Functional analysis

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Chapter 1

Linear spaces

Functional analysis can best be characterized as infinite dimensional linear algebra. We will use some real analysis, complex analysis, and algebra, but functional analysis is not really an extension of any one of these.

1.1 Definitions

We start with a field $F$, which for us will always be the reals or the complex numbers. Elements of $F$ will be called scalars.

A linear space is a set $X$ together with two operations, addition (denoted “$x + y$”) mapping $X \times X$ into $X$ and scalar multiplication (denoted “$ax$”) mapping $F \times X$ into $X$, having the following properties.

1. $x + y = y + x$ for all $x, y \in X$;
2. $x + (y + z) = (x + y) + z$ for all $x, y, z \in X$;
3. there is an element of $X$ denoted $0$ such that $x + 0 = 0 + x = x$ for all $x \in X$;
4. for each $x \in X$ there is an element $-x$ in $X$ such that $x + (-x) = 0$;
5. $a(bx) = (ab)x$ whenever $a, b \in F$ and $x \in F$;
6. $a(x + y) = ax + ay$ and $(a + b)x = ax + bx$ whenever $x, y \in X$ and $a, b \in F$;
7. $1x = x$ for all $x \in X$ where $1$ is the identity for $F$.

A vector space is the same thing as a linear space.

Under the operation of addition we see that (1)–(4) says that $X$ is an
Abelian group.

We use $x - y$ for $x + (-y)$.

By the same proofs as in the finite dimensional case, we have the following.

**Lemma 1.1** (1) $0x = 0$ and
(2) $(-1)x = -x$.

**Proof.** First write $0x = (0 + 0)x = 0x + 0x$ and subtract $0x$ from both sides to get (1). Then write

$$0 = 0x = (1)x + (-1)x = x + (-1)x$$

and subtract $x$ from both sides to get (2). □

We give a number of examples of linear spaces. We leave to the reader the verification that these satisfy the definition of linear spaces.

**Example 1.2** Let $X = \mathbb{R}^n$ be the set of $n$-tuples of real numbers. This is a linear space over the reals. We have

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

and

$$a(x_1, \ldots, x_n) = (ax_1, \ldots, ax_n).$$

**Example 1.3** Let $X = \mathbb{C}^n$ be the set of $n$-tuples of complex numbers. This is a linear space over the complex numbers. We define addition and scalar multiplication as in Example 1.2.

**Example 1.4** Let $X$ be the collection of all infinite sequences $(x_1, x_2, \ldots)$ of real numbers with addition being coordinate-wise and scalar multiplication also being coordinate-wise, that is,

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots)$$

and similarly for scalar multiplication.
1.1. DEFINITIONS

Example 1.5 Let $S$ be a set and let $X$ be the collection of real-valued bounded functions on $S$. We define $f + g$ by

$$(f + g)(s) = f(s) + g(s)$$

(1.1)

for each $s \in S$ and $af$ by

$$(af)(s) = af(s)$$

(1.2)

for each $s \in S$. A closely related example is to let $X$ be the collection of complex-valued bounded functions on $S$.

Example 1.6 Let $S$ be a topological space (so that the notion of continuous functions from $S$ to $\mathbb{R}$ or to $\mathbb{C}$ makes sense) and let $X = C(S)$, the collection of real-valued continuous functions on $S$, with addition and scalar multiplication being defined by (1.1) and (1.2).

Example 1.7 Let $X = C^k(\mathbb{R})$, the set of $k$ times continuously differentiable functions on $\mathbb{R}$, where addition and scalar multiplication being defined by (1.1) and (1.2).

Example 1.8 Let $\mu$ be a $\sigma$-finite measure and let $X = L^p(X, \mu)$, the set of functions $f$ such that $|f|^p$ is integrable with respect to the measure $\mu$. Addition and scalar multiplication are again given by (1.1) and (1.2).

Example 1.9 We can let $X$ be the set of complex-valued functions that are analytic on the unit disk.

Example 1.10 Let $X$ be the set of finite signed measures on a measurable space.

If $X$ is a linear space and $Y \subseteq X$, then we say $Y$ is a linear subspace of $X$ if $ay \in Y$ and $x + y \in Y$ whenever $x, y \in Y$ and $a \in F$. This definition is the obvious generalization of the one given in linear algebra courses.

Let $Y$ be a subset of $X$, not necessarily a linear subspace. Consider the collection

$$\{Z_\alpha : Z_\alpha \text{ is a linear subspace of } X, S \subseteq Z_\alpha\}.$$ 

It is easy to check that $\cap_\alpha Z_\alpha$ is a subspace of $X$, and it is called the linear span of $S$. 
Proposition 1.11 The linear span of $S$ is equal to

$$W = \left\{ \sum_{i=1}^{n} a_i x_i : a_i \in F, x_i \in S, n \geq 1 \right\}.$$

Proof. $W$ is clearly a linear subspace of $X$ containing $Y$, therefore the span of $Y$ is contained in $W$. If $Z_\alpha$ is any linear subspace containing $Y$, then $Z_\alpha$ must contain $W$, therefore $\cap_\alpha Z_\alpha$ contains $W$. \hfill $\square$

1.2 Normed linear spaces

A norm is a map from $X \to \mathbb{R}$, denoted $\|x\|$, such that

1. $\|0\| = 0$;
2. $\|x\| > 0$ if $x \neq 0$;
3. $\|x + y\| \leq \|x\| + \|y\|$ whenever $x, y \in X$; and
4. $\|ax\| = |a| \|x\|$ whenever $x \in X$ and $a \in F$.

A linear space together with its norm is called a normed linear space.

If we define $d(x, y) = \|x - y\|$, then $d$ is easily seen to be a metric, and we can use all the terminology of topology. Here are a few terms we will need right away. We define the open ball of radius $r$ about $x$ by

$$B(x, r) = \{ y \in X : \|y - x\| < r \}.$$ 

The topology generated by the metric $d$ is the smallest collection of subsets of $X$ that contains all the open balls, has the property that the intersection of two elements in the topology is again in the topology, and has the property that the arbitrary union of elements of the topology is again in the topology. We write $x_n \to x$ and say $x_n$ converges to $x$ if $\|x_n - x\| \to 0$. A subset $Y$ of $X$ is closed if $y \in Y$ whenever $y_n \in Y$ for $n = 1, 2, \ldots$ and $y_n \to y$. A sequence \{\{y_n\}\} of elements of $X$ is a Cauchy sequence if given $\varepsilon > 0$ there exists $N$ such that $d(y_n, y_m) < \varepsilon$ whenever $n, m \geq N$. A metric space $X$ is complete if every Cauchy sequence converges to a point in $X$. The space $X$ is separable if there exists a countable subset of $X$ that is dense in $X$, that is, such that the smallest closed set containing this countable subset is $X$ itself.
Two norms \( \|x\|_1 \) and \( \|x\|_2 \) are equivalent if there exist constants \( c_1 \) and \( c_2 \) such that
\[
c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1, \quad x \in X.
\]
Equivalent norms give rise to the same topology.

A subspace of a normed linear space is again a normed linear space.

For many purposes it is important to know whether a subspace is closed or not, closed meaning that the subspace is closed in the topological sense given above. Here is an example of a subspace that is not closed. Let \( X = \ell^2 \), the set of all sequences \( \{x = (x_1, x_2, \ldots)\} \) with \( \|x\| = (\sum_{j=1}^{\infty} |x_j|^2)^{1/2} < \infty \). Let \( Y \) be the collection of points in \( X \) such that all but finitely many coordinates are zero. Clearly \( Y \) is a linear subspace. Let \( y_1 = (1, 0, \ldots), y_2 = (1, \frac{1}{2}, 0, \ldots), y_3 = (1, \frac{1}{2}, \frac{1}{4}, 0, \ldots) \) and so on. Each \( y_k \in Y \). But it is easy to see that \( |y_k - y| \to 0 \) as \( k \to \infty \), where \( y = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots) \) and \( y \notin Y \). Thus \( Y \) is not closed.

For another example, let \( X = C(\mathbb{R}) \) and \( Y = C^1(\mathbb{R}) \), and define \( \|f\| = \sup_{r \in \mathbb{R}} |f(r)| \), the supremum norm. Clearly \( Y \) is a subspace of \( X \), but we can find a sequence of continuously differentiable functions converging in the supremum norm to a function that is continuous but not everywhere differentiable.

1.3 Examples

We give some examples of normed linear spaces. A Banach space is a normed linear space that is complete.

**Example 1.12** Let \( X \) be the collection of infinite sequences \( x = \{a_1, a_2, \ldots\} \) with each \( a_i \in \mathbb{C} \) and \( \sup_i |a_i| < \infty \). Another name for such a space \( X \) is \( \ell^\infty \). We define \( \|x\|_\infty = \sup_j |a_j| \). This is a normed linear space from a result in real analysis, because we can identify \( \ell^\infty \) with \( L^\infty(\mathbb{N}, \mu) \), where \( \mathbb{N} \) is the set of natural numbers and \( \mu \) is counting measure, that is, \( \mu(A) \) is equal to the number of elements of \( A \). In fact \( \ell^\infty \) is a Banach space.

**Example 1.13** If \( 1 \leq p < \infty \), \( \ell^p \) is the collection of infinite sequences...
CHAPTER 1. LINEAR SPACES

\[ x = (a_1, a_2, \ldots) \text{ for which} \]

\[ \|x\|_p = \left( \sum_j |a_j|^p \right)^{1/p} \]

is finite. This is a complete normed linear space, hence a Banach space, because we can identify \( \ell^p \) with \( L^p(\mathbb{N}, \mu) \), where \( \mathbb{N} \) and \( \mu \) are as in Example 1.12.

**Example 1.14** If \( S \) is a set, the collection of bounded functions on \( S \) with \( \|f\|_{\infty} = \sup_s |f(s)| \) is a complete normed linear space. This is a well known result from undergraduate analysis. Most of the examples above are separable metric spaces. However the collection of bounded functions on \( S \) is separable if and only if \( S \) is countable - look at the collection \( \{\chi_{\{y\}}\}, y \in Y \), where \( \chi_{\{y\}}(x) \) equals 1 if \( y = x \) and 0 otherwise.

**Example 1.15** If \( S \) is a topological space, then the collection of continuous bounded functions with \( \|f\| = \sup_s |f(s)| \) is also a Banach space.

**Example 1.16** The \( L^p \) spaces are complete normed linear spaces.

**Example 1.17** (Sobolev spaces) First consider one dimension. For \( f \in C^\infty(\mathbb{R}) \) we can define

\[ \|f\|_{k,p} = \left( \int_\mathbb{R} |f|^p + \int_\mathbb{R} |f'|^p + \cdots + \int_\mathbb{R} |f^{(k)}|^p \right)^{1/p}, \]

where \( f^{(k)} \) is the \( k^{th} \) derivative of \( f \). The set of \( C^\infty \) functions with compact support is not complete under this norm. We will discuss this in detail later.

In higher dimensions, let \( E \) be a domain in \( \mathbb{R}^n \) and consider the \( C^\infty \) functions on \( E \) with

\[ \int_E |D^j f(x)|^p \, dx \]

finite for all \( |j| \leq k \). Here \( j = (j_1, \ldots, j_n) \),

\[ D^j = \frac{\partial^{j_1}}{\partial x_1^{j_1}} \cdots \frac{\partial^{j_n}}{\partial x_n^{j_n}}, \]
and $|j| = j_1 + \cdots + j_n$. For a norm, we take

$$|f|_{k,p} = \left( \sum_{|j| \leq k} \int |D^j f(x)|^p \, dx \right)^{1/p}.$$ 

This is not a complete space, but its completion is denoted $W^{k,p}$ and is called a Sobolev space.

**Example 1.18** If $X$ is the set of finite signed measures $\mu$ on a measurable space, setting $\|\mu\|$ equal to the total variation of $\mu$ makes this into a normed linear space.

We will discuss Banach spaces in more detail in Chapter 3.

### 1.4 Direct sums

If $Y$ and $Z$ are subspaces of $X$, we write $X = Y \oplus Z$ if for each $x \in X$, there exist a unique $y \in Y$ and $z \in Z$ such that $x = y + z$. The decomposition must be unique, i.e., there is only one $y$ and one $z$ that works for any particular $x$. Of course, $y$ and $z$ depend on $x$. In this case we say that $X$ is the **direct sum** of $Y$ and $Z$.

As an example, let $X = \mathbb{R}^3$, $Y = \{(x,0,0) : x \in \mathbb{R}\}$. There are lots of possibilities for $Z$, in fact, any plane in $\mathbb{R}^3$ that passes through the origin and does not contain the $x$ axis. Given any choice of $Z$, though, there is only one way to write a given $x$ as $y + z$.

We will frequently use Zorn’s lemma, which is equivalent to the axiom of choice.

Suppose we have a **partially ordered** set $S$, which means that there is an order relation such that

1. $a \leq a$ for all $a \in S$,
2. if $a \leq b$ and $b \leq a$, then $a = b$, and
3. if $a \leq b$ and $b \leq c$, then $a \leq c$.

A subset is **totally ordered** if for every pair $x, y$ in the subset, either $x \leq y$ or $y \leq x$. An element $u$ of a partially ordered set is an **upper bound** for a subset of $S$ if $x \leq u$ for every $x$ in the subset. An element $x$ of a partially ordered set is **maximal** if $y \geq x$ implies $y = x$. 


**Lemma 1.19** (Zorn’s lemma) Let $X$ be a partially ordered set. If every totally ordered subset of $X$ has an upper bound in $X$, then $X$ has a maximal element.

Note that it is not required that the upper bound for a totally ordered subset be in the subset.

**Lemma 1.20** Suppose $Y$ is a subspace of a linear space $X$. Then there exists a linear subspace $Z$ such that $X = Y \oplus Z$.

**Proof.** Look at $\{Z : Z \text{ a subspace of } X, Z \cap Y = \{0\}\}$. We partially order this collection by inclusion: $Z_\alpha \leq Z_\beta$ if $Z_\alpha \subset Z_\beta$. If $\{Z_\alpha\}$ is a totally ordered subcollection, then $\bigcup_\alpha Z_\alpha$ is an upper bound in the collection. Let $Z_0$ be the maximal element guaranteed by Zorn’s lemma.

Suppose there is a point $x \in X$ that is not in $Y \oplus Z_0$. We adjoin $x$ to $Z_0$ to form $Z_1$ as follows: $Z_1 = \{ax + z : z \in Z_0, a \in \mathbb{R}\}$. $Z_1$ is a subspace of $X$ that is strictly bigger than $Z_0$. We argue that $Z_1 \cap Y = \{0\}$, a contradiction to the fact that $Z_0$ is maximal.

$x$ is not in the direct sum of $Y$ and $Z_0$, so $x \notin Y$, or else we could write $x = x + 0$. If $w \neq 0$ and $w \in Z_1 \cap Y$, then there exist $a \in \mathbb{R}$ and $z \in Z_0$ such that $w = ax + z$. One possibility is that $a = 0$; but then $w = z \in Z_0 \cap Y$, which isn’t possible since $w$ is nonzero. The other possibility is that $a \neq 0$. But $w \in Y$, so

$$x = \frac{w}{a} + \frac{-z}{a} \in Y \oplus Z_0,$$

also a contradiction. □

If $Z$ and $U$ are normed linear spaces, we can make $Z \oplus U$ into a normed linear space by defining $|(z, u)| = |z| + |u|$.

### 1.5 The unit ball in infinite dimensions

In finite dimensions, the closed unit ball is always compact, but this is not the case in infinite dimensions. As an example, consider $l^2$. If $e_i$ is the sequence
which has a one in the $i$th place and 0 everywhere else, then $\|e_i - e_j\| = \sqrt{2}$ if $i \neq j$. But then $\{e_i\}$ is a sequence contained in the unit ball that has no convergent subsequence, hence the unit ball is not compact.

In fact, the closed unit ball $B = \{x : |x| \leq 1\}$ is never compact in infinite dimensions.

First we define what infinite dimensional means. Elements $x_1, x_2, \ldots, x_n$ of $X$ are said to be linearly dependent if there exist $a_1, \ldots, a_n$ in $F$, not all equal to 0, such that $a_1x_1 + \cdots + a_nx_n = 0$. If $x_1, \ldots, x_n$ are not linearly dependent, they are linearly independent. A linear space $X$ is finite dimensional if there are finitely many nonzero elements whose linear span is all of $X$. If $X$ is not finite dimensional, it is infinite dimensional. We write $\dim X = n$ if there exist $n$ linearly independent nonzero elements of $X$ whose linear span is equal to $X$.

The key to proving that the unit ball in an infinite dimensional space is not compact is the following proposition.

**Proposition 1.21** Suppose $Y$ is a finite dimensional subspace of $X$ that is not all of $X$. There exists $v$ such that $\|v\| = 1$ and $\inf_{z \in Y} \|v - z\| \geq 1/2$.

**Proof.** Since $Y$ is finite dimensional, it is closed. It is not all of $X$, so there exists $x \in X \setminus Y$. Let $d = \inf_{y \in Y} \|y - x\|$. We claim that $d > 0$. If not, then there exists a sequence $y_n \in Y$ such that $\|y_n - x\| \to 0$. This means that $y_n$ converges to $x$. But $Y$ is closed, so $x \in Y$, a contradiction.

Choose $w \in Y$ such that $\|x - w\| < 2d$. Let $z = x - w$ so $\|z\| < 2d$. If $y \in Y$, then $y + w \in Y$ and

$$\|z - y\| = \|x - (y + w)\| \geq d.$$  

If we let $v = z/\|z\|$, then for all $y \in Y$

$$\|v - y\| = \left\| \frac{z}{\|z\|} - y \right\| = \frac{1}{\|z\|} \|z - \|z\|y\| \geq \frac{d}{2d} = \frac{1}{2}$$

since $\|z\|y \in Y$.

**Theorem 1.22** Let $X$ be an infinite dimensional normed linear space. Then the closed unit ball is not compact.
**Proof.** Choose $y_1$ such that $\|y_1\| = 1$. Given $y_1, \ldots, y_{n-1}$, let $Y_n$ be the linear span. Since $Y_n$ is finite dimensional, it is closed. By Proposition 1.21 there exists $y_n$ such that $\|y_n\| = 1$ and $\inf_{y \in Y_n} \|y - y_n\| \geq 1/2$. We continue by induction and find a sequence $\{y_n\}$ contained in the closed unit ball such that $\|y_j - y_n\| \geq 1/2$ if $j < n$, hence which has no convergent subsequence. □
Chapter 2

Linear maps

2.1 Basic notions

Let $X$ and $Y$ be linear spaces over $F$. A map $M : X \to U$ is linear or is a linear map or is a linear operator if $M(x + y) = M(x) + M(y)$ and $M(ax) = aM(x)$ for all $x, y \in X$ and all $a \in F$.

Here are some examples of linear operators.

(1) Let $X = L^1(\mu)$, $g$ be bounded and measurable, and define

$$Mf = \int f(x)g(x) \mu(dx).$$

Here $M$ maps $X$ into $\mathbb{R}$.

(2) Let $X$ be the space of bounded functions on a set $S$, fix points $x_1, \ldots, x_n \in S$, and let $Mf = (f(x_1), \ldots, f(x_n))$. Here $M$ maps $X$ to $\mathbb{R}^n$.

(3) Let $X$ be $\mathbb{R}^m$, let $Y$ be $\mathbb{R}^n$, let $a_{ij} \in \mathbb{R}$ for $1 \leq i \leq n, 1 \leq j \leq m$, and define the $i$th coordinate of $Mx$ to be $\sum_{j=1}^n a_{ij}x_j$. This is just matrix multiplication.

In fact all linear maps in finite dimensions can be viewed in this way. To see this, let $e_1, \ldots, e_m$ be linearly independent nonzero elements of $X$ and $f_1, \ldots, f_n$ linearly independent nonzero elements of $Y$. Since $\{f_1, \ldots, f_n\}$ spans $Y$, there exist elements $a_{i1}, \ldots, a_{in}$ of $F$ such that $Me_i = \sum_{j=1}^n a_{ij}f_j$ for $i = 1, \ldots, m$. 

11
(4) Let $X$ be the set of bounded sequences, suppose that
\[ \sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty, \]
and define the $i$th coordinate of $Mx$ to be $\sum_{j=1}^{\infty} a_{ij}x_j$.

(5) Let $X$ be the set of bounded measurable functions on some measure space with finite measure $\mu$, and suppose $K(x, y)$ is jointly measurable and bounded. Define $Mf$ by
\[ Mf(x) = \int K(x, y) \mu(dy). \]

If $M$ and $N$ are linear maps from $X$ into $Y$ and $a$ is a scalar, we define
\[ (M + N)(x) = M(x) + N(x), \quad (aM)(x) = aM(x). \]

Thus the set of linear maps from $X$ into $Y$ is a linear space, and we denote it by $\mathcal{L}(X,Y)$.

If $M : X \to Y$ and $N : Y \to Z$, we define $(NM)(x) = N(M(x))$.

An exercise is to show this is associative but not necessarily commutative. (Multiplication by matrices is an example to show commutativity need not hold.) It is distributive:
\[ M(N + K) = MN + MK, \quad (M + K)N = MN + KN. \]

We usually write $Mx$ for $M(x)$.

Define the identity $I : X \to X$ by $Ix = x$. We will also write $I_X$ when we want to emphasize the space.

We say $M : X \to Y$ is invertible if there exists $M^{-1} : Y \to X$ such that $M^{-1}M = I_X$, $MM^{-1} = I_Y$.

If $M$ is linear and invertible, then $M^{-1}$ is also linear. To see this, suppose $y_1 = Mx_1$ and $y_2 = Mx_2$ are elements of $Y$ with $x_1, x_2 \in X$. Then $y_1 + y_2 = Mx_1 + Mx_2 = M(x_1 + x_2)$. Hence $M^{-1}(y_1 + y_2) = x_1 + x_2 = M^{-1}y_1 + M^{-1}y_2$. Similarly $M^{-1}(ay) = aM^{-1}y$. 
2.2. BOUNDEDNESS AND CONTINUITY

Two linear spaces are said to be *isomorphic* if there exists a one-to-one linear mapping from one space onto the other.

Given two linear spaces $Y$ and $Z$, we can define a new space $X = \{(y, z) : y \in Y, z \in Z\}$ and define $(y_1, z_1) + (y_2, z_2) = (y_1 + y_2, z_1, z_2)$ and define scalar multiplication similarly. Clearly $Y$ is isomorphic to $Y' = \{(y, 0) : y \in Y\}$ and $Z$ is isomorphic to $Z' = \{(0, z) : z \in Z\}$. Moreover we see that $X = Y' \oplus Z'$. If $Y$ and $Z$ are normed linear spaces, then $X$ is also if we define $\| (y, z) \| = \| y \| + \| z \|.$

The *null space* or *kernel* of $M$ is $N_M = \{ x \in X : Mx = 0 \}$ and the *range* of $M$ is $R_M = \{ Mx : x \in X \}$.

Observe that $N_M \subset X$ and $R_M \subset Y$.

Some easily checked facts: $N_M$ and $R_M$ are linear subspaces, and if $L, M$ are invertible, then $(LM)^{-1} = M^{-1}L^{-1}$.

### 2.2 Boundedness and continuity

A linear map $M$ from a normed linear space $X$ into a normed linear space $Y$ is a *bounded linear map* if

$$
\| M \| = \sup \{ \| Mx \| : \| x \| = 1 \} \tag{2.1}
$$

is finite.

A linear map $M$ from a normed linear space $X$ to a Banach space $Y$ is continuous if $x_n \to x$ implies $Mx_n \to Mx$.

**Proposition 2.1** $M$ is continuous if and only if it is bounded.

**Proof.** If $M$ is bounded,

$$
\| M(x_n) - M(x) \| = \| M(x_n - x) \| \leq c \| x_n - x \| \to 0
$$

for some $c < \infty$, and so it is continuous.

Suppose $M$ is continuous but not bounded. Then there exist $x_n$ such that $\| M(x_n) \| > n \| x_n \|$. If

$$
y_n = \frac{1}{\sqrt{n}} x_n
$$

for some large $n$, then

$$
\| y_n \| = \frac{1}{\sqrt{n}} \| x_n \| \to 0.
$$
then $\|y_n - 0\| = \|y_n\| \to 0$, but

$$
\|M(y_n)\| > \frac{1}{\sqrt{n}} \|x_n\| = \sqrt{n},
$$

which does not tend to $0 = M(0)$, a contradiction. \hfill \Box

Define

$$
\|M\| = \sup_{x \neq 0} \frac{\|Mx\|}{\|x\|},
$$

or what is the same,

$$
\|M\| = \sup_{\|x\|=1} \|Mx\|.
$$

**Proposition 2.2** Suppose $M$ and $N$ are linear maps from a normed linear space $X$ into a normed linear space $Y$. Then $\|aM\| = |a| \|M\|$, $\|M\| \geq 0$ and equals 0 if and only if $M = 0$, and $\|M + N\| \leq \|M\| + \|N\|$.

The proofs are easy.

**Proposition 2.3** $N_M$ is closed.

**Proof.** $\{0\}$ is closed, $M$ is a continuous function from one metric space into another, so $N_M = M^{-1}(\{0\})$ is closed. \hfill \Box

**Proposition 2.4** Suppose $X, Y$, and $Z$ are normed linear spaces, $M$ is a linear map from $X$ to $Y$, and $N$ is a linear map from $Y$ to $Z$. Then $\|NM\| \leq \|N\| \|M\|$.

**Proof.** For $\|x\| = 1$,

$$
\|NMx\| \leq \|N\| \|Mx\| \leq \|N\| \|M\| \|x\|.
$$

\hfill \Box
2.3 Quotient spaces

Let $X$ be a linear space and $Y$ a subspace. We write $x_1 \equiv x_2$ and say that $x_1$ is equivalent to $x_2$ if $x_1 - x_2 \in Y$. This is an equivalence relation. Let $\overline{x}$ denote the equivalence class containing $x$. The collection of all such equivalence classes is denoted $X/Y$ and called the quotient space of $X$ with respect to $Y$.

Let’s make $X/Y$ into a linear space. If $x_1, x_2$ are in $X/Y$, define $x_1 + x_2$ to be $x_1 + x_2$. To see that this is well defined, if $z_1, z_2$ are any two elements of $\overline{x}_1, \overline{x}_2$, resp., then $(x_1 + x_2) - (z_1 + z_2) = (x_1 - z_1) + (x_2 - z_2)$, the sum of two elements of $Y$, hence an element of $Y$. Hence $x_1 + x_2 = z_1 + z_2$, and it doesn’t matter in the definition which elements of $\overline{x}_1$ and $\overline{x}_2$ we choose. We similarly define $a \overline{x} = \overline{ax}$. It is now routine to verify that $X/Y$ is a linear space.

We define the codimension of $Y$ by

$$\text{codim } Y = \dim X/Y.$$ 

Let’s look at an example. Let $X = \mathbb{R}^5$ and suppose $Y = \{ (x, y, 0, 0, 0) : x, y \in \mathbb{R} \}$. $x_1 \equiv x_2$ if and only if the $3^{rd}$ through $5^{th}$ coordinates of $x_1$ and $x_2$ agree. Therefore $X/Y$ is (essentially - at least it is isomorphic to) the $3^{rd}$ through $5^{th}$ coordinates of points in $\mathbb{R}^5$, hence isomorphic to $\mathbb{R}^3$. We see $\text{codim } Y = \dim X/Y = 3$, while $\dim Y = 2$.

**Proposition 2.5** If $X = Y \oplus Z$, then $X/Y$ is isomorphic to $Z$.

**Proof.** If $\overline{x} \in X/Y$, then $x \in X$ and we can write $x = z + y$, where $z \in Z$ and $y \in Y$. Define $M \overline{x} = z$. We will show that $M$ is an isomorphism.

First we need to show $M$ is well defined. If $x'$ is another element of $\overline{x}$, we can write $x' = z' + y'$ with $z' \in Z$ and $y' \in Y$. Then $x - x' = (z - z') + (y - y')$. Since we can also write $x - x' = 0 + (x - x')$ and we can write each element of $X$ as a sum of elements of $Z$ and $Y$ in only one way, we must have $z - z' = 0$, or $z = z'$.

Next we show $M$ is linear. If $x_1 + x_2 \in \overline{x_1 + x_2}$, then $x_1 = z_1 + y_1, x_2 = z_2 + y_2$, and then $x_1 + x_2 = (z_1 + z_2) + (y_1 + y_2)$. So $M(\overline{x_1 + x_2}) = \overline{z_1 + z_2}$.
$z_1 + z_2 = M\bar{x}_1 + M\bar{x}_2$. The linearity with respect to scalar multiplication is similar.

We show $M$ is one-to-one. If $M\bar{x} = M\bar{x}'$, and we write $x = z + y$, $x' = z' + y'$, then $z = M\bar{x} = M\bar{x}' = z'$. Hence

$$x - x' = (z - z') + (y - y') = y - y' \in Y,$$

so $\bar{x} = \bar{x}'$.

Finally, $M$ is onto, because if $z \in Z$, then $z = z + 0 \in X$ and $M\bar{x} = z$. □

Let $M : X \rightarrow Y$. We will need the fact that

**Proposition 2.6** $X/N_M$ is isomorphic to $R_M$.

**Proof.** If $\bar{x} \in X/N_M$, we define $\tilde{M}\bar{x}$ to be $Mx$ for any $x \in \bar{x}$. If $x'$ is any other element of $\bar{x}$, then $x - x' \in N_M$, or $M(x - x') = 0$, or $Mx = Mx'$. So the map $\tilde{M}$ is well defined. It is routine to check that $\tilde{M}$ is linear.

To show $\tilde{M}$ is one-to-one, if $\tilde{M}\bar{x} = \tilde{M}\bar{y}$, then $Mx = My$, or $M(x - y) = 0$, or $x - y \in N_M$, so $\bar{x} = \bar{y}$. To show $\tilde{M}$ is onto, if $y \in R_M$, then $y = Mx$ for some $x \in X$. Then $\tilde{M}\bar{x} = Mx = y$. □

### 2.4 Convex sets

A set $K \subset X$ is convex if $\lambda x + (1 - \lambda)y \in K$ whenever $x, y \in K$ and $\lambda \in [0, 1]$.

A convex combination of $x_1, \ldots, x_m$ is a sum of the form

$$\sum_{i=1}^n a_i x_i,$$

where $\sum_{i=1}^n a_i = 1$, $n$ is a positive integer, and all the $a_i$ are non-negative.

If $K$ is convex and $x_1, \ldots, x_n \in K$, each $a_i \geq 0$, and $\sum_{i=1}^n a_i = 1$, then $\sum_{i=1}^n a_i x_i \in K$. This can be proved easily using induction on $n$. 


Lemma 2.7  Linear subspaces are convex. Intersections of convex sets are convex. If \( M : X \to Y \) is linear and \( K \subset X \) is convex, then \( \{ M(x) : x \in K \} \) is convex.

The proof is left as an exercise.

If \( S \subset X \), the convex hull of \( S \) is the intersection of all the convex sets containing \( S \).

Proposition 2.8  The convex hull of \( S \) is equal to the set of all convex combinations of points of \( S \).

Proof. Let \( H \) be the convex hull of \( S \) and \( C \) the set of all convex combinations of points of \( S \). If \( y \) is in \( C \), then \( y = \sum_{i=1}^{n} a_i x_i \) where each \( x_i \in S \), each \( a_i \geq 0 \), and \( \sum_{i=1}^{n} a_i = 1 \). Therefore \( y \) is in each convex set containing \( S \), hence \( y \in H \). Thus \( C \subset H \).

Suppose \( y = \sum_{i=1}^{n} a_i x_i \), where \( x_i \in S \), each \( a_i \geq 0 \), and \( \sum_{i=1}^{n} a_i = 1 \) and similarly \( z = \sum_{j=1}^{m} b_j x'_j \). If \( m < n \), we can set \( b_{m+1}, \ldots, b_n = 0 \) and \( x_{m+1}, \ldots, x_n \) any point in \( S \), and can thus assume \( m \geq n \). Similarly we may without loss of generality assume \( n \geq m \), hence \( m = n \). If \( \lambda \in [0,1] \), we have

\[
\lambda y + (1-\lambda) z = \sum_{k=1}^{2n} c_k w_k,
\]

where \( c_k = \lambda a_k \) and \( w_k = x_k \) if \( k \leq n \) and \( c_k = (1-\lambda) b_{k-n} \) and \( w_k = x'_{k-n} \) if \( k > n \). Then each \( c_k \geq 0 \) and

\[
\sum_{k=1}^{n} c_k + \sum_{k=n+1}^{2n} c_k = \lambda \sum_{k=1}^{n} a_k + (1-\lambda) \sum_{k=1}^{n} b_k = \lambda + (1-\lambda) = 1.
\]

This proves that \( C \) is convex, and since \( C \) contains \( S \) we have \( H \subset C \). \( \Box \)

If \( K \) is convex, and \( E \subset K \), then \( E \) is an extreme subset of \( K \) if
(1) \( E \) is convex and non-empty, and
(2) if \( x \in E \) and \( x = \frac{y+z}{2} \) with \( y, z \in K \), then \( y, z \in E \).

If \( E \) is a single point, then the point is called an extreme point of \( K \).

For an example, consider the case where \( K \) is a polygon (plus the interior) in \( \mathbb{R}^2 \). Each edge of \( K \) is an extreme subset. Each vertex is an extreme point.
Proposition 2.9  If $E$ is an extreme subset of $F$ and $F$ is an extreme subset of $G$, then $E$ is an extreme subset of $G$.

Proof. Suppose $x \in E$ and $x = (y + z)/2$ for $y, z \in G$. Since $x \in E \subset F$ and $F$ is an extreme subset of $G$, then $y, z \in F$. But then since $E$ is an extreme subset of $F$, we must have $y, z \in E$.

\[ \blacksquare \]

2.5  Hahn-Banach theorem

A linear functional $\ell$ is a linear map from $X$ to $F$. In this section we will take $F$ to be the reals. In a later section we will consider the case when $F$ is the complex numbers.

The Hahn-Banach theorem is a tool that lets us assert that there is a plentiful supply of linear functionals.

We will be working with a sublinear functional $p(x)$ in the statement of the theorem. Suppose $p : X \to \mathbb{R}$ is a sublinear functional if

1. $p(ax) = ap(x)$ whenever $a > 0$ and $x \in X$ and
2. $p(x + y) \leq p(x) + p(y)$ if $x, y \in X$.

One example is to let $p(x) = c\|x\|$, where $c > 0$. This example requires $X$ to be a normed linear space.

Here is another example that applies to linear spaces, whether or not they are normed linear spaces.

A point $x_0 \in S \subset X$ is interior to $S$ if for all $y \in X$, there exists $\varepsilon$ (depending on $y$) such that $x_0 + ty \in S$ if $-\varepsilon < t < \varepsilon$.

Let $K$ be a convex set with 0 as an interior point. Define

$$p_K(x) = \inf \left\{ b > 0 : \frac{x}{b} \in K \right\}.$$

$p_K$ is sometimes called the gauge of $K$.

Proposition 2.10  $p_K$ is a sublinear functional.
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Proof. It is clear that $p_K(ax) = ap_K(x)$ if $a > 0$. Let $x, y \in X$. If $p_K(x)$ or $p_K(y)$ is infinite, there is nothing to prove. So suppose both are finite and let $\varepsilon > 0$. Choose $p_K(x) < a < p_K(x) + \varepsilon$ and $p_K(y) < b < p_K(y) + \varepsilon$. Then $\frac{x}{a}$ and $\frac{y}{b}$ are in $K$. Letting $\lambda = \frac{a}{a+b}$, then by the convexity of $K$

$$\lambda \frac{x}{a} + (1-\lambda) \frac{y}{b} = \frac{x+y}{a+b}$$

is in $K$. So

$$p_K(x+y) \leq a + b \leq p_K(x) + p_K(y) + 2\varepsilon.$$ 

Since $\varepsilon$ is arbitrary, we are done. \hfill \Box

Here is the Hahn-Banach theorem for real-valued linear functionals.

**Theorem 2.11** Suppose $p : X \to \mathbb{R}$ satisfies $p(ax) = ap(x)$ if $a > 0$ and $p(x+y) \leq p(x) + p(y)$ if $x, y \in X$. Suppose $Y$ is a linear subspace, $\ell$ is a linear functional on $Y$, and $\ell(y) \leq p(y)$ for all $y \in Y$. Then $\ell$ can be extended to a linear functional on $X$ satisfying $\ell(x) \leq p(x)$ for all $x \in X$.

**Proof.** If $Y$ is not all of $X$, pick $z \in X \setminus Y$. Look at $Y_1 = \{y + az : y \in Y, a \in \mathbb{R}\}$. We want to define $\ell(z)$ to be some real number with the property that if we set

$$\ell(y + az) = \ell(y) + a\ell(z),$$

we would have $\ell(y) + a\ell(z) \leq p(y + az)$ for all $y \in Y$ and $a \in \mathbb{R}$. This would give us an extension of $\ell$ from $Y$ to $Y_1$.

For all $y, y' \in Y$,

$$\ell(y') + \ell(y) = \ell(y' + y) \leq p(y' + y) = p((y+z) + (y'-z))$$

$$\leq p(y + z) + p(y' - z).$$

So

$$\ell(y') - p(y' - z) \leq p(y + z) - \ell(y).$$

This is true for all $y, y' \in Y$. So choose $\ell(z)$ to be a number between $\sup_{y'}[\ell(y') - p(y' - z)]$ and $\inf_y[p(y + z) - \ell(y)]$. Therefore

$$\ell(y') - p(y' - z) \leq \ell(z) \leq p(y + z) - \ell(y),$$

$$\ell(y') \leq p(y' + z) - \ell(y).$$

or

$$
\ell(y) + \ell(z) \leq p(y + z), \quad \ell(y') - \ell(z) \leq p(y' - z).
$$

If $a > 0$,

$$
\ell(y + az) = a\ell\left(\frac{y}{a} + z\right) \leq ap\left(\frac{y}{a} + z\right) = p(y + az).
$$

Similarly $\ell(y' - az) \leq p(y' - az)$ if $a > 0$.

So we have extended $\ell$ from $Y$ to $Y_1$, a larger space. Let $\{(Y_\alpha, \ell_\alpha)\}$ be the collection of all extensions of $(Y, \ell)$. We partially ordered this collection by saying $(Y_\alpha, \ell_\alpha) \leq (Y_\beta, \ell_\beta)$ if $Y_\alpha \subset Y_\beta$ and $\ell_\beta$ is an extension of $\ell_\alpha$. If $\{(Y_\beta, \ell_\beta)\}$ is a totally ordered subset, define $\ell$ on $\bigcup_\beta Y_\beta$ by setting $\ell(z) = \ell_\beta(z)$ if $z \in Y_\beta$.

