Lumer-Phillips) Let A be densely defined in a Hilbert space B and suppose $(0,\infty) \subset \rho(A)$. Then $\|R_\lambda\| \le 1/\lambda$ if and only if $\operatorname{Re}\langle x, Ax\rangle \le 0$ for all $x \in D(A)$.*

If the last property holds, we say A is dissipative. An example is the Laplacian:

$$\langle f, Af\rangle = \int f(x)\Delta f(x)\, dx = -\int |\nabla f(x)|^2\, dx \le 0$$

by integration by parts. Another example is if

$$Af(x) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(\cdot)\frac{\partial f}{\partial x_j}(\cdot)\Big)(x).$$

To verify this, we again use integration by parts.

Proof. Suppose

$$\|(\lambda I - A)^{-1}u\|^2 \le \frac{1}{\lambda^2}\|u\|^2.$$

Let $x = (\lambda I - A)^{-1}u$. So

$$\langle x, x\rangle \le \frac{1}{\lambda^2}\langle \lambda x - Ax, \lambda x - Ax\rangle.$$

This becomes

$$2\mathrm{Re}\,\langle x, Ax\rangle = \langle x, Ax\rangle + \langle Ax, x\rangle \le \frac{1}{\lambda}\|Ax\|^2.$$

This is true for all λ, so let $\lambda \to \infty$.

For the converse,

$$\langle x, Ax\rangle + \langle Ax, x\rangle = 2\mathrm{Re}\,\langle x, Ax\rangle \le 0 \le \frac{1}{\lambda}\|Ax\|^2$$

for all $\lambda > 0$. Now reverse the above steps. □

Theorem 13.9 *(Trotter) Suppose A is the infinitesimal generator of a semigroup of contractions in a Hilbert space. Let B be a densely defined dissipative operator such that $D(A) \subset D(B)$ and there exist $b > 0$ and $a \in (0, 1)$ such that*

$$\|Bx\| \le a\|Ax\| + b\|x\|, \qquad x \in D(A).$$

Then $A + B$ (defined on $D(A)$) is the generator of a contraction semigroup.

Proof. First, $A + B$ is closed: Let $x_n \to x$ and $y_n = (A + B)x_n \to y$. So

$$A(x_n - x_m) = y_n - y_m - B(x_n - x_m),$$

and

$$\|A(x_n - x_m)\| \le \|y_n - y_m\| + a\|A(x_n - x_m)\| + b\|x_n - x_m\|.$$

Since $a < 1$, then Ax_n converges. Therefore Bx_n converges. A is closed, so $Ax_n \to Ax$. If $x \in D(A) \subset D(B)$,

$$\|Bx_n - Bx\| \le a\|A_n x - Ax\| + b\|x_n - x\| \to 0.$$

Then $(A + B)x_n \to (A + B)x$.

Next, $\lambda \in \rho(A+B)$: By the Lumer-Phillips lemma, A is dissipative. B is also. So $A+B$ is dissipative. By Lumer-Phillips,

$$\|x\| \leq \frac{1}{\lambda}\|(\lambda I - (A+B))x\|.$$

One immediate consequence of this is that the operator $\lambda - (A+B)$ is one-to-one. Another is that the range of $\lambda - (A+B)$ is closed, because if y_n is in the range and $y_n \to y$, then $y_n = (\lambda - (A+B))x_n$ for some x_n. The inequality shows that $\|x_n - x_m\| \to 0$. If $x_n \to x$, then $y = (\lambda - (A+B))x$, since $A+B$ is a closed operator. Therefore the range of $(A+B) - \lambda I$ is closed.

The range is X: if not, there exists $v \neq 0$ perpendicular to the range. $A - \lambda I$ is invertible, so there exists $x \in D(A)$ such that $(A - \lambda I)x = v$. Then $v + Bx$ is in the range, or $\langle v + Bx, v\rangle = 0$. So $\|v\|^2 + \langle Bx, v\rangle = 0$, or

$$\|v\|^2 \leq \|Bx\|\,\|v\|,$$

and so $\|v\| \leq \|Bx\|$. Then

$$\|Ax - \lambda x\| \leq \|Bx\| \leq a\|Ax\| + b\|x\|.$$

Squaring and use the fact that A is dissipative,

$$\|Ax\|^2 + \lambda^2\|x\|^2 \leq a^2\|Ax\|^2 + 2ab\|Ax\|\,\|x\| + b^2\|x\|^{\cdot}$$

This holds for all $\lambda > 0$, so for λ large enough, $\|x\| = 0$. So $x = 0$ and the range is the whole space.