By Zorn’s lemma, there is a maximal extension. This maximal extension must be all of $X$, or else by the above we could extend it.

**Corollary 2.12** Suppose that $X$ is a normed linear space $X$, $Y$ is a linear subspace of $X$, and $\ell$ is a bounded linear functional on $Y$. Then $\ell$ can be extended to a bounded linear functional on $X$ with the same norm.

**Proof.** Let $M$ be the norm of $\ell$ as a bounded linear map on $Y$ and set $p(x) = M\|x\|$ for $x \in X$. Our assumption tells us that $\ell(y) \leq p(y)$ for $y \in Y$.

Take the extension guaranteed by Theorem 2.11, and then $\ell(x) \leq M\|x\|$ for all $x \in X$. Applying this also to $-x$, we have $-\ell(x) = \ell(-x) \leq M\|x\|$, hence $|\ell(x)| \leq M\|x\|$.

This corollary is the version of the Hahn-Banach theorem one learns in real analysis courses.

### 2.6 Complex linear functionals

Now we turn to linear spaces (not necessarily normed) over the complex numbers.

**Theorem 2.13** Let $X$ be a linear space over $\mathbb{C}$. Suppose $p \geq 0$ satisfies $p(ax) = |a|p(x)$ for all $x \in X, a \in \mathbb{C}$, and $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$. If $Y$ is a subspace of $X$, $\ell$ is a linear functional on $Y$, and $|\ell(y)| \leq p(y)$ for all $y \in Y$, then $\ell$ can be extended to a linear functional on $X$ with $|\ell(x)| \leq p(x)$ for all $x$. 

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For normed linear spaces an example would be \( p(x) = M\|x\| \).

**Proof.** Write \( \ell \) as \( \ell(y) = \ell_1(y) + i\ell_2(y) \), the real and imaginary parts of \( \ell \). Since \( \ell \) is linear,
\[
i\ell(y) = \ell_1(iy) + i\ell_2(iy).
\]
On the other hand
\[
i\ell(y) = i\ell_1(y) - \ell_2(y)
\]
by substituting in for \( \ell(y) \) and multiplying by \( i \). Equating the real parts,
\[
\ell_1(iy) = -\ell_2(y).
\]
One can work this in reverse to see that if \( \ell_1 \) is a linear functional over the reals, and we define \( \ell(x) = \ell_1(x) - i\ell_1(ix) \), we get a linear functional over the complexes.

To extend \( \ell \), we have
\[
\ell_1(y) \leq |\ell(y)| \leq p(y).
\]
Use Hahn-Banach to extend \( \ell_1 \) to all of \( X \) and set \( \ell(x) = \ell_1(x) - i\ell_1(ix) \).

We need to show that \( |\ell(x)| \leq p(x) \) for all \( x \). Fix \( x \) and write \( \ell(x) = ar \), where \( r \) is real and \( |a| = 1 \). Then
\[
|\ell(x)| = r = a^{-1}\ell(x) = \ell(a^{-1}x).
\]
Since \( \ell(a^{-1}x) = |\ell(x)| \), it is real with no imaginary part, and therefore equals
\[
\ell_1(a^{-1}x) \leq p(a^{-1}x) = |a^{-1}|p(x) = p(x).
\]

\( \square \)

2.7 Positive linear functionals

Let \( S \) be an arbitrary set and let \( X \) be the collection of real-valued bounded functions on \( S \). We say \( x \leq y \) if \( x(s) \leq y(s) \) for all \( s \in S \). (We’ll use \( x \geq y \) if \( y \leq x \).) A function \( x \) is non-negative if \( 0 \leq x \). Let \( Y \) be a linear subspace of \( X \). \( \ell \) is a **positive linear functional** on \( Y \) if \( \ell(y) \geq 0 \) whenever \( y \geq 0 \). Note that if \( x \leq y \), then \( 0 \leq \ell(y - x) = \ell(y) - \ell(x) \), so \( \ell(x) \leq \ell(y) \).
One example is to take \( \ell(y) = y(s_0) \) for some point \( s_0 \) in \( S \). Or we could take a linear combination \( \sum c_i y(s_i) \) provided all the \( c_i \geq 0 \). Another example is to take \((S, \mu)\) to be a measure space and let \( \ell(y) = \int y(s) \mu(dx) \).

**Proposition 2.14** Let \( Y \) be a linear subspace and suppose there exists \( y_0 \in Y \) such that \( y_0(s) \geq 1 \) for all \( s \). Let \( \ell \) be a positive linear functional on \( Y \). Then \( \ell \) can be extended to a positive linear functional on \( X \).

**Proof.** Define 
\[
p(x) = \inf \{ \ell(y) : y \in Y, y \geq 0, y \geq x \}.
\]
Since \( -cy_0 \leq x \leq cy_0 \) if \( x \) is bounded by \( c \), we are not taking the infimum of an empty set. Since \( x \leq cy_0 \), then \( p(x) \leq c \ell(y_0) < \infty \).

It is clear that \( p(ax) = ap(x) \) if \( x \in X \) and \( a > 0 \). To show that \( p \) is a sublinear functional, suppose \( x_1, x_2 \in X \), let \( \varepsilon > 0 \), and choose \( y_1, y_2 \in Y \) with \( x_1 \leq y_1, x_2 \leq y_2, 0 \leq y_1, 0 \leq y_2, \ell(y_1) \leq p(x_1) + \varepsilon \), and \( \ell(y_2) \leq p(x_2) + \varepsilon \). Then \( y_1 + y_2 \in Y, y_1 + y_2 \geq 0, y_1 + y_2 \geq x_1 + x_2 \), and so
\[
p(x_1 + x_2) \leq \ell(y_1 + y_2) = \ell(y_1) + \ell(y_2) \leq p(x_1) + p(x_2) + 2\varepsilon.
\]
Since \( \varepsilon \) is arbitrary, this proves sublinearity.

If \( y \in Y, y' \geq 0 \), and \( y' \geq y \) is any other element in \( Y \), then \( \ell(y) \leq \ell(y') \), so \( p(y) \geq \ell(y) \).

We now use Theorem 2.11 to extend \( \ell \) to all of \( B \). If \( x \geq 0 \), then \( -x \leq 0 \), so
\[
\ell(-x) \leq p(-x) \leq l(0) = 0
\]
since \( 0 \in Y \), and then \( \ell(x) = -\ell(-x) \geq 0 \).

The additional assumption here is that there is a function in \( Y \) that is bounded below by a positive number. (If \( y_0 \) is bounded below by \( \delta > 0 \), look at \( y_0/\delta \).)

### 2.8 Separating hyperplanes

If \( \ell \) is a real-valued linear functional on a linear space over the reals, then \( \{ x : \ell(x) = c \} \) is a hyperplane. This splits \( X \) into two parts, those \( x \) for which \( \ell(x) > c \) and those for which \( \ell(x) < c \).
Recall the definition of the gauge $p_K$ of a convex set from (2.2).

**Proposition 2.15**  
(1) If $K$ is convex, 0 is interior to $K$, and $x \in K$, then $p_K(x) \leq 1$. If $K$ is convex and $x$ is interior to $K$, then $p_K(x) < 1$.

(2) Let $p$ be a positive sublinear functional. Then $\{x : p(x) < 1\}$ is convex and 0 is an interior point. Also $\{x : p(x) \leq 1\}$ is convex.

We leave the proof to the reader. $\Box$

We now prove the hyperplane separation theorem.

**Theorem 2.16**  
Suppose $K$ is a nonempty convex subset of a linear space $X$ over the reals and all points of $K$ are interior. If $y \not\in K$, then there exist $\ell$ and $c$ such that $\ell(x) < c$ for all $x \in K$ and $\ell(y) = c$.

**Proof.** Without loss of generality, assume $0 \in K$. Note $p_K(x) < 1$ for all $x \in K$. Set $\ell(y) = 1$ and $\ell(ay) = a$. If $a \leq 0$, $\ell(ay) \leq 0 \leq p_K(ay)$. If $a > 0$, then since $y \not\in K$, $p_K(y) \geq 1$, and so $p_K(ay) \geq a = \ell(ay)$.

We let $Y = \{ay\}$ and use Hahn-Banach to extend $\ell$ to all of $X$. We have $\ell(x) \leq p_K(x) < 1$ if $x \in K$ and $\ell(y) = 1$. We take $c = 1$. $\Box$

**Corollary 2.17**  
If $K$ is convex with at least one interior point and $y \not\in K$, there exists $\ell \neq 0$ such that $\ell(x) \leq \ell(y)$ for all $x \in K$.

$A + B$ is defined to be $\{a + b : a \in A, b \in B\}$.

**Corollary 2.18**  
Let $H$ and $M$ be disjoint convex sets, with at least one having an interior point. Then there exist $\ell$ and $c$ such that $\ell(u) \leq c \leq \ell(v)$, $u \in H, v \in M$.

**Proof.** $-M$ is convex, so $K = H + (-M)$ is convex. $K$ must have an interior point. $H \cap M = \emptyset$, so $0 \not\in K$. Let $y = 0$. There exists $\ell$ such that $\ell(x) \leq \ell(0) = 0$ $x \in K$.

If $u, v \in H, M$, resp., then $x = u - v \in K$, so $\ell(x) \leq 0$, and hence $\ell(u) \leq \ell(v)$. $\Box$
Chapter 3

Banach spaces

3.1 Preliminaries

A Banach space is a complete normed linear space.

If $X$ and $Y$ are metric spaces, a map $\varphi : X \to Y$ is an isometry if $d_Y(\varphi(x), \varphi(y)) = d_X(x, y)$ for all $x, y \in X$, where $d_X$ is the metric for $X$ and $d_Y$ the one for $Y$. A metric space $X^*$ is the completion of a metric space $X$ if there is an isometry $\varphi$ of $X$ into $X^*$ such that $\varphi(X)$ is dense in $X^*$ and $X^*$ is complete.

Recall that any metric space can be embedded in a complete metric space. See [1] for a proof of the following theorem.

**Theorem 3.1** If $X$ is a metric space, then it has a completion $X^*$.

Of course, if $X$ is already complete, its completion is $X$ itself and $\varphi$ is the identity map.

We have already seen some examples of Banach space. We looked at

1. $\ell^\infty$, the collection of infinite sequences $\{a_1, a_2, \ldots\}$ with each $a_i \in \mathbb{C}$ and $\sup_i |a_i| < \infty$. We define $\|x\|_\infty = \sup_j |a_j|$.

2. If $1 \leq p < \infty$, $\ell^p$ is the collection of infinite sequences for which

$$\|x\|_p = \left( \sum_j |a_j|^p \right)^{1/p}$$
is finite.

(3) If \( S \) is a set, the collection of bounded functions on \( S \) with \( \| f \|_\infty = \sup_s |f(s)| \).

(4) If \( S \) is a topological space, then the collection of continuous bounded functions with \( \| f \| = \sup_s |f(s)| \).

(5) The \( L^p \) spaces.

(6) We defined for \( f \in C^\infty(\mathbb{R}) \)
\[
\| f \|_{k,p} = \left( \int_\mathbb{R} |f|^p + \int_\mathbb{R} |f'|^p + \cdots + \int_\mathbb{R} |f^{(k)}|^p \right)^{1/p},
\]
where \( f^{(k)} \) is the \( k^{th} \) derivative of \( f \). The set of \( C^\infty \) functions with compact support is not complete under this norm, but we can take its completion and that will be a Banach space.

In higher dimensions, let \( D \) be a domain in \( \mathbb{R}^n \) and consider the \( C^\infty \) functions on \( D \) with
\[
\int_D |\partial^\alpha f(x)|^p \, dx
\]
finite for all \( |\alpha| \leq k \). Here \( \partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \) and \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). For a norm, we take
\[
\| f \|_{k,p} = \left( \sum_{|\alpha| \leq k} \int_D |\partial^\alpha f(x)|^p \, dx \right)^{1/p}.
\]
This is not a complete space, but its completion is denoted \( W^{k,p} \) and is called a Sobolev space.

We wrote \( \mathcal{L}(X,Y) \) for the set of linear maps from a Banach space \( X \) to a Banach space \( Y \).

**Proposition 3.2** \( \mathcal{L} \) is itself a Banach space.

**Proof.** Let \( M_n \) be a Cauchy sequence. For each \( x \in X \), \( M_n x \) is a Cauchy sequence in \( Y \). Since \( Y \) is complete, \( M_n x \) converges, say to a point \( N x \). Let \( \varepsilon > 0 \). Note
\[
|N x - M_n x| \leq \limsup_{m \to \infty} |M_m x - M_n x|,
\]
which will be less than \( \varepsilon \) if \( n \) is large enough, independently of \( x \). Thus \( M_n \) converges to \( N \) uniformly. Showing that \( N \) is a linear map is easy. \( \square \)
3.2 Baire’s theorem

We turn now to the Baire category theorem and some of its consequences. Recall that if $A$ is a set, we use $\overline{A}$ for the closure of $A$ and $A^\circ$ for the interior of $A$. A set $A$ is dense in $X$ if $\overline{A} = X$ and $A$ is nowhere dense if $(\overline{A})^\circ = \emptyset$.

The Baire category theorem is the following. Completeness of the metric space is crucial to the proof.

**Theorem 3.3** Let $X$ be a complete metric space.
(1) If $G_n$ are open sets dense in $X$, then $\bigcap_n G_n$ is dense in $X$.
(2) $X$ cannot be written as the countable union of nowhere dense sets.

**Proof.** We first show that (1) implies (2). Suppose we can write $X$ as a countable union of nowhere dense sets, that is, $X = \bigcup_n E_n$ where $(E_n)^\circ = \emptyset$.

Let $F_n = E_n^c$, which is open. Since $F_n^\circ = \emptyset$, then $G_n = F_n^c = \emptyset$. Starting with $X = \bigcup_n F_n$ and taking complements, we see that $\emptyset = \bigcap_n G_n$, a contradiction to (1).

We must prove (1). Suppose $G_1, G_2, \ldots$ are open and dense in $X$. Let $H$ be any non-empty open set in $X$. We need to show there exists a point in $H \cap (\bigcap_n G_n)$. We will construct a certain Cauchy sequence $\{x_n\}$ and the limit point, $x$, will be the point we seek.

Let $B(z, r) = \{y \in X : d(z, y) < r\}$, where $d$ is the metric. Since $G_1$ is dense in $X$, $H \cap G_1$ is non-empty and open, and we can find $x_1$ and $r_1$ such that $B(x_1, r_1) \subseteq H \cap G_1$ and $0 < r_1 < 1$. Suppose we have chosen $x_{n-1}$ and $r_{n-1}$ for some $n \geq 2$. Since $G_n$ is dense, then $G_n \cap B(x_{n-1}, r_{n-1})$ is open and non-empty, so there exists $x_n$ and $r_n$ such that $B(x_n, r_n) \subseteq G_n \cap B(x_{n-1}, r_{n-1})$ and $0 < r_n < 2^{-n}$. We continue and get a sequence $x_n$ in $X$. If $m, n > N$, then $x_m$ and $x_m$ both lie on $B(x_N, r_N)$, and so $d(x_m, x_n) < 2r_N < 2^{-N+1}$. Therefore $x_n$ is a Cauchy sequence, and since $X$ is complete, $x_n$ converges to a point $x \in X$.

It remains to show that $x \in H \cap (\bigcap_n G_n)$. Since $x_n$ lies in $\overline{B(x_N, r_N)}$ if $n > N$, then $x$ lies in each $\overline{B(x_N, r_N)}$, and hence in each $G_N$. Therefore $x \in \bigcap_n G_n$. Also,

$$x \in \overline{B(x_n, r_n)} \subseteq B(x_{n-1}, r_{n-1}) \subseteq \cdots \subseteq B(x_1, r_1) \subseteq H.$$
Thus we have found a point \( x \) in \( H \cap (\cap_n G_n) \).

A set \( A \subset X \) is called meager or of the first category if it is the countable union of nowhere dense sets; otherwise it is of the second category.

### 3.3 Uniform boundedness theorem

An important application of the Baire category theorem is the Banach-Steinhaus theorem, also called the uniform boundedness theorem.

**Theorem 3.4** Suppose \( X \) is a Banach space and \( Y \) is a normed linear space. Let \( A \) be an index set and let \( \{M_\alpha : \alpha \in A\} \) be a collection of bounded linear maps from \( X \) into \( Y \). Then either there exists a positive real number \( N < \infty \) such that \( \|M_\alpha\| \leq N \) for all \( \alpha \in A \) or else \( \sup_\alpha \|M_\alpha x\| = \infty \) for some \( x \).

**Proof.** Let \( \ell(x) = \sup_\alpha \|M_\alpha x\| \). Let \( G_n = \{ x : \ell(x) > n \} \). We argue that \( G_n \) is open. The map \( x \to \|M_\alpha x\| \) is a continuous function for each \( \alpha \) since \( M_\alpha \) is a bounded linear functional. This implies that for each \( \alpha \), the set \( \{ x : \|M_\alpha x\| > n \} \) is open. Since \( x \in G_n \) if and only if for some \( \alpha \in A \) we have \( \|M_\alpha x\| > n \), we conclude \( G_n \) is the union of open sets, hence is open.

Suppose there exists \( N \) such that \( G_N \) is not dense in \( X \). Then there exists \( x_0 \) and \( r \) such that \( B(x_0, r) \cap G_N = \emptyset \). This can be rephrased as saying that if \( \|x - x_0\| \leq r \), then \( \|M_\alpha(x)\| \leq N \) for all \( \alpha \in A \). If \( \|y\| \leq r \), we have \( y = (x_0 + y) - x_0 \). Then \( \|(x_0 + y) - x_0\| = \|y\| \leq r \), and hence \( \|M_\alpha(x_0 + y)\| \leq N \) for all \( \alpha \). Also, of course, \( \|x_0 - x_0\| = 0 \leq r \), and thus \( \|M_\alpha(x_0)\| \leq N \) for all \( \alpha \). We conclude that if \( \|y\| \leq r \) and \( \alpha \in A \),

\[
\|M_\alpha y\| = \|M_\alpha((x_0 + y) - x_0)\| \leq \|M_\alpha(x_0 + y)\| + \|M_\alpha x_0\| \leq 2N.
\]

Consequently, \( \sup_\alpha \|M_\alpha\| \leq N \) with \( N = 2N/r \).

The other possibility, by the Baire category theorem, is that every \( G_n \) is dense in \( X \), and in this case \( \cap_n G_n \) is dense in \( X \). But \( \ell(x) = \infty \) for every \( x \in \cap_n G_n \). \( \square \)
3.4 Open mapping theorem

The following theorem is called the open mapping theorem. It is important that $M$ be onto. A mapping $M : X \to Y$ is open if $M(U)$ is open in $Y$ whenever $U$ is open in $X$. For a measurable set $A$, we let $M(A) = \{Mx : x \in A\}$.

**Theorem 3.5** Let $X$ and $Y$ be Banach spaces. A bounded linear map $M$ from $X$ onto $Y$ is open.

**Proof.** We need to show that if $B(x, r) \subset X$, then $M(B(x, r))$ contains a ball in $Y$. We will show $M(B(0, r))$ contains a ball centered at 0 in $Y$. By linearity, to show that $M(B(0, r))$ contains a ball centered at 0, it suffices to show that $M(B(0, 1))$ contains a ball centered at 0 in $Y$.

**Step 1.** We show that there exists $r$ such that $B(0, r2^{-n}) \subset M(B(0, 2^{-n}))$ for each $n$. Since $M$ is onto, $Y = \bigcup_{n=1}^{\infty} M(B(0, n))$. The Baire category theorem tells us that at least one of the sets $M(B(0, n))$ cannot be nowhere dense. Since $M$ is linear, $M(B(0, 1))$ cannot be nowhere dense. Thus there exist $y_0$ and $r$ such that $B(y_0, 4r) \subset M(B(0, 1))$.

Pick $y_1 \in M(B(0, 1))$ such that $\|y_1 - y_0\| < 2r$ and let $z_1 \in B(0, 1)$ be such that $y_1 = Mz_1$. Then $B(y_1, 2r) \subset B(y_0, 4r) \subset M(B(0, 1))$. Thus if $\|y\| < 2r$, then $y + y_1 \in B(y_1, 2r)$, and so

$$y = -Mz_1 + (y + y_1) \in M(-z_1 + B(0, 1)).$$

Since $z_1 \in B(0, 1)$, then $-z_1 + B(0, 1) \subset B(0, 2)$, hence

$$y \in M(-z_1 + B(0, 1)) \subset M(B(0, 2)).$$

By the linearity of $M$, if $\|y\| < r$, then $y \in M(B(0, 1))$. It follows by linearity that if $\|y\| < r2^{-n}$, then $y \in M(B(0, 2^{-n}))$. This can be rephrased as saying that if $\|y\| < r2^{-n}$ and $\varepsilon > 0$, then there exists $x$ such that $\|x\| < 2^{-n}$ and $\|y - Mx\| < \varepsilon$.

**Step 2.** Suppose $\|y\| < r/2$. We will construct a sequence $\{x_j\}$ by induction such that $y = M(\sum_{j=1}^{\infty} x_j)$. By Step 1 with $\varepsilon = r/4$, we can find $x_1 \in$
$B(0, 1/2)$ such that $\|y - Mx_1\| < r/4$. Suppose we have chosen $x_1, \ldots, x_{n-1}$ such that
\[
\left\| y - \sum_{j=1}^{n-1} Mx_j \right\| < r2^{-n}.
\]

Let $\varepsilon = r2^{-(n+1)}$. By Step 1, we can find $x_n$ such that $\|x_n\| < 2^{-n}$ and
\[
\left\| y - \sum_{j=1}^{n} Mx_j \right\| = \left\| (y - \sum_{j=1}^{n-1} Mx_j) - Mx_n \right\| < r2^{-(n+1)}.
\]

We continue by induction to construct the sequence $\{x_j\}$. Let $w_n = \sum_{j=1}^{n} x_j$. Since $\|x_j\| < 2^{-j}$, then $w_n$ is a Cauchy sequence. Since $X$ is complete, $w_n$ converges, say, to $x$. But then $\|x\| < \sum_{j=1}^{\infty} 2^{-j} = 1$, and since $M$ is continuous, $y = Mx$. That is, if $y \in B(0, r/2)$, then $y \in M(B(0, 1))$. \hfill \Box

**Corollary 3.6** $M$ maps open sets onto open sets.

**Corollary 3.7** If $M$ is one-to-one, onto, and bounded, then $M^{-1}$ is bounded.

**Proof.** By the open mapping theorem, there exists $d$ such that $B(0, d) \subset M(B(0, 1))$. If $u \in U$ with $|u| = d/2$, there exists $x$ with $|x| < 1$ and $Mx = u$. By homogeneity, if $u \in B_U(0, 1)$, there exists $x \in X$ with $Mx = u$ and $|x| < 2|u|/d$. So $x = M^{-1}u$ and
\[
\|M^{-1}u\| = \|x\| < 2\|u\|/d,
\]
so $|M^{-1}| < 2/d$. \hfill \Box

### 3.5 Closed graph theorem

A map $M : X \to U$ is **closed** if whenever $x_n \to x$ and $MX_n \to u$, then $Mx = u$. This is equivalent to the graph $\{x, Mx\}$ being a closed set.

If $M$ is continuous, it is closed. If $D$ is the differentiation operator on the set of differentiable functions on $[0, 1]$, then $D$ is closed, but not continuous.
3.5. CLOSED GRAPH THEOREM

**Theorem 3.8** (Closed graph theorem) If $X$ and $U$ are Banach spaces and $M$ a closed linear map, then $M$ is continuous.

**Proof.** Let $G = \{g = (x, Mx)\}$ with norm $\|g\| = \|x\| + \|Mx\|$. It is easy to see that $\|g\|$ is a norm, and we use the closedness of $M$ to show that $G$ is complete: If $g_n = (x_n, Mx_n)$ is a Cauchy sequence in $G$, then $\|x_n - x_m\| \leq \|g_n - g_m\|$ is a Cauchy sequence, so $x_n$ converges, say to $x$. $\|Mx_n - Mx_m\| \leq \|g_n - g_m\|$, so $Mx_n$ is a Cauchy sequence in $U$, and hence converges, say to $y$. Since $G$ is closed, then $y = Mx$. Therefore $g_n$ converges to $(x, Mx)$.

Define $P : G \to X$ by $P(x, Mx) = x$, so that $P$ is a projection onto the first coordinate.

$\|Pg\| = \|x\| \leq \|x\| + \|Mx\| = \|g\|$, so $P$ is bounded with norm less than or equal to 1. $P$ is linear and one-to-one, and onto, so $P^{-1}$ is bounded, i.e., there exists $c$ such that $c\|Pg\| \geq \|g\|$. So $(c - 1)\|x\| \geq \|Mx\|$, which proves $M$ is bounded.

**Corollary 3.9** Suppose $X$ has two norms such that if $\|x_n - x\|_1 \to 0$ and $\|x_n - y\|_2 \to 0$, then $x = y$. Suppose $X$ is complete with respect to both norms. Then the norms are equivalent.

**Proof.** Let $X_1 = (X, \|\cdot\|_1)$ and similarly $X_2$. Let $I : X_1 \to X_2$. The hypothesis is equivalent to $I$ being closed. Therefore $I$ and $I^{-1}$ are bounded. □
Chapter 4

Hilbert spaces

Hilbert spaces are complete normed linear spaces that have an inner product. This added structure allows one to talk about orthonormal sets. We will give the definitions and basic properties. As an application we briefly discuss Fourier series.

4.1 Inner products

Recall that if $a$ is a complex number, then $\overline{a}$ represents the complex conjugate. When $a$ is real, $\overline{a}$ is just $a$ itself.

Definition 4.1 Let $H$ be a vector space where the set of scalars $F$ is either the real numbers or the complex numbers. $H$ is an inner product space if there is a map $\langle \cdot, \cdot \rangle$ from $H \times H$ to $F$ such that
(1) $\langle y, x \rangle = \langle x, y \rangle$ for all $x, y \in H$;
(2) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in H$;
(3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$, for $x, y \in H$ and $\alpha \in F$;
(4) $\langle x, x \rangle \geq 0$ for all $x \in H$;
(5) $\langle x, x \rangle = 0$ if and only if $x = 0$.

We define $\|x\| = \langle x, x \rangle^{1/2}$, so that $\langle x, x \rangle = \|x\|^2$. From the definitions it follows easily that $\langle 0, y \rangle = 0$ and $\langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$. 

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The following is the *Cauchy-Schwarz inequality*. The proof is the same as the one usually taught in undergraduate linear algebra classes, except for some complications due to the fact that we allow the set of scalars to be the complex numbers.

**Theorem 4.2** For all \( x, y \in H \), we have

\[
\langle x, y \rangle \leq \|x\| \|y\|.
\]

**Proof.** Let \( A = \|x\|^2 \), \( B = |\langle x, y \rangle| \), and \( C = \|y\|^2 \). If \( C = 0 \), then \( y = 0 \), hence \( \langle x, y \rangle = 0 \), and the inequality holds. If \( B = 0 \), the inequality is obvious. Therefore we will suppose that \( C > 0 \) and \( B \neq 0 \).

If \( \langle x, y \rangle = Re^{i\theta} \), let \( \alpha = e^{i\theta} \), and then \( |\alpha| = 1 \) and \( \alpha \langle y, x \rangle = |\langle x, y \rangle| = B \). Since \( B \) is real, we have that \( \bar{\alpha} \langle x, y \rangle \) also equals \( |\langle x, y \rangle| \).

We have for real \( r \)

\[
0 \leq \|x - r\alpha y\|^2 \\
= \langle x - r\alpha y, x - r\alpha y \rangle \\
= \langle x, x \rangle - r\alpha \langle y, x \rangle - r\alpha \bar{\alpha} \langle x, y \rangle + r^2 \langle y, y \rangle \\
= \|x\|^2 - 2r|\langle x, y \rangle| + r^2 \|y\|^2.
\]

Therefore

\[
A - 2Br + Cr^2 \geq 0
\]

for all real numbers \( r \). Since we are supposing that \( C > 0 \), we may take \( r = B/C \), and we obtain \( B^2 \leq AC \). Taking square roots of both sides gives the inequality we wanted. \( \square \)

From the Cauchy-Schwarz inequality we get the *triangle inequality*:

**Proposition 4.3** For all \( x, y \in H \) we have

\[
\|x + y\| \leq \|x\| + \|y\|.
\]

**Proof.** We write

\[
\|x + y\|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
\leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2,
\]

\[
\|x + y\| \leq \|x\| + \|y\|.
\]
The triangle inequality implies
\[ \|x - z\| \leq \|x - y\| + \|y - z\|. \]
Therefore \(\|\cdot\|\) is a norm on \(H\), and so if we define the distance between \(x\) and \(y\) by \(\|x - y\|\), we have a metric space.

**Definition 4.4** A Hilbert space \(H\) is an inner product space that is complete with respect to the metric \(d(x, y) = \|x - y\|\).

**Example 4.5** Let \(\mu\) be a positive measure on a set \(X\), let \(H = L^2(\mu)\), and define
\[ \langle f, g \rangle = \int f \overline{g} \, d\mu. \]
As is usual, we identify functions that are equal a.e. \(H\) is easily seen to be a Hilbert space. The completeness is a result of real analysis.

If we let \(\mu\) be counting measure on the natural numbers, we get what is known as the space \(\ell^2\). An element of \(\ell^2\) is a sequence \(a = (a_1, a_2, \ldots)\) such that \(\sum_{n=1}^{\infty} |a_n|^2 < \infty\) and if \(b = (b_1, b_2, \ldots)\), then
\[ \langle a, b \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}. \]

We get another common Hilbert space, \(n\)-dimensional Euclidean space, by letting \(\mu\) be counting measure on \(\{1, 2, \ldots, n\}\).

**Proposition 4.6** Let \(y \in H\) be fixed. Then the functions \(x \to \langle x, y \rangle\) and \(x \to \|x\|\) are continuous.

**Proof.** By the Cauchy-Schwarz inequality,
\[ |\langle x, y \rangle - \langle x', y \rangle| = |\langle x - x', y \rangle| \leq \|x - x'\| \|y\|, \]
which proves that the function \(x \to \langle x, y \rangle\) is continuous. By the triangle inequality, \(\|x\| \leq \|x - x'\| + \|x'\|\), or
\[ \|x\| - \|x'\| \leq \|x - x'\|. \]
The same holds with \( x \) and \( x' \) reversed, so
\[
|\|x\|-\|x'\|| \leq \|x-x'\|,
\]
and thus the function \( x \to \|x\| \) is continuous.

## 4.2 Subspaces

**Definition 4.7** A subset \( M \) of a vector space is a **subspace** if \( M \) is itself a vector space with respect to the same operations of addition and scalar multiplication. A **closed subspace** is a subspace that is closed relative to the metric given by \( \langle \cdot, \cdot \rangle \).

For an example of a subspace that is not closed, consider \( \ell^2 \) and let \( M \) be the collection of sequences for which all but finitely many elements are zero. \( M \) is clearly a subspace. Let \( x_n = (1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots) \) and \( x = (1, \frac{1}{2}, \frac{1}{3}, \ldots) \). Then each \( x_n \in M, x \notin M \), and we conclude \( M \) is not closed because
\[
\|x_n - x\|^2 = \sum_{j=n+1}^{\infty} \frac{1}{j^2} \to 0
\]
as \( n \to \infty \).

Since \( \|x+y\|^2 = \langle x+y, x+y \rangle \) and similarly for \( \|x-y\|^2, \|x\|^2, \) and \( \|y\|^2 \), a simple calculation yields the **parallelogram law**:
\[
\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.
\] (4.1)

A set \( E \subset H \) is **convex** if \( \lambda x + (1 - \lambda x) \in E \) whenever \( 0 \leq \lambda \leq 1 \) and \( x, y \in E \).

**Proposition 4.8** Each non-empty closed convex subset \( E \) of \( H \) has a unique element of smallest norm.

**Proof.** Let \( \delta = \inf \{\|x\| : x \in E\} \). Dividing (4.1) by 4, if \( x, y \in E \), then
\[
\frac{1}{4}\|x-y\|^2 = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \left\| \frac{x+y}{2} \right\|^2.
\]
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Since $E$ is convex, if $x, y \in E$, then $(x + y)/2 \in E$, and we have

$$\|x - y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - 4\delta^2. \quad (4.2)$$

Choose $y_n \in E$ such that $\|y_n\| \to \delta$. Applying (4.2) with $x$ replaced by $y_n$ and $y$ replaced by $y_m$, we see that

$$\|y_n - y_m\|^2 \leq 2\|y_n\|^2 + 2\|y_m\|^2 - 4\delta^2,$$

and the right hand side tends to 0 as $m$ and $n$ tend to infinity. Hence $y_n$ is a Cauchy sequence, and since $H$ is complete, it converges to some $y \in H$. Since $y_n \in E$ and $E$ is closed, $y \in E$. Since the norm is a continuous function, $\|y\| = \lim \|y_n\| = \delta$.

If $y'$ is another point with $\|y'\| = \delta$, then by (4.2) with $x$ replaced by $y'$ we have $\|y - y'\| = 0$, and hence $y = y'$.

We say $x \perp y$, or $x$ is orthogonal to $y$, if $\langle x, y \rangle = 0$. Let $x^\perp$, read “$x$ perp,” be the set of all $y$ in $X$ that are orthogonal to $x$. If $M$ is a subspace, let $M^\perp$ be the set of all $y$ that are orthogonal to all points in $M$. The subspace $M^\perp$ is called the orthogonal complement of $M$. It is clear from the linearity of the inner product that $x^\perp$ is a subspace of $H$. The subspace $x^\perp$ is closed because it is the same as the set $f^{-1}(\{0\})$, where $f(x) = \langle x, y \rangle$, which is continuous. Also, it is easy to see that $M^\perp$ is a subspace, and since

$$M^\perp = \cap_{x \in M} x^\perp,$$

$M^\perp$ is closed. We make the observation that if $z \in M \cap M^\perp$, then

$$\|z\|^2 = \langle z, z \rangle = 0,$$

so $z = 0$.

**Lemma 4.9** Let $M$ be a closed subspace of $H$ with $M \neq H$. Then $M^\perp$ contains a non-zero element.

**Proof.** Choose $x \in H$ with $x \notin M$. Let $E = \{w - x : w \in M\}$. It is routine to check that $E$ is a closed and convex subset of $H$. By Lemma 4.8, there exists an element $y \in E$ of smallest norm.
Note $y + x \in M$ and we conclude $y \neq 0$ because $x \notin M$.

We show $y \in M^\perp$ by showing that if $w \in M$, then $\langle w, y \rangle = 0$. This is obvious if $w = 0$, so assume $w \neq 0$. We know $y + x \in M$, so for any real number $t$ we have $tw + (y + x) \in M$, and therefore $tw + y \in E$. Since $y$ is the element of $E$ of smallest norm,

$$\langle y, y \rangle = \|y\|^2 \leq \|tw + y\|^2$$

$$= \langle tw + y, tw + y \rangle$$

$$= t^2 \langle w, w \rangle + 2t\text{Re} \langle w, y \rangle + \langle y, y \rangle,$$

which implies

$$t^2 \langle w, w \rangle + 2t\text{Re} \langle w, y \rangle \geq 0$$

for each real number $t$. Choosing $t = -\text{Re} \langle w, y \rangle / \langle w, w \rangle$, we obtain

$$\frac{|\text{Re} \langle w, y \rangle|^2}{\langle w, w \rangle} \geq 0,$$

from which we conclude $\text{Re} \langle w, y \rangle = 0$.

Since $w \in M$, then $iw \in M$, and if we repeat the argument with $w$ replaced by $iw$, then we get $\text{Re} \langle iw, y \rangle = 0$, and so

$$\text{Im} \langle w, y \rangle = -\text{Re} (i\langle w, y \rangle) = -\text{Re} \langle iw, y \rangle = 0.$$

Therefore $\langle w, y \rangle = 0$ as desired. \hfill \Box

If in the proof above we set $Px = y + x$ and $Qx = -y$, then $Px \in M$ and $Qx \in M^\perp$, and we can write $x = Px + Qx$. We call $Px$ and $Qx$ the orthogonal projections of $x$ onto $M$ and $M^\perp$, resp. If we write $x = w + z = w' + z'$, with $w, w' \in M$ and $z, z' \in M^\perp$, then $w - w' = z' - z$ is in $M$ and $M^\perp$, hence 0. Therefore each element of $H$ can be written as the sum of an element of $M$ and an element of $M^\perp$ in exactly one way.

**Proposition 4.10** $P$ and $Q$ are linear operators.

**Proof.** Since

$$x + y = (Px + Qx) + (Py + Qy)$$
and also \( x + y = P(x + y) + Q(x + y) \), we have
\[
P x + P y - P(x + y) = Q(x + y) - Q x - Q y.
\]
The left hand side is in \( M \), the right hand side in \( M^\perp \), so \( P x + P y = P(x + y) \) and similarly for \( Q \). To show \( P(kx) = kPx \) and \( Q(kx) = kQx \) is similar, so \( P \) and \( Q \) are linear.

Proposition 4.11 Suppose \( M \) is a closed subspace of \( H \). Then
1) \( M^\perp \) is a closed linear subspace of \( H \).
2) \( H = M \oplus M^\perp \).
3) \( (M^\perp)^\perp = M \).

Proof. 1) (We don’t need \( M \) closed for this first part.) That \( M^\perp \) is a linear subspace is easy. We already showed \( M^\perp \) is closed.

2) We proved this: write \( x = Px + Qx \).

3) If \( y \in M \), then for any \( v \in M^\perp \) we have \( (y, v) = 0 \), and hence \( y \in (M^\perp)^\perp \). We thus need to show \( (M^\perp)^\perp \subset M \).

By 2), \( H = M \oplus M^\perp \). If \( y \in (M^\perp)^\perp \), we can write \( y = v + z \) with \( z \in M \subseteq (M^\perp)^\perp \) and \( v \in M^\perp \). Then \( v = y - z \in (M^\perp)^\perp \). Since \( v \) is also in \( M^\perp \), we see \( v = 0 \), or \( y = z \in M \).

4.3 Riesz representation theorem

The following is sometimes called the Riesz representation theorem, although usually that name is reserved for the theorem of real analysis about linear functionals on the set of continuous functions. To motivate the theorem, consider the case where \( H \) is \( n \)-dimensional Euclidean space. Elements of \( \mathbb{R}^n \) can be identified with \( n \times 1 \) matrices and linear maps from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) can be represented by multiplication on the left by a \( m \times n \) matrix \( A \). For bounded linear functionals on \( H \), \( m = 1 \), so \( A \) is \( 1 \times n \), and the \( y \) of the next theorem is the vector associated with the transpose of \( A \).
Theorem 4.12 If $L$ is a bounded linear functional on $H$, then there exists a unique $y \in H$ such that $Lx = \langle x, y \rangle$.

Proof. The uniqueness is easy. If $Lx = \langle x, y \rangle = \langle x, y' \rangle$, then $\langle x, y - y' \rangle = 0$ for all $x$, and in particular, when $x = y - y'$.

We now prove existence. If $Lx = 0$ for all $x$, we take $y = 0$. Otherwise, let $M = \{x : Lx = 0\}$, take $z \neq 0$ in $M^\perp$, and let $y = \alpha z$ where $\alpha = \frac{Lz}{\langle z, z \rangle}$.

Notice $y \in M^\perp$, $Ly = \frac{Lz}{\langle z, z \rangle}Lz = \frac{|Lz|^2}{\langle z, z \rangle} = \langle y, y \rangle$, and $y \neq 0$.

If $x \in H$ and $w = x - \frac{Lx}{\langle y, y \rangle} y$, then $Lw = 0$, so $w \in M$, and hence $\langle w, y \rangle = 0$. Then

$$\langle x, y \rangle = \langle x - w, y \rangle = Lx$$

as desired. \qed

4.4 Lax-Milgram lemma

Theorem 4.13 (Lax-Milgram lemma) Let $H$ be a Hilbert space and suppose

(1) for each $y$, $B(x, y)$ is linear in $x$;
(2) for each $x$, $B(x, y_1 + y_2) = B(x, y_1) + B(x, y_2)$ and $B(x, cy) = cB(x, y)$;
(3) there exists $c$ such that $|B(x, y)| \leq c\|x\|\|y\|$;
(4) there exists $b$ such that $|B(y, y)| \geq b\|y\|^2$ for all $y$.

(We do not assume $B(x, y) \neq B(y, x)$.) Then every bounded linear functional $\ell$ is of the form $\ell(x) = B(x, y)$ for some unique $y$.

Proof. For each $y$, $B(x, y)$ is a bounded linear functional of $x$, so there exists $z = z(y)$ such that $B(x, y) = \langle x, z \rangle$ for all $x$, and $z$ is unique.
4.5. ORTHONORMAL SETS

If \( Z = \{ z : z = z(y) \text{ for some } y \in H \} \), then \( Z \) is a linear space.

\( Z \) is closed: setting \( x = y \), and letting \( z = z(y) \),

\[
b ||y||^2 \leq B(y, y) = (y, z) \leq c ||y|| \cdot ||z||,
\]
or

\[
b ||y|| \leq ||z||.
\]

If \( z_n \in Z \) and \( z_n \to z \), let \( y_n \) be a point such that \( z_n = z(y_n) \). Then \( B(x, y_n) = (x, z_n) \). So \( B(x, y_n - y_m) = (x, z_n - z_m) \), hence \( b ||y_n - y_m|| \leq ||z_n - z_m|| \), and therefore \( y_n \) is a Cauchy sequence. \( H \) is complete; let \( y \) be the limit. Since \( B(x, y_n) \to B(x, y) \) and \( (x, z_n) \to (x, z) \), we have \( B(x, y) = (x, z) \), and hence \( z \in Z \).

\( Z = H \): For each \( y \), there exists \( z(y) \) such that \( B(x, y) = (x, z) \) for all \( x \). If \( Z \neq H \), there exists \( x \in Z^\perp \). Applying the above with \( y = x \), there exists \( z(x) \) such that \( B(x, x) = (x, z(x)) \). Since \( x \in Z^\perp \) and \( z(x) \in Z \),

\[
b ||x||^2 \leq B(x, x) = (x, z(x)) = 0.
\]

So \( x = 0 \).

Existence: given \( \ell \), there exists \( y \) such that \( \ell(x) = (x, y) \) for all \( x \). Then \( \ell(x) = B(x, z(y)) \).

Uniqueness: if there are two such \( z \), then \( B(x, z - z') = B(x, z) - B(x, z') = \ell(x) - \ell(x) = 0 \). Now set \( x = z - z' \).

4.5 Orthonormal sets

A subset \( \{ u_\alpha \}_{\alpha \in A} \) of \( H \) is orthonormal if \( ||u_\alpha|| = 1 \) for all \( \alpha \) and \( \langle u_\alpha, u_\beta \rangle = 0 \) whenever \( \alpha, \beta \in A \) and \( \alpha \neq \beta \).

The Gram-Schmidt procedure from linear algebra also works in infinitely many dimensions. Suppose \( \{ x_n \}_{n=1}^\infty \) is a linearly independent sequence, i.e., no finite linear combination of the \( x_n \) is 0. Let \( u_1 = x_1/||x_1|| \) and define inductively

\[
v_N = x_N - \sum_{i=1}^{N-1} \langle x_N, u_i \rangle u_i,
\]

\[
u_N = v_N/||v_N||.
\]

We have \( \langle v_N, u_i \rangle = 0 \) if \( i < N \), so \( u_1, \ldots, u_N \) are orthonormal.
Proposition 4.14 If \( \{u_\alpha\}_{\alpha \in A} \) is an orthonormal set, then for each \( x \in H \),
\[
\sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2.
\] (4.3)

This is called Bessel’s inequality. This inequality implies that only finitely many of the summands on the left hand side of (4.3) can be larger than \( 1/n \) for each \( n \), hence only countably many of the summands can be non-zero.

Proof. Let \( F \) be a finite subset of \( A \). Let
\[
y = \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha.
\]
Then
\[
0 \leq \|x - y\|^2 = \|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2.
\]
Now
\[
\langle y, x \rangle = \left\langle \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha, x \right\rangle = \sum_{\alpha \in F} \langle x, u_\alpha \rangle \langle u_\alpha, x \rangle = \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2.
\]
Since this is real, then \( \langle y, x \rangle = \langle y, x \rangle \). Also
\[
\|y\|^2 = \langle y, y \rangle = \left\langle \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha, \sum_{\beta \in F} \langle x, u_\beta \rangle u_\beta \right\rangle
\]
\[
= \sum_{\alpha, \beta \in F} \langle x, u_\alpha \rangle \overline{\langle x, u_\beta \rangle} \langle u_\alpha, u_\beta \rangle
\]
\[
= \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2,
\]
where we used the fact that \( \{u_\alpha\} \) is an orthonormal set. Therefore
\[
0 \leq \|y - x\|^2 = \|x\|^2 - \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2.
\]
Rearranging,
\[
\sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2
\]
4.5. ORTHONORMAL SETS

when \( F \) is a finite subset of \( A \). If \( N \) is an integer larger than \( n\|x\|^2 \), it is not possible that \( |\langle x, u_\alpha \rangle|^2 > 1/n \) for more than \( N \) of the \( \alpha \). Hence \( |\langle x, u_\alpha \rangle|^2 \neq 0 \) for only countably many \( \alpha \). Label those \( \alpha \)'s as \( \alpha_1, \alpha_2, \ldots \). Then

\[
\sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 = \sum_{j=1}^\infty |\langle x, u_{\alpha_j} \rangle|^2 = \lim_{J \to \infty} \sum_{j=1}^J |\langle x, u_{\alpha_j} \rangle|^2 \leq \|x\|^2,
\]

which is what we wanted.

Proposition 4.15 Suppose \( \{u_\alpha\}_{\alpha \in A} \) is orthonormal. Then the following are equivalent.

1. If \( \langle x, u_\alpha \rangle = 0 \) for each \( \alpha \in A \), then \( x = 0 \).
2. \( \|x\|^2 = \sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 \) for all \( x \).
3. For each \( x \in H \), \( x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha \).

We make a few remarks. When (1) holds, we say the orthonormal set is complete. (2) is called Parseval’s identity. In (3) the convergence is with respect to the norm of \( H \) and implies that only countably many of the terms on the right hand side are non-zero.

Proof. First we show (1) implies (3). Let \( x \in H \). By Bessel’s inequality, there can be at most countably many \( \alpha \) such that \( |\langle x, u_\alpha \rangle|^2 \neq 0 \). Let \( \alpha_1, \alpha_2, \ldots \) be an enumeration of those \( \alpha \). By Bessel’s inequality, the series \( \sum_i |\langle x, u_{\alpha_i} \rangle|^2 \) converges. Using that \( \{u_\alpha\} \) is an orthonormal set,

\[
\left\| \sum_{j=m}^n \langle x, u_{\alpha_j} \rangle u_{\alpha_j} \right\|^2 = \sum_{j,k=m}^n \langle x, u_{\alpha_j} \rangle \overline{\langle x, u_{\alpha_k} \rangle} \langle u_{\alpha_j}, u_{\alpha_k} \rangle = \sum_{j=m}^n |\langle x, u_{\alpha_j} \rangle|^2 \to 0
\]
as \( m,n \to \infty \). Thus \( \sum_{j=1}^n \langle x, u_{\alpha_j} \rangle u_{\alpha_j} \) is a Cauchy sequence, and hence converges. Let \( z = \sum_{j=1}^\infty \langle x, u_{\alpha_j} \rangle u_{\alpha_j} \). Then \( \langle z - x, u_{\alpha_j} \rangle = 0 \) for each \( \alpha_j \). By (1), this implies \( z - x = 0 \).

We see that (3) implies (2) because

\[
\|x\|^2 - \sum_{j=1}^n |\langle x, u_{\alpha_j} \rangle|^2 = \left\| x - \sum_{j=1}^n \langle x, u_{\alpha_j} \rangle u_{\alpha_j} \right\|^2 \to 0.
\]
That (2) implies (1) is clear. \(\square\)

**Example 4.16** Take \(H = \ell^2 = \{x = (x_1, x_2, \ldots) : \sum |x_i|^2 < \infty\}\) with \(\langle x, y \rangle = \sum x_i \overline{y}_i\). Then \(\{e_i\}\) is a complete orthonormal system, where \(e_i = (0, 0, \ldots, 0, 1, 0, \ldots)\), i.e., the only non-zero coordinate of \(e_i\) is the \(i^{th}\) one.

If \(K\) is a subset of a Hilbert space \(H\), the set of finite linear combinations of elements of \(K\) is called the span of \(K\).

A collection of elements \(\{e_\alpha\}\) is a basis for \(H\) if the set of finite linear combinations of the \(e_\alpha\) is dense in \(H\). A basis, then, is an orthonormal subset of \(H\) such that the closure of its span is all of \(H\).

**Proposition 4.17** Every Hilbert space has an orthonormal basis.

This means that (3) in Proposition 4.15 holds.

**Proof.** If \(B = \{u_\alpha\}\) is orthonormal, but not a basis, let \(V\) be the closure of the linear span of \(B\), that is, the closure with respect to the norm in \(H\) of the set of finite linear combinations of elements of \(B\). Choose \(x \in V^\perp\), and if we let \(B' = B \cup \{x/\|x\|\}\), then \(B'\) is a basis that is strictly bigger than \(B\).

It is easy to see that the union of an increasing sequence of orthonormal sets is an orthonormal set, and so there is a maximal one by Zorn’s lemma. By the preceding paragraph, this maximal orthonormal set must be a basis, for otherwise we could find a larger basis. \(\square\)

### 4.6 Fourier series

An interesting application of Hilbert space techniques is to *Fourier series*, or equivalently, to *trigonometric series*. For our Hilbert space we take \(H = L^2([0, 2\pi])\) and let

\[
u_n = \frac{1}{\sqrt{2\pi}} e^{inx}\]
for $n$ an integer. ($n$ can be negative.) Recall that

$$\langle f, g \rangle = \int_{0}^{2\pi} f(x)\overline{g(x)} \, dx$$

and $\|f\|^2 = \int_{0}^{2\pi} |f(x)|^2 \, dx$.

It is easy to see that $\{u_n\}$ is an orthonormal set:

$$\int_{0}^{2\pi} e^{inx} e^{-imx} \, dx = \int_{0}^{2\pi} e^{i(n-m)x} \, dx = 0$$

if $n \neq m$ and equals $2\pi$ if $n = m$.

Let $F$ be the set of finite linear combinations of the $u_n$, i.e., the span of $\{u_n\}$. We want to show that $F$ is a dense subset of $L^2([0, 2\pi])$. The first step is to show that the closure of $F$ with respect to the supremum norm is equal to the set of continuous functions $f$ on $[0, 2\pi]$ with $f(0) = f(2\pi)$. We will accomplish this by using the Stone-Weierstrass theorem.

We identify the set of continuous functions on $[0, 2\pi)$ that take the same value at 0 and $2\pi$ with the continuous functions on the circle. To do this, let $S = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$ be the unit circle in $\mathbb{C}$. If $f$ is continuous on $[0, 2\pi)$ with $f(0) = f(2\pi)$, define $\tilde{f} : S \to \mathbb{C}$ by $\tilde{f}(e^{i\theta}) = f(\theta)$. Note $\tilde{u}_n(e^{i\theta}) = e^{in\theta}$.

Let $\tilde{F}$ be the set of finite linear combinations of the $\tilde{u}_n$. $S$ is a compact metric space. Since the complex conjugate of $\tilde{u}_n$ is $\tilde{u}_{-n}$, then $\tilde{F}$ is closed under the operation of taking complex conjugates. Since $\tilde{u}_n \cdot \tilde{u}_m = \tilde{u}_{n+m}$, it follows that $F$ is closed under the operation of multiplication. That it is closed under scalar multiplication and addition is obvious. $\tilde{u}_0$ is identically equal to 1, so $\tilde{F}$ vanishes at no point. If $\theta_1, \theta_2 \in S$ and $\theta_1 \neq \theta_2$, then $\theta_1 - \theta_2$ is not an integer multiple of $2\pi$, so

$$\frac{\tilde{u}_1(\theta_1)}{\tilde{u}_1(\theta_2)} = e^{i(\theta_1-\theta_2)} \neq 1,$$

or $\tilde{u}_1(\theta_1) \neq \tilde{u}_1(\theta_2)$. Therefore $F$ separates points. By the Stone-Weierstrass theorem, the closure of $F$ with respect to the supremum norm is equal to the set of continuous complex-valued functions on $S$.

If $f \in L^2([0, 2\pi])$, then

$$\int |f - f_{\chi_{[1/m, 2\pi-1/m]}}|^2 \to 0$$
by the dominated convergence theorem as \( m \to \infty \). Recall that any function in \( L^2([1/m, 2\pi - 1/m]) \) can be approximated in \( L^2 \) by continuous functions which have support in the interval \([1/m, 2\pi - 1/m]\). By what we showed above, a continuous function with support in \([1/m, 2\pi - 1/m]\) can be approximated uniformly on \([0, 2\pi]\) by elements of \( \mathcal{F} \). Finally, if \( g \) is continuous on \([0, 2\pi]\) and \( g_m \to g \) uniformly on \([0, 2\pi]\), then \( g_m \to g \) in \( L^2([0, 2\pi]) \) by the dominated convergence theorem. Putting all this together proves that \( \mathcal{F} \) is dense in \( L^2([0, 2\pi]) \).

It remains to show the completeness of the \( u_n \). If \( f \) is orthogonal to each \( u_n \), then it is orthogonal to every finite linear combination, that is, to every element of \( \mathcal{F} \). Since \( \mathcal{F} \) is dense in \( L^2([0, 2\pi]) \), we can find \( f_n \in \mathcal{F} \) tending to \( f \) in \( L^2 \). Then

\[
\|f\|^2 = |\langle f, \mathcal{F} \rangle| \leq |\langle f - f_n, \mathcal{F} \rangle| + |\langle f_n, \mathcal{F} \rangle|.
\]

The second term on the right of the inequality sign is 0. The first term on the right of the inequality sign is bounded by \( \|f - f_n\| \|f\| \) by the Cauchy-Schwarz inequality, and this tends to 0 as \( n \to \infty \). Therefore \( \|f\|^2 = 0 \), or \( f = 0 \), hence the \( \{u_n\} \) are complete. Therefore \( \{u_n\} \) is a complete orthonormal system.

Given \( f \) in \( L^2([0, 2\pi]) \), write

\[
c_n = \langle f, u_n \rangle = \int_0^{2\pi} f u_n \, dx = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)e^{-inx} \, dx,
\]

the Fourier coefficients of \( f \). Parseval’s identity says that

\[
\|f\|^2 = \sum_n |c_n|^2.
\]

For any \( f \) in \( L^2 \) we also have

\[
\sum_{|n| \leq N} c_n u_n \to f
\]

as \( N \to \infty \) in the sense that

\[
\left\| f - \sum_{|n| \leq N} c_n u_n \right\|_2 \to 0
\]
4.7. THE RADON-NIKODYM THEOREM

as $N \to \infty$.

Using $e^{inx} = \cos nx + i \sin nx$, we have

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = A_0 + \sum_{n=1}^{\infty} B_n \cos nx + \sum_{n=1}^{\infty} C_n \sin nx,$$

where $A_0 = c_0$, $B_n = c_n + c_{-n}$, and $C_n = i(c_n - c_{-n})$. Conversely, using $\cos nx = (e^{inx} + e^{-inx})/2$ and $\sin nx = (e^{inx} - e^{-inx})/2i$,

$$A_0 + \sum_{n=1}^{\infty} B_n \cos nx + \sum_{n=1}^{\infty} C_n \sin nx = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

if we let $c_0 = A_0$, $c_n = B_n/2 + C_n/2i$ for $n > 0$ and $c_n = B_n/2 - C_n/2i$ for $n < 0$. Thus results involving the $u_n$ can be transferred to results for series of sines and cosines and vice versa.

If $c_n$ are the Fourier coefficients of $f$, then

$$\sum_{-N}^{N} c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) k_N(\theta) d\theta,$$

where

$$k_N(\theta) = \sum_{n=-N}^{N} e^{-in\theta} = e^{iN\theta} \left( \frac{e^{-(2N+1)i\theta} - 1}{e^{-i\theta} - 1} \right)$$

$$= \frac{e^{-(N+1)i\theta} - e^{iN\theta}}{e^{-i\theta} - 1}$$

$$= \frac{e^{-(N+\frac{1}{2})i\theta} - e^{i(N+\frac{1}{2})\theta}}{e^{-i\theta/2} - e^{i\theta/2}}$$

$$= \frac{\sin(N + \frac{1}{2})\theta}{\sin \theta/2}.$$ 

4.7 The Radon-Nikodym theorem

We can use Hilbert space techniques to give an alternate proof of the Radon-Nikodym theorem.
Suppose $\mu$ and $\nu$ are finite measures on a space $S$ and we have the condition $\nu(A) \leq \mu(A)$ for all measurable $A$. For $f \in L^2(\mu)$, define

$$\ell(f) = \int f \, d\nu.$$ 

Our condition implies $\int h \, d\nu \leq \int h \, d\mu$ if $h \geq 0$. We use this with $h = |f|$, use Cauchy-Schwarz, and obtain

$$|\ell(f)| = \left| \int f \, d\nu \right| \leq \int |f| \, d\nu \leq \int |f| \, d\mu \leq \left( \mu(S) \right)^{1/2} \left( \int f^2 \, d\mu \right)^{1/2} \leq c\|f\|_{L^2(\mu)}.$$ 

There exists $g$ such that $\ell(f) = \langle f, g \rangle$, which translates to

$$\int f \, d\nu = \int fg \, d\mu.$$ 

Letting $f = \chi_A$, we get $\nu(A) = \int_A g \, d\mu$.

If $\nu$ is absolutely continuous with respect to $\mu$, we let $\rho = \mu + \nu$ and apply the above to $\nu$ and $\rho$ and also to $\mu$ and $\rho$. The absolute continuity implies that $d\mu/d\rho > 0$ a.e., and we use

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\rho} / \frac{d\mu}{d\rho}.$$ 

### 4.8 The Dirichlet problem

Let $D$ be a bounded domain in $\mathbb{R}^n$, contained in $B(0, K)$, say, where this is the ball of radius $K$ about $0$. Let $\langle f, g \rangle$ be the usual $L^2$ scalar product for real valued functions. It is easy to see that if $C_0^\infty(D)$ is the set of $C^\infty$ functions that vanish on the boundary of $D$, then the completion of $C^\infty(D)$ with respect to the $L^2$ norm is simply $L^2(D)$. Define

$$\mathcal{E}(f, g) = \int_D \langle \nabla f(x), \nabla g(x) \rangle \, dx.$$ 

Clearly $\mathcal{E}$ is bilinear and symmetric.
4.8. THE DIRICHLET PROBLEM

If we start with

\[ f(x_1, \ldots, x_n) = \int_{-K}^{x_1} \frac{\partial f}{\partial x_1}(y, x_2, \ldots, x_n) \, dy \]

and apply Cauchy-Schwarz, we have

\[ |f(x_1, \ldots, x_n)|^2 \leq \int_{-K}^{K} 1 \, dy \int_{-K}^{K} |\nabla f(y, x_2, \ldots, x_n)|^2 \, dy. \]

Integrating over \((x_2, \ldots, x_n) \in [-K,K]^{n-1}\) we obtain

\[ \int_D |f(x)|^2 \, dx \leq c \int_D |\nabla f(x)|^2 \, dx, \]

or in other words,

\[ \langle f, f \rangle \leq c \mathcal{E}(f, f). \]

If \(\mathcal{E}(f, f) = 0\), then \(\langle f, f \rangle = 0\), and so \(f = 0\) (a.e., of course). This proves that \(\mathcal{E}\) is positive. We let \(H_0^1\) be the completion of \(C_0^\infty(D)\) with respect to the norm induced by \(\mathcal{E}\). The superscript 1 refers to the fact we are working with first derivatives, the subscript 0 to the fact that our functions vanish on the boundary. \(\mathcal{E}\) is an example of a Dirichlet form.

Recall the divergence theorem:

\[ \int_{\partial D} (F, n) \, d\sigma = \int_D \text{div} \, F \, dx, \]

where \(D\) is a reasonably smooth domain, \(\partial D\) is the boundary of \(D\), \(n\) is the outward pointing unit normal, and \(\sigma\) is surface measure. In three dimensions, this is also known as Gauss’ theorem, and along with Green’s theorem and Stokes’ theorem are consequences of the fundamental theorem of calculus.

If we apply the divergence theorem to \(F = u \nabla v\), then

\[ \frac{\partial}{\partial x_1} F_1 = \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} + u \frac{\partial^2 v}{\partial x_1^2}, \]

and so

\[ \text{div} \, F = \langle \nabla u, \nabla v \rangle + u \Delta v, \]
where $\Delta v$ is the Laplacian. Also,
\[ \langle \text{div } F, n \rangle = u \frac{\partial v}{\partial n}, \]
where $\frac{\partial v}{\partial n}$ is the normal derivative of $v$. We then get Green’s first identity:
\[ \int_D u \Delta v + \int_D \langle \nabla u, \nabla v \rangle = \int_{\partial D} u \frac{\partial v}{\partial n}. \]

Our goal is to solve the equation $\Delta v = g$ in $D$ with $v = 0$ on the boundary of $D$. This is Poisson’s equation, while the Dirichlet problem more properly refers to the equation $\Delta v = 0$ in $D$ with $v$ equal to some pre-specified function $f$ on the boundary of $D$.

If we have a solution $v$ and $u \in C^\infty_0(D)$, then by Green’s identity we get
\[ \int_D u(x)g(x) \, dx = -\int_D \langle \nabla u(x), \nabla v(x) \rangle \, dx. \]
So one way of formulating a (weak) solution to Poisson’s equation is: given $g \in L^2(D)$, find $v \in H^1_0$ such that
\[ \mathcal{E}(u, v) = -\langle u, g \rangle \]
for all $u \in C^\infty_0(D)$.

After all this, it is easy to find a weak solution to the Poisson equation. Suppose $g \in H^1_0$. Define $\ell(u) = -\langle u, g \rangle$. Then
\[ |\ell(u)| \leq \|g\| \|u\| \leq c\|g\| \mathcal{E}(u, u)^{1/2}. \]
By the Riesz representation theorem for Hilbert spaces, there exists $v \in H^1_0$ such that $\ell(u) = \mathcal{E}(u, v)$ for all $u$. So
\[ \mathcal{E}(u, v) = \ell(u) = -\langle u, g \rangle, \]
and $v$ is the desired solution.
Chapter 5

Duals of normed linear spaces

5.1 Bounded linear functionals

If $X$ is a normed linear space, a linear functional $\ell$ is a linear map from $X$ to $F$, the field of scalars. $\ell$ is continuous if $\|x_n - x\| \to 0$ implies $\ell(x_n) \to \ell(x)$. $\ell$ is bounded if there exists $c$ such that $|\ell(x)| \leq c\|x\|$ for all $x$.

Theorem 5.1 A linear functional $\ell$ is continuous if and only if it is bounded.

This is a special case of Proposition 2.1.

The collection of all continuous linear functionals of $X$ is called the dual of $X$, written $X'$ or $X^*$.

Note $N_\ell = \ell^{-1}([0])$ is closed, since $\ell$ is continuous.

Define

$$\|\ell\| = \sup_{\|x\| \neq 0} \frac{|\ell(x)|}{\|x\|}.$$ 

By linearity, this is the same as $\sup_{\|x\|=1} |\ell(x)|$.

Proposition 5.2 $X^*$ is a Banach space.

This is Proposition 3.2.
5.2 Extensions of bounded linear functionals

**Proposition 5.3** Let $X$ be a normed linear space, $Y$ a subspace, $\ell$ a linear functional on $Y$ with $|\ell(y)| \leq c\|y\|$ for all $y \in Y$. Then $\ell$ can be extended to a bounded linear functional on $X$ with the same bound on $X$ as on $Y$.

**Proof.** This is the Hahn-Banach theorem with $p(x) = c\|x\|$. \hfill $\square$

$y_1, \ldots, y_N$ are said to be linearly independent if $\sum_{i=1}^N c_i y_i = 0$ implies all the $c_i$ are zero.

**Theorem 5.4** Suppose $y_1, \ldots, y_N$ are linearly independent and $a_1, \ldots, a_N$ are scalars. Then there exists a bounded linear functional $\ell$ such that $\ell(y_j) = a_j$.

**Proof.** Let $Y$ be the span of $y_1, \ldots, y_N$. If $y \in Y$, then $y$ can be written as $\sum b_j y_j$ in only one way, for if $\sum b'_j y_j$ is another way, then

$$\sum (b_j - b'_j) y_j = y - y = 0,$$

and so $b_j = b'_j$ for all $j$. Define

$$\ell\left(\sum b_j y_j\right) = \sum a_j b_j.$$

Now use the preceding theorem to extend $\ell$ to all of $X$. \hfill $\square$

**Theorem 5.5** If $X$ is a normed linear space, then

$$\|y\| = \max_{\|\ell\| = 1} |\ell(y)|.$$

**Proof.** $|\ell(y)| \leq \|\ell\| \|y\|$, so the maximum on the right hand side is less than or equal to $y$.

If $y \in X$, let $Y = \{ay\}$ and define $\ell(ay) = a\|y\|$. Then the norm of $\ell$ on $Y$ is 1. Now extend $\ell$ to all of $X$ so as to have norm 1. \hfill $\square$
**Theorem 5.6** *(Spanning criterion)* Let $Y$ be the closed linear span of $\{y_j\}$. Suppose that whenever $\ell$ is a bounded linear functional such that $\ell(y_j) = 0$ for all $j$, then $\ell(z) = 0$. We conclude that $z \in Y$.

**Proof.** If $\ell(y_j) = 0$ for all $j$, then $\ell(y)$ for all $y$ of the form $\sum a_j y_j$, and by continuity of $\ell$, for all $y \in Y$.

Suppose $z \notin Y$. Then

$$\inf_{y \in Y} \|z - y\| = d > 0.$$ 

Let $Z = \{y + az : y \in Y\}$. Define $\ell_0$ on $Z$ by $\ell_0(y + az) = a$. We note that $\ell$ is well defined, since if $y + az = y' + a'z$, then $(a' - a)z = y - y'$. Since $z \notin Y$ and $y - y' \in Y$, then $a' - a = 0$, and it follows that $y = y'$. Then

$$\|y + az\| = |a| \| - \frac{y}{a} + z \| \geq d \|a\|.$$ 

Therefore on $Z$, $\ell_0$ is bounded by $d^{-1}$. Extend $\ell_0$ to all of $X$. But then $\ell_0(y_j) = 0$ while $\ell_0(z) = 1$.

---

### 5.3 Uniform boundedness

**Theorem 5.7** Let $X$ be a Banach space and $\{\ell_\nu\}$ a collection of bounded functionals such that $|\ell_\nu(x)| \leq M(x)$ for all $\nu$ and each $x$. Then there exists $c$ such that $\|\ell_\nu\| \leq c$.

In other words, if the $\ell_\nu$ are bounded pointwise, they are bounded uniformly.

This is just a special case of the uniform boundedness principle (Banach-Steinhaus theorem).

### 5.4 Reflexive spaces

If $x \in X$, define the linear functional $L_x$ on $X^*$ by

$$L_x(\ell) = \ell(x).$$
It is clear that $L_x$ is linear. Since

$$|L_x(\ell)| = |\ell(x)| \leq \|\ell\| \|x\|,$$

we see that $\|L_x\| \leq \|x\|$. Define $\ell'$ on $Y = \{ax\}$ by $\ell'(ax) = a\|x\|$. Note the norm of $\ell'$ on $Y$ is 1. Use Hahn-Banach to extend this to a linear functional on $X$. Then

$$|L_x(\ell')| = |\ell'(x)| = \|x\|,$$

and since $\|\ell'\| = 1$, we conclude $\|L_x\| = \|x\|$. So we can isomorphically embed $X$ into $X^{**}$.

**Corollary 5.8** Let $X$ be a normed linear space, $\{x_\nu\}$ a subset such that for all $\ell \in X^*$ we have

$$|\ell(x_\nu)| \leq M(\ell) \quad \text{for all } x_\nu.$$

Then there exists $c$ such that $|x_\nu| \leq c$ for all $x_\nu$.

**Proof.** Write $L_\nu(\ell) = \ell(x_\nu)$. So each $x_\nu$ acts as a bounded linear functional on $X^*$.

A Banach space is reflexive if $X^{**} = X$.

If $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the dual of $L^p$ is isomorphic to $L^q$. Hence the $L^p$ spaces are reflexive.

**Theorem 5.9** Hilbert spaces are reflexive.

**Proof.** Recall $X^* = X$, and the result follows from this. To see $X^* = X$, if $\ell$ is a linear functional, there exists $y$ such that $\ell(x) = \langle x, y \rangle$ for all $x$. If we show $|\ell| = \|y\|$, this gives an isometry between $X$ and $X^*$. By Cauchy-Schwarz, $|\ell(x)| = |\langle x, y \rangle| \leq \|x\| \|y\|$, so $\|\ell\| \leq \|y\|$. Taking $x = y$, $\ell(y) = \|y\|^2$, hence $\|\ell\| \geq \|y\|$.

**Proposition 5.10** If $X$ is a normed linear space over $\mathbb{C}$ and $X^*$ is separable, then $X$ is separable.
5.5. WEAK CONVERGENCE

Proof. Since $X^*$ is separable, there is a countable dense subset $\{\ell_n\}$. Recall $\|\ell_n\| = \sup_{\|x\|=1} |\ell_n(x)|$. So for each $n$ there exists $x_n \in X$ such that $\|x_n\| = 1$ and $\ell_n(x_n) > \frac{1}{2} \|\ell_n\|$

We claim the linear span of $\{x_n\}$ is dense in $X$. To prove this, we start by showing that if $\ell$ is a linear functional on $X$ that vanishes on $\{x_n\}$, then $\ell$ vanishes identically.

Suppose not and that there exists $\ell$ such that $\ell(x_n) = 0$ for all $x_n$ but $\ell \neq 0$. We can normalize so that $\|\ell\| = 1$. Since the $\ell_n$ are dense in $X^*$, there exists $\ell_n$ such that $\|\ell - \ell_n\| < 1/3$. Therefore $\|\ell_n\| > 2/3$. Then

$$
\frac{1}{3} > |(\ell - \ell_n)(x_n)| = |\ell_n(x_n)| > \frac{1}{2} \|\ell_n\| > \frac{1}{2} \cdot \frac{2}{3},
$$

a contradiction.

If $z \in X$, any linear functional that vanishes on $\{x_n\}$ is identically zero. By the spanning criterion, $z$ is in the closed linear span of the $x_n$, and thus the closed linear span is all of $X$. Then the set of finite linear combinations of the $x_n$ where all the coefficients have rational coordinates, is also dense in $X$, and is countable.

By the Riesz representation theorem from real analysis, the dual of $X = C([0,1])$ is the set of finite signed measures on $[0,1]$. $X$ is separable, but $X^*$ is not, since $\|\delta_x - \delta_y\| = 2$ whenever $x \neq y$. It follows that $X$ is not reflexive: if it were, we would have $X^{**}$ separable, but $X^*$ not, contradicting the previous proposition.

5.5 Weak convergence

Let $X$ be a normed linear space. We say $x_n$ converges to $x$ weakly, written $w - \lim x_n = x$ or $x_n \xrightarrow{w} x$ if $\ell(x_n) \to \ell(x)$ for all $\ell \in X^*$.

$x_n$ converges to $x$ strongly, written $s - \lim x_n = x$ or $x_n \xrightarrow{s} x$ if $\|x_n - x\| \to 0$.

Strong convergence implies weak convergence because

$$
|\ell(x_n) - \ell(x)| = |\ell(x_n - x)| \leq \|\ell\| \|x_n - x\| \to 0.
$$
As an example where we have weak convergence but not strong convergence, let $X = \ell^2$ and let $e_n$ be the element whose $n^{th}$ coordinate is 1 and all other coordinate coordinates are 0. Since $\|e_n\| = 1$, then $e_n$ does not converge strongly to 0. But it does converge weakly to 0. To see this, if $\ell$ is any bounded linear functional on $X$, then $\ell$ is of the form $\ell(x) = \langle x, y \rangle$ for some $y \in X$, which means $y = (b_1, b_2, \ldots)$ with $\sum_j |b_j|^2 < \infty$. In particular, $b_j \to 0$. Then $\ell(e_n) = b_n \to 0 = \ell(0)$.

This example stretches to any Hilbert space. If $\{x_n\}$ is an orthonormal sequence in the space, $\ell(x_n) = \langle x_n, y \rangle$ for some $y$. By Bessel’s inequality, $\sum |\langle x_n, y \rangle|^2 \leq \|y\|^2$, so $\langle x_n, y \rangle \to 0$.

**Proposition 5.11** Let $X$ be a normed linear space and suppose $x_n$ converges weakly to $x$. Then $\|x\| \leq \liminf \|x_n\|$.

**Proof.** There exists $\ell$ such that $\|\ell\| = 1$ and $|\ell(x)| = \|x\|$. Then $|\ell(x)| = \lim |\ell(x_n)|$ and $|\ell(x_n)| \leq \|\ell\| \|x_n\| = \|x_n\|$.

### 5.6 Weak$^*$ convergence

We say $u_n \in X^*$ is weak$^*$ convergent to $u$ if $\lim u_n(x) = u(x)$ for all $x \in X$.

If $X$ is reflexive, then weak$^*$ convergence is the same as weak convergence.

Weak convergence in probability theory can be identified as weak$^*$ convergence in functional analysis.

As an example, if $S$ is a compact Hausdorff space and $X = C(S)$, then $X^*$ is the collection of finite signed measures. Saying a sequence of measures $\nu_n$ converges in the weak$^*$ sense means that $\int f \, d\nu_n$ converges for each continuous function $f$.

A set is weak$^*$ sequentially compact if every sequence in the set has a subsequence which converges in the weak$^*$ sense to an element of the set.
5.7 Approximating the $\delta$ function

Let $k_n$ be a sequence of integrable functions on the interval $[{-1,1}]$. They approximate the $\delta$ function (or are an approximation to the identity) if

$$\int_{-1}^{1} f(t)k_n(t) \, dt \to f(0) \quad (5.1)$$

as $n \to \infty$ for all $f$ continuous on $[-1,1]$.

As an example, we could take $k_n(t) = n\chi_{[0,1/n]}(t)$.

**Theorem 5.12** $k_n$ approximates the $\delta$ function on $[-1,1]$ if and only if the following three properties hold.

1. $\int_{-1}^{1} k_n(t) \, dt \to 1$.
2. If $g$ is $C^\infty$ and 0 in a neighborhood of 0, then
   $$\int_{-1}^{1} g(t)k_n(t) \, dt \to 0$$
as $n \to \infty$.
3. There exists $c$ such that $\int_{-1}^{1} |k_n(t)| \, dt \leq c$ for all $n$.

**Proof.** If (1)–(3) hold, write $f = (f - f(0)) + f(0)$, and we may suppose without loss of generality that $f(0) = 0$. Choose $g \in C^\infty$ such that $g$ is 0 in a neighborhood of 0 and $\|g - f\| < \varepsilon$. We have

$$\left| \int_{-1}^{1} (f-g)k_n \right| \leq \varepsilon \int |k_n| \leq c\varepsilon$$

and

$$\int gk_n \to 0.$$

So

$$\limsup \left| \int f_kn \right| \leq c\varepsilon.$$

Since $\varepsilon$ is arbitrary, this shows (5.1).
If (5.1) holds, then (1) holds by taking \( f \) identically 1 and (2) holds by taking \( f \) equal to \( g \). So we must show (3). If \( X \) is the set \( C \) of continuous functions on \([-1, 1]\), then \( X^* \) is the collection of finite signed measures (by the Riesz representation theorem of real analysis). Let \( m_n(dt) = k_n(t) \, dt \) and \( m_0(dt) = \delta_0(dt) \). Then (5.1) says that \( m_n(f) \to m_0(f) \) for all \( f \in C \), or \( m_n \) converges to \( m_0 \) in the sense of weak-* convergence. \( \limsup |m_n(f)| < \infty \), so \( |m_n(f)| \leq M(f) \) for all \( f \), and by the uniform boundedness principle, \( \|m_n\| \leq c \). Note by the proof of the Riesz representation theorem, \( \|m_n\| \) is the total mass of \( m_n \), which is \( \int_{-1}^{1} |k_n(t)| \, dt \).

From the approximation of the \( \delta \)-function, we can show that there exists a continuous function \( f \) whose Fourier series diverges at 0.

We look at the set of continuous functions on \( S^1 \), the unit circle. We say \( f(\theta) \) has Fourier series \( \sum_{-\infty}^{\infty} a_n e^{in\theta} \) with

\[
a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} \, d\theta.
\]

The Fourier series converges at 0 if

\[
\lim_{N \to \infty} \sum_{-N}^{N} a_n = f(0).
\]

Recall from Section 4.6 that

\[
\sum_{-N}^{N} a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) k_N(\theta) \, d\theta,
\]

where

\[
k_N(\theta) = \frac{\sin(N + \frac{1}{2})\theta}{\sin \theta/2}.
\]

So the convergence of the Fourier series at 0 is equivalent to \( k_N \) being an approximation to the \( \delta \) function. And if (3) fails, then \( \sum_{-N}^{N} a_n \) does not converge for some \( f \).