Now use the Hille-Yosida theorem. □

13.4 Groups of unitary operators

We prove *Stone's theorem.*

Theorem 13.10 *(1) Suppose A is self-adjoint and H is a Hilbert space. There exists a strongly continuous group $U(t)$ of unitary operators with infinitesimal generator iA.*

(2) Given a strongly continuous group of unitary operators, the generator is of the form iA where A is self-adjoint.

Proof. (1) We saw in our proof that the spectrum of an unbounded self-adjoint operator is real that $\|(z-A)^{-1}\| \le 1/|\mathrm{Im}\, z|$. So if $\lambda > 0$ and $z = -i\lambda$, then

$$\|(\lambda - iA)^{-1}\| = \|(iz - iA)^{-1}\| = \|(z-A)^{-1}\| \le \frac{1}{|\mathrm{Im}\,(iz)|} = \frac{1}{\lambda}.$$

The resolvent set of iA contains the positive reals. So iA and $-iA$ satisfy the Hille-Yosida theorem. Let $U(t), V(t)$ be the respective semigroups.

V and U are inverses:

$$\frac{d}{dt}U(t)V(t) = U(t)iAV(t)x - U(t)iAV(t)x = 0.$$

So $U(t)V(t)x$ is independent of t. When $t = 0$, we get x. So $U(t)V(t)x = x$ if $x \in D(A)$. But $D(A)$ is dense.

Both U and V are contractions. Since $U(t)V(t) = I$, they must be norm preserving. This is because

$$\|x\| = \|U(t)V(t)x\| \le \|V(t)x\| \le \|x\|,$$

so $\|x\| = \|V(t)x\|$ and similarly with U. Since they are invertible, they are unitary. Define $U(t) = V(-t)$ for $t < 0$.

(2) Let $V(t) = U(-t)$. Then $U(t)$ and $V(t)$ are strongly continuous semigroups of contractions, and the infinitesimal generators are additive inverses. So the generators are $B, -B$.

Since both $B, -B$ are infinitesimal generators, all real numbers except 0 are in the resolvent set of B. Take $x \in D(B)$.

$$\|U(t)x\|^2 = (U(t)x, U(t)x) = \|x\|^2.$$

Take the derivative with respect to t:

$$(Bx, x) + (x, Bx) = 0.$$

Letting $A = -iB$ so that $B = iA$, we see that

$$\langle Ax, x\rangle = \langle x, Ax\rangle \tag{13.4}$$

for all $x \in D(A)$. Using (13.4) with x replaced by $x + y$ and with x replaced by y, we obtain

$$\langle Ax, y\rangle + \langle Ay, x\rangle = \langle x, Ay\rangle + \langle y, Ax\rangle. \tag{13.5}$$

Replacing y by iy in (13.5),

$$-i\langle Ax, y\rangle + i\langle Ay, x\rangle = -i\langle x, Ay\rangle + i\langle y, Ax\rangle.$$

Dividing this by i and subtracting from (13.5) we have

$$\langle Ax, y\rangle = \langle x, Ay\rangle.$$

Therefore A is symmetric and A^* is an extension of A. It follows that B^* is an extension of $-B$. We showed in the previous chapter that the adjoint of $(\lambda - B)^{-1}$ was $(\overline{\lambda} - B^*)^{-1}$, and it follows that $\rho(B^*) = \overline{\rho(B)}$. If $z \neq 0$ and $z \in \mathbb{R}$, then $z \in \rho(B)$, so $z \in \rho(B^*)$. Also $z \in \rho(-B)$. By Proposition 13.6 B^* cannot be a proper extension of $-B$, hence $B^* = -B$, and so $A^* = A$, or A is self-adjoint. □

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