Since \( |\sin x| \leq |x| \), then

\[
\left| \frac{1}{\sin x/2} \right| \geq \frac{2}{|x|}.
\]
and therefore
\[
\int_{-\pi}^{\pi} |k_N(\theta)| \, d\theta \geq 2 \int_{-\pi}^{\pi} |\sin(N + \frac{1}{2})\theta| \frac{d\theta}{|\theta|}
\]
\[
= 4 \int_0^{(N+\frac{1}{2})\pi} |\sin x| \frac{dx}{x}
\]
\[
\geq c \log N
\]

5.8 Weak and weak* topologies

The weak topology is the coarsest topology (i.e., fewest sets) in which all bounded linear functionals are continuous.

Bounded linear functionals are continuous in the usual norm topology (also called the strong topology), so the weak topology is coarser than the strong topology.

Recall that a topology is a collection of sets that contain \( \emptyset \) and \( X \) and which is closed under arbitrary unions and finite intersections. Elements of the topology are called open sets. A subcollection of a topology is a basis if every open set can be written as the union of elements of the subcollection. A subcollection of the topology is a subbasis is the set of finite intersections of elements of the subcollection is a basis.

Let \( S \) be the collection of sets of the form
\[
\{ x : a < \ell(x) < b \}
\]
for reals \( a < b \) and \( \ell \in X^* \).

Proposition 5.13 If we are considering real-valued bounded linear functionals, then \( S \) is a subbasis for the weak topology.

Proof. Since \( \ell \) is continuous in the weak topology and \( \{ x : a < \ell(x) < b \} = \ell^{-1}((a,b)) \), then \( S \) is a subcollection of the weak topology.

Any topology with respect to which all bounded linear functionals are continuous must contain \( S \), and the smallest such is the topology generated by \( S \). \( \square \)
Finite intersections of sets in $S$ are unbounded if $X$ is infinite dimensional, so every open set in the weak topology when $X$ is infinite dimensional is unbounded. The open unit ball ($\{x : \|x\| < 1\}$) is a set that is open in the strong topology but not the weak topology.

### 5.9 The Alaoglu theorem

Consider $X^*$, where $X$ is a Banach space. For $x \in X$, define $L_x : X^* \to \mathbb{R}$ by $L_x(\ell) = \ell(x)$. The weak$^*$ topology is the coarsest topology on $X^*$ with respect to which all the $L_x$ with $x \in X$ are continuous. As above, a subbasis for the weak$^*$ topology is the collection of sets $\{\ell : a < \ell(x) < b\}$, where $a < b \in \mathbb{R}$ and $x \in X$.

**Lemma 5.14** Suppose $f$ is a function from a topological space $(X, T)$ to a topological space $(Y, U)$. Let $S$ be a subbase for $Y$. If $f^{-1}(G) \in T$ whenever $G \in S$, then $f$ is continuous.

**Proof.** Let $B$ be the collection of finite intersections of elements of $S$. By the definition of subbase, $B$ is a base for $Y$. Suppose $H = G_1 \cap G_2 \cap \cdots \cap G_n$ with each $G_i \in S$. Since $f^{-1}(H) = f^{-1}(G_1) \cap \cdots \cap f^{-1}(G_n)$ and $T$ is closed under the operation of finite intersections, then $f^{-1}(H) \in T$. If $J$ is an open subset of $Y$, then $J = \bigcup_{\alpha \in I} H_\alpha$, where $I$ is a non-empty index set and each $H_\alpha \in B$. Then $f^{-1}(J) = \bigcup_{\alpha \in I} f^{-1}(H_\alpha)$, which proves $f^{-1}(J) \in T$. That is what we needed to show.

Recall that the product topology on $\prod_{\alpha \in I} X_\alpha$ is the one generated by the sets $\{\pi_\alpha^{-1}(G) : \alpha \in I, G \text{ open in } X_\alpha\}$. Here $\pi_\beta$ is the projection of $\prod_{\alpha \in I} X_\alpha$ onto $X_\beta$, that is, if $x = \{x_\alpha\}$ so that $x_\alpha$ is the $\alpha^{th}$ coordinate of $x$, then $\pi_\beta(x) = x_\beta$.

**Theorem 5.15** (Alaoglu theorem) The closed unit ball $B$ in $X^*$ is compact in the weak$^*$ topology.

There is a connection with the Prohorov theorem of probability theory. Let $X = C(S)$ where $S$ is a compact Hausdorff space. If $\mu_n$ is a sequence of
probability measures, then the $\mu_n$ are elements of the closed unit ball in $X^*$. The Alaoglu theorem implies there must be a subsequence which converges in the weak* sense.

**Proof.** If $\ell \in B$, then $|\ell(x)| \leq \|x\|$. Let

$$P = \prod_{x \in X} I_x,$$

where $I_x = [-\|x\|, \|x\|]$. Map $B$ into $P$ by setting $\varphi(\ell) = \{\ell(x)\}$, the function whose $x^{th}$ coordinate is $\ell(x)$.

We will show that $\varphi$ is one-to-one, continuous, and onto $\varphi(B)$. Hence to show $B$ is compact, it suffices to show $\varphi(B)$ is compact. By Tychonov’s theorem, $P$ is compact. So it suffices to show that $\varphi(B)$ is closed.

That $\varphi$ is one-to-one is obvious. To show $\varphi$ is continuous, we use the lemma and show that $\varphi^{-1}(G)$ is open when $G$ is a subbasic open set in the product topology. The collection of sets $\{\{f \in P : a < f(x) < b\}, x \in X, a < b \in \mathbb{R}\}$ is a subbasis for the product topology. If $G$ is such a set, then

$$\varphi^{-1}(G) = \{\ell \in B : a < \ell(x) < b\} = \{\ell \in B : a < L_x(\ell) < b\}$$

is open in the weak* topology. To show $\varphi$ is open, we need to show, using the lemma, that $\varphi(G)$ is relatively open in $P$ if $G$ is of the form $\{\ell : a < \ell(x) < b\}$. But then $\varphi(G) = \{f \in \varphi(B) : a < f(x) < b\}$ is relatively open in $P$.

Let $p$ be in the closure of $\varphi(B)$. We will show $p = \varphi(\ell)$ for some $\ell \in B$. Fix $x, y \in X$ and $c \in \mathbb{R}$. If we show that $p(x + y) = p(x) + p(y)$ and $p(cx) = cp(x)$, then $p$ will be equal to $\varphi(\ell)$ for a linear functional on $X$. Since $p \in \varphi(B) \subset P$, then $|p(x)| \leq \|x\|$ for each $x$, and $p$ will be equal to $\varphi(\ell)$ for $\ell \in B$, which finishes the proof.

We prove that $p(x + y) = p(x) + p(y)$. For each $m$, the set

$$\{f \in P : p(x) - 2^{-m} < f(x) < p(x) + 2^{-m}, p(y) - 2^{-m} < f(y) < p(y) + 2^{-m},$$

$$p(x + y) - 2^{-m} < f(x + y) < p(x + y) + 2^{-m}\}$$

is the intersection of three subbasic sets in the product topology, and hence is open. Since $p$ is a limit point of $\varphi(B)$, there exists $q_m$ in $B$ such that $\varphi(q_m)$ is in this set. We conclude $\varphi(q_m)(x) \to p(x)$, and similarly with $x$ replaced by $y$ and by $x + y$. Since $q_m$ is a bounded linear functional,
\[ q_m(x + y) = q_m(x) + q_m(y). \] Passing to the limit, \( p(x + y) = p(x) + p(y) \) as required. \( \square \)

### 5.10 Transpose of a bounded linear map

Suppose \( M : X \to U \). We define the \textit{transpose} \( M' \) (or \( M^* \)) as follows. \( M' : U^* \to X^* \). If \( \ell \in U^* \), \( \ell(Mx) \) is a linear functional on \( X \), and we call this linear functional \( M'\ell \).

Sometimes the notation \( \ell(u) = \langle u, \ell \rangle \). With this notation,

\[
\langle Mx, \ell \rangle = \ell(Mx) = M'\ell(x) = \langle x, M'\ell \rangle,
\]

which justifies the name adjoint or transpose.

**Proposition 5.16**

(1) \( M' \) is bounded and \( \| M' \| = \| M \|. \)

(2) \((M + N)' = M' + N'. \)

(3) If \( M : X \to U \) and \( N : U \to W \) are linear, then \((NM)' = M'N'. \)

**Proof.** (1) \( \| M' \| = \sup_{\| \ell \|=1} \| M'\ell \| \) (recall \( M'\ell \in X^* \)) and

\[
\| M'\ell \| = \sup_{\| x \|=1} |M'\ell(x)| = \sup_{\| x \|=1} |\ell(Mx)|.
\]

So

\[
\| M' \| = \sup_{\| \ell \|=1, \| x \|=1} |\ell(Mx)| = \sup_{\| x \|=1} \| Mx \| = \| M \|.
\]

(2) is easy.

To prove (3), we write

\[
m(NM)x = (N'm)(Mx) = (M'N'm)(x),
\]

so \( m(NM) = M'N'm \), and our result follows. \( \square \)
Chapter 6

Convexity

6.1 Locally convex topological spaces

We look at topologies other than those defined in terms of linear functionals.

A topological linear space is a linear space over the reals with a Hausdorff topology satisfying

1) \((x, y) \rightarrow x + y\) is a continuous mapping from \(X \times X \rightarrow \mathbb{R}\).
2) \((k, x) \rightarrow kx\) is a continuous mapping from \(F \times X \rightarrow X\).

If in addition,

3) Every open set containing the origin contains a convex open set containing the origin

then we have a locally convex topological linear space (LCT).

It is an exercise to show that the weak and weak* topologies are locally convex (i.e., satisfy (3)).

Topological linear spaces just satisfying (1) and (2) are not satisfactory, because there may not be enough linear functionals. If we have (3), then we can use the hyperplane separation theorem to produce linear functionals. Thus convexity is important.

**Proposition 6.1** In a LCT linear space,

1) if \(T\) is open, so are \(T - x_0, kT, \) and \(-T\).
2) Every point of an open set \(T\) is interior to \(T\).
Proof. (1) The map \( \varphi : y \to x_0 + y \) is the composition of the maps \( y \to (x_0, y) \) and \( (x_0, y) \to x_0 + y \). If \( A \) and \( B \) are open in \( X \), the inverse image of \( A \times B \) under the first map is \( B \) if \( x_0 \in A \) and \( \emptyset \) if \( x_0 \notin A \), which is open in either case. By Lemma 5.14, since the inverse image of subbasic sets is open, the first map is continuous. Therefore \( \varphi \) is continuous. \( T - x \) is the inverse image of \( T \) under \( \varphi \). \( kT \) is similar.

(2) Suppose \( 0 \in T \). Fix \( x \in X \). \( k \to kx \) is continuous, so \( \{ k : kx \in T \} \) is open. Since \( 0 \in T \), then \( 0 \) is in this set, and therefore there exists an interval about \( 0 \) such that \( kx \in T \) if \( k \) is in this interval. This is true for all \( x \), and therefore \( 0 \) is an interior point. Use translation if the point we are interested in is other than \( 0 \).

\[ \square \]

6.2 Separation of points

In a LCT space, we can talk about continuous linear functionals, but not bounded linear functionals.

Proposition 6.2 Continuous linear functionals in a LCT linear space \( X \) separate points: if \( y \neq z \), there exists \( \ell \) such that \( \ell(y) \neq \ell(z) \).

Proof. Without loss of generality assume \( y = 0 \). There exists an open set \( T \) that contains 0 but not \( z \), since the topology is Hausdorff. We can take \( T \) to be convex. By looking at \( T \cap (-T) \), we may assume that \( T \) is symmetric, that is, \( T = -T \). \( 0 \in T \) is interior, so the gauge function \( p_T \) is finite. Recall \( p_T(u) < 1 \) if \( u \in T \).

By the hyperplane separation theorem, there exists \( \ell \) such that \( \ell(z) = 1 \) and \( \ell(x) \leq p_T(x) \) for all \( x \). Since \( \ell(y) = \ell(0) \), then \( \ell \) separates.

It remains to prove that \( \ell \) is continuous.

We first show \( H = \{ w : \ell(w) < c \} \) is open. If \( w \in H \) and \( u \in T \), let \( r = c - \ell(w) \). Then

\[ \ell(w + ru) = \ell(w) + r\ell(u) \leq \ell(w) + r p_T(u) < \ell(w) + c - \ell(w) = c, \]

so \( w + ru \in H \). Therefore the inverse image under \( \ell \) of \( (-\infty, c) \) contains \( w + rT \), an open neighborhood of \( w \). Hence \( H \) is open.
A similar argument shows $J = \{ w : \ell(w) > d \}$ is open. Let $w \in J$, $u \in T$, and $r = d - \ell(w)$. Since $r$ is negative,

$$\ell(w - ru) = \ell(w) + r\ell(-u) \geq \ell(w) + rp_T(-u) > \ell(w) + r = d,$$

so $w - ru \in J$. As above, $J$ is open.

Since the inverse images of $(-\infty, c)$ and $(d, \infty)$ are open and the collection of such sets is a subbasis for the topology of the real line, $\ell$ is continuous. □

Using the extended hyperplane separation theorem, we have

**Corollary 6.3** Let $K$ be a closed convex set in a LCT space, $z \notin K$. There exists a continuous linear functional $\ell$ such that $\ell(y) \leq c$ for $y \in K$ and $\ell(z) > c$.

### 6.3 Krein-Milman theorem

We will use the easy fact that if $E$ is an extreme subset of a convex set $K$ and $p$ is an extreme point for $E$, then $p$ is an extreme point for $K$.

**Theorem 6.4** (Krein-Milman) Let $K$ be a nonempty, compact, convex subset of a LCT linear space $X$. Then

1. $K$ has at least one extreme point.
2. $K$ is the closure of the convex hull of its extreme points.

**Proof.** (1) Let $\{E_j\}$ be the collection of all nonempty closed extreme subsets of $K$. It is nonempty because it contains $K$. We partially order by reverse inclusion: $E \leq F$ if $E \supset F$. We show that if we have a totally ordered subcollection, $\cap_j E_j$ is an upper bound with respect to $\leq$, and hence by Zorn’s lemma a maximal element, which means that it contains no strictly smaller extreme subset.

The intersection of any finite totally ordered subcollection $\{E_j\}$ is just the smallest one. Since $K$ is compact, by the finite intersection property, the intersection of any totally ordered subcollection is nonempty. (If $\cap E_j = \emptyset$, then $\{E_j\}$ forms an open cover of $K$, so there is a finite subcover, and
then the intersection of those finitely many \( E_j \) is empty, a contradiction.) The intersection of closed sets is closed, and it is easy to check that the intersection of extreme sets is extreme.

By Zorn’s lemma, there is a maximal element \( E \), an extreme subset that contains no strictly smaller extreme subset. We claim \( E \) is a single point. If not, there exists a continuous linear functional \( \ell \) that separates 2 of the points of \( E \). Let \( \mu \) be the maximum value of \( \ell \) on \( E \). Since \( E \) is compact, this maximum value is attained. Let \( M = \{ x \in E : \ell(x) = \mu \} \). \( M \neq E \) since \( \ell \) is not constant. \( \ell \) is continuous and \( E \) is closed, so \( M \) is closed. \( \ell^{-1}\{\mu\} \) is the inverse image of an extreme set and we can check that it therefore is itself extreme, so \( M \) is extreme in \( E \), and since \( E \) is extreme in \( K \), \( M \) is extreme in \( K \). But this contradicts the fact that \( E \) was a minimal extreme subset.

(2) Let \( K_e \) be the extreme points of \( K \). We’ll show that if \( z \) is not in the closure of the convex hull, then \( z \notin K \). There exists a continuous linear functional \( \ell \) such that \( \ell(y) \leq c \) for \( y \in K_e \) and \( \ell(z) > c \). \( K \) is compact and \( \ell \) is continuous, so \( \ell \) achieves it maximum on a closed subset \( E \) of \( K \). \( E \) is extreme, and \( E \) must contain an extreme point \( p \). Since \( p \in E \subseteq K_e \), then \( \ell(p) \leq c \). Since \( \ell(p) = \max_K \ell(x) \), then \( \ell(x) \leq \ell(p) \leq c \) for all \( x \in K \). Since \( \ell(z) > c \), then \( z \notin K \). \( \square \)

### 6.4 Choquet’s theorem

Here is a theorem of Choquet.

**Theorem 6.5** Suppose \( K \) is a nonempty compact convex subset of a LCT linear space \( X \). Let \( K_e \) be the set of extreme points. If \( u \in K \), there exists a measure \( m_u \) of total mass 1 on \( K_e \) such that

\[
u = \int_{K_e} e_m u (de)
\]

in the weak sense.

A measure with total mass one is a probability measure, but this theorem has nothing to do with probability.
6.4. CHOQUET’S THEOREM

The equation holding in the weak sense means

\[ \ell(u) = \int_{K_e} \ell(e) m_{u}(de) \]

for all continuous linear functionals \( \ell \).

**Proof.** Let \( m, M \) be the minimum and maximum of \( \ell \) on \( K \). \( K \) is compact, so these values are achieved. Then \( \{ x \in K : \ell(x) = m \} \) is an extreme subset of \( K \) and similarly with \( m \) replaced by \( M \). They each contain extreme points. So if \( u \in K \),

\[ \min_{p \in K_e} \ell(p) \leq \ell(u) \leq \max_{p \in K_e} \ell(p). \]  \( \text{(6.1)} \)

If \( \ell_1 \) and \( \ell_2 \) are equal on \( K_e \), then applying the above to \( \ell_1 - \ell_2 \) shows they are equal on \( K \).

Let \( L \) be the class of continuous functions on \( K_e \) that are the restriction of a continuous linear functional. Fix \( u \). Define \( r \) on \( L \) by setting

\[ r(\ell) = \ell(u). \]

If \( L \) contains the constant function 1, then by (6.1) we have \( r(\ell) = \ell(u) = 1 \). If \( L \) does not contain the constant functions, adjoin the constant function \( f_0 = 1 \) to \( L \) and set \( r(f_0) = 1 \). The set \( L \) is a linear subspace of \( C(K_e) \).

Check that \( r \) is a positive linear functional on \( L \).

Now use Hahn-Banach to extend \( r \) from \( L \) to \( C(K_e) \).

\( K_e \) is a closed subset of \( K \), hence compact. \( r \) is a positive linear functional on \( C(K_e) \). By the Riesz representation theorem from measure theory, there exists a measure \( m \) such that

\[ r(f) = \int_{K_e} f dm. \]

Since \( r(f_0) = 1 \), then \( m(K_e) = 1 \).

An example: in \( \mathbb{R}^3 \), let \( K \) be the unit circle in the \((x, y)\) plane together with \( \{(1, 0, z) : |z| \leq 1\} \). Then \( (1, 0, 0) \not\in K_e \), so the collection of extreme points is not closed.

Choquet proved an important extension of his theorem in that we can take the integral to be over \( K_e \) rather than its closure, provided \( K \) is metrizable.

We write \( b(\mu) \) for \( \int e \mu(de) \), the barycenter of \( \mu \).
Theorem 6.6 (Choquet) If $K$ is convex, compact, and metrizable, and $x \in K$, there exists a probability measure $\mu$ supported on the extreme points of $K$ such that $x = b(\mu)$.

A function $f$ is concave if $-f$ is convex. Let $\mathcal{S}$ be the set of continuous concave functions on $K$. $\mathcal{S}$ is closed under the operation $\wedge$ (the operation of taking the greatest lower bound) and is closed under addition. It is an exercise to show that $\mathcal{S} - \mathcal{S}$ is closed under $\wedge$ and $\vee$. $\mathcal{S} - \mathcal{S}$ contains constants, and since it contains linear functions, it separates points. By the Stone-Weierstrass theorem, $\mathcal{S} - \mathcal{S}$ is dense in $C(K)$.

Let us write $\lambda \prec \mu$ if $\int f \, d\lambda \geq \int f \, d\mu$ for all $f \in \mathcal{S}$, where $\mu$ and $\lambda$ are probability measures on $K$. The idea of the existence of an extremal measure is the following. If $x$ is not extremal, it can be written as $\sum_i a_i x_i$. Any $x_i$ that is not an extreme point has a similar representation. The measure $\sum_i a_i \delta_{x_i}$ is “closer” to the boundary than $\delta_x$ and we will see this means $\sum_i a_i \delta_{x_i} \succ \delta_x$. The desired representation will come if we find the measure that is maximal with respect to $\prec$.

Proposition 6.7 There exists $\mu$ such that $\delta_x \prec \mu$ and $\mu \succ \lambda$ whenever $\lambda \succ \delta_x$.

Proof. We use Zorn’s lemma. Suppose $I$ is a totally ordered set and $\mu_i$ is a probability measure on $K$ for each $i \in I$ with $\mu_i \prec \mu_j$ if $i < j$. If $f \in \mathcal{S}$, then $\int f \, d\mu_i$ decreases as $i$ increases. Let $\ell(f)$ denote the limit. In this context, this means that given $\varepsilon > 0$, there exists $i_0 \in I$ such that $|\ell(f) - \int f \, d\mu_i| < \varepsilon$ if $i > i_0$. Because $\int f \, d\mu_i$ decreases in $i$, it is easy to see that $\ell(f) = \inf_{i \in I} \int f \, d\mu_i$. Define $\ell$ on $\mathcal{S} - \mathcal{S}$ by $\ell(f) = \lim_i \int f \, d\mu_i$. Because all the $\mu_i$ have total mass 1, $\ell$ is a bounded linear operator, and we can then extend $\ell$ to $C(K)$, the continuous functions on $K$. By the Riesz representation theorem, there exists a measure $\mu$ such that $\ell(f) = \int f \, d\mu$ for all $f \in C(K)$. If $f \in \mathcal{S} - \mathcal{S}$ is nonnegative, $\ell(f) = \lim_i \int f \, d\mu_i \geq 0$, or $\mu$ is a positive measure. Since $\ell(1) = 1$, $\mu$ is a probability measure. Because $\int f \, d\mu = \ell(f) = \inf_i \int f \, d\mu_i$ if $f \in \mathcal{S}$, $\mu \succ \mu_i$ for all $i \in I$. Thus $\mu$ is an upper bound for the $\mu_i$. By Zorn’s lemma, then, $\{\lambda : \lambda \succ \delta_x\}$ has a maximal element. 

We must now show that $\mu$ has barycenter $x$ and that $\mu$ is supported on the extreme points of $K$. 

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Proposition 6.8 If \( \mu \succ \delta_x \), then \( b(\mu) = x \).

Proof. Suppose \( \ell \) is any linear functional on \( K \). Then \( \ell \in S \) and \( \ell \in -S \). Thus \( \int \ell \, d\mu \leq \int \ell \, d\delta_x \) and \( \int (-\ell) \, d\mu \leq \int (-\ell) \, d\delta_x \), or \( \int \ell \, d\mu = \int \ell \, d\delta_x \) for all \( \ell \) linear. That implies \( b(\mu) = x \).

If \( f \in C(K) \), the continuous functions on \( K \), let

\[
\tilde{f} = \inf \{ g \in S, g \geq f \}. \tag{6.2}
\]

\( \tilde{f} \) is the least concave majorant of \( f \). Note \( f \leq \tilde{f} \) and that \( \tilde{f} \) is bounded, since the constant function \( g \equiv \sup_K f \) is continuous and dominates \( f \).

The existence part is completed by the following.

Proposition 6.9 If \( \mu \) is maximal with respect to \( \succ \), then \( \mu \) is supported on the extreme points of \( K \).

Proof. Suppose \( \mu \) is maximal. We first show

\[
\int f \, d\mu = \int \tilde{f} \, d\mu \tag{6.3}
\]

for all \( f \in C(K) \). Let \( E = C(K) \) and define

\[
P(f) = \int \tilde{f} \, d\mu. \tag{6.4}
\]

Since \( \tilde{f} + g \leq \tilde{f} + \tilde{g} \), \( P \) is clearly sublinear. Suppose \( \int f \, d\mu \neq \int \tilde{f} \, d\mu \) for some \( f \in C(K) \). Since \( S - S \) is dense in \( C(K) \), there exists \( f \in S \) such that \( \int f \, d\mu \neq \int \tilde{f} \, d\mu \). Let \( F = \{ cf : c \in \mathbb{R} \} \) and let \( \ell(f) = P(f) = \int \tilde{f} \, d\mu \).

\[
0 \leq P(f) + P(-f) \text{ by sublinearity, so } \ell(-f) = -P(f) \leq P(-f), \text{ or } \ell \text{ is dominated by } P \text{ on } F. \text{ Use the Hahn-Banach theorem to extend } \ell \text{ to } E. \text{ Since } \ell \text{ is a linear functional on } C(K), \text{ there exists a measure } \nu \text{ such that } \ell g = \int g \, d\nu \text{ for all } g \in C(K) \text{ by the Riesz representation theorem. We claim } \nu \text{ is a probability. } \ell(1) = P(1) = 1, \text{ and } -\ell(-1) = \ell(-1) \leq P(-1) = -1, \text{ or } \ell(1) = 1. \text{ If } g \geq 0, -\ell(g) = \ell(-g) \leq P(-g) = \int (-g) \, d\mu \leq 0, \text{ or } \ell(g) \geq 0. \text{ Thus } \nu \text{ is a positive measure with total mass 1, which proves the claim.}
If \( h \in \mathcal{S} \), then \( \tilde{h} = h \) and \( \int h \, d\nu = \ell(h) \leq P(h) = \int \tilde{h} \, d\mu = \int h \, d\mu \), or \( \nu \succ \mu \). Moreover, \( \int f \, d\nu = \ell(f) = P(f) = \int \tilde{f} \, d\mu > \int f \, d\mu \), or \( \nu \not\succ \mu \). This contradicts the maximality of \( \mu \). Therefore \( \int f \, d\mu = \int \tilde{f} \, d\mu \) for all \( f \in C(K) \).

Since (6.3) holds and \( f \leq \tilde{f} \), \( \mu \) must be concentrated on \( B_f = \{ x \in K : f(x) = \tilde{f}(x) \} \) for all \( f \in -\mathcal{S} \). Since \( K \) is metrizable, \( C(K) \) has a countable dense subset. Since \( \mathcal{S} - \mathcal{S} \) is dense in \( C(K) \), we can find a countable sequence \( f_n \in -\mathcal{S} \) that separate points. We may normalize so that \( \|f_n\|_\infty \leq 1 \). Let \( f = \sum_{n=1}^{\infty} (f_n)^2 / 2^n \). Then \( f \) is convex, continuous, and also strictly convex. Since \( f \) is concave, \( \tilde{f}(x) > f(x) \) for all \( x \) in \( K \) that are not extremal. So \( B_f \) is contained in the set of extreme points of \( K \), which proves the proposition. \( \square \)
Chapter 7

Sobolev spaces

7.1 Weak derivatives

Let $C^\infty_K$ be the set of $C^\infty$ functions on $\mathbb{R}^n$ that have compact support and have partial derivatives of all orders. For $j = (j_1, \ldots, j_n)$, write

$$D^j f = \frac{\partial^{j_1+\cdots+j_n} f}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}},$$

and set $|j| = j_1 + \cdots + j_n$. We use the convention that $\partial^0 f/\partial x_i^0$ is the same as $f$.

Let $f, g$ be locally integrable. We say that $D^j f = g$ in the weak sense or $g$ is the weak $j^{th}$ order partial derivative of $f$ if

$$\int f(x) D^j \varphi(x) \, dx = (-1)^{|j|} \int g(x) \varphi(x) \, dx$$

for all $\varphi \in C^\infty_K$. Note that if $g = D^j f$ in the usual sense, then integration by parts shows that $g$ is also the weak derivative of $f$.

Let

$$W^{k,p}(\mathbb{R}^n) = \{ f : f \in L^p, D^j f \in L^p \text{ for each } j \text{ such that } |j| \leq k \}.$$ 

Set

$$\| f \|_{W^{k,p}} = \sum_{\{j \geq 0 \ : \ |j| \leq k\}} \| D^j f \|_p,$$
where we set $D^0 f = f$.

This is slightly different than

$$\|f\|_0 = \left( \sum_{|j| \leq k} \int |D^j f|^p \right)^{1/p},$$

which is the definition we used for the $W^{k,p}$ norm earlier. However they are equivalent norms. To see this, we use the inequality

$$(a + b)^p \leq c_p a^p + c_p b^p, \quad a, b \geq 0,$$

where $c_p = 2^{p-1}$ when $p \geq 1$ and $c_p = 1$ when $p < 1$. An induction argument leads to

$$\left( \sum_{i=1}^N a_i \right)^p \leq c(p,N) \sum_{i=1}^N a_i^p$$

if each $a_i \geq 0$.

We have

$$\|f\|_{W^{k,p}}^p = \left( \sum_{|j| \leq k} \|D^j f\|_p \right)^p \leq c(p,k) \sum_{j \leq k} \|D^j f\|_p^p = c(p,k) \|f\|_0^p$$

for one direction. For the other,

$$\|f\|_0 \leq c(1/p,k) \sum_{|j| \leq k} \left( \int |D^j f|^p \right)^{1/p} = \|f\|_{W^{k,p}}.$$

**Theorem 7.1** The space $W^{k,p}$ is complete.

**Proof.** Let $f_m$ be a Cauchy sequence in $W^{k,p}$. For each $j$ such that $|j| \leq k$, we see that $D^j f_m$ is a Cauchy sequence in $L^p$. Let $g_j$ be the $L^p$ limit of $D^j f_m$. Let $f$ be the $L^p$ limit of $f_m$. Then

$$\int f_m D^j \varphi = (-1)^{|j|} \int (D^j f_m) \varphi \to (-1)^{|j|} \int g_j \varphi$$

for all $\varphi \in C_0^\infty$. On the other hand, $\int f_m D^j \varphi \to \int f D^j \varphi$. Therefore

$$(-1)^{|j|} \int g_j \varphi = \int f D^j \varphi$$

for all $\varphi \in C_0^\infty$. We conclude that $g_j = D^j f$ a.e. for each $j$ such that $|j| \leq k$. We have thus proved that $D^j f_m$ converges to $D^j f$ in $L^p$ for each $j$ such that $|j| \leq k$, and that suffices to prove the theorem. ∎
7.2 Sobolev inequalities

**Lemma 7.2** If $k \geq 1$ and $f_1, \ldots, f_k \geq 0$, then

$$\int f_1^{1/k} \cdots f_k^{1/k} \leq \left( \int f_1 \right)^{1/k} \cdots \left( \int f_k \right)^{1/k}.$$

**Proof.** We will prove

$$\left( \int f_1^{1/k} \cdots f_k^{1/k} \right)^k \leq \left( \int f_1 \right) \cdots \left( \int f_k \right). \quad (7.1)$$

We will use induction. The case $k = 1$ is obvious. Suppose (7.1) holds when $k$ is replaced by $k - 1$ so that

$$\left( \int f_1^{1/(k-1)} \cdots f_{k-1}^{1/(k-1)} \right)^{k-1} \leq \left( \int f_1 \right) \cdots \left( \int f_{k-1} \right). \quad (7.2)$$

Let $p = k/(k-1)$ and $q = k$ so that $p^{-1} + q^{-1} = 1$. Using Hölder’s inequality,

$$\int (f_1^{1/k} \cdots f_k^{1/k}) f_k^{1/k} \leq \left( \int f_1^{1/(k-1)} \cdots f_{k-1}^{1/(k-1)} \right)^{(k-1)/k} \left( \int f_k \right)^{1/k}.$$

Taking both sides to the $k^{th}$ power, we obtain

$$\left( \int (f_1^{1/k} \cdots f_k^{1/k}) f_k^{1/k} \right)^k \leq \left( \int f_1^{1/(k-1)} \cdots f_{k-1}^{1/(k-1)} \right)^{(k-1)/k} \left( \int f_k \right)^{1/k}.$$

Using (7.2), we obtain (7.1). Therefore our result follows by induction. \qed

Let $C^1_K$ be the continuously differentiable functions with compact support. The following theorem is sometimes known as the Gagliardo-Nirenberg inequality.

**Theorem 7.3** There exists a constant $c_1$ depending only on $n$ such that if $u \in C^1_K$, then

$$\| u \|_{n/(n-1)} \leq c_1 \| \nabla u \|_1.$$
We observe that \( u \) having compact support is essential; otherwise we could just let \( u \) be identically equal to one and get a contradiction. On the other hand, the constant \( c_1 \) does not depend on the support of \( u \).

**Proof.** For simplicity of notation, set \( s = 1/(n - 1) \). Let \( K_{j_1\ldots j_m} \) be the integral of \( |\nabla u(x_1, \ldots, x_n)| \) with respect to the variables \( x_{j_1}, \ldots, x_{j_m} \). Thus

\[
K_1 = \int |\nabla u(x_1, \ldots, x_n)| \, dx_1
\]

and

\[
K_{23} = \int \int |\nabla u(x_1, \ldots, x_n)| \, dx_2 \, dx_3.
\]

Note \( K_1 \) is a function of \( (x_2, \ldots, x_n) \) and \( K_{23} \) is a function of \( (x_1, x_4, \ldots, x_n) \).

If \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), then since \( u \) has compact support,

\[
|u(x)| = \left| \int_{-\infty}^{x_1} \frac{\partial u}{\partial x_1}(y_1, x_2, \ldots, x_n) \, dy_1 \right|
\leq \int_{\mathbb{R}} |\nabla u(y_1, x_2, \ldots, x_n)| \, dy_1
= K_1.
\]

The same argument shows that \( |u(x)| \leq K_i \) for each \( i \), so that

\[
|u(x)|^{n/(n-1)} = |u(x)|^{s} \leq K_1^s K_2^s \cdots K_n^s.
\]

Since \( K_1 \) does not depend on \( x_1 \), Lemma 7.2 shows that

\[
\int |u(x)|^{ns} \, dx_1 \leq K_1^s \int K_2^s \cdots K_n^s \, dx_1
\leq K_1^s \left( \int K_2 \, dx_1 \right)^s \cdots \left( \int K_n \, dx_1 \right)^s.
\]

Note that

\[
\int K_2 \, dx_1 = \int \left( \int |\nabla u(x_1, \ldots, x_n)| \, dx_2 \right) \, dx_1 = K_{12},
\]
and similarly for the other integrals. Hence

\[
\int |u|^{ns} \, dx_1 \leq K_1^s K_{12}^s \cdots K_{1n}^s.
\]
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Next, since $K_{12}$ does not depend on $x_2$,

$$
\int |u(x)|^{ns} \, dx_1 \, dx_2 \leq K_{12}^{s} \int K_1^{s} K_2^{s} \cdots K_n^{s} \, dx_2
$$

$$
\leq K_{12}^{s} \left( \int K_1 \, dx_2 \right)^{s} \left( \int K_2 \, dx_2 \right)^{s} \cdots \left( \int K_n \, dx_2 \right)^{s}
$$

$$
= K_{12}^{s} K_{123}^{s} \cdots K_{12n}^{s}.
$$

We continue, and get

$$
\int |u(x)|^{ns} \, dx_1 \, dx_2 \, dx_3 \leq K_{123}^{s} K_{123}^{s} K_{123}^{s} \cdots K_{123n}^{s}
$$

and so on, until finally we arrive at

$$
\int |u(x)|^{ns} \, dx_1 \cdots \, dx_n \leq \left( K_{12 \cdots n}^{s} \right)^{n} = K_{12 \cdots n}^{ns}.
$$

If we then take the $ns = n/(n - 1)$ roots of both sides, we get the inequality we wanted.

From this we can get the Sobolev inequalities.

**Theorem 7.4** Suppose $1 \leq p < n$ and $u \in C_1^1$. Then there exists a constant $c_1$ depending only on $n$ such that

$$
\|u\|_{np/(n-p)} \leq c_1 \|\nabla u\|_p.
$$

**Proof.** The case $p = 1$ is the case above, so we assume $p > 1$. The case when $u$ is identically equal to 0 is obvious, so we rule that case out. Let

$$
r = \frac{p(n-1)}{n-p},
$$

and note that $r > 1$ and

$$
r - 1 = \frac{np - n}{n - p}.
$$

Let $w = |u|^r$. Since $r > 1$, then $x \to |x|^r$ is continuously differentiable, and by the chain rule, $w \in C^1$. We observe that

$$
|\nabla w| \leq c_2 |u|^{r-1} |\nabla u|.
$$
Applying Theorem 7.3 to $w$ and using Hölder’s inequality with $q = \frac{p}{p-1}$, we obtain

$$
\left( \int |w|^{n/(n-1)} \right)^{\frac{n-1}{n}} \leq c_3 \int |\nabla w| \\
\leq c_4 \int |u|^{(np-n)/(n-p)} |\nabla u| \\
\leq c_5 \left( \int |u|^{np/(n-p)} \right)^{\frac{p-1}{p}} \left( \int |\nabla u|^p \right)^{1/p}.
$$

The left hand side is equal to

$$
\left( \int |u|^{np/(n-p)} \right)^{\frac{n-1}{n}}.
$$

Divide both sides by

$$
\left( \int |u|^{np/(n-p)} \right)^{\frac{p-1}{p}}.
$$

Since

$$
\frac{n-1}{n} - \frac{p-1}{p} = \frac{1}{p} - \frac{1}{n} = \frac{n-p}{pn},
$$

we get our result.

We can iterate to get results on the $L^p$ norm of $f$ in terms of the $L^q$ norm of $D^k f$ when $k > 1$.

**Theorem 7.5** Suppose $k \geq 1$. Suppose $p < n/k$ and we define $q$ by $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. Then there exists $c_1$ such that

$$
\|f\|_q \leq c \left\| \sum_{\{j:|j|=k\}} |D^k f| \right\|_p.
$$

### 7.3 Morrey’s inequality

Morrey’s inequality shows that if $f \in L^p$ for large enough $p$, then $f$ is Hölder continuous.

Let

$$
\|f\|_{C^{\gamma}} = \|f\|_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma}.
$$
Theorem 7.6 Suppose $p > n$ and $u \in C^1(\mathbb{R}^n)$ with compact support. Let $\gamma = 1 - \frac{n}{p}$. Then there exists a constant $c$ depending only on $p$ and $n$ such that
\[ \|u\|_{C^\gamma} \leq c\|u\|_{W^{1,p}}. \]

Proof. We will prove the Hölder estimate first and then do the $L^\infty$ estimate. Let's take $x = 0$.

If $v \in \partial B(0, 1)$ and $0 < s < r$, then
\[
|u(x + sv) - u(x)| = \left| \int_0^s \frac{d}{dt} u(x + tv) \ dt \right| = \left| \int_0^s \langle \nabla u(x + tv), v \rangle \ dt \right|
\leq \int_0^s |\nabla u(x + tv)| \ dt.
\]
Integrating over $v \in \partial B(0, 1)$, if $\sigma$ is surface measure,
\[
\int_{\partial B(0,1)} |u(x + sv) - u(x)| \ d\sigma(v) \leq \int_0^s \int_{\partial B(0,1)} |\nabla u(x + tv)| \ d\sigma(v) \ dt.
\]

We change to rectangular coordinates, with $x + tv = y$, so that $t = |y - x|$ and $d\sigma(v) = t^{-n+1} \ d\sigma(y)$:
\[
\int_{\partial B(0,1)} |u(x + sv) - u(x)| \ d\sigma(v) \leq \int_{B(0,s)} \frac{|\nabla u(y)|}{y - x|^{n-1}} \ d\sigma(y).
\]

If we change the integral on the right to being over $B(0, r)$, this just makes the integral larger.

Now multiply by $s^{n-1}$ and integrate over $s$ from 0 to $r$ to get
\[
\int_{B(0,r)} |u(y) - u(x)| \ dy \leq \frac{r^n}{n} \int_{B(0,r)} \frac{|\nabla u(y)|}{|y - x|^{n-1}} \ dy.
\]

If $x, y \in \mathbb{R}^n$, set $r = |y - x|$ and let $W = B(x, r) \cap B(y, r)$. We see that if $z \in W$, then
\[
|u(x) - u(y)| \leq |u(x) - u(z)| + |u(z) - u(y)|,
\]
and then integrating over such $z$,

$$|u(x) - u(y)| \leq \frac{1}{|W|} \int_W |u(x) - u(z)|
\leq \frac{1}{|W|} \int_W |u(y) - u(z)|
dz,$$

where $|A|$ is the Lebesgue measure of $A$.

We estimate the first integral, the second being almost identical. Note $|B(x, r)|/|W|$ does not depend on $r$. By Hölder’s inequality,

$$\frac{1}{|W|} \int_W |u(x) - u(z)|
dz \leq c \left( \int_{B(x, r)} |\nabla u(z)|^p \, dz \right)^{\frac{1}{p}} \left( \int_{B(x, r)} \frac{1}{|x - z|^{\frac{p(n-1)}{p-1}}} \right)^{\frac{p-1}{p}}
\leq c \left( r^{n-\frac{p(n-1)}{p-1}} \right)^{\frac{p-1}{p}} \|\nabla u\|_p
\leq cr^{1-\frac{p}{n}} \|\nabla u\|_p
= c_1 |x - y|^{1-\frac{p}{n}} \|\nabla u\|_p.$$

This argument works no matter what $x$ is.

Now we turn to the $L^\infty$ estimate. Suppose $\|u\|_{W^{1,p}} = 1$. If there exists $x$ such that $|u(x)| \geq M$ (where $M$ will be chosen in a moment), then from

$$|u(x) - u(y)| \leq c_1 |x - y|^{1-\frac{p}{n}} \|\nabla u\|_p \leq c_1 |x - y|^{1-\frac{p}{n}},$$

we see that $|u(y)| \geq M/2$ in $B(x, 1)$ as long as $M \geq 2c_1$. But

$$1 = \|u\|_{W^{1,p}} \geq \|u\|_p \geq \left( \int_{B(x, 1)} |u(y)|^p \, dy \right)^{\frac{1}{p}} \geq c_2 M.$$ 

Take $M = \max(2c_1, 2/c_2)$. We then get a contradiction to the assumption that there exists $x$ with $|u(x)| \geq M$.

If $r \geq 0$ is an integer and $\alpha \in (0, 1)$, define

$$\|f\|_{C^{r,\alpha}} = \sum_{|j| \leq r} \sup_x |D^j f(x)| + \sum_{|j| = r} \sup_{x \neq y} \frac{|D^j f(x) - D^j f(y)|}{|x - y|^\alpha},$$

and let $C^{r,\alpha}$ be the set of functions whose norm is finite.

Also part of the Sobolev embedding theorem is the following.
Theorem 7.7  If

\[
\frac{k - r - \alpha}{n} = \frac{1}{p},
\]

then \( W^{k,p}(\mathbb{R}^n) \subset C^{r,\alpha} \) and for \( f \in C^{k+\alpha} \) with compact support,

\[
\|f\|_{C^{r,\alpha}} \leq c\|f\|_{W^{k,p}}.
\]

This follows easily from the Morrey inequality.
Chapter 8

Distributions

For simplicity of notation, in this chapter we restrict ourselves to dimension one, but everything we do can be extended to $\mathbb{R}^n$, $n > 1$, although in some cases a more complicated proof is necessary.

8.1 Definitions and examples

We use $C^\infty_K$ for the set of $C^\infty$ functions on $\mathbb{R}$ with compact support. Let $Df = f'$, the derivative of $f$, $D^2f = f''$, the second derivative, and so on, and we make the convention that $D^0f = f$.

If $f$ is a continuous function on $\mathbb{R}$, let supp$(f)$ be the support of $f$, the closure of the set $\{x : f(x) \neq 0\}$. If $f_j, f \in C^\infty_K$, we say $f_j \to f$ in the $C^\infty_K$ sense if there exists a compact subset $K$ such that supp$(f_j) \subset K$ for all $j$, $f_j$ converges uniformly to $f$, and $D^m f_j$ converges uniformly to $D^m f$ for all $m$.

We have not claimed that $C^\infty_K$ with this notion of convergence is a Banach space, so it doesn’t make sense to talk about bounded linear functionals. But it does make sense to consider continuous linear functionals. A map $F : C^\infty_K \to \mathbb{C}$ is a continuous linear functional on $C^\infty_K$ if $F(f + g) = F(f) + F(g)$ whenever $f, g \in C^\infty_K$, $F(cf) = cF(f)$ whenever $f \in C^\infty_K$ and $c \in \mathbb{C}$, and $F(f_j) \to F(f)$ whenever $f_j \to f$ in the $C^\infty_K$ sense.

A distribution is defined to be a complex-valued continuous linear functional on $C^\infty_K$. 

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Here are some examples of distributions.

**Example 8.1** If $g$ is a continuous function, define

$$G_g(f) = \int_{\mathbb{R}} f(x)g(x) \, dx, \quad f \in C^\infty_K. \quad (8.1)$$

It is routine to check that $G_g$ is a distribution.

Note that knowing the values of $G_g(f)$ for all $f \in C^\infty_K$ determines $g$ uniquely up to almost everywhere equivalence. Since $g$ is continuous, $g$ is uniquely determined at every point by the values of $G_g(f)$.

**Example 8.2** Set $\delta(f) = f(0)$ for $f \in C^\infty_K$. This distribution is the Dirac delta function.

**Example 8.3** If $g$ is integrable and $k \geq 1$, define

$$F(f) = \int_{\mathbb{R}} D^k f(x)g(x) \, dx, \quad f \in C^\infty_K.$$

**Example 8.4** If $k \geq 1$, define $F(f) = D^k f(0)$ for $f \in C^\infty_K$.

There are a number of operations that one can perform on distributions to get other distributions. Here are some examples.

**Example 8.5** Let $h$ be a $C^\infty$ function, not necessarily with compact support. If $F$ is a distribution, define $M_h(F)$ by

$$M_h(F)(f) = F(fh), \quad f \in C^\infty_K.$$

It is routine to check that $M_h(F)$ is a distribution.

Example 8.1 shows how to consider a continuous function $g$ as a distribution. Defining $G_g$ by (8.1),

$$M_h(G_g)(f) = G_g(fh) = \int (fh)g = \int f(hg) = G_{hg}(f).$$

Therefore we can consider the operator $M_h$ we just defined as an extension of the operation of multiplying continuous functions by a $C^\infty$ function $h.$
Example 8.6 If $F$ is a distribution, define $D(F)$ by

$$D(F)(f) = F(-Df), \quad f \in C_K^\infty.$$ 

Again it is routine to check that $D(F)$ is a distribution.

If $g$ is a continuously differentiable function and we use (8.1) to identify the function $g$ with the distribution $G_g$, then

$$D(G_g)(f) = G_g(-Df) = \int (-Df)(x)g(x) \, dx$$

$$= \int f(x)(Dg)(x) \, dx = G_{Dg}(f), \quad f \in C_K^\infty,$$

by integration by parts. Therefore $D(G_g)$ is the distribution that corresponds to the function that is the derivative of $g$. However, $D(F)$ is defined for any distribution $F$. Hence the operator $D$ on distributions gives an interpretation to the idea of taking the derivative of any continuous function.

Example 8.7 Let $a \in \mathbb{R}$ and define $T_a(F)$ by

$$T_a(F)(f) = F(f-a), \quad f \in C_K^\infty,$$

where $f-a(x) = f(x+a)$. If $G_g$ is given by (8.1), then

$$T_a(G_g)(f) = G_g(f-a) = \int f-a(x)g(x) \, dx$$

$$= \int f(x)g(x-a) \, dx = G_{g(a)}(f), \quad f \in C_K^\infty,$$

by a change of variables, and we can consider $T_a$ as the operator that translates a distribution by $a$.

Example 8.8 Define $R$ by

$$R(F)(f) = F(Rf), \quad f \in C_K^\infty,$$

where $Rf(x) = f(-x)$. Similarly to the previous examples, we can see that $R$ reflects a distribution through the origin.
Example 8.9 Finally, we give a definition of the convolution of a distribution with a continuous function $h$ with compact support. Define $C_h(F)$ by

$$C_h(F)(f) = F(f * Rh), \quad f \in C_K^\infty,$$

where $Rh(x) = h(-x)$. To justify that this extends the notion of convolution, note that

$$C_h(G_g)(f) = G_g(f * Rh) = \int g(x)(f * Rh)(x) \, dx$$

$$= \int \int g(x)f(y)h(y - x) \, dy \, dx = \int f(y)(g * h)(y) \, dy$$

$$= G_g * h(f),$$

or $C_h$ takes the distribution corresponding to the continuous function $g$ to the distribution corresponding to the function $g * h$.

One cannot, in general, define the product of two distributions or quantities like $\delta(x^2)$.

8.2 Distributions supported at a point

We first define the support of a distribution. We then show that a distribution supported at a point is a linear combination of derivatives of the delta function.

Let $G$ be open. A distribution $F$ is zero on $G$ if $F(f) = 0$ for all $C_K^\infty$ functions $f$ for which $\text{supp}(f) \subset G$.

Lemma 8.10 If $F$ is zero on $G_1$ and $G_2$, then $F$ is zero on $G_1 \cup G_2$.

Proof. This is just the usual partition of unity proof. Suppose $f$ has support in $G_1 \cup G_2$. We will write $f = f_1 + f_2$ with $\text{supp}(f_1) \subset G_1$ and $\text{supp}(f_2) \subset G_2$. Then $F(f) = F(f_1) + F(f_2) = 0$, which will achieve the proof.

Fix $x \in \text{supp}(f)$. Since $G_1, G_2$ are open, we can find $h_x$ such that $h_x$ is non-negative, $h_x(x) > 0$, $h_x$ is in $C_K^\infty$, and the support of $h_x$ is contained either in $G_1$ or in $G_2$. The set $B_x = \{y : h_x(y) > 0\}$ is open and contains $x$. 
8.2. DISTRIBUTIONS SUPPORTED AT A POINT

By compactness we can cover \( \text{supp } f \) by finitely many sets \( \{B_{x_1}, \ldots, B_{x_m}\} \). Let \( h_1 \) be the sum of those \( h_{x_i} \) whose support is contained in \( G_1 \) and let \( h_2 \) be the sum of those \( h_{x_i} \) whose support is contained in \( G_2 \). Then let

\[
f_1 = \frac{h_1}{h_1 + h_2} f, \quad f_2 = \frac{h_2}{h_1 + h_2} f.
\]

Clearly \( \text{supp } (f_1) \subset G_1, \text{supp } (f_2) \subset G_2, f_1 + f_2 > 0 \) on \( G_1 \cup G_2 \), and \( f = f_1 + f_2 \).

If we have an arbitrary collection of open sets \( \{G_\alpha\} \), \( F \) is zero on each \( G_\alpha \), and \( \text{supp } (f) \) is contained in \( \cup_\alpha G_\alpha \), then by compactness there exist finitely many of the \( G_\alpha \) that cover \( \text{supp } (f) \). By Lemma 8.10, \( F(f) = 0 \).

The union of all open sets on which \( F \) is zero is an open set on which \( F \) is zero. The complement of this open set is called the support of \( F \).

**Example 8.11** The support of the Dirac delta function is \( \{0\} \). Note that the support of \( D^k \delta \) is also \( \{0\} \).

Define

\[
\|f\|_{C_N(K)} = \max_{0 \leq k \leq N} \sup_{x \in K} |D^k f(x)|.
\]

**Proposition 8.12** Let \( F \) be a distribution and \( K \) a fixed compact set. There exist \( N \) and \( c \) depending on \( F \) and \( K \) such that if \( f \in C_\infty^\infty \) has support in \( K \), then

\[
|F(f)| \leq c \|f\|_{C_N(K)}.
\]

**Proof.** Suppose not. Then for each \( m \) there exists \( f_m \in C_\infty^\infty \) with support contained in \( K \) such that \( F(f_m) = 1 \) and \( \|f\|_{C^m(K)} \leq 1/m \). Therefore \( f_m \to 0 \) in the sense of \( C_\infty^\infty \). However \( F(f_m) = 1 \) while \( F(0) = 0 \), a contradiction. \( \square \)

**Proposition 8.13** Suppose \( F \) is a distribution and \( \text{supp } (F) = \{0\} \). There exists \( N \) such that if \( f \in C_\infty^\infty \) and \( D^j f(0) = 0 \) for \( j \leq N \), then \( F(f) = 0 \).
**Proof.** Let $\varphi \in C^\infty$ be 0 on $[-1, 1]$ and 1 on $|x| > 2$. Let $g = (1 - \varphi)f$. Note $\varphi f = 0$ on $[-1, 1]$, so $F(\varphi f) = 0$ because $F$ is supported on $\{0\}$. Then

$$F(g) = F(f) - F(\varphi f) = F(f).$$

Thus is suffices to show that $F(g) = 0$ whenever $g \in C^\infty_K$, $\text{supp}(g) \subset [-3, 3]$, and $D^j g(0) = 0$ for $0 \leq j \leq N$.

Let $K = [-3, 3]$. By Proposition 8.12 there exist $N$ and $c$ depending only on $F$ such that $|F(g)| \leq c\|g\|_{C^N(K)}$. Define $g_m(x) = \varphi(mx)g(x)$. Note that $g_m(x) = g(x)$ if $|x| > 2/m$.

Suppose $|x| < 2/m$ and $g \in C^\infty_K$ with support in $[-3, 3]$ and $D^j g(0) = 0$ for $j \leq N$. By Taylor’s theorem, if $j \leq N$,

$$D^j g(x) = D^j g(0) + D^{j+1} g(0)x + \cdots + D^N g(0) \frac{x^{N-j}}{(N-j)!} + R$$

$$= R,$$

where the remainder $R$ satisfies

$$|R| \leq \sup_{y \in \mathbb{R}} |D^{N+1} g(y)| \frac{|x|^{N+1-j}}{(N+1-j)!}.$$  

Since $|x| < 2/m$, then

$$|D^j g(x)| = |R| \leq c_1 m^{j-1-N}$$  \hspace{1cm} (8.2)

for some constant $c_1$.

By the definition of $g_m$ and (8.2),

$$|g_m(x)| \leq c_2 |g(x)| \leq c_3 m^{-N-1},$$

where $c_2$ and $c_3$ are constants. Again using (8.2),

$$|Dg_m(x)| \leq |\varphi(mx)||Dg(x)| + m|g(x)||D\varphi(mx)| \leq c_4 m^{-N}.$$  

Continuing, repeated applications of the product rule show that if $k \leq N$, then

$$|D^k g_m(x)| \leq c_5 m^{k-1-N}$$

for $k \leq N$ and $|x| \leq 2/m$, where $c_5$ is a constant.
Recalling that $g_m(x) = g(x)$ if $|x| > 2/m$, we see that $D^i g_m(x) \to D^i g(x)$ uniformly over $x \in [-3,3]$ if $j \leq N$. We conclude

$$F(g_m - g) = F(g_m) - F(g) \to 0.$$ 

However, each $g_m$ is 0 in a neighborhood of 0, so by the hypothesis, $F(g_m) = 0$; thus $F(g) = 0$. 

By Example 8.6, $D^i \delta$ is the distribution such that

$$D^i \delta(f) = (-1)^i D^i f(0).$$ 

**Theorem 8.14** Suppose $F$ is a distribution supported on $\{0\}$. Then there exist $N$ and constants $c_i$ such that

$$F = \sum_{i=0}^{N} c_i D^i \delta.$$ 

**Proof.** Let $P_i(x)$ be a $C^\infty_K$ function which agrees with the polynomial $x^i$ in a neighborhood of 0. Taking derivatives shows that $D^i P_i(0) = 0$ if $i \neq j$ and equals $i!$ if $i = j$. Then $D^i \delta(P_i) = (-1)^i i!$ if $i = j$ and 0 otherwise.

Use Proposition 8.13 to determine the integer $N$. Suppose $f \in C^\infty_K$. By Taylor’s theorem, $f$ and the function

$$g(x) = \sum_{i=0}^{N} D^i f(0) P_i(x)/i!$$

agree at 0 and all the derivatives up to order $N$ agree at 0. By the conclusion of Proposition 8.13 applied to $f - g$,

$$F\left(f - \sum_{i=0}^{N} \frac{D^i f(0)}{i!} P_i\right) = 0.$$ 

Therefore

$$F(f) = \sum_{i=0}^{N} \frac{D^i f(0)}{i!} F(P_i) = \sum_{i=0}^{N} (-1)^i \frac{D^i \delta(f)}{i!} F(P_i)$$

$$= \sum_{i=0}^{N} c_i D^i \delta(f)$$

if we set $c_i = (-1)^i F(P_i)/i!$ Since $f$ was arbitrary and the $c_i$ do not depend on $f$, this proves the theorem. 

\[ \Box \]
8.3 Distributions with compact support

In this section we consider distributions whose supports are compact sets.

**Theorem 8.15** If \( F \) has compact support, there exist a non-negative integer \( L \) and continuous functions \( g_j \) such that

\[
F = \sum_{j \leq L} D^j G_{g_j},
\]

where \( G_{g_j} \) is defined by Example 8.1.

**Example 8.16** The delta function is the derivative of \( h \), where \( h \) is 0 for \( x < 0 \) and 1 for \( x \geq 0 \). In turn \( h \) is the derivative of \( g \), where \( g \) is 0 for \( x < 0 \) and \( g(x) = x \) for \( x \geq 0 \). Therefore \( \delta = D^2 G_g \).

**Proof.** Let \( h \in C^\infty_K \) and suppose \( h \) is equal to 1 on the support of \( F \). Then \( F((1-h)f) = 0 \), or \( F(f) = F(hf) \). Therefore there exist \( N \) and \( c_1 \) such that

\[
|F(hf)| \leq c_1 \|hf\|_{C^N(K)}.
\]

By the product rule,

\[
|D(hf)| \leq |h(Df)| + |(Dh)f| \leq c_2 \|f\|_{C^N(K)},
\]

and by repeated applications of the product rule,

\[
\|hf\|_{C^N(K)} \leq c_3 \|f\|_{C^N(K)}.
\]

Hence

\[
|F(f)| = |F(hf)| \leq c_4 \|f\|_{C^N(K)}.
\]

Let \( K = [-x_0, x_0] \) be a closed interval containing the support of \( F \). Let \( C^N(K) \) be the \( N \) times continuously differentiable functions whose support is contained in \( K \). We will use the fact that \( C^N(K) \) is a complete metric space with respect to the metric \( \|f - g\|_{C^N(K)} \).
Define
\[ \|f\|_{H^M} = \left( \sum_{k \leq M} \int |D^k f|^2 \, dx \right)^{1/2}, \quad f \in C^\infty_K, \]
and let \( H^M \) be the completion of \( \{ f \in C^\infty_K : \text{supp}(f) \subset K \} \) with respect to this norm. It is routine to check that \( H^M \) is a Hilbert space.

Suppose \( M = N + 1 \) and \( x \in K \). Then using the Cauchy-Schwarz inequality and the fact that \( K = [-x_0, x_0] \),
\[ |D^j f(x)| = |D^j f(x) - D^j f(-x_0)| = \left| \int_{-x_0}^x D^{j+1} f(y) \, dy \right| \]
\[ \leq |2x_0|^{1/2} \left( \int_{-x_0}^x |D^{j+1} f(y)|^2 \, dy \right)^{1/2} \]
\[ \leq c_5 \left( \int_{-x_0}^x |D^{j+1} f(y)|^2 \, dy \right)^{1/2}. \]
This holds for all \( j \leq N \), hence
\[ \|u\|_{C^N(K)} \leq c_6 \|u\|_{H^M}. \quad (8.4) \]

Recall the definition of completion. If \( g \in H^M \), there exists \( g_m \in C^N(K) \) such that \( \|g_m - g\|_{H^M} \to 0 \). In view of (8.4), we see that \( \{g_m\} \) is a Cauchy sequence with respect to the norm \( \|\cdot\|_{C^N(K)} \). Since \( C^N(K) \) is complete, then \( g_m \) converges with respect to this norm. The only possible limit is equal to \( g \) a.e. We may therefore conclude \( g \in C^N(K) \) whenever \( g \in H^M \).

Since \( |F(f)| \leq c_4 \|f\|_{C^N(K)} \leq c_4 c_6 \|f\|_{H^M} \), then \( F \) can be viewed as a bounded linear functional on \( H^M \). By the Riesz representation theorem for Hilbert spaces (Theorem 4.12), there exists \( g \in H^M \) such that
\[ F(f) = \langle f, g \rangle_{H^M} = \sum_{k \leq M} \langle D^k f, D^k g \rangle, \quad f \in H^M. \]

Now if \( g_m \to g \) with respect to the \( H^M \) norm and each \( g_m \in C^N(K) \), then
\[ \langle D^k f, D^k g \rangle = \lim_{m \to \infty} \langle D^k f, D^k g_m \rangle = \lim_{m \to \infty} (-1)^k \langle D^{2k} f, g_m \rangle \]
\[ = (-1)^k \langle D^{2k} f, g \rangle = (-1)^k G_g(D^{2k} f) \]
\[ = (-1)^k D^{2k} G_g(f) \]
if \( f \in C^\infty_K \), using integration by parts and the definition of the derivative of a distribution. Therefore

\[
F = \sum_{k \leq M} (-1)^k D^{2k} G_{g_k},
\]

which gives our result if we let \( L = 2M \), set \( g_j = 0 \) if \( j \) is odd, and set \( g_{2k} = (-1)^k g \).

\[\square\]

### 8.4 Tempered distributions

Let \( \mathcal{S} \) be the class of complex-valued \( C^\infty \) functions \( u \) such that \( |x^j D^k u(x)| \to 0 \) as \( |x| \to \infty \) for all \( k \geq 0 \) and all \( j \geq 1 \). \( \mathcal{S} \) is called the \textit{Schwartz class}. An example of an element in the Schwartz class that is not in \( C^\infty_K \) is \( e^{-x^2} \).

Define

\[
\|u\|_{j,k} = \sup_{x \in \mathbb{R}} |x|^j |D^k u(x)|.
\]

We say \( u_n \in \mathcal{S} \) converges to \( u \in \mathcal{S} \) in the sense of the Schwartz class if \( \|u_n - u\|_{j,k} \to 0 \) for all \( j, k \).

A continuous linear functional on \( \mathcal{S} \) is a function \( F : \mathcal{S} \to \mathbb{C} \) such that \( F(f + g) = F(f) + F(g) \) if \( f, g \in \mathcal{S} \), \( F(cf) = cF(f) \) if \( f \in \mathcal{S} \) and \( c \in \mathbb{C} \), and \( F(f_m) \to F(f) \) whenever \( f_m \to f \) in the sense of the Schwartz class. A \textit{tempered distribution} is a continuous linear functional on \( \mathcal{S} \).

Since \( C^\infty_K \subset \mathcal{S} \) and \( f_n \to f \) in the sense of the Schwartz class whenever \( f_n \to f \) in the sense of \( C^\infty_K \), then any continuous linear functional on \( \mathcal{S} \) is also a continuous linear functional on \( C^\infty_K \). Therefore every tempered distribution is a distribution.

Any distribution with compact support is a tempered distribution. If \( g \) grows slower than some power of \( |x| \) as \( |x| \to \infty \), then \( G_g \) is a tempered distribution, where \( G_g(f) = \int f(x)g(x) \, dx \).

For \( f \in \mathcal{S} \), recall that we defined the Fourier transform \( \mathcal{F}f = \hat{f} \) by

\[
\hat{f}(u) = \int f(x)e^{ixu} \, dx.
\]
Theorem 8.17 \( \mathcal{F} \) is a continuous map from \( S \) into \( S \).

**Proof.** For elements of \( S \), \( D^k(f) = \mathcal{F}((ix)^k f) \). If \( f \in S \), \( |x^k f(x)| \) tends to zero faster than any power of \( |x|^{-1} \), so \( x^k f(x) \in L^1 \). This implies \( D^k \mathcal{F} f \) is a continuous function, and hence \( \mathcal{F} f \in C^\infty \).

By an exercise,

\[
 u^j D^k(\mathcal{F} f)(u) = i^{k+j} \mathcal{F}(D^j(x^k f))(u). \tag{8.5}
\]

Using the product rule, \( D^j(x^k f) \) is in \( L^1 \). Hence \( u^j D^k \mathcal{F} f(u) \) is continuous and bounded. This implies that every derivative of \( \mathcal{F} f(u) \) goes to zero faster than any power of \( |u|^{-1} \). Therefore \( \mathcal{F} f \in S \).

Finally, if \( f_m \to f \) in the sense of the Schwartz class, it follows by the dominated convergence theorem that \( \mathcal{F} f_m(u) \to \mathcal{F} f(u) \) uniformly over \( u \in \mathbb{R} \) and moreover \( |u|^k D^j(\mathcal{F} f_m) \to |u|^k D^j(\mathcal{F} f) \) uniformly over \( \mathbb{R} \) for each \( j \) and \( k \).

If \( F \) is a tempered distribution, define \( \mathcal{F} F \) by

\[ \mathcal{F} F(f) = F(\hat{f}) \]

for all \( f \in S \). We verify that \( \mathcal{F} G_g = G_{\hat{g}} \) if \( g \in S \) as follows:

\[
 \mathcal{F}(G_g)(f) = G_g(\hat{f}) = \int \hat{f}(x) g(x) \, dx = \int \int e^{ixy} f(y)g(x) \, dy \, dx = \int f(y) \hat{g}(y) \, dy = G_{\hat{g}}(f)
\]

if \( f \in S \).

Note that for the above equations to work, we used the fact that \( \mathcal{F} \) maps \( S \) into \( S \). Of course, \( \mathcal{F} \) does not map \( C^\infty_K \) into \( C^\infty_K \). That is why we define the Fourier transform only for tempered distributions rather than all distributions.

**Theorem 8.18** \( \mathcal{F} \) is an invertible map on the class of tempered distributions and \( \mathcal{F}^{-1} = (2\pi)^{1/2} \mathcal{F} R \). Moreover \( \mathcal{F} \) and \( R \) commute.
Proof. We know
\[ f(x) = (2\pi)^{-1/2} \int \hat{f}(-u)e^{ixu} \, du, \quad f \in \mathcal{S}, \]
so \( f = (2\pi)^{-1/2} \mathcal{F} \mathcal{R} f \), and hence \( \mathcal{F} \mathcal{R} f = (2\pi)^{1/2} I \), where \( I \) is the identity. Then if \( H \) is a tempered distribution,
\[ (2\pi)^{-1/2} \mathcal{F} \mathcal{R} f = R \mathcal{F} H((2\pi)^{-1/2} f) = \mathcal{F} H((2\pi)^{-1/2} R \mathcal{F} f) = H(f). \]
Thus
\[ (2\pi)^{-1/2} \mathcal{F} \mathcal{R} f = H, \]
or
\[ (2\pi)^{-1/2} \mathcal{F} \mathcal{R} f = I. \]
We conclude \( A = (2\pi)^{-1/2} \mathcal{F} \mathcal{R} \) is a left inverse of \( \mathcal{F} \) and \( B = (2\pi)^{-1/2} R \mathcal{F} \) is a right inverse of \( \mathcal{F} \). Hence \( B = (A \mathcal{F})B = A(\mathcal{F} B) = A \), or \( \mathcal{F} \) has an inverse, namely, \( (2\pi)^{-1/2} \mathcal{F} \), and moreover \( R \mathcal{F} = \mathcal{F} \mathcal{R} \). \( \square \)
Chapter 9

Banach algebras

9.1 Normed algebras

An algebra is a linear space over $+$ and a ring over $\cdot$. We assume there is an identity for the multiplication, which we call $I$. Our algebras will be over the scalar field $\mathbb{C}$; the reasons will be very apparent shortly.

An algebra is a normed algebra if the linear space is normed and $\|NM\| \leq \|N\| \|M\|$ and $\|I\| = 1$. If the normed algebra is complete, it is called a Banach algebra.

One example is to let $\mathcal{L} = \mathcal{L}(X, X)$, the set of linear maps from $X$ into $X$. Another is to let $\mathcal{L}$ be the collection of bounded continuous functions on some set. A third example is to let $\mathcal{L}$ be the collection of bounded functions that are analytic in the unit disk.

An element $M$ of $\mathcal{L}$ is invertible if there exists $N \in \mathcal{L}$ such that $NM = MN = I$.

$M$ has a left inverse $A$ if $AM = I$ and a right inverse $B$ if $MB = I$. If it has both, then $B = AMB = A$, and so $M$ is invertible.

**Proposition 9.1** (1) If $M$ and $K$ are invertible, then

$$(MK)^{-1} = K^{-1}M^{-1}.$$

(2) If $M$ and $K$ commute and $MK$ is invertible, then $M$ and $K$ are...
invertible.

**Proof.** (1) is easy. For (2), let \( N = (MK)^{-1} \). Then \( MKN = I \), so \( KN \) is a right inverse for \( M \). Also, \( I = NMK = NKM \), so \( NK \) is a left inverse for \( M \). Since \( M \) has a left and right inverse, it is invertible. The argument for \( K \) is similar. \( \square \)

**Proposition 9.2** If \( K \) is invertible, then so is \( L = K - A \) provided \( \|A\| < 1/\|K^{-1}\| \).

**Proof.** First we suppose \( K = I \). If \( \|B\| < 1 \), then

\[
\left\| \sum_{m} B^i \right\| \leq \sum_{m} \|B^i\| \leq \sum_{m} \|B\|^i
\]

is a Cauchy sequence, so \( S = \sum_{i=0}^{\infty} B^i \) converges. We see \( BS = \sum_{i=1}^{\infty} B^i = S - I \), so \( (I - B)S = I \). Similarly \( S(I - B) = I \).

For the general case, write \( K - A = K(I - K^{-1}A) \), and let \( B = K^{-1}A \). Then \( \|B\| \leq \|K^{-1}\| \|A\| < 1 \), and

\[
(K - A)^{-1} = (I - K^{-1}A)^{-1}K^{-1}.
\]

\( \square \)

We note for future reference the equation

\[
(I - B)^{-1} = \sum_{i=0}^{\infty} B^i.
\] (9.1)

The *resolvent* set of \( M \), \( \rho(M) \), is the set of \( \lambda \in \mathbb{C} \) such that \( \lambda I - M \) is invertible. The *spectrum* of \( M \), \( \sigma(M) \), is the set of \( \lambda \) for which \( \lambda I - M \) is not invertible. We frequently write \( \lambda - M \) for \( \lambda I - M \). We will also use \( R_{\lambda} \) for \( (\lambda I - M)^{-1} \).

Let \( F : G \to X \), where \( G \subseteq \mathbb{C} \), and write \( F_z \) for \( F(z) \). \( F_z \) is *strongly analytic* if

\[
\lim_{h \to 0} \frac{F_{z+h} - F_z}{h}
\]
exists in the norm topology for all \( z \in G \), that is, there exists an element \( F_z' \) such that the norm of \( (F_{z+h} - F_z)/h - F_z' \) tends to 0 as \( h \to 0 \).

One can check that much of complex analysis can be extended to strongly analytic functions. There are two approaches one could follow to show this. One is to recall that much of complex analysis is derived from Cauchy’s theorem, and that in turn is based on the fact that \( \int_C c z \, dz = 0 \) and \( \int_C c \, dz = 0 \) when \( C \) is the boundary of a rectangle. If we replace \( c \) by \( M \in \mathcal{L} \), the same argument goes through.

The other argument is that if \( \ell \) is a bounded linear functional on \( \mathcal{L} \), then \( f(z) = \ell(F_z) \) is analytic in the usual sense, and by Riemann sum approximations one can show that

\[
\ell \left( \int_C F_z \, dz \right) = \int_C \ell(F_z) \, dz = 0
\]

for suitable closed curves \( C \). This is true for every \( \ell \), and so \( \int_C F_z \, dz = 0 \). Many of the other theorems of complex analysis can be proved in a similar way.

**Proposition 9.3**

(1) \( \rho(M) \) is open in \( \mathbb{C} \).

(2) \( (z - M)^{-1} \) is an analytic function of \( z \) in \( \rho(M) \).

**Proof.** If \( \lambda \in \rho(M) \), letting \( K = \lambda I - M \) and \( A = -hI, \ K - A = (\lambda+h) - M \) is invertible if \( |h| = \|A\| < 1/\|K^{-1}\| \), which happens if \( h \) is small. So \( \lambda + h \in \rho(M) \).

For (2),

\[
\lambda - M + hI = (\lambda - M)(I + hR_\lambda),
\]

and so

\[
(\lambda - M + hI)^{-1} = \left( \sum_{i=0}^{\infty} (-1)^i(\lambda - M)^i h^i \right)(\lambda - M)^{-1}.
\]

Therefore

\[
((\lambda + h) - M)^{-1} = \sum_{n=0}^{\infty} (-1)^n(\lambda - M)^{n-1}h^n = \left( \sum_{i=0}^{\infty} (-hR_\lambda)^i \right) R_\lambda
\]
for $h$ small. So the resolvent can be expanded in a power series in $h$ which is valid if $|h| < \|\lambda - M\|^{-1}$. We then have

$$\left\| \frac{R_{\lambda+h} - R_{\lambda}}{h} - (R_{\lambda}^2) \right\| \to 0$$

as $h \to 0$. \hfill \square

We write

$$r(M) = \sup_{\lambda \in \sigma(M)} |\lambda|,$$

and call this the *spectral radius* of $M$.

**Theorem 9.4** $\sigma(M)$ is closed, bounded, and nonempty.

**Proof.** $\rho(M)$ is open, so $\sigma(M)$ is closed.

$$(zI - M)^{-1} = z^{-1}(I - Mz)^{-1} = \sum_{n=0}^{\infty} M^n z^{-n-1}$$

converges if $\|z^{-1}M\| < 1$, or equivalently, $|z| > \|M\|$. Therefore, if $|z| > \|M\|$, then $z \in \rho(M)$. Hence the spectrum is contained in $B_{\|M\|}(0)$.

Suppose $\sigma(M)$ is empty. For $z > \|M\|$ we have

$$R_z = (z - M)^{-1} = z^{-1}(I - z^{-1}M)^{-1}.$$  

Let $\ell$ be any bounded linear functional on $\mathcal{L}$. We conclude that $f(z) = \ell(R_z)$ is analytic and $f(z) = \ell(R_z) \to 0$ as $|z| \to \infty$.

We thus know that $f$ is analytic on $\mathbb{C}$, i.e., it is an entire function, and that $f(z)$ tends to 0 as $|z| \to \infty$. Therefore $f$ is a bounded entire function. By Liouville’s theorem from complex analysis, $f$ must be constant. Since $f$ tends to 0 as $|z|$ tends to infinity, that constant must be 0. This holds for all $\ell$, so $R_z$ must be equal to 0 for all $z$. But then we have $I = (z - M)R_z = 0$, a contradiction. \hfill \square

A key result is the *spectral radius formula*. First we need a consequence of the uniform boundedness principle.


Lemma 9.5 If $\mathcal{B}$ is a Banach space and $\{x_n\}$ a subset of $\mathcal{B}$ such that $\sup_n |f(x_n)|$ is finite for each bounded linear functional $f$, then $\sup_n \|x_n\|$ is finite.

**Proof.** For each $x \in \mathcal{B}$, define a linear functional $L_x$ on $\mathcal{B}^*$, the dual space of $\mathcal{B}$, by $L_x(f) = f(x)$, $f \in \mathcal{B}^*$. We already have shown that $\|L_x\| = \|f\|$.

Since $\sup_n |L_{x_n}(f)| = \sup_n |f(x_n)|$ is finite for each $f \in \mathcal{B}^*$, by the uniform boundedness principle, $\sup_n \|L_{x_n}\| < \infty$.

Since $\|L_{x_n}\| = \|x_n\|$, we obtain our result. □

Theorem 9.6 *(Spectral radius formula)*

$$r(M) = \lim_{k \to \infty} \|M^k\|^{1/k}.$$ 

**Proof.** Fix $k$ for the moment. If we write $n = kq + r$,

$$\left\| \sum_{n=0}^\infty \frac{M^n}{z^{n+1}} \right\| \leq \sum \frac{\|M^n\|}{|z|^{n+1}} \leq \sum_{n=0}^{k-1} \frac{\|M\|^r}{|z|^{r+1}} \sum_q \left( \frac{\|M\|}{|z|^k} \right)^q.$$

So $\sum M^n|z|^{-n-1}$ converges absolutely if $\|M\|^k/|z|^k < 1$, or if $|z| > \|M\|^k/1/k$.

If $|z| > \|M\|^k/1/k$, then $z \in \rho(M)$. Hence if $\lambda \in \sigma(M)$, then $|\lambda| \leq \|M\|^k/1/k$. This is true for all $k$, so $r(M) \leq \liminf_{k \to \infty} \|M^k\|^{1/k}$.

For the other direction, if $z \in \mathbb{C}$ with $|z| < 1/r(M)$, then $|1/z| > r(M)$, and thus $1/z \notin \sigma(M)$ by the definition of $r(M)$. Hence $I - zM = z(z^{-1}I - M)$ if invertible if $z \neq 0$. Clearly $I - zM$ is invertible when $z = 0$ as well.

Suppose $\ell$ a linear functional on $\mathcal{L}$. The function $F(z) = \ell((I - zM)^{-1})$ is analytic in $B(0, 1/r(M)) \subset \mathbb{C}$. We know from complex analysis that a function has a Taylor series that converges absolutely in any disk on which the function is analytic. Therefore $F$ has a Taylor series which converges absolutely at each point of $B(0, 1/r(M))$. 

Let us identify the coefficients of the Taylor series. If $|z| < 1/\|M\|$, then
we see that
\[
F(z) = \ell \left( \sum_{n=0}^{\infty} z^n M^n \right) = \sum_{n=0}^{\infty} \ell(M^n) z^n.
\] (9.2)

Therefore $F^{(n)}(0) = n! \ell(M^n)$, where $F^{(n)}$ is the $n^{th}$ derivative of $F$. We conclude that the Taylor series for $F$ in $B(0, 1/r(M))$ is
\[
F(z) = \sum_{n=0}^{\infty} \ell(M^n) z^n.
\] (9.3)

The difference between (9.2) and (9.3) is that the former is valid in the ball $B(0, 1/\|M\|)$ while the latter is valid in $B(0, 1/r(M))$.

We conclude from this that
\[
\sum_{n=0}^{\infty} \ell(z^n M^n)
\]
converges absolutely for $z$ in the ball $B(0, 1/r(M))$, and consequently
\[
\lim_{n \to \infty} |\ell(z^n M^n)| = 0
\]
if $|z| < 1/r(M)$. By Lemma 9.5 there exists a real number $K$ such that
\[
\sup_n \|z^n M^n\| \leq K
\]
for all $n \geq 1$ and all $z \in B(0, 1/r(M))$. This implies that
\[
|z| \|M^n\|^{1/n} \leq K^{1/n},
\]
and hence
\[
|z| \limsup_{n \to \infty} \|M^n\|^{1/n} \leq 1
\]
if $|z| < 1/r(M)$. Thus
\[
\limsup_{n \to \infty} \|M^n\|^{1/n} \leq r(M),
\]
which completes the proof. \qed
9.2. FUNCTIONAL CALCULUS

We say \( \lambda \) is an eigenvalue for \( M \) with associated eigenvector \( x \neq 0 \) if \( Mx = \lambda x \). Note that not every element of \( \sigma(M) \) is an eigenvalue of \( M \). For example, if \( M : \ell^2 \to \ell^2 \) is defined by

\[
M(x_1, x_2, \ldots) = (x_1, x_2/2, x_3/4, \ldots),
\]

then \( 1, 1/2, 1/4, \ldots \) are eigenvalues. Since the spectrum is closed, then \( 0 \in \sigma(M) \), but \( 0 \) is not an eigenvalue for \( M \).

9.2 Functional calculus

We can define \( p(M) = \sum_{i=1}^{n} a_i M^i \) for any polynomial \( p \). Suppose \( f(z) = \sum_{j=0}^{\infty} a_j z^j \) is an analytic function with radius of convergence \( R \). If \( r(M) < R \), then there exists \( r(M) < S < R \), and for \( j \) sufficiently large, \( \|M^j\| \leq S^j \). Since

\[
\left\| \sum_{j=m+1}^{n} a_j M^j \right\| \leq \sum_{j=m+1}^{n} |a_j| \|M^j\| \leq \sum_{j=m+1}^{n} |a_j| S^j
\]

for \( m \) and \( n \) large enough and \( \sum |a_j| S^j \) converges (since \( S \) is less than the radius of convergence for \( f \)), then we can define \( f(M) \) for for any analytic function whose power series’ radius of convergence is larger than the spectral radius of \( M \) as the limit of the polynomials \( \sum_{j=0}^{n} a_j M^j \).

Let \( G \) be a domain containing \( \sigma(M) \), \( f \) analytic in \( G \), \( C \) a closed curve in \( G \cap \rho(M) \) whose winding number is 1 about each point in \( \sigma(M) \) and 0 about each point of \( G^c \). Define

\[
f(M) = \frac{1}{2\pi i} \int_C (z - M)^{-1} f(z) \, dz.
\]

By Cauchy’s theorem, this is independent of the contour chosen.

**Theorem 9.7**

(1) If \( f \) is a polynomial, the two definitions agree.

(2) Suppose \( f \) and \( g \) are analytic functions defined on a ball containing \( B(0, r(M)) \). Then \( f(M)g(M) = (fg)(M) \).
**Proof.** (1) Take a $c$ larger than $r(M)$ and let $C = \{|z| = c\}$. Expanding $(z - M)^{-1}$ in a power series, which is valid if $|z| = c$, we have
\[
\frac{1}{2\pi i} \int_{C} (z - M)^{-1} z^n \, dz = \frac{1}{2\pi i} \sum_{j=0}^{\infty} M^j \int_{C} z^{n-j-1} \, dz = M^n.
\]

(2) The radius of convergence of $fg$ is at least as large as the smaller of the radii of convergence of $f$ and $g$, and hence is larger than $r(M)$. (2) follows easily by approximating $f$ and $g$ by polynomials.

Here is the spectral mapping theorem for polynomials.

**Theorem 9.8** Suppose $A$ is a bounded linear operator and $P$ is a polynomial. Then $\sigma(P(A)) = P(\sigma(A))$.

By $P(\sigma(A))$ we mean the set $\{P(\lambda) : \lambda \in \sigma(A)\}$.

**Proof.** We first suppose $\lambda \in \sigma(P(A))$ and prove that $\lambda \in P(\sigma(A))$. Factor
\[
\lambda - P(x) = c(x - a_1) \cdots (x - a_n).
\]
Since $\lambda \in \sigma(P(A))$, then $\lambda - P(A)$ is not invertible, and therefore for at least one $i$ we must have that $A - a_i$ is not invertible. That means that $a_i \in \sigma(A)$. Since $a_i$ is a root of the equation $\lambda - P(x) = 0$, then $\lambda = P(a_i)$, which means that $\lambda \in P(\sigma(A))$.

Now suppose $\lambda \in P(\sigma(A))$. Then $\lambda = P(a)$ for some $a \in \sigma(A)$. We can write
\[
P(x) = \sum_{i=0}^{n} b_i x^i
\]
for some coefficients $b_i$, and then
\[
P(x) - P(a) = \sum_{i=1}^{n} b_i (x^i - a^i) = (x - a)Q(x)
\]
for some polynomial $Q$, since $x - a$ divides $x^i - a^i$ for each $i \geq 1$. We then have
\[
P(A) - \lambda = P(A) - P(a) = (A - a)Q(A).
\]
If $P(A) - \lambda$ were invertible, then by an earlier lemma we would have that $A - a$ is invertible, a contradiction. Therefore $P(A) - \lambda$ is not invertible, i.e., $\lambda \in \sigma(P(A))$. \hfill \square

9.3 Commutative Banach algebras

We look at commutative Banach algebras with a unit. Commutative means $MN = NM$ for all $M, N \in \mathcal{L}$.

$p$ is a multiplicative functional on $\mathcal{L}$ if $p$ is a homomorphism from $\mathcal{L}$ into $\mathbb{C}$.

**Proposition 9.9** Every homomorphism is a contraction.

**Proof.** $M = IM$, so $p(M) = p(IM) = p(I)p(M)$, or $p(I) = 1$. If $K$ is invertible,

$$p(K)p(K^{-1}) = p(KK^{-1}) = p(I) = 1,$$

so $p(K) \neq 0$. Suppose $|p(M)| > \|M\|$ for some $M$. Then if $B = M/p(M)$, we have $\|B\| < 1$, so $K = I - B$ is invertible. But

$$p(K) = p(I) - p(M/p(M)) = 1 - 1 = 0,$$

a contradiction.

Our goal in this section is to show that if $p(K) \neq 0$ for all homomorphisms, then $K$ is invertible.

$\mathcal{I} \subset \mathcal{L}$ is an ideal if $\mathcal{I}$ is a linear subspace, $\mathcal{I} \neq \{0\}$, and if $M \in \mathcal{L}$ and $J \in \mathcal{I}$, then $MJ \in \mathcal{I}$. $\mathcal{I}$ is a proper ideal if $\mathcal{I} \neq \mathcal{L}$.

As an example, let $\mathcal{L} = C(S)$, let $r \in S$, and let $\mathcal{I} = \{f : f(r) = 0\}$.

If $I \in \mathcal{I}$, then $\mathcal{I} = \mathcal{L}$. If $\mathcal{I}$ contains an invertible element, then $\mathcal{I}$ contains the identity, and hence equals $\mathcal{L}$.

**Lemma 9.10** Let $q$ be a homomorphism from $\mathcal{L}$ onto $\mathcal{A}$, but where $q$ is not an isomorphism and $q(\mathcal{L}) \neq 0$. Then
(1) \( \{ K \in \mathcal{L} : q(K) = 0 \} \) is a proper ideal. (This set is called the kernel of \( q \).

(2) If \( \mathcal{I} \) is a proper ideal, then \( \mathcal{I} \) is the kernel of some non-trivial homomorphism.

**Proof.** (1) is easy. For (2), let \( \mathcal{A} = \mathcal{L}/\mathcal{I} \). Let \( q \) map \( M \) into the equivalence class containing \( M \). Then the kernel of \( q \) is \( \mathcal{I} \).

**Proposition 9.11** If \( K \in \mathcal{L}, \ K \neq 0 \), and \( K \) not invertible, then \( K \) lies in some proper ideal.

**Proof.** Look at \( K\mathcal{L} = \{ KM : M \in \mathcal{L} \} \). Note \( K\mathcal{L} \) does not contain the identity.

**Lemma 9.12** Every proper ideal is contained in a maximal proper ideal.

**Proof.** Let \( \mathcal{J} \) be a proper ideal. Order the set of proper ideals that contain \( \mathcal{J} \) by inclusion. The union of a totally ordered subcollection will be an upper bound. (Note that if \( I \notin \mathcal{I}_\alpha \) for all \( \alpha \), then \( I \notin \bigcup_\alpha \mathcal{I}_\alpha \).) Then use Zorn’s lemma to find a maximal element of this collection. This element will be in the collection, and hence will be a proper ideal containing \( \mathcal{J} \).

A *division algebra* is one where every nonzero element is invertible.

**Proposition 9.13** If \( \mathcal{M} \) is a maximal proper ideal of \( \mathcal{L} \), then \( \mathcal{A} = \mathcal{L}/\mathcal{M} \) is a division algebra.

**Proof.** Suppose \( C \in \mathcal{A}, \ C \neq 0 \), and \( C \) is not invertible. Then \( \mathcal{J} = C\mathcal{A} = \{ CM : M \in \mathcal{A} \} \) is a proper ideal contained in \( \mathcal{A} \). Let \( q : \mathcal{L} \to \mathcal{L}/\mathcal{M} = \mathcal{A} \) be the usual map. \( \mathcal{R} = q^{-1}(\mathcal{J}) \) is easily checked to be a proper ideal in \( \mathcal{L} \). If \( M \in \mathcal{M} \), then \( q(M) = 0 \). So \( \mathcal{M} = q^{-1}(\{0\}) \) is contained in \( \mathcal{R} = q^{-1}(\mathcal{J}) \). \( \mathcal{M} \) is a proper subset of \( \mathcal{R} \) because \( \mathcal{J} \neq \{0\} \). This contradicts that \( \mathcal{M} \) is maximal.
Lemma 9.14 The closure of a proper ideal is a proper ideal.

Proof. The only thing to prove is that \( I \notin \overline{I} \). We know \( I \notin \mathcal{I} \), and so if \( N \in B(I,1) \), the ball of radius 1 about \( I \), then \( N \) is invertible, and hence not in \( \mathcal{I} \). So \( B(I,1) \) is an open set about \( I \) that is disjoint from \( \mathcal{I} \). Therefore \( I \notin \overline{I} \). \( \square \)

Lemma 9.15 If \( \mathcal{M} \) is a maximal proper ideal, then \( \mathcal{M} \) is closed.

Proof. If not, \( \overline{\mathcal{M}} \) is a proper ideal strictly larger than \( \mathcal{M} \). \( \square \)

Lemma 9.16 If \( \mathcal{I} \) is a closed ideal in \( \mathcal{L} \), then \( \mathcal{L}/\mathcal{I} \) is a Banach algebra.

Proposition 9.17 If \( \mathcal{A} \) is a Banach algebra with unit that is a division algebra, then \( \mathcal{A} \) is isomorphic to \( \mathbb{C} \).

Proof. If \( K \in \mathcal{A} \), there exists \( \kappa \in \sigma(K) \). So \( \kappa I - K \) is not invertible. Therefore \( \kappa I - K = 0 \), or \( K = \kappa I \). The map \( K \to \kappa \) is the desired isomorphism. \( \square \)

Theorem 9.18 \( K \in \mathcal{L} \) is invertible if and only if \( p(K) \neq 0 \) for all homomorphisms \( p \) of \( \mathcal{L} \) into \( \mathbb{C} \).

Proof. Suppose \( K \) is not invertible. \( K \) is in some maximal proper ideal \( \mathcal{M} \). Then \( \mathcal{M} \) is closed, \( \mathcal{L}/\mathcal{M} \) is a division algebra, and is isomorphic to \( \mathbb{C} \).

\[
p : \mathcal{L} \to \mathcal{L}/\mathcal{M} \to \mathbb{C}
\]

is a homomorphism onto \( \mathbb{C} \), and its null space is \( \mathcal{M} \). Since \( K \in \mathcal{M} \), then \( p(K) = 0 \). \( \square \)
9.4 Absolutely convergent Fourier series

Let \( \mathcal{L} \) be the set of continuous functions from the unit circle \( S^1 \) to the complex functions such that \( f(\theta) = \sum c_n e^{in\theta} \) with \( \sum |c_n| < \infty \). We let the norm of \( f \) be \( \sum |c_n| \).

We check that \( \mathcal{L} \) is a Banach algebra. To do that, we use the fact that the Fourier coefficients for \( fg \) are the convolution of those for \( f \) and those for \( g \), and that the convolution of two \( \ell^1 \) functions is in \( \ell^1 \), so \( fg \in \mathcal{L} \). Here is the verification. If \( f \) has Fourier coefficients \( a_n \) and \( g \) has Fourier coefficients \( b_n \), let

\[
c_n = \sum_j a_j b_{n-j}.
\]

Then to see that \( c_n \) are the Fourier coefficients of \( fg \), write

\[
\sum_n c_n e^{inx} = \sum_n \sum_j a_j b_{n-j} e^{i(n-j)x} e^{ijx} = \sum_k \sum_j a_j b_k e^{ikx} e^{ijx} = f(x)g(x).
\]

To see that the norm of \( fg \) is less than or equal to the norm of \( f \) times the norm of \( g \),

\[
\sum_n |c_n| \leq \sum_n \sum_j |a_j| |b_{n-j}| = \sum_k \sum_j |a_j| |b_k|.
\]

If \( w \in S^1 \), set \( p_w(f) = f(w) \). \( p_w \) is a homomorphism from \( \mathcal{L} \) to \( \mathbb{C} \).

**Proposition 9.19** If \( p \) is a homomorphism from \( \mathcal{L} \) to \( \mathbb{C} \), then there exists \( w \) such that \( p(f) = f(w) \) for all \( f \in \mathcal{L} \).

**Proof.** \( p(I) = 1 \) and \( |p(M)| \leq \|M\| \), so \( p \) has norm 1. Then

\[
|p(e^{i\theta})| \leq 1, \quad |p(e^{-i\theta})| \leq 1,
\]

and

\[
1 = p(1) = p(e^{i\theta})p(e^{-i\theta}).
\]

We must have \( |p(e^{i\theta})| = 1 \), or we would have inequality in the above.
Therefore there exists \( w \) such that \( p(e^{i\theta}) = e^{iw} \). Since \( p \) is a homomorphism, by induction \( p(e^{in\theta}) = e^{inw} \). By linearity,

\[
p\left(\sum_{n=-N}^{N} c_n e^{in\theta}\right) = \sum_{n=-N}^{N} c_n e^{inw}.
\]

If \( f \in \mathcal{L} \), since \( p \) is continuous and \( \sum |c_n| < \infty \), we have \( p(f) = f(w) \).

**Theorem 9.20** Suppose \( f \) has an absolutely convergent Fourier series and \( f \) is never 0 on \( S^1 \). Then \( 1/f \) also has an absolutely convergent Fourier series.

**Proof.** If \( p \) is a homomorphism on \( \mathcal{L} \), then \( p(f) = f(w) \) for some \( w \). Since \( f \) is never 0, \( p(f) \neq 0 \) for all non-trivial homomorphisms \( p \). This implies \( f \) is invertible in \( \mathcal{L} \). \( \square \)
Chapter 10

Compact maps

10.1 Basic properties

A subset $S$ is precompact if $\overline{S}$ is compact. Recall that if $A$ is a subset of a metric space, $A$ is precompact if and only if every sequence in $A$ has a subsequence which converges in $\overline{A}$. Also, $A$ is compact if and only if $A$ is complete and totally bounded. Write $B_1$ for the unit ball in $X$.

A map $K$ from a Banach space $X$ to a Banach space $U$ is compact if $K(B_1)$ is precompact in $U$.

One example is if $K$ is degenerate, so that $R_K$ is finite dimensional. The identity on $\ell^2$ is not compact.

The following facts are easy:

1. If $C_1, C_2$ are precompact subsets of a Banach space, then $C_1 + C_2$ is precompact.
2. If $C$ is precompact, so is the convex hull of $C$.
3. If $M : X \to U$ and $C$ is precompact in $X$, then $M(C)$ is precompact in $U$.

**Proposition 10.1** (1) If $K_1$ and $K_2$ are compact maps, so is $kK_1 + K_2$.

(2) If $X \xrightarrow{L} U \xrightarrow{M}$, where $M$ is bounded and $L$ is compact, then $ML$ is compact.
CHAPTER 10. COMPACT MAPS

(3) In the same situation as (2), if \( L \) is bounded and \( M \) is compact, then \( MLM \) is compact.

(4) If \( K_n \) are compact maps and \( \lim \|K_n - K\| = 0 \), then \( K \) is compact.

Proof. (1) For the sum, \((K_1 + K_2)(B_1) \subset K_1(B_1) + K_2(B_1)\), and the multiplication by \( k \) is similar.

(2) \( ML(B_1) \) will be compact because \( L(B_1) \) is compact and \( M \) is continuous.

(3) \( L(B_1) \) will be contained in some ball, so \( ML(B_1) \) is precompact.

(4) Let \( \varepsilon > 0 \). Choose \( n \) such that \( \|K_n - K\| < \varepsilon \). \( K_n(B_1) \) can be covered by finitely many balls of radius \( \varepsilon \), so \( K(B_1) \) is covered by the set of balls with the same centers and radius \( 2\varepsilon \). Therefore \( K(B_1) \) is totally bounded. \( \Box \)

We can use (4) to give a more complicated example of a compact operator. Let \( X = U = \ell^2 \) and define

\[ K(a_1, a_2, \ldots) = (a_1/2, a_2/2^2, a_3/2^3, \ldots). \]

It is the limit in norm of \( K_n \), where

\[ K_n(a_1, a_2, \ldots) = (a_1/2, a_2/2^2, \ldots, a_n/2^n, 0, \ldots). \]

Note that any bounded operator \( K \) on \( \ell^2 \) maps \( B_1 \) into a set of the form \([−M, M]^\mathbb{N}\). By Tychonoff, this is compact in the product topology. However it is not necessarily compact in the topology of the space \( \ell^2 \).

Proposition 10.2 If \( X \) and \( Y \) are Banach spaces and \( K : X \to Y \) is compact and \( Z \) is a closed subspace of \( X \), then the map \( K |_Z \) is compact.

Let \( A \) be a bounded linear operator on a Banach space. If \( z \) is a complex number and \( I \) is the identity operator, then \( zI - A \) is a bounded linear operator on \( H \) which might or might not be invertible. We define the spectrum of \( A \) by

\[ \sigma(A) = \{ z \in \mathbb{C} : zI - A \text{ is not invertible} \}. \]

We sometimes write \( z - A \) for \( zI - A \). The resolvent set for \( A \) is the set of complex numbers \( z \) such that \( z - A \) is invertible. A non-zero element \( z \) is an eigenvector for \( A \) with corresponding eigenvalue \( \lambda \) if \( Az = \lambda z \).
10.2 Compact symmetric operators

If $A$ is a bounded operator on $H$, a Hilbert space over the complex numbers, the adjoint of $A$, denoted $A^*$, is the operator on $H$ such that $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all $x$ and $y$.

It follows from the definition that the adjoint of $cA$ is $cA^*$ and the adjoint of $A^n$ is $(A^*)^n$. If $P(x) = \sum_{j=0}^{n} a_j x^j$ is a polynomial, the adjoint of $P(A) = \sum_{j=0}^{n} a_j A^j$ will be $P(A^*) = \sum_{j=0}^{n} \overline{a_j} P(A^*)$.

The adjoint operator always exists.

**Proposition 10.3** If $A$ is a bounded operator on $H$, there exists a unique operator $A^*$ such that $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all $x$ and $y$.

**Proof.** Fix $y$ for the moment. The function $f(x) = \langle Ax, y \rangle$ is a linear functional on $H$. By the Riesz representation theorem for Hilbert spaces, there exists $z_y$ such that $\langle Ax, y \rangle = \langle x, z_y \rangle$ for all $x$. Since

$$\langle x, z_{y_1} + z_{y_2} \rangle = \langle Ax, y_1 + y_2 \rangle = \langle Ax, y_1 \rangle + \langle Ax, y_2 \rangle = \langle x, z_{y_1} \rangle + \langle x, z_{y_2} \rangle$$

for all $x$, then $z_{y_1} + z_{y_2} = z_{y_1} + z_{y_2}$ and similarly $z_{cy} = cz_y$. If we define $A^* y = z_y$, this will be the operator we seek.

If $A_1$ and $A_2$ are two operators such that $\langle x, A_1 y \rangle = \langle Ax, y \rangle = \langle x, A_2 y \rangle$ for all $x$ and $y$, then $A_1 y = A_2 y$ for all $y$, so $A_1 = A_2$. Thus the uniqueness assertion is proved.

A bounded linear operator $A$ mapping $H$ into $H$ is called symmetric if $\langle Ax, y \rangle = \langle x, Ay \rangle$ (10.1) for all $x$ and $y$ in $H$. Other names for symmetric are *Hermitian* or *self-adjoint*. When $A$ is symmetric, then $A^* = A$, which explains the name “self-adjoint.”
Example 10.4 For an example of a symmetric bounded linear operator, let \((X, \mathcal{A}, \mu)\) be a measure space with \(\mu\) a \(\sigma\)-finite measure, let \(H = L^2(X)\), and let \(F(x, y)\) be a jointly measurable function from \(X \times X\) into \(\mathbb{C}\) such that \(F(y, x) = \overline{F(x, y)}\) and
\[
\int \int F(x, y)^2 \mu(dx) \mu(dy) < \infty. \tag{10.2}
\]
Define \(A : H \to H\) by
\[
Af(x) = \int F(x, y) f(y) \mu(dy). \tag{10.3}
\]
You can check that \(A\) is a bounded symmetric operator.

Here is an example of a compact symmetric operator.

Example 10.5 Let \(H = L^2([0, 1])\) and let \(F : [0, 1]^2 \to \mathbb{R}\) be a continuous function with \(F(x, y) = \overline{F(y, x)}\) for all \(x, y\). Define \(K : H \to H\) by
\[
Kf(x) = \int_0^1 F(x, y) f(y) dy.
\]
We discussed in Example 10.4 the fact that \(K\) is a bounded symmetric operator. Let us show that it is compact.

If \(f \in L^2([0, 1])\) with \(\|f\| \leq 1\), then
\[
|Kf(x) - Kf(x')| = \left| \int_0^1 [F(x, y) - F(x', y)] f(y) dy \right|
\leq \left( \int_0^1 |F(x, y) - F(x', y)|^2 dy \right)^{1/2} \|f\|,
\]
using the Cauchy-Schwarz inequality. Since \(F\) is continuous on \([0, 1]^2\), which is a compact set, then it is uniformly continuous there. Let \(\varepsilon > 0\). There exists \(\delta\) such that
\[
\sup_{|x - x'| < \delta} \sup_y |F(x, y) - F(x', y)| < \varepsilon.
\]
Hence if \(|x - x'| < \delta\), then \(|Kf(x) - Kf(x')| < \varepsilon\) for every \(f\) with \(|f| \leq 1\). In other words, \(\{Kf : |f| \leq 1\}\) is an equicontinuous family.
Since $F$ is continuous, it is bounded, say by $N$, and therefore
\[ |Kf(x)| \leq \int_{0}^{1} N|f(y)| \, dy \leq N\|f\|, \]
again using the Cauchy-Schwarz inequality. If $Kf_n$ is a sequence in $K(B_1)$, then $\{Kf_n\}$ is a bounded equicontinuous family of functions on $[0, 1]$, and by the Ascoli-Arzelà theorem, there is a subsequence which converges uniformly on $[0, 1]$. It follows that this subsequence also converges with respect to the $L^2$ norm. Since every sequence in $K(B_1)$ has a subsequence which converges, the closure of $K(B_1)$ is compact. Thus $K$ is a compact operator.

We have the following proposition.

**Proposition 10.6** Suppose $A$ is a bounded symmetric operator.
(1) $(Ax, x)$ is real for all $x \in H$.
(2) The function $x \to \langle Ax, x \rangle$ is not identically 0 unless $A = 0$.
(3) $\|A\| = \sup_{\|x\|=1} |\langle Ax, x \rangle|$.

**Proof.** (1) This one is easy since
\[ \langle Ax, x \rangle = \langle x, Ax \rangle = \overline{\langle Ax, x \rangle}, \]
where we use $\overline{z}$ for the complex conjugate of $z$.

(2) If $\langle Ax, x \rangle = 0$ for all $x$, then
\[
0 = \langle A(x + y), x + y \rangle = \langle Ax, x \rangle + \langle Ay, y \rangle + \langle Ax, y \rangle + \langle Ay, x \rangle = \langle Ax, y \rangle + \langle y, Ax \rangle = \langle Ax, y \rangle + \overline{\langle Ax, y \rangle}.
\]
Hence $\text{Re} \langle Ax, y \rangle = 0$. Replacing $x$ by $ix$ and using linearity,
\[
\text{Im} (\langle Ax, y \rangle) = -\text{Re} (i\langle Ax, y \rangle) = -\text{Re} (\langle A(ix), y \rangle) = 0.
\]
Therefore $\langle Ax, y \rangle = 0$ for all $x$ and $y$. We conclude $Ax = 0$ for all $x$, and thus $A = 0$.

(3) Let $\beta = \sup_{\|x\|=1} |\langle Ax, x \rangle|$. By the Cauchy-Schwarz inequality,
\[ |\langle Ax, x \rangle| \leq \|Ax\| \|x\| \leq \|A\| \|x\|^2, \]
so $\beta \leq \|A\|$. 

To get the other direction, let $\|x\| = 1$ and let $y \in H$ such that $\|y\| = 1$ and $\langle y, Ax \rangle$ is real. Then

$$\langle y, Ax \rangle = \frac{1}{2}(\langle x + y, A(x + y) \rangle - \langle x - y, A(x - y) \rangle).$$

We used that $\langle y, Ax \rangle = \langle Ay, x \rangle = \langle Ax, y \rangle = \langle x, Ay \rangle$ since $\langle y, Ax \rangle$ is real and $A$ is symmetric. Then

$$16|\langle y, Ax \rangle|^2 \leq \beta^2(\|x + y\|^2 + \|x - y\|^2)^2 = 4\beta^2(\|x\|^2 + \|y\|^2)^2 = 16\beta^2.$$

We used the parallelogram law (equation $(4.1)$) in the first equality. We conclude $|\langle y, Ax \rangle| \leq \beta$.

If $\|y\| = 1$ but $\langle y, Ax \rangle = re^{i\theta}$ is not real, let $y' = e^{-i\theta}y$ and apply the above with $y'$ instead of $y$. We then have

$$|\langle y, Ax \rangle| = |\langle y', Ax \rangle| \leq \beta.$$ Setting $y = Ax/\|Ax\|$, we have $\|Ax\| \leq \beta$. Taking the supremum over $x$ with $\|x\| = 1$, we conclude $\|A\| \leq \beta$. \hfill \Box

If $\langle Ax, x \rangle \geq 0$ for all $x$, we say $A$ is positive, and write $A \geq 0$. Writing $A \leq B$ means $B - A \geq 0$. For matrices, one uses the words “positive definite.”

Now suppose $A$ is compact.

**Proposition 10.7** If $x_n \xrightarrow{w} x$, then $Ax_n \xrightarrow{s} Ax$.

**Proof.** If $x_n \xrightarrow{w} x$, then $Ax_n \xrightarrow{w} Ax$, since $\langle Ax_n, y \rangle = \langle x_n, Ay \rangle \to \langle x, Ay \rangle = \langle Ax, y \rangle$. If $x_n$ converges weakly, then $\|x_n\|$ is bounded so $Ax_n$ lies in a precompact set.

Any subsequence of $Ax_n$ has a further subsequence which converges strongly. The limit must be $Ax$. \hfill \Box

We will use the following easy lemma repeatedly.
Lemma 10.8 If $K$ is a compact operator and $\{x_n\}$ is a sequence with $\|x_n\| \leq 1$ for each $n$, then $\{Kx_n\}$ has a convergent subsequence.

Proof. Since $\|x_n\| \leq 1$, then $\{\frac{1}{2}x_n\} \subset B_1$. Hence $\{\frac{1}{2}Kx_n\} = \{K(\frac{1}{2}x_n)\}$ is a sequence contained in $\overline{K(B_1)}$, a compact set and therefore has a convergent subsequence. \qed

We now prove the spectral theorem for compact symmetric operators.

Theorem 10.9 Suppose $H$ is a separable Hilbert space over the complex numbers and $K$ is a compact symmetric linear operator. There exist a sequence $\{z_n\}$ in $H$ and a sequence $\{\lambda_n\}$ in $\mathbb{R}$ such that

1. $\{z_n\}$ is an orthonormal basis for $H$,
2. each $z_n$ is an eigenvector with eigenvalue $\lambda_n$, that is, $Kz_n = \lambda_n z_n$,
3. for each $\lambda_n \neq 0$, the dimension of the linear space $\{x \in H : Kx = \lambda_n x\}$ is finite,
4. the only limit point, if any, of $\{\lambda_n\}$ is 0; if there are infinitely many distinct eigenvalues, then 0 is a limit point of $\{\lambda_n\}$.

Note that part of the assertion of the theorem is that the eigenvalues are real. (3) is usually phrased as saying the non-zero eigenvalues have finite multiplicity.

Proof. If $K = 0$, any orthonormal basis will do for $\{z_n\}$ and all the $\lambda_n$ are zero, so we suppose $K \neq 0$. We first show that the eigenvalues are real, that eigenvectors corresponding to distinct eigenvalues are orthogonal, the multiplicity of non-zero eigenvalues is finite, and that 0 is the only limit point of the set of eigenvalues. We then show how to sequentially construct a set of eigenvectors and that this construction yields a basis.

If $\lambda_n$ is an eigenvalue corresponding to a eigenvector $z_n \neq 0$, we see that

$$\lambda_n \langle z_n, z_n \rangle = \langle \lambda_n z_n, z_n \rangle = \langle Kz_n, z_n \rangle = \langle z_n, Kz_n \rangle$$

$$= \langle z_n, \lambda_n z_n \rangle = \lambda_n \langle z_n, z_n \rangle,$$

which proves that $\lambda_n$ is real.
If \( \lambda_n \neq \lambda_m \) are two distinct eigenvalues corresponding to the eigenvectors \( z_n \) and \( z_m \), we observe that

\[
\lambda_n \langle z_n, z_m \rangle = \langle K z_n, z_m \rangle = \langle z_n, \lambda_m z_m \rangle = \lambda_m \langle z_n, z_m \rangle,
\]

using that \( \lambda_m \) is real. Since \( \lambda_n \neq \lambda_m \), we conclude \( \langle z_n, z_m \rangle = 0 \).

Suppose \( \lambda_n \neq 0 \) and that there are infinitely many orthonormal vectors \( x_k \) such that \( K x_k = \lambda_n x_k \). Then

\[
\| x_k - x_j \|^2 = \langle x_k - x_j, x_k - x_j \rangle = \| x_k \|^2 - 2 \langle x_k, x_j \rangle + \| x_j \|^2 = 2
\]

if \( j \neq k \). But then no subsequence of \( \lambda_n x_k = K x_k \) can converge, a contradiction to Lemma 10.8. Therefore the multiplicity of \( \lambda_n \) is finite.

Suppose we have a sequence of distinct non-zero eigenvalues converging to a real number \( \lambda \neq 0 \) and a corresponding sequence of eigenvectors each with norm one. Since \( K \) is compact, there is a subsequence \( \{n_j\} \) such that \( K z_{n_j} \) converges to a point in \( H \), say \( w \). Then

\[
z_{n_j} = \frac{1}{\lambda_{n_j}} K z_{n_j} \to \frac{1}{\lambda} w,
\]

or \( \{z_{n_j}\} \) is an orthonormal sequence of vectors converging to \( \lambda^{-1} w \). But as in the preceding paragraph, we cannot have such a sequence.

Since \( \{\lambda_n\} \subset \overline{B(0, r(K))} \), a bounded subset of the complex plane, if the set \( \{\lambda_n\} \) is infinite, there will be a subsequence which converges. By the preceding paragraph, 0 must be a limit point of the subsequence.

We now turn to constructing eigenvectors. By Lemma 10.6(3), we have

\[
\|K\| = \sup_{\|x\|=1} |\langle K x, x \rangle|.
\]

We claim the maximum is attained. If \( \sup_{\|x\|=1} \langle K x, x \rangle = \|K\| \), let \( \lambda = \|K\| \); otherwise let \( \lambda = -\|K\| \). Choose \( x_n \) with \( \|x_n\| = 1 \) such that \( \langle K x_n, x_n \rangle \) converges to \( \lambda \). There exists a subsequence \( \{n_j\} \) such that \( K x_{n_j} \) converges, say to \( z \). Since \( \lambda \neq 0 \), then \( z \neq 0 \), for otherwise \( \lambda = \lim_{j \to \infty} \langle K x_{n_j}, x_{n_j} \rangle = 0 \). Now

\[
\| (K - \lambda I) z \|^2 = \lim_{j \to \infty} \| (K - \lambda I) K x_{n_j} \|^2 \\
\leq \|K\|^2 \lim_{j \to \infty} \| (K - \lambda I) x_{n_j} \|^2
\]
and
\[
\|(K - \lambda I)x_n\|^2 = \|Kx_n\|^2 + \lambda^2\|x_n\|^2 - 2\lambda \langle x_n, Kx_n \rangle \\
\leq \|K\|^2 + \lambda^2 - 2\lambda \langle x_n, Kx_n \rangle \\
\rightarrow \lambda^2 + \lambda^2 - 2\lambda^2 = 0.
\]

Therefore \((K - \lambda I)z = 0\), or \(z\) is an eigenvector for \(K\) with corresponding eigenvalue \(\lambda\).

Suppose we have found eigenvalues \(z_1, z_2, \ldots, z_n\). Let \(X_n\) be the linear subspace spanned by \(\{z_1, \ldots, z_n\}\) and let \(Y = X_n^\perp\) be the orthogonal complement of \(X_n\), that is, the set of all vectors orthogonal to every vector in \(X_n\). If \(x \in Y\) and \(k \leq n\), then
\[
\langle Kx, z_k \rangle = \langle x, Kz_k \rangle = \bar{\lambda}_k \langle x, z_k \rangle = 0,
\]
or \(Kx \in Y\). Hence \(K\) maps \(Y\) into \(Y\). It is an exercise to show that \(K|_Y\) is a compact symmetric operator. If \(Y\) is non-zero, we can then look at \(K|_Y\), and find a new eigenvector \(z_{n+1}\).

It remains to prove that the set of eigenvectors forms a basis. Suppose \(y\) is orthogonal to every eigenvector. Then
\[
\langle Ky, z_k \rangle = \langle y, Kz_k \rangle = \langle y, \lambda_k z_k \rangle = 0
\]
if \(z_k\) is an eigenvector with eigenvalue \(\lambda_k\), so \(Ky\) is also orthogonal to every eigenvector. Suppose \(X\) is the closure of the linear subspace spanned by \(\{z_k\}\), \(Y = X^\perp\), and \(Y \neq \{0\}\). If \(y \in Y\), then \(\langle Ky, z_k \rangle = 0\) for each eigenvector \(z_k\), hence \(\langle Ky, z \rangle = 0\) for every \(z \in X\), or \(K : Y \rightarrow Y\). Thus \(K|_Y\) is a compact symmetric operator, and by the argument already given, there exists an eigenvector for \(K|_Y\). This is a contradiction since \(Y\) is orthogonal to every eigenvector. \(\square\)

**Remark 10.10** If \(\{z_n\}\) is an orthonormal basis of eigenvectors for \(K\) with corresponding eigenvalues \(\lambda_n\), let \(E_n\) be the projection onto the subspace spanned by \(z_n\), that is, \(E_n x = \langle x, z_n \rangle z_n\). A vector \(x\) can be written as \(\sum_n \langle x, z_n \rangle z_n\), thus \(Kx = \sum_n \lambda_n \langle x, z_n \rangle z_n\). We can then write
\[
K = \sum_n \lambda_n E_n.
\]
For general bounded symmetric operators there is a related expansion where the sum gets replaced by an integral, which we’ll do later on.

**Remark 10.11** If $z_n$ is an eigenvector for $K$ with corresponding eigenvalue $\lambda_n$, then $Kz_n = \lambda_n z_n$, so

$$K^2 z_n = K(Kz_n) = K(\lambda_n z_n) = \lambda_n Kz_n = (\lambda_n)^2 z_n.$$  

More generally, $K^j z_n = (\lambda_n)^j z_n$. Using the notation of Remark 10.10, we can write

$$K^j = \sum_n (\lambda_n)^j E_n.$$  

If $Q$ is any polynomial, we can then use linearity to write

$$Q(K) = \sum_n Q(\lambda_n) E_n.$$  

It is a small step from here to make the definition

$$f(K) = \sum_n f(\lambda_n) E_n$$  

for any bounded and Borel measurable function $f$.

If $\alpha_1 \geq \alpha_2 \geq \cdots > 0$ and $Az_n = \alpha_n z_n$, then our construction shows that

$$\alpha_N = \max_{x\perp z_1, \ldots, z_{N-1}} \frac{\langle Ax, x \rangle}{\|x\|^2}.$$  

This is known as the **Rayleigh principle**.

Let

$$R_A(x) = \frac{\langle Ax, x \rangle}{\|x\|^2}.$$  

**Proposition 10.12** Let $A$ be compact and symmetric and let $\alpha_k$ be the non-negative eigenvalues with $\alpha_1 \geq \alpha_2 \geq \cdots$. Then

(1) (Fisher’s principle)

$$\alpha_N = \max_{S_N} \min_{x \in S_N} R_A(x),$$
where the maximum is over all linear subspaces $S_N$ of dimension $N$.

(2) (Courant’s principle)

$$\alpha_N = \min_{S_{N-1}} \max_{x \perp S_{N-1}} R_A(x),$$

where the minimum is over all linear subspaces of dimension $N - 1$.

**Proof.** Let $z_1, \ldots, z_N$ be eigenvectors with corresponding eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N$. Let $T_N$ be the linear subspace spanned by $\{z_1, \ldots, z_N\}$. If $y \in T_N$, we have $y = \sum_{j=1}^{N} c_j z_j$ for some complex numbers $c_j$ and then

$$\langle Ay, y \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} c_i \overline{c_j} \langle Az_i, z_j \rangle = \sum_{i} \sum_{j} c_i \overline{c_j} \alpha_i \langle z_i, z_j \rangle$$

$$= \sum_{i} |c_i|^2 \alpha_i \geq \sum_{i} |c_i|^2 \alpha_N$$

$$= \langle y, y \rangle$$

using the fact that the $z_i$ are orthogonal by our construction.

(1) Let $z_k$ be the eigenvectors. Let $S_N$ be a subspace of dimension $N$. There exists $y \in S_N$ such that $\langle y, z_k \rangle = 0$ for $k = 1, \ldots, N - 1$. Since

$$\alpha_N = \max_{x \perp z_1, \ldots, z_{N-1}} R_A(x),$$

then $y$ is one of the vectors over which the max is being taken, so $R_A(y) \leq \alpha_N$ for this $y$. So $\min_{x \in S_N} R_A(x) \leq \alpha_N$. This is true for all spaces of dimension $N$. So the right hand side is less than or equal to $\alpha_N$.

Now we show the right hand side is greater than or equal to $\alpha_N$. Let $S_N$ be the linear span of $\{z_1, \ldots, z_N\}$. By the first paragraph of the proof, $R_A(x) \geq \alpha_N$ for every $x \in S_N$, and $R_A(x) = \alpha_N$ when $x = z_N$. So $\min_{x \in S_N} R_A(x) = \alpha_N$. The maximum over all subspaces of dimension $N$ will be larger than the value for this particular subspace, so the right hand side is at least as large as $\alpha_N$.

(2) Let $S_{N-1}$ be a subspace of dimension $N - 1$ and let $T_N$ be the span of $\{z_1, \ldots, z_N\}$. Since the dimension of $T_N$ is larger than that of $S_{N-1}$,
there must be a vector \( y \in T_N \) perpendicular to \( S_{N-1} \). Since \( y \in T_N \), then \( R_A(y) \geq \alpha_N \) by the first paragraph of this proof, so

\[
\max_{x \perp S_{N-1}} R_A(x) \geq R_A(y) \geq \alpha_N.
\]

Taking the minimum over all spaces \( S_{N-1} \) shows that right hand side is greater than or equal to \( \alpha_N \).

If \( x \perp T_{N-1} \), then \( x = \sum_{j=N+1}^{\infty} c_j z_j \), and then

\[
\langle Ax, x \rangle = \sum_{j=N}^{\infty} \sum_{k=N}^{\infty} c_j \overline{c_k} \alpha_j \langle z_j, z_k \rangle
\]

\[
= \sum_{j=N}^{\infty} \alpha_j |c_j|^2 \leq \alpha_N \sum_{j=N}^{\infty} |c_j|^2
\]

\[
= \alpha_N \langle x, x \rangle.
\]

Therefore \( R_A(x) \leq \alpha_N \). This leads to

\[
\min_{S_{N-1}} \max_{x \perp S_{N-1}} R_A(x) \leq \max_{x \perp T_{N-1}} R_A(x) \leq \alpha_N,
\]

since \( T_{N-1} \) is a particular subspace of dimension \( N - 1 \).

\[ \square \]

**Proposition 10.13** Suppose \( A \leq B \) with eigenvalues \( \alpha_k, \beta_k \), resp., ordered to be decreasing. Then \( \alpha_k \leq \beta_k \) for all \( k \).

**Proof.** \( A \leq B \) implies \( \langle Ax, x \rangle \leq \langle Bx, x \rangle \), so \( R_A(x) \leq R_B(x) \). Now use either Fisher’s or Courant’s principle. \[ \square \]

**10.3 Mercer’s theorem**

We will need to use **Dini’s theorem** from analysis.

**Proposition 10.14** Suppose \( g_n \) are continuous functions on \([0,1]\) with \( g_n(x) \leq g_{n+1}(x) \) for each \( n \) and \( x \) and \( g_\infty(x) = \lim_{n \to \infty} g_n(x) \) is continuous. Then \( g_n \) converges to \( g \) uniformly.
10.3. MERCER’S THEOREM

Proof. Let $f_n = g_n - g$, so the $f_n$ are continuous and decrease to 0. Let $\varepsilon > 0$. If $G_n(x) = \{x \in [0,1] : f_n(x) < \varepsilon\}$, then $G_n$ is an open set (with respect to the relative topology on $[0,1]$), since $f_n$ is continuous. Since $f_n(x) \rightarrow 0$, each $x$ will be in some $G_n$. Thus $\{G_n\}$ is an open cover for $[0,1]$. Let $G_{n_1}, \ldots, G_{n_m}$ be a finite subcover. If $n \geq \max(n_1, \ldots, n_m)$ and $x \in [0,1]$, then $x$ is in some $G_{n_j}$ and $f_n(x) \leq f_{n_j}(x) < \varepsilon$. Thus the convergence is uniform.

Define $K : L^2[0,1] \rightarrow L^2[0,1]$ by

$$ Ku(x) = \int_0^1 K(x,y)u(y) \, dy. $$

$K^*$ has kernel $\overline{K(y,x)}$.

Suppose $K$ is continuous, symmetric, and real-valued. Then $K$ is compact, as we showed before. Therefore there exists a complete orthonormal system $\{e_j\}$ of eigenvectors. Let $\kappa_j$ be the eigenvalue corresponding to $e_j$. $K : L^2 \rightarrow C[0,1]$, so $e_j = \kappa_j^{-1}Ke_j$ is continuous if $\kappa_j \neq 0$.

**Theorem 10.15** (Mercer) Suppose $K$ is real-valued, symmetric, and continuous. Suppose $K$ is positive: $\langle Ku, u \rangle \geq 0$ for all $u \in H$. Then

$$ K(x,y) = \sum_j \kappa_j e_j(x)e_j(y), $$

and the series converges uniformly and absolutely.

An example is to let $K = P_t$, the transition density of absorbing or reflecting Brownian motion.

Proof. First we observe that the $\kappa_j$ are non-negative. To see this, let $u = e_j$, and we have

$$ 0 \leq \langle e_j, Ke_j \rangle = \kappa_j \langle e_j, e_j \rangle. $$

$K \geq 0$ on the diagonal: Suppose $K(r,r) < 0$ for some $r$. Then $K(x,y) < 0$ if $|x-r|, |y-r| < \delta$ for some $\delta$. Take $u = \chi_{[r-\delta/2,r+\delta/2]}$. Then

$$ \langle Ku, u \rangle = \int \int K(x,y)u(y)x(s) \, ds \, dt < 0, $$
Let $K_N(x, y) = \sum_{j=1}^{N} \kappa_j e_j(x)e_j(y)$. If $f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k$, we have

$$K_N f(x) = \int_{0}^{1} \sum_{j=1}^{N} \kappa_j e_j(x)e_j(y) \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k(y) \, dy = \sum_{j=1}^{N} \kappa_j \langle f, e_j \rangle e_j(x).$$

We have

$$K f(x) = \sum_{j=1}^{\infty} \langle f, e_j \rangle Ke_j(x) = \sum_{j=1}^{\infty} \langle f, e_j \rangle \kappa_j e_j(x).$$

We conclude that $K - K_N$ is a positive operator, since

$$\langle f, (K - K_N)f \rangle = \sum_{k=1}^{\infty} \sum_{j=1}^{N} \kappa_j |\langle f, e_j \rangle|^2 \langle e_k, e_j \rangle = \sum_{j=1}^{N} \kappa_j |\langle f, e_j \rangle|^2 \geq 0.$$}

As above, $K - K_N$ is non-negative on the diagonal, which implies that

$$\sum_{j=1}^{N} \kappa_j |e_j(x)|^2 \leq K(x, x).$$

Each term is non-negative, so the sum converges for each $x$. Let $J(x)$ be the limit.

Let $M = \sup_{x, y \in [0, 1]} |K(x, y)|$. By Cauchy-Schwarz,

$$|K_N(x, y)| \leq \left( \sum_{j=1}^{N} \kappa_j |e_j(x)|^2 \right)^{1/2} \left( \sum_{j=1}^{N} \kappa_j |e_j(y)|^2 \right)^{1/2} = (K_N(x, x))^{1/2} (K_N(y, y))^{1/2}.$$
10.3. MERCER’S THEOREM

Fix $x$. By the same argument,

$$\left| \sum_{j=m}^{n} \kappa_j e_j(x)e_j(y) \right|$$

$$\leq \left( \sum_{j=m}^{n} \kappa_j |e_j(x)|^2 \right)^{1/2} \left( \sum_{j=m}^{n} \kappa_j |e_j(y)|^2 \right)^{1/2}$$

$$\leq \left( \sum_{j=m}^{n} \kappa_j |e_j(x)|^2 \right)^{1/2} M^{1/2}.$$ 

The last line goes to $0$ as $m, n \to \infty$ since $K_N(x,x) \to J(x) \leq M$. Therefore, for each $x$, the functions $K_N(x, \cdot)$ converge uniformly. Let’s call the limit $L(x,y)$. Then $L(x,y)$ will be continuous in $y$ for each $x$.

Given $f$, let

$$f_N(x) = \sum_{j=1}^{N} \langle f, e_j \rangle e_j(x).$$

Note

$$Kf_N(x) = \sum_{j=1}^{N} \langle f, e_j \rangle Ke_j(x)$$

$$= \sum_{j=1}^{N} \langle f, e_j \rangle \kappa_j e_j(x)$$

$$= K_N f(x).$$

We have

$$\|f - f_N\|^2 = \sum_{j=N+1}^{\infty} |\langle f, e_j \rangle|^2 \to 0$$

as $N \to \infty$ by Bessel’s inequality, so

$$|Kf(x) - Kf_N(x)| \leq \int_{0}^{1} |K(x,y)||f(y) - f_N(y)| dy \leq M \|f - f_N\|$$

by Cauchy-Schwarz. Therefore $K_N f(x) \to Kf(x)$ as $N \to \infty$.

By dominated convergence,

$$K_N f(x) = \int_{0}^{1} K_N(x,y)f(y) dy \to \int_{0}^{1} L(x,y)f(y) dy.$$
We therefore have
\[ \int_0^1 L(x,y) f(y) \, dy = Kf(x) \]
for all \( f \in L^2[0,1] \). This implies that (\( x \) is still fixed) \( K(x,y) = L(x,y) \) for almost every \( y \). With \( x \) fixed, both sides are continuous functions of \( y \), hence they are equal for every \( y \).

This is true for each \( x \), and \( K(x,y) \) is continuous, hence \( L \) is continuous. We now can apply Dini’s theorem to conclude that \( K_N(x,x) \) converges to \( L(x,x) = J(x) \) uniformly. Finally, again by Cauchy-Schwarz,
\[
\sum_{j=m}^{n} \kappa_j |e_j(x)| |\overline{e_j(y)}| \\
\leq \left( \sum_{j=m}^{n} \kappa_j |e_j(x)|^2 \right)^{1/2} \left( \sum_{j=m}^{n} \kappa_j |e_j(y)|^2 \right)^{1/2},
\]
and this proves that \( K_N(x,y) \) converges to \( K \) uniformly and absolutely. \( \square \)

### 10.4 Positive compact operators

We’ll do the *Krein-Rutman theorem*, which is a generalization of the Perron-Frobenius theorem for matrices.

**Theorem 10.16** Suppose \( Q \) is compact and Hausdorff and \( X = C(Q) \), the complex-valued continuous functions on \( Q \). Suppose \( K : C(Q) \to C(Q) \) and \( K \) is compact. Suppose further than \( K \) maps real-valued functions to real-valued functions. Finally, suppose that whenever \( f \geq 0 \) and \( f \) is not identically zero, then \( Kf \) is strictly positive. Then \( K \) has a positive eigenvalue \( \sigma \) of multiplicity one, the associated eigenfunction is positive, and all the other eigenvalues of \( K \) are strictly smaller in absolute value than \( \sigma \).

Examples include matrices with all positive entries, the semigroup \( P_t \) when \( t = 1 \) for reflecting Brownian motion on a bounded interval, and
\[
Kf(x) = \int K(x,y) f(y) \, \mu(dy),
\]
where $K$ is jointly continuous, positive, and $\mu$ is a finite measure. We have seen that the operator $K$ is compact.

**Proof.** If $f \leq g$ and $f \neq g$, then $g - f \geq 0$, so $K(g - f) > 0$, or $Kf < Kg$.

*Step 1.* We show there exists a non-zero eigenvalue. Let $f$ be the identically one function. Since $Kf$ is continuous and everywhere positive, there exists a positive number $b$ such that $Kf \geq b = bf$.

If $f$ and $b$ are any pair such that $f \geq 0$, and $Kf \geq bf$, then

$$b^2 f \leq bKf = K(bf) \leq K(Kf) = K^2 f,$$
and continuing,

$$b^n f \leq K^n f.$$

Since $f \geq 0$,

$$b^n \|f\| \leq \|K^n f\| \leq \|K^n\| \|f\|,$$
so

$$r(K) = \lim \|K^n\|^{1/n} \geq b.$$ Therefore $r(K)$ is strictly positive. Since $K$ is compact, the set of eigenvalues of $K$ is nonempty. We have shown that there exists a non-zero eigenvalue for $K$. Moreover, any $b$ that satisfies $Kf \geq bf$ for some $f \geq 0$ is less than or equal to $r(K)$.

*Step 2.* $K$ is compact, so there exists an eigenvalue $\lambda$ and an eigenfunction $g$ such that $Kg = \lambda g$, $|\lambda| = r(K)$. Let $\lambda$ and $g$ be any pair with $|\lambda| = r(K)$.

(a) We claim: if $f = |g|$ and $\sigma = |\lambda|$, then $\sigma f \leq Kf$.

*Proof.* Let $x \in Q$. Multiply $g$ by $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $\alpha \lambda g(x)$ is real and non-negative. Of course $\alpha$ depends on $x$. Write $g = u + iv$. Then

$$Ku(x) + iKv(x) = Kg(x) = \lambda g(x).$$

Looking at the real part,

$$\lambda g(x) = (Ku)(x).$$

Next, $u \leq |g| = f$, and

$$|\lambda|f(x) = |\lambda g(x)| = Ku(x) \leq (Kf)(x). \quad (10.4)$$
Although $g$ depends on $\alpha$, which depends on $x$, neither $\sigma$ nor $f$ depend on $x$. Since $x$ was arbitrary, the inequality (10.5) holds for all $x$.

(b) We claim

$$\sigma f = Kf.$$ 

Proof: If not, there exists $x$ such that $\sigma f(x) < Kf(x)$. By continuity, there exists a neighborhood $N$ about $x$ such that

$$\sigma f(s) + \delta \leq Kf(s), \quad s \in N.$$ 

Let $h > 0$ in $N$, $0$ outside of $N$, and $s \in N$, so $Kh > 0$.

We will find $c, \varepsilon > 0$ and set $F = f + \varepsilon h$, $\kappa = \sigma + c\varepsilon$, and get $\kappa F \leq KF$. This will be a contradiction to Step 1: if $bf \leq Kf$, then we know $b \leq r(K)$; use this with $b$ replaced by $\kappa$ and $f$ replaced by $F$.

(i) Now $Kh > 0$, so there exists $c \leq 1$ such that $cf \leq Kh$. If $s \in N$,

$$KF(s) = Kf(s) + \varepsilon Kh(s) \geq Kf(s) + \varepsilon cf(s) \geq \sigma f(s) + \delta + \varepsilon cf(s).$$

Then

$$\kappa F(s) = (\sigma + c\varepsilon)(f + \varepsilon h)(s) = \sigma f(s) + \varepsilon cf(s) + \sigma \varepsilon h(s) + c\varepsilon^2 h(s) \leq KF(s) - \delta + \varepsilon cf(s) + \sigma \varepsilon h(s) + c\varepsilon^2 h(s).$$

Since $h$ is bounded above, we can take $\varepsilon$ small enough so that the last line is less than or equal to $KF(s)$.

(ii) If $s \notin N$, then $h(s) = 0$ and

$$\kappa F(s) = \kappa f(s) = (\sigma + c\varepsilon)f(s) = \sigma f(s) + \varepsilon cf(s) \leq Kf(s) + \varepsilon Kh(s) = KF(s),$$

using that $cf \leq Kh$.

Step 3. We next show that any other eigenvalue that has absolute value $\sigma$ is in fact equal to $\sigma$. Let $G$ be any eigenfunction corresponding to $\lambda$ with
\(|\lambda| = \sigma\). Fix \(x \in Q\). As before, we may assume \(\lambda G(x) \geq 0\). As before, write \(G = u + iv\) and then \(\lambda G(x) = Ku(x)\). We have \(u \leq |G| = f\).

Suppose \(u < f\) at some point \(y \in Q\). Then \(u \leq f\) and \(u < f\) at one point means that we have \(Ku < Kf\) at every point, and so

\[|\lambda|f(x) = |\lambda G(x)| = \lambda G(x) = Ku(x) < Kf(x).\]

So \(\sigma f(x) < Kf(x)\). But we showed \(\sigma f = Kf\). Therefore \(u\) is identically equal to \(f\). This implies that \(G\) is real and positive, and then it follows that \(\lambda\) is real and positive. Since \(G = \sigma^{-1}KG\), \(G\) is strictly positive.

\(\text{Step 4.}\) Finally, we show \(\sigma\) has multiplicity 1. If not, there exist distinct real eigenfunctions \(f_1, f_2\). But some linear combination \(H\) of \(f_1, f_2\) will be real, take the value 0, but not be identically zero. As before \(|H|\) will be an eigenfunction that is non-negative, and must also take the value 0. Moreover the corresponding eigenvalue is \(\sigma\). But then \(0 < K|H| = \sigma|H|\), a contradiction to \(|H|\) taking the value 0. \(\square\)
Chapter 11

Spectral theory

11.1 Preliminaries

Suppose $A$ is a bounded linear operator over a complex-valued Hilbert space. If $y \in H$ is fixed, then $\ell(x) = \langle Ax, y \rangle$ is a bounded linear functional on $H$. Therefore there exists $z = z_y \in H$ such that $\ell(x) = \langle x, z \rangle$ for all $x$. We define $A^*y = z_y$, so we have $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all $x$ and $y$.

Note

$$\langle x, A^*(y_1 + y_2) \rangle = \langle Ax, y_1 + y_2 \rangle = \langle Ax, y_1 \rangle + \langle Ax, y_2 \rangle = \langle x, A^*y_1 \rangle + \langle x, A^*y_2 \rangle = \langle x, A^*y_1 + A^*y_2 \rangle.$$

We conclude $A^*(y_1 + y_2) = A^*y_1 + A^*y_2$. Similarly $A^*(cy) = cA^*y$, so $A^*$ is a linear operator.

If $x, y \in H$, we have

$$\|x, A^*y\| = |\langle Ax, y \rangle| \leq \|A\| \|x\| \|y\|.$$

Taking the supremum over $\|x\| = 1$, we have $\|A^*y\| \leq \|A\| \|y\|$, and taking the supremum over $\|y\| = 1$, we get $\|A^*\| \leq \|A\|$. Replacing $A$ by $A^*$ we obtain $\|A^{**}\| \leq \|A^*\|$, and noticing that $A^{**} = A$, we have

$$\|A^*\| = \|A\|.$$
CHAPTER 11. SPECTRAL THEORY

It is easy to check that \((A + B)^* = A^* + B^*\), and since 

\[
\langle cAx, y \rangle = c\langle Ax, y \rangle = c\langle x, A^*y \rangle = \langle x, cA^*y \rangle
\]

for all \(x\) and \(y\), we have \((cA)^* = cA^*\). We note that 

\[
\langle A^2x, y \rangle = \langle Ax, A^*y \rangle = \langle x, (A^*)^2y \rangle,
\]

so \((A^2)^* = (A^*)^2\). This holds for all positive powers \(n\) by an induction argument. If \(P(z) = \sum_{j=0}^{n} c_j z^j\), let \(\overline{P}(z) = \sum_{j=0}^{n} \overline{c_j} z^n\). We then have that \(P(A)^* = \overline{P}(A^*)\).

In the case that \(H = \mathbb{C}^n\), we can identify vectors in \(\mathbb{C}^n\) with \(n \times 1\) matrices and an operator \(A\) is identified with a \(n \times n\) matrix. Then \(\langle X, Y \rangle = X^TY\), where \(B^T\) is the transpose of a matrix \(B\). Saying \(\langle AX, Y \rangle = \langle X, A^*Y \rangle\) is the same as saying that \(X^TA^TY = (AX)^TY\) is equal to \(X^TA^TY\) for all \(X\) and \(Y\). Hence \(A^T = \overline{A}^T\), or \(A^* = A^T\), the conjugate transpose of \(A\).

We say \(A\) is a symmetric operator over a complex-valued Hilbert space if \(A = A^*\), or equivalently, if \(\langle Ax, y \rangle = \langle x, Ay \rangle\) for all \(x\) and \(y \in H\).

If \(A\) is compact, we can write \(x = \sum a_n e_n\) and \(Ax = \sum \lambda_n a_n e_n\). Let \(E_n\) be the projection onto the eigenspace with eigenvector \(\lambda_n\), so \(x = \sum E_n x\) and \(Ax = \sum \lambda_n E_n(x)\).

If we define a projection-valued measure \(E(S)\) by 

\[
E(S) = \sum_{\lambda_n \in S} E_n
\]

for \(S\) a Borel subset of \(\mathbb{R}\), then \(x = \int E(d\lambda)x\) and \(Ax = \int \lambda E(d\lambda)x\).

Here \(E\) is a pure point measure. In general, we get the same result, but \(E\) might not be pure point.

When we move away from compact operators, the spectrum can become much more complicated. Let us look at an instructive example.

**Example 11.1** Let \(H = L^2([0, 1])\) and define \(A : H \to H\) by \(Af(x) = xf(x)\). There is no difficulty seeing that \(A\) is bounded and symmetric.

We first show that no point in \([0, 1]^c\) is in the spectrum of \(A\). If \(z\) is a fixed complex number and either has a non-zero imaginary part or has a real
part that is not in \([0,1]\), then \(z - A\) has the inverse \(Bf(x) = \frac{1}{z-x}f(x)\). It is obvious that \(B\) is in fact the inverse of \(z - A\) and it is a bounded operator because \(1/|z-x|\) is bounded on \(x \in [0,1]\).

If \(z \in [0,1]\), we claim \(z-A\) does not have a bounded inverse. The function that is identically equal to 1 is in \(L^2([0,1])\). The only function \(g\) that satisfies \((z-A)g = 1\) is \(g = 1/(z-x)\), but \(g\) is not in \(L^2([0,1])\), hence the range of \(z-A\) is not all of \(H\).

We conclude that \(\sigma(A) = [0,1]\). We show now, however, that no point in \([0,1]\) is an eigenvalue for \(A\). If \(z \in [0,1]\) were an eigenvalue, then there would exist a non-zero \(f\) such that \((z-A)f = 0\). Since our Hilbert space is \(L^2\), saying \(f\) is non-zero means that the set of \(x\) where \(f(x) \neq 0\) has positive Lebesgue measure. But \((z-A)f = 0\) implies that \((z-x)f(x) = 0\) a.e., which forces \(f = 0\) a.e. Thus \(A\) has no eigenvalues.

We have shown that the spectrum of a bounded symmetric operator is closed and bounded and never empty because the collection of bounded symmetric operators is a Banach algebra, although not a commutative one.

We proved the spectral radius formula when we studied Banach algebras:

\[
\|A\| = \lim_{n \to \infty} \|A^n\|^{1/n}.
\]

We have the following important corollary.

**Proposition 11.2** If \(A\) is a symmetric operator, then

\[
\|A\| = r(A).
\]

**Proof.** It suffices to show that \(\|A^n\| = \|A\|^n\) when \(n\) is a power of 2. We show this for \(n = 2\) and the general case follows by induction.

On the one hand, \(\|A^2\| \leq \|A\|^2\). On the other hand,

\[
\|A\|^2 = \left( \sup_{\|x\| = 1} \|Ax\| \right)^2 = \sup_{\|x\| = 1} \|Ax\|^2 = \sup_{\|x\| = 1} \langle Ax, Ax \rangle = \sup_{\|x\| = 1} \langle A^2 x, x \rangle \leq \|A^2\|
\]
by the Cauchy-Schwarz inequality.

The following corollary will be important in the proof of the spectral theorem.

**Corollary 11.3** Let $A$ be a symmetric bounded linear operator.
(1) If $P$ is a polynomial with real coefficients, then
\[ \|P(A)\| = \sup_{z \in \sigma(A)} |P(z)|. \]

(2) If $P$ is a polynomial with complex coefficients, then
\[ \|P(A)\| \leq 2 \sup_{z \in \sigma(A)} |P(z)|. \]

A later proposition will provide an improvement of assertion (2).

**Proof.** (1) Since $P$ has real coefficients, then $P(A)$ is symmetric and
\[
\|P(A)\| = r(P(A)) = \sup_{z \in \sigma(P(A))} |z|
= \sup_{z \in P(\sigma(A))} |z| = \sup_{w \in \sigma(A)} |P(w)|,
\]
where we used Corollary 11.2 for the first equality and the spectral mapping theorem for the third.

(2) If $P(z) = \sum_{j=0}^{n} (a_{j} + ib_{j})z^{j}$, let $Q(z) = \sum_{j=0}^{n} a_{j}z^{j}$ and $R(z) = \sum_{j=0}^{m} b_{j}z^{n}$. By (1),
\[ \|P(A)\| \leq \|Q(A)\| + \|R(A)\| \leq \sup_{z \in \sigma(A)} |Q(z)| + \sup_{z \in \sigma(A)} |R(z)|, \]
and (2) follows.

We will also need the fact that the spectrum of a bounded symmetric operator is real. We know that each eigenvalue of a bounded symmetric operator is real, but as we have seen, not every element of the spectrum is an eigenvalue.
Proposition 11.4 If $A$ is bounded and symmetric, then $\sigma(A) \subseteq \mathbb{R}$.

Proof. Suppose $\lambda = a + ib, b \neq 0$. We want to show that $\lambda$ is not in the spectrum.

If $r$ and $s$ are real numbers, rewriting the inequality $(r - s)^2 \geq 0$ yields the inequality $2rs \leq r^2 + s^2$. By the Cauchy-Schwarz inequality

$$2a\langle x, Ax \rangle \leq 2|a| \, \|x\| \, \|Ax\| \leq a^2 \|x\|^2 + \|Ax\|^2.$$  

We then obtain the inequality

$$\| (\lambda - A)x \|^2 = \langle (a + bi - A)x, (a + bi - A)x \rangle$$
$$= (a^2 + b^2) \|x\|^2 + \|Ax\|^2 - (a + bi) \langle Ax, x \rangle$$
$$- (a - bi) \langle x, Ax \rangle$$
$$= (a^2 + b^2) \|x\|^2 + \|Ax\|^2 - 2a \langle Ax, x \rangle$$
$$\geq b^2 \|x\|^2. \quad (11.1)$$

This inequality shows that $\lambda - A$ is one-to-one, for if $(\lambda - A)x_1 = (\lambda - A)x_2$, then

$$0 = \| (\lambda - A)(x_1 - x_2) \| \geq b^2 \|x_1 - x_2\|^2.$$  

Suppose $\lambda$ is in the spectrum of $A$. Since $\lambda - A$ is one-to-one but not invertible, it cannot be onto. Let $R$ be the range of $\lambda - A$. We next argue that $R$ is closed.

If $y_k = (\lambda - A)x_k$ and $y_k \to y$, then (11.1) shows that

$$b^2 \|x_k - x_m\|^2 \leq \|y_k - y_m\|^2,$$

or $x_k$ is a Cauchy sequence. If $x$ is the limit of this sequence, then

$$(\lambda - A)x = \lim_{n \to \infty} (\lambda - A)x_n = \lim_{n \to \infty} y_k = y.$$  

Therefore $R$ is a closed subspace of $H$ but is not equal to $H$. Choose $z \in R^\perp$. For all $x \in H$,

$$0 = \langle (\lambda - A)x, z \rangle = \langle x, (\lambda - A)z \rangle.$$
This implies that \((\lambda - A)z = 0\), or \(\lambda\) is an eigenvalue for \(A\) with corresponding eigenvector \(z\). However we know that all the eigenvalues of a bounded symmetric operator are real, hence \(\lambda\) is real. This shows \(\lambda\) is real, a contradiction.

11.2 Functional calculus

Let \(f\) be a continuous function on \(\mathbb{C}\) and let \(A\) be a bounded symmetric operator on a separable Hilbert space over the complex numbers. We describe how to define \(f(A)\).

We have shown that the spectrum of \(A\) is a closed and bounded subset of \(\mathbb{C}\), hence a compact set. By the Stone-Weierstrass theorem we can find polynomials \(P_n\) (with complex coefficients) such that \(P_n\) converges to \(f\) uniformly on \(\sigma(A)\). Then

\[
\sup_{z \in \sigma(A)} |(P_n - P_m)(z)| \to 0
\]
as \(n, m \to \infty\). By Corollary 11.3

\[
\|(P_n - P_m)(A)\| \to 0
\]
as \(n, m \to \infty\), or in other words, \(P_n(A)\) is a Cauchy sequence in the space \(\mathcal{L}\) of bounded symmetric linear operators on \(H\). We call the limit \(f(A)\).

The limit is independent of the sequence of polynomials we choose. If \(Q_n\) is another sequence of polynomials converging to \(f\) uniformly on \(\sigma(A)\), then

\[
\lim_{n \to \infty} \|P_n(A) - Q_n(A)\| \leq 2 \sup_{z \in \sigma(A)} |(P_n - Q_n)(z)| \to 0,
\]
so \(Q_n(A)\) has the same limit \(P_n(A)\) does.

We record the following facts about the operators \(f(A)\) when \(f\) is continuous.

**Proposition 11.5** Let \(f\) be continuous on \(\sigma(A)\).

(1) \(\langle f(A)x, y \rangle = \langle x, f(A)y \rangle\) for all \(x, y \in H\).
(2) If $f$ is equal to 1 on $\sigma(A)$, then $f(A) = I$, the identity.
(3) If $f(z) = z$ on $\sigma(A)$, then $f(A) = A$.
(4) $f(A)$ and $A$ commute.
(5) If $f$ and $g$ are two continuous functions, then $(f + g)(A) = f(A) + g(A)$ and $f(A)g(A) = (fg)(A)$.
(6) $\|f(A)\| \leq \sup_{z \in \sigma(A)} |f(z)|$.

**Proof.** The proofs of (1)-(4) are routine and follow from the corresponding properties of $P_n(A)$ when $P_n$ is a polynomial. Let us prove (5) and (6) and leave the proofs of the others to the reader.

(5) Let $P_n$ and $Q_n$ be polynomials converging uniformly on $\sigma(A)$ to $f$ and $g$, respectively. Then $P_nQ_n$ will be polynomials converging uniformly to $fg$. The second assertion of (5) now follows from

$$(fg)(A) = \lim_{n \to \infty} (P_nQ_n)(A) = \lim_{n \to \infty} P_n(A)Q_n(A) = f(A)g(A).$$

The limits are with respect to the norm on bounded operators on $H$. The first assertion of (5) is similar.

(6) Since $f$ is continuous on $\sigma(A)$, so is $g = |f|^2$. Let $P_n$ be polynomials with real coefficients converging to $g$ uniformly on $\sigma(A)$. By Corollary 11.3(1),

$$\|g(A)\| = \lim_{n \to \infty} \|P_n(A)\| \leq \lim_{n \to \infty} \sup_{z \in \sigma(A)} |P_n(z)| = \sup_{z \in \sigma(A)} |g(z)|.$$

If $\|x\| = 1$, using (1) and (5),

$$\|f(A)x\|^2 = \langle f(A)x, f(A)x \rangle = \langle x, \overline{f}(A)f(A)x \rangle = \langle x, g(A)x \rangle \leq \|x\| \|g(A)x\| \leq \|g(A)\| \leq \sup_{z \in \sigma(A)} |g(z)|$$

$$= \sup_{z \in \sigma(A)} |f(z)|^2.$$

Taking the supremum over the set of $x$ with $\|x\| = 1$ yields

$$\|f(A)\|^2 \leq \sup_{z \in \sigma(A)} |f(z)|^2,$$

and (6) follows. $\square$

We have the spectral mapping theorem for continuous functions.
**Theorem 11.6** Suppose $A$ is symmetric and suppose $f$ is continuous on $\sigma(A)$. Then

$$\sigma(f(A)) = f(\sigma(A)).$$

Here $f(\sigma(A)) = \{f(\lambda) : \lambda \in \sigma(A)\}$.

**Proof.** Step 1. Suppose $\mu \notin f(\sigma(A))$; we show $\mu \notin f(\sigma(A))$ and then conclude $\sigma(f(A)) \subset f(\sigma(A))$. If $\mu \neq f(\lambda)$ for some $\lambda \in \sigma(A)$, then $f(z) - \mu$ does not vanish on $\sigma(A)$. So we can find a continuous function $g$ that agrees with $(f(z) - \mu)^{-1}$ on $\sigma(A)$.

If we take polynomials $P_n$ converging to $f - \mu$ and polynomials converging to $g$ uniformly on $\sigma(A)$ and let $R_n(z) = (P_n(z) - \mu)((Q_n(z)) - 1$, then $R_n$ converges uniformly to 0 on $\sigma(A)$. Therefore $R_n(A)$ converges to 0 in norm. So

$$[f(A) - \mu I]g(A) = I.$$

Therefore $g(A)$ is the inverse of $f(A) - \mu I$, and hence $\mu \notin f(\sigma(A))$.

Step 2. We show $f(\sigma(A)) \subset f(\sigma(A))$. Suppose $\lambda \in \sigma(A)$. We need to show $f(\lambda) \in f(\sigma(A))$. Suppose not, that is, $f(\lambda) - f(A)$ is invertible. Let $P_n$ be polynomials converging to $f$ uniformly on $\sigma(A)$. We use the fact that $K - B$ is invertible if $K$ is invertible and the norm of $B$ is less than $1/\|K^{-1}\|$. We set $K = f(\lambda) - f(A)$ and

$$B = f(\lambda) - P_n(z) - f(A) + P_n(A).$$

Then if $n$ is sufficiently large and $z$ is sufficiently close to $\lambda$, then $P_n(z) - P_n(A)$ will be invertible. Thus for such $n$ and $z$, we see that $P_n(z) \notin \sigma(P_n(A)) = P_n(\sigma(A))$. Since $P_n$ converges uniformly to $f$ on $\sigma(A)$, we conclude $f(\lambda) \notin f(\sigma(A))$, a contradiction. $\Box$

$A$ is a positive operator if $\langle Ax, x \rangle \geq 0$ for all $x$.

**Proposition 11.7** Let $A$ be bounded and symmetric. $A$ is positive if and only if $\sigma(A) \geq 0$.

**Proof.** If $\sigma(A) \geq 0$, then $f(\lambda) = \sqrt{\lambda}$ is a continuous real-valued function for $\lambda \geq 0$, and so $N = f(A) = \sqrt{A}$ exists and is a symmetric operator because $N^* = \overline{f}(A) = f(A) = N$. Then

$$\langle Ax, x \rangle = \langle N^2 x, x \rangle = \langle N x, N x \rangle \geq 0.$$
Now suppose $A$ is positive. If $\lambda \in \sigma(A)$ is strictly negative,
\[ \|x\| \|(A - \lambda)x\| \geq \langle x, (A - \lambda)x \rangle = \langle x, Ax \rangle - \lambda \|x\|^2 \geq -\lambda \|x\|^2, \]
using Cauchy-Schwarz. Dividing both sides by $\|x\|$, we have
\[ \|(\lambda - A)x\| \geq (-\lambda)\|x\|. \]
Similarly to the proof that the spectrum of a symmetric operator is contained in the reals, we see that $\lambda - A$ is one-to-one, its range $R$ is closed, and $\overline{\lambda}$ is an eigenvalue for $A$. But then
\[ \langle Az, z \rangle = \langle \lambda z, z \rangle = \lambda \|z\|^2 < 0, \]
a contradiction, where we used the fact that $\lambda$ is real and hence $\overline{\lambda} = \lambda$. □

Corollary 11.8 Every positive symmetric operator has a positive symmetric square root.

11.3 Riesz representation theorem

The Riesz representation theorem for positive linear functionals on $\mathcal{C}(X)$ is proved in real analysis. We will need the version for complex-valued bounded linear functionals. See [1] for a proof.

Theorem 11.9 If $S$ is a compact metric space and $I$ is a bounded complex-valued linear functional on $\mathcal{C}(X)$, there exists a unique finite complex-valued measure $\mu$ on the Borel $\sigma$-algebra such that
\[ I(f) = \int f \, d\mu \]
for each $f \in \mathcal{C}(X)$. Moreover the total variation of $\mu$ is
\[ \sup \left\{ \int f \, d\mu : \sup_{z \in S} |f(z)| \leq 1 \right\}. \]
11.4 Spectral resolution

We now want to define $f(A)$ when $f$ is a bounded Borel measurable function on $\mathbb{C}$. Fix $x, y \in H$. If $f$ is a continuous function on $\mathbb{C}$, let

$$L_{x,y}f = \langle f(A)x, y \rangle. \quad (11.2)$$

It is easy to check that $L_{x,y}$ is a bounded linear functional on $C(\sigma(A))$, the set of continuous functions on $\sigma(A)$. By the Riesz representation theorem for complex-valued linear functionals, there exists a complex measure $\mu_{x,y}$ such that

$$\langle f(A)x, y \rangle = L_{x,y}f = \int_{\sigma(A)} f(z) \mu_{x,y}(dz)$$

for all continuous functions $f$ on $\sigma(A)$.

Define

$$L_{x,y}f = \int f(z) \mu_{x,y}(dz)$$

for all $f$ that are bounded and Borel measurable on $\sigma(A)$.

We have the following properties of $\mu_{x,y}$.

**Proposition 11.10**

1. $\mu_{x,y}$ is linear in $x$.
2. $\mu_{y,x} = \overline{\mu_{x,y}}$.
3. The total variation of $\mu_{x,y}$ is less than or equal to $\|x\| \|y\|$.

**Proof.** (1) The linear functional $L_{x,y}$ defined in (11.2) is linear in $x$ and

$$\int f d(\mu_{x,y} + \mu_{x',y}) = L_{x,y}f + L_{x',y}f = L_{x+x',y}f = \int f d\mu_{x+x',y}.$$

By the uniqueness of the Riesz representation, $\mu_{x+x',y} = \mu_{x,y} + \mu_{x'+y}$. The proof that $\mu_{cx,y} = c\mu_{x,y}$ is similar.

(2) follows from the fact that if $f$ is continuous on $\sigma(A)$, then

$$\int f d\mu_{y,x} = L_{y,x}f = \langle f(A)y, x \rangle = \langle y, \overline{f(A)}x \rangle = \overline{\langle f(A)x, y \rangle} = \overline{L_{x,y}} = \int \overline{f} d\mu_{x,y} = \int f d\overline{\mu_{x,y}}.$$
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Now use the uniqueness of the Riesz representation.

(3) This follows from the Riesz representation theorem.

If $f$ is a bounded Borel measurable function on $\mathbb{C}$, then $L_{y,x}\overline{f}$ is linear in $y$. Note that

$$|L_{y,x}f| \leq \sup_{z \in \sigma(A)} |f(z)| \|\mu_{x,y}\|_{TV} \leq \sup_{z \in \sigma(A)} |f(z)| \|x\| \|y\|,$$

where $TV$ stands for “total variation.” Thus $L_{y,x}$ is a bounded linear functional on $\mathbb{C}$ with norm bounded by $\|x\| \|y\|$. By the Riesz representation theorem for Hilbert spaces, there exists $w_x \in H$ such that $L_{y,x}\overline{f} = \langle y, w_x \rangle$ for all $y \in H$. We then have that for all $y \in H$,

$$L_{x,y}f = \int_{\sigma(A)} f(z) \mu_{x,y}(dz) = \int_{\sigma(A)} f(z) \overline{\mu_{y,x}(dz)}$$

$$= \int_{\sigma(A)} \overline{f(z)} \mu_{y,x}(dz) = \overline{L_{y,x}f}$$

$$= \langle y, w_x \rangle = \langle w_x, y \rangle.$$

Since

$$\langle y, w_{x_1+x_2} \rangle = L_{y,x_1+x_2} \overline{f} = L_{y,x_1} \overline{f} + L_{y,x_2} \overline{f} = \langle y, w_{x_1} \rangle + \langle y, w_{x_2} \rangle$$

for all $y$ and

$$\langle y, w_{cx} \rangle = L_{y,cx} \overline{f} = \overline{c} L_{y,x} \overline{f} = \overline{c} \langle y, w_x \rangle = \langle y, cw_x \rangle$$

for all $y$, we see that $w_x$ is linear in $x$. We define $f(A)$ to be the linear operator on $H$ such that $f(A)x = w_x$.

In the particular case when $A$ is also compact, if $(\lambda_n, \varphi_n)$ are the eigenvalue/eigenvector pairs with $\{\varphi_n\}$ an orthonormal basis, we have

$$x = \sum_{n=1}^{\infty} \langle x, \varphi_n \rangle \varphi_n$$

and

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \varphi_n \rangle \varphi_n.$$
Then
\[ A^2 x = \sum_{n=1}^{\infty} \lambda_n \langle x, \varphi_n \rangle A^2 \varphi_n = \sum_{n=1}^{\infty} \lambda_n^2 \langle x, \varphi_n \rangle \varphi_n. \]

Generalizing this, we have
\[ P(A) x = \sum_{n=1}^{\infty} P(\lambda_n) \langle x, \varphi_n \rangle \varphi_n, \]
and passing to the limit for continuous functions, and then for bounded and Borel measurable \( f \),
\[ f(A) x = \sum_{n=1}^{\infty} f(\lambda_n) \langle x, \varphi_n \rangle \varphi_n. \]

Specializing further to matrices, if \( A \) is a diagonal matrix with diagonal entries \( A_{jj} = \lambda_j \), then \( f(A) \) is the diagonal matrix with diagonal entries \( f(\lambda_j) \).

If \( C \) is a Borel measurable subset of \( \mathbb{C} \), we let
\[ E(C) = \chi_C(A). \quad (11.3) \]

**Remark 11.11** Later on we will write the equation
\[ f(A) = \int_{\sigma(A)} f(z) E(dz). \quad (11.4) \]
Let us give the interpretation of this equation. If \( x, y \in H \), then
\[ \langle E(C) x, y \rangle = \langle \chi_C(A)x, y \rangle = \int_{\sigma(A)} \chi_C(z) \mu_{x,y}(dz). \]

Therefore we identify \( \langle E(dz)x, y \rangle \) with \( \mu_{x,y}(dz) \). With this in mind, (11.4) is to be interpreted to mean that for all \( x \) and \( y \),
\[ \langle f(A)x, y \rangle = \int_{\sigma(A)} f(z) \mu_{x,y}(dz). \]
Theorem 11.12 (1) $E(C)$ is symmetric.
(2) $\|E(C)\| \leq 1$.
(3) $E(\emptyset) = 0, E(\sigma(A)) = I$.
(4) If $C, D$ are disjoint, $E(C \cup D) = E(C) + E(D)$.
(5) $E(C \cap D) = E(C)E(D)$.
(6) $E(C)$ and $A$ commute.
(7) $E(C)^2 = E(C)$, so $E(C)$ is a projection. If $C, D$ are disjoint, then $E(C)E(D) = 0$.
(8) $E(C)$ and $E(D)$ commute.

Proof. (1) This follows from
\[ \langle x, E(C)y \rangle = \langle E(C)y, x \rangle = \int \chi_C(z) \mu_{y,x}(dz) \]
\[ = \int \chi_C(z) \mu_{x,y}(dz) = \langle E(C)x, y \rangle. \]

(2) Since the total variation of $\mu_{x,y}$ is bounded by $\|x\|\|y\|$, we obtain (2).
(3) $\mu_{x,y}(\emptyset) = 0$, so $E(\emptyset) = 0$. If $f$ is identically equal to 1, then $f(A) = I$, and
\[ \langle x, y \rangle = \int_{\sigma(A)} \mu_{x,y}(dz) = \langle E(\sigma(A))x, y \rangle. \]
This is true for all $y$, so $x = E(\sigma(A))x$ for all $x$.

(4) holds because $\mu_{x,y}$ is a measure, hence finitely additive.

(5) If we prove that
\[ f(A)g(A) = (fg)(A) \quad (11.5) \]
for $f$ and $g$ bounded and Borel measurable on $\sigma(A)$, we can apply this with $f = \chi_C$ and $g = \chi_D$. Then $fg = \chi_{C \cap D}$, and we get (5).

Now
\[ \langle f_n(A)g_m(A)x, y \rangle = \langle (f_n g_m)(A)x, y \rangle \quad (11.6) \]
when $f_n$ and $g_m$ are continuous. The right hand side equals
\[ \int (f_n g_m)(z) \mu_{x,y}(dz), \]
which converges to
\[ \int (f_n g)(z) \mu_{x,y}(dz) = \langle (f_n g)(A)x, y \rangle \]
when \( g_m \to g \) boundedly and a.e. with respect to \( \mu_{x,y} \). The left hand side of (11.6) equals
\[ \langle g_m(A)x, \overline{f_n(A)y} \rangle = \int g_m(z) \mu_{A,x,y}(dz) \]
which converges to
\[ \int g(z) \mu_{A,x,y}(dz) = \langle g(A)x, \overline{f_n(A)y} \rangle \]
as long as \( g_m \) also converges a.e. with respect to \( \mu_{A,x,y} \). So we have
\[ \langle f_n(A)g(A)x, y \rangle = \langle (f_n g)(A)x, y \rangle. \] (11.7)

If we let \( f_n \) converge to \( f \) boundedly and a.e. with respect to \( \mu_{x,y} \), the right hand side converges as in the previous paragraph to \( \langle (fg)(A)x, y \rangle \). The right hand side of (11.7) is equal to
\[ \langle g(A)x, \overline{f_n(A)y} \rangle = \overline{\langle f_n(A)y, g(A)x \rangle}. \] (11.8)

If \( f_n \) converges to \( f \) a.e with respect to \( \mu_{A,x,y} \), the right hand side of (11.8) converges by arguments similar to the above to
\[ \overline{\langle f(A)y, g(A)x \rangle} = \langle g(A)x, \overline{f(A)y} \rangle = \langle f(A)g(A)x, y \rangle. \]

(6) Let \( h(z) = z \) and apply (11.5) with \( f = \chi_C \) and \( g = h \) to get \( \chi_C(A)A = (\chi_C h)(A) \). Then apply (11.5) with \( f = h \) and \( g = \chi_C \) to get \( A \chi_C(A) = (h \chi_C)(A) \).

(7) Setting \( C = D \) in (5) shows \( E(C) = E(C)^2 \), so \( E(C) \) is a projection. If \( C \cap D = \emptyset \), then \( E(C)E(D) = E(\emptyset) = 0 \), as required.

(8) Writing
\[ E(C)E(D) = E(C \cap D) = E(D \cap C) = E(D)E(C) \]
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proves (8).

The family \(\{E(C)\}\), where \(C\) ranges over the Borel subsets of \(\mathbb{C}\) is called the spectral resolution of the identity. We explain the name in just a moment.

Here is the spectral theorem for bounded symmetric operators.

**Theorem 11.13** Let \(H\) be a separable Hilbert space over the complex numbers and \(A\) a bounded symmetric operator. There exists an operator-valued measure \(E\) satisfying (1)–(8) of Theorem 11.12 such that

\[
f(A) = \int_{\sigma(A)} f(z) E(dz),
\]

(11.9)

for bounded Borel measurable functions \(f\). Moreover, the measure \(E\) is unique.

**Remark 11.14** When we say that \(E\) is an operator-valued measure, here we mean that (1)–(8) of Theorem 11.12 hold. We use Remark 11.11 to give the interpretation of (11.9).

**Remark 11.15** If \(f\) is identically one, then (11.9) becomes

\[
I = \int_{\sigma(A)} E(d\lambda),
\]

which shows that \(\{E(C)\}\) is a decomposition of the identity. This is where the name “spectral resolution” comes from.

**Proof of Theorem 11.13.** Given Remark 11.14, the only part to prove is the uniqueness, and that follows from the uniqueness of the measure \(\mu_{x,y}\). □

**Proposition 11.16** Suppose \(A_1, \ldots, A_m\) are pairwise disjoint. Then

\[
\left\| \sum_{i=1}^{m} c_i E(A_i) \right\| = \max_{1 \leq i \leq m} |c_i|.
\]
Proof. By letting $A_{m+1} = \sigma(A) \setminus \bigcup_{i=1}^{m} A_i$ and setting $c_{m+1} = 0$, we may suppose without loss of generality that the union of the $A_i$ is $\sigma(A)$. Let $r = \max_i |c_i|$. Given $x$, let $x_i = E(A_i)x$. Then

$$\langle x_i, x_j \rangle = \langle E(A_i)x, E(A_j)x \rangle = \langle x, E(A_i)E(A_j)x \rangle = 0$$

if $i \neq j$. We have

$$\left\| \sum_i c_i E(A_i)x \right\|^2 = \left\langle \sum_i c_i E(A_i)x, \sum_j c_j E(A_j)x \right\rangle = \left\langle \sum_i c_i x_i, \sum_j c_j x_j \right\rangle$$

$$= \sum_i |c_i|^2 \| x_i \|^2 \leq r^2 \sum_i \| x_i \|^2 \leq r^2 \| x \|^2.$$

Therefore the operator norm is less than or equal to $r$. If $j$ is such that $|c_j| = r$, then take $x$ in the range of $E(A_j)$, and then $\sum_i c_i E(A_i)x = c_j x$, which implies that the norm is equal to $r$. \qed

Suppose $\chi_B(A)$ is defined for every Borel measurable subset $B$ of $\sigma(A)$. If $f$ is simple, i.e., $f = \sum c_i \chi_{A_i}$, where the $A_i$ are disjoint, we could define $f(A) = \sum c_i E(A_i)$. By the previous proposition, we know

$$\|f(A)\| = \sup_{\lambda \in \sigma(A)} |f(\lambda)|$$

if $f$ is simple. If $f$ is bounded and measurable, we can take $f_n$ simple converging to $f$ uniformly. Then

$$\|f_n(A) - f_m(A)\| = \sup_{\lambda \in \sigma(A)} |f_n - f_m|,$$

since $f_n - f_m$ is simple, and therefore $f_n(A)$ is a Cauchy sequence. We define $f(A)$ to be the limit of $f_n(A)$. This allows us to define $f(A)$ for $f$ bounded and Borel measurable provided we know how to define $\chi_C(A)$.

Proposition 11.17

$$\|f(A)x\|^2 = \int |f(\lambda)|^2 m_{x,x}(d\lambda).$$

Proof. If $f$ is bounded and measurable

$$\|f(A)x\|^2 = \langle f(A)x, f(A)x \rangle = \langle |f|^2(A)x, x \rangle = \int |f^2(\lambda)| m_{x,x}(d\lambda).$$

\qed
11.5 Normal operators

We need a simple lemma.

**Lemma 11.18** If $U$ is a bounded linear operator on $H$, then $\|U^*U\| = \|U\|^2$.

**Proof.** On the one hand

$$\|U^*U\| \leq \|U^*\| \|U\|,$$

and we saw at the beginning of this chapter that $\|U^*\| = \|U\|$.

On the other hand,

$$\|Ux\|^2 = \langle Ux, Ux \rangle = \langle U^*Ux, x \rangle \leq \|U^*U\| \|x\|^2.$$

Taking the sup over $\|x\| = 1$, we get our result. □

Let $F$ be a Banach algebra with unit.

**Proposition 11.19** If $Q \in F$, then

$$\sigma(Q) = \{p(Q) : p \text{ a homomorphism of } F \text{ into } \mathbb{C}\}.$$

**Proof.** $\lambda \in \sigma(Q)$ if and only if $\lambda I - Q$ is not invertible, which happens if and only if $p(\lambda I - Q) = 0$ for some homomorphism $p$. Since $p(I) = 1$, this happens if and only if $\lambda = p(Q)$ for some $p$. □

**Proposition 11.20** If $p$ is a homomorphism, then $p(T^*) = \overline{p(T)}$.

**Proof.** Let $A = (T + T^*)/2$ and $B = (T - T^*)/2$. Then $A^* = A$, $B^* = -B$, $T = A + B$, $T^* = A - B$, and so $p(T) = p(A) + p(B)$ and similarly with $T$ replaced by $T^*$.

It will suffice to show $p(A)$ is real and $p(B)$ is imaginary, for then we have

$$p(T^*) = p(A) - p(B), \quad p(T) = p(A) + p(B),$$
and these two are complex conjugates of each other.

Write $p(A) = a + ib$ and let $U = A + itI$, so that $U^* = A - itI$. Then

$$U^*U = A^2 + t^2I.$$  

We have $p(U) = a + i(b + t)$, so $|p(U)|^2 = a^2 + (b + t)^2$. We know from Chapter 9 that homomorphisms are contractions, so $|p(U)| \leq \|U\|$, and hence 

$$a^2 + (b + t)^2 = |p(U)|^2 \leq \|U\|^2 = \|U^*U\| \leq \|A\|^2 + t^2$$

for all $t$, which can only happen if $b = 0$. (If $b > 0$, take $t$ large positive, and $t$ large negative if $b < 0$.) The operator $iB$ is symmetric, so apply the above to $iB$. 

An alternate proof that $p(A)$ is real is that since $A$ is symmetric, its spectrum is contained in the real line and we know $p(A) \in \sigma(A)$.

**Proposition 11.21** If $T$ and $T^*$ commute, then $\|T\| = r(T)$.

**Proof.** We already know this if $T$ is symmetric. Note $T^*T$ is always symmetric.

For general $T$,

$$\|T\|^2 = \|T^*T\| = r(T^*T) = \sup_{\lambda \in \sigma(T^*T)} |\lambda|$$

$$= \sup_{p \in \mathcal{F}} |p(T^*T)| = \sup_{p \in \mathcal{F}} |p(T)|^2$$

$$= \left( \sup_{p \in \mathcal{F}} |p(T)| \right)^2 = \left( \sup_{\lambda \in \sigma(T)} |\lambda| \right)^2$$

$$= (r(T))^2.$$  

An operator $N$ is **normal** if $N^*N = NN^*$.

We will see that there is a spectral resolution for the identity as for symmetric operators, but now the spectrum is not necessarily real. In the case of matrices, normal matrices are diagonalizable, but the eigenvalues can be complex.
Lemma 11.22 Let $R(x, y)$ be a polynomial, $N$ normal, and $Q = R(N, N^*)$. Then

$$\sigma(Q) = \{R(\lambda, \overline{\lambda}) : \lambda \in \sigma(N)\}.$$ 

Proof. Operators of the form $R(N, N^*)$ are a commutative algebra with unit. Let $\mathcal{F}$ be the closure in the operator norm.

Since $N$ and $N^*$ commute, they each commute with $Q$, and so $Q$ and $Q^*$ commute. Now $p(Q) = R(p(N), p(N^*)) = R(p(N), \overline{p(N)})$. Then $\sigma(Q)$ is equal to the set of points $R(p(N), p(N))$ where $p$ is a homomorphism, which is the same as the set of $R(\lambda, \overline{\lambda})$ where $\lambda \in \sigma(N)$. \qed

Theorem 11.23 Let $N$ be normal. There exists an orthogonal projection valued measure $E$ on $\sigma(N)$ such that $I = \int_{\sigma(N)} dE$ and $N = \int_{\sigma(N)} \lambda E(d\lambda)$.

Proof. Let $q(x, y)$ be a polynomial in $x$ and $y$. If we let $w = x + yi \in \mathbb{C}$, we can let $x = (w + \overline{w})/2$, $y = (w - \overline{w})/2$, and write $q(x, y) = R(w, \overline{w})$ for some polynomial $R$. Set $Q = R(N, N^*)$. By the above lemma we have $\sigma(Q) = R(\lambda, \overline{\lambda})$ for $\lambda \in \sigma(N)$. We have $\|Q\| = r(Q)$, since $Q$ and $Q^*$ commute. Therefore

$$\|Q\| = \sup_{\lambda \in \sigma(N)} |R(\lambda, \overline{\lambda})|.$$ 

Also, $R(\lambda, \overline{\lambda}) = q(\frac{1}{2}(\lambda + \overline{\lambda}), \frac{1}{2}(\lambda - \overline{\lambda}))$. Now we can define $f(N)$ as the limit of polynomials, and the rest of the proof is as before. \qed

11.6 Unitary operators

$U$ is a unitary operator if it is linear, isometric, one-to-one, and onto. (Cf. rotations) So $\|Ux\| = \|x\|$, or $\langle Ux, Ux \rangle = \langle x, x \rangle$. By polarization, $\langle Ux, Uy \rangle = \langle x, y \rangle$, so $\langle x, U^*Uy \rangle = \langle x, y \rangle$, which implies $U^*U = I$. $U$ is invertible, since it is one-to-one and onto, and thus $U^{-1} = U^*$.

$U^*U = I = UU^*$, so unitary operators are also normal operators.

Proposition 11.24 If $U$ is unitary, then $\sigma(U) \subset \{|z| = 1\}$. 

Proof. \((\lambda I - U) = \lambda(I - U/\lambda)\). Since \(U\) is an isometry, then \(\|U\| = 1\). Then \(I - \frac{1}{\lambda}U\) is invertible if \(\frac{1}{|\lambda|}\|U\| < 1\), or if \(|\lambda| > 1\).

Now suppose \(|\lambda| < 1\). \((\lambda I - U) = U(\lambda U^{-1} - I) = U(\lambda U^* - I)\). Since \(\|\lambda U^*\| = |\lambda| < 1\), then \(I - \lambda U^*\) is invertible.

\[\Box\]

Proposition 11.25 Suppose \(T\) is a bounded normal operator.

(1) If \(\sigma(T) \subset \mathbb{R}\), then \(T\) is symmetric.

(2) If \(\sigma(T) \subset \{|z| = 1\}\), then \(T\) is unitary.

Proof. (1) Let \(q(\lambda, \bar{\lambda}) = \lambda - \bar{\lambda}\). Then

\[T - T^* = q(T, T^*),\]

and

\[\|T - T^*\| = \sup_{\lambda \in \sigma(T)} q(\lambda, \bar{\lambda}) = 0\]

since \(\lambda = \bar{\lambda}\) if \(\lambda\) is real.

(2) Let \(q(\lambda, \bar{\lambda}) = \bar{\lambda}\lambda - 1\). Then

\[T^*T - I = q(T, T^*),\]

and

\[\|T^*T - I\| = \sup_{\lambda \in \sigma(T)} q(\lambda, \bar{\lambda}) \leq \sup_{|\lambda| = 1} (|\lambda|^2 - 1) = 0.\]

\[\Box\]
Chapter 12

Unbounded operators

12.1 Definitions

Let $D$ be a subspace of a Hilbert space $H$. In this chapter $D$ will almost never be closed. An unbounded operator $T$ in $H$ with domain $D$ is a linear mapping from $D$ into $H$. We will write $D(T)$ for the domain of $T$. $T$ is densely defined if $D(T)$ is dense in $H$.

For an example, let $H = L^2[0,1]$, let $D = C^1[0,1]$, and let $Tf = f'$. Note $T$ is not a bounded operator. For another example, let $D = \{ f \in C^2 : f(0) = f(1) = 0 \}$ and $Uf = f''$. Then one can show that $\{-n^2\pi^2\}$ are eigenvalues.

Recall that $G(T)$, the graph of $T$, is the set $\{(x,Tx) : x \in D(T)\}$. If $U$ is an extension of $T$, that means that $D(T) \subset D(U)$ and $Ux = Tx$ if $x \in D(T)$. Note $U$ will be an extension of $T$ if and only if $G(T) \subset G(U)$. One often writes $T \subset U$ to mean that $U$ is an extension of $T$.

A closed operator in $H$ is one whose graph is a closed subspace of $H \times H$. This is equivalent to saying that whenever $x_n \to x$ and $Tx_n \to y$, then $x \in D(T)$ and $y = Tx$.

**Proposition 12.1** If $D(T) = H$ and $T$ is closed, then $T$ is a bounded operator.

**Proof.** Recall the closed graph theorem, which says that if $M$ is a closed
linear map from a Banach space to itself, then $M$ is bounded. The proposition follows immediately from this.

Given a densely defined operator $T$, we want to define its adjoint $T^*$. First we define $D(T^*)$ to be the set of $y \in H$ such that the linear functional $\ell(x) = \langle Tx, y \rangle$ is continuous (i.e., bounded) on $D(T)$. If $y \in D(T^*)$, the Hahn-Banach theorem allows us to extend $\ell$ to a bounded linear functional on $H$. By the Riesz representation theorem for Hilbert spaces, there exists $z_y \in H$ such that

$$\ell(x) = \langle x, z_y \rangle, \quad x \in D(T).$$

Of course $z_y$ depends on $y$. We then define $T^* y = z_y$.

Since $T$ is densely defined, it is routine to check that $T^*$ is well defined and also that $T^*$ is an operator in $H$, that is, $D(T^*)$ is a subspace of $H$ and $T^*$ is linear.

For an example, let $H = L^2[0, 1]$, $D(T) = \{ f \in C^1 : f(0) = f(1) = 0 \}$, and $T f = f'$. If $f \in D(T)$ and $g \in C^1$, then

$$\langle Tf, g \rangle = \int_0^1 f'(x) \overline{g}(x) \, dx = f(1) \overline{g}(1) - f(0) \overline{g}(0) - \int_0^1 f(x) \overline{g'}(x) \, dx = \langle f, -g' \rangle$$

by integration by parts. Thus $|\langle Tf, g \rangle| \leq \|f\| \|g'\|$ is a bounded linear functional, and we see that $C^1 \subset D(T^*)$ and $T^* g = -g'$ if $g \in C^1$.

Some care is needed for the sum and composition of unbounded operators. We define

$$D(S + T) = D(S) \cap D(T)$$

and

$$D(ST) = \{ x \in D(T) : Tx \in D(S) \}.$$

**Proposition 12.2** If $S$, $T$, and $ST$ are densely defined operators in $H$, then

$$T^* S^* \subset (ST)^*.$$  \hspace{1cm} (12.1)

If in addition $S$ is bounded, then

$$T^* S^* = (ST)^*.$$
12.1. DEFINITIONS

Proof. Suppose \( x \in D(ST) \) and \( y \in D(T^*S^*) \). Since \( x \in D(T) \) and \( S^*y \in D(T^*) \), then
\[
\langle Tx, S^* y \rangle = \langle x, T^* S^* y \rangle.
\]
Since \( Tx \in D(S) \) and \( S^*y \in D(T^*) \), then
\[
\langle STx, y \rangle = \langle Tx, S^* y \rangle.
\]
Therefore
\[
\langle STx, y \rangle = \langle x, T^* S^* y \rangle.
\]
Assume now that \( S \) is bounded and \( y \in D((ST)^*) \). Then \( S^* \) is also bounded and \( D(S^*) \) is therefore equal to \( H \). Hence
\[
\langle Tx, S^* y \rangle = \langle STx, y \rangle = \langle x, (ST)^* y \rangle
\]
for every \( x \in D(ST) \). Thus \( S^* y \in D(T^*) \), and so \( y \in D(T^*S^*) \). Now combine with (12.1).

An operator \( T \) in \( H \) is symmetric if \( \langle Tx,y \rangle = \langle x,Ty \rangle \) whenever \( x,y \) are both in \( D(T) \). Thus a densely defined symmetric operator \( T \) is one such that \( T \subset T^* \). If \( T = T^* \), we say \( T \) is self-adjoint. Note that the domains of \( T \) and \( T^* \) are crucial here. This is not an issue with bounded operators because every symmetric bounded operator is self-adjoint.

Let us look at some examples. These will all be the same operator, but with different domains. Let \( H = L^2[0,1] \). Let \( D(S) \) be the set of absolutely continuous functions \( f \) on \([0,1]\) such that \( f' \in L^2 \). Let \( D(T) \) be the set of \( f \in D(S) \) such that in addition \( f(0) = f(1) \), and let \( D(U) \) be the set of functions in \( D(S) \) such that \( f(0) = f(1) = 0 \). Note that if \( f' \in L^2 \), then
\[
|f(t) - f(s)| = \left| \int_s^t f'(x) \, dx \right| \leq \|f'\|_{L^2}|t-s|^{1/2}
\]
by Cauchy-Schwarz, so functions in any of these domains can be well defined at points.

The operator will be the same in each case: \( Sf = if' \), and the same for \( Tf \) and \( Uf \) provided \( f \) is in the appropriate domain. We see that \( U \subset T \subset S \). We will show that \( T \) is self-adjoint, \( U \) is symmetric but not self-adjoint, and \( S \) is not symmetric.
By integration by parts,
\[
\langle Tf, g \rangle = \int_0^1 (if')\overline{g} \quad (12.2)
\]
\[
= if(1)\overline{g}(1) - if(0)\overline{g}(0) - \int_0^1 if(\overline{g})'
\]
\[
= if(1)\overline{g}(1) - if(0)\overline{g}(0) + \int_0^1 f(ig').
\]
Thus if \( f, g \in D(T) \), we have \( \langle Tf, g \rangle = \langle f, Tg \rangle \), since \( f(1) = f(0) \) and \( g(1) = g(0) \) for \( f, g \in D(T) \).

The same calculation with \( T \) replaced by \( S \) shows that \( S \) is not symmetric. The calculation with \( T \) replaced by \( U \) shows that \( U \) is symmetric. Moreover \((12.2)\) shows that \( U \subset S^\ast \).

Suppose \( g \in D(T^\ast) \) and \( \phi = T^\ast g \). Let \( \Phi(x) = \int_0^x \phi(y) \, dy \). If \( f \in D(T) \), then
\[
\int_0^1 if'\overline{g} = \langle Tf, g \rangle = \langle f, \phi \rangle = f(1)\overline{\Phi}(1) - \int_0^1 f'\overline{\Phi},
\]
the last equality by integration by parts. Since \( D(T) \) contains non-zero constants, take \( f \) identically equal to 1 to conclude that \( \Phi(1) = 0 \). Therefore we have
\[
\int_0^1 f'G = 0
\]
whenever \( f \in D(T) \) and
\[ G = ig - \Phi. \]
Taking the complex conjugate and replacing \( f \) by \( \overline{f} \),
\[
\int_0^1 f'G = 0
\]
if \( f \in D(T) \).

We claim \( G \) is constant (a.e.). Suppose \( a < b \) is such that \( [a, a+h], [b, b+h] \) are both subsets of \([0, 1] \) and take \( f \) such that
\[
f' = \frac{1}{h} \chi_{[a,a+h]} - \frac{1}{h} \chi_{[b,b+h]}.
\]
Then \( f \in D(T) \) and so
\[
\frac{1}{h} \int_{a}^{a+h} G(x) \, dx - \frac{1}{h} \int_{b}^{b+h} G(x) \, dx = 0.
\]

There is a set \( N \) of Lebesgue measure 0 such that if \( y \notin N \), then
\[
\frac{1}{h} \int_{y}^{y+h} G(x) \, dx \to G(y).
\]

So if \( a, b \notin N \), taking the limit shows \( G(a) = G(b) \). Since we are on \( L^2 \), we can modify \( G \) on a set of Lebesgue measure 0 and take \( G \) constant.

This implies that \( g = -i\Phi + c \) is absolutely continuous and \( g' = -i\phi \in L^2 \). Also, \( g(0) = -i\Phi(0) + c = -i\Phi(1) + c \), hence \( g \in D(T) \). Thus \( T^* \subset T \).

In the case of \( U \): if \( g \in D(U^*) \) and \( f \in D(U) \), then \( f(1) = 0 \) and so
\[
\int_{0}^{1} i f' \overline{g} = f(1) \overline{\Phi}(1) - \int_{0}^{1} f' \Phi = - \int_{0}^{1} f' \overline{\Phi}.
\]

If \( G = ig - \Phi \), then \( \int_{0}^{1} f' G = 0 \). As before \( G \) is constant, so \( g = -i\Phi + c \), but now we no longer know that \( \Phi(1) = 0 \). So \( g(1) \) might not equal \( g(0) \). Therefore \( U^* \subset S \).

If \( g \in D(S) \) and \( f \in D(U) \), we have
\[
\langle Uf, g \rangle = if(1)\overline{g}(1) - if(0)\overline{g}(0) + \int_{0}^{1} f' \overline{g} = \langle f, Ug \rangle.
\]

Hence \( g \in D(U^*) \). Thus \( S \subset U^* \), and with the above \( U^* = S \). Hence \( U \) is not self-adjoint.

**Proposition 12.3** Let \( H \) be a Hilbert space over \( \mathbb{C} \), \( A \) self-adjoint. Then \( A \) is closed.

**Proof.** \( A \) is closed: if \( x_n \to x \) and \( Ax_n \to u \), then
\[
\langle Ax_n, y \rangle = \langle x_n, Ay \rangle \to \langle x, Ay \rangle = \langle Ax, y \rangle.
\]

Also \( \langle Ax_n, y \rangle \to \langle u, y \rangle \). This is true for all \( y \), so \( Ax = u \). \( \square \)
If $A$ is defined on all of $H$ and is self-adjoint, we conclude that $A$ is bounded.

We say $z$ is in the resolvent set of $A$ if $A - zI$ maps $D$ one-to-one onto $H$.

**Proposition 12.4** If $z$ is not real, then $z$ is in the resolvent set. Equivalently, $\sigma(A) \subset \mathbb{R}$.

**Proof.** (1) $R = \text{Range} \ (A - zI)$ is a closed subspace.

$R$ is equal to the set of all vectors $u$ of the form $Av - zv = u$ for some $v \in D$. Then $\langle Av, v \rangle - z \langle v, v \rangle = \langle u, v \rangle$. $A$ is self-adjoint, so $\langle Av, v \rangle = \langle v, Av \rangle = \langle Av, v \rangle$ is real. Looking at the imaginary parts,

$$-\text{Im} \ (z\|v\|^2) = \text{Im} \ \langle u, v \rangle,$$

so $|\text{Im} \ z\|v\|^2| \leq \|u\|\|v\|$, or

$$\|v\| \leq \frac{1}{|\text{Im} \ z|}\|u\|.$$

If $u_n \in R$ and $u_n \to u$, then $\|v_n - v_m\| \leq (1/|\text{Im} \ z|)\|u_n - u_m\|$, so $v_n$ is a Cauchy sequence, and hence converges to some point $v$.

Since $Av_n - zv_n = u_n \to u$ and $zv_n$ converges to $zv$, then $Av_n$ converges to $u + zv$. Since $A$ is self-adjoint, it is closed, and so $v \in D(A)$. Since $\langle Av_n, w \rangle = \langle v_n, Aw \rangle$ for $w \in D$, then $\langle u + zv, w \rangle = \langle v, Aw \rangle$, which implies and $Av = u + zv$, or $u = (A - z)v \in R$.

(2) $R = H$. If not, there exists $x \neq 0$ such that $x$ is orthogonal to $R$, and then

$$\langle Av - zv, x \rangle = \langle Av, x \rangle - z\langle v, x \rangle = 0$$

for all $v \in D$. Then $\langle Av, x \rangle = z\langle v, x \rangle$, so $x \in D$ and $Ax = zx$. But then $\langle x, Ax \rangle = z\langle x, x \rangle$ is not real, a contradiction.

(3) $A - zI$ is one-to-one. If not, there exists $x \in D$ such that $(A - zI)x = 0$. But then $\|x\| \leq (1/|\text{Im} \ z|)\|0\| = 0$, or $x = 0$. \qed

If we set $R(z) = (A - zI)^{-1}$ the resolvent, we have

$$\|R(z)\| \leq \frac{1}{|\text{Im} \ z|}.$$
If \( u, w \in H \) and \( v = R(z)u \), then \( (A - z)v = u \), and
\[
\langle u, R(\bar{z})w \rangle = \langle (A - z)v, R(\bar{z})w \rangle = \langle v, ((A - \bar{z})R(\bar{z})w \rangle = \langle v, w \rangle = \langle R(z)u, w \rangle.
\]
So the adjoint of \( R(z) \) is \( R(\bar{z}) \).

**Theorem 12.5** Let \( A \) be a symmetric operator. \( A \) is self-adjoint if and only if \( \sigma(A) \subseteq \mathbb{R} \).

**Proof.** That \( A \) self-adjoint implies that all non-real \( z \) are in the resolvent set has already been proved. We thus have to show that if \( A \) is symmetric and \( \sigma(A) \subseteq \mathbb{R} \), then \( A \) is self-adjoint.

If \( x, y \in D(A) \),
\[
\langle (A - z)x, y \rangle = \langle x, (A - \bar{z})y \rangle.
\]
If \( z \) is not real, then \( z \notin \sigma(A) \), so \( z - A \) is invertible and \( A - z \) and \( A - \bar{z} \) map \( D(A) \) one-to-one and onto \( H \). For \( f, g \in H \), we can define \( x = (A - z)^{-1}f \) and \( y = (A - \bar{z})g \), and we note that \( x \) and \( y \) are both in \( D(A) \). We then have
\[
\langle f, (A - \bar{z})^{-1}g \rangle = \langle (A - z)^{-1}f, g \rangle
\]
for all \( f, g \in H \).

Now we show \( A \) is self-adjoint. Take \( z \) non-real and suppose \( v \in D(A^*) \). Set \( w = A^*v \in H \). We have
\[
\langle Ax, v \rangle = \langle x, A^*v \rangle
\]
for all \( x \in D(A) \). Subtract \( z \langle x, v \rangle \) from both sides:
\[
\langle (A - z)x, v \rangle = \langle x, (A^* - \bar{z})v \rangle.
\]
Let \( g = (A^* - \bar{z})v \) and \( f = (A - z)x \). Then
\[
\langle f, v \rangle = \langle (A - z)x, v \rangle = \langle x, (A^* - \bar{z})v \rangle = \langle (A - z)^{-1}f, g \rangle = \langle f, (A - \bar{z})^{-1}g \rangle.
\]
The set of \( f \) of the form \( (A - z)x \) for \( x \in D(A) \) is all of \( H \), hence \( v = (A - \bar{z})^{-1}g \), which is in \( D(A) \). In particular, \( D(A^*) \subseteq D(A) \). We have
\[
(A - \bar{z})v = g = (A^* - \bar{z})v, \text{ so } A^*v = Av.
\]
12.2 Cayley transform

Define
\[ U = (A - i)(A + i)^{-1}. \]
This is the image of the operator \( A \) under the function
\[ F(z) = \frac{z - i}{z + i}, \tag{12.3} \]
which maps the real line to \( \partial B(0,1) \setminus \{1\} \), and is called the Cayley transform of \( A \).

**Proposition 12.6** \( U \) is a unitary operator.

**Proof.** \( A + i \) and \( A - i \) each map \( D(A) \) one-to-one onto \( H \), so \( U \) maps \( H \) onto itself.

\( U \) is norm preserving: Let \( u \in H, v = (A+i)^{-1}u, w = Uu \). So \((A+i)v = u, (A - i)v = w \). We need to show \( \|u\| = \|w\| \).

We have
\[ \|u\|^2 = \langle (A + i)v, (A + i)v \rangle = \|Av\|^2 + \|v\|^2 + i\langle v, Av \rangle - i\langle Av, v \rangle \]
\[ = \|Av\|^2 + \|v\|^2, \]
and similarly
\[ \|w\| = \langle (A - i)v, (A - i)v \rangle = \|Av\|^2 + \|v\|^2. \]

\( \square \)

**Proposition 12.7** Given \( A \) and \( U \) as above and \( E \) the spectral resolution for \( U \), \( E(\{1\}) = 0 \).

**Proof.** Write \( E_1 \) for \( E(\{1\}) \). If \( E_1 \neq 0 \), there exists \( z \neq 0 \) in the range of \( E_1 \), so \( z = E_1 w \). Then
\[ Uz = \int_{\sigma(U)} \lambda E(d\lambda)z = \int_{\sigma(U)} \lambda (E - E_1)(d\lambda)z + \int_{\{1\}} \lambda E_1(d\lambda)z. \]
The first integral is zero since \((E - E_1)(A)\) and \(E_1\) are orthogonal for all \(A\). The second integral is equal to
\[
E_1z = E_1E_1w = E_1w = z
\]
since \(E_1\) is a projection.

We conclude \(z\) is an eigenvector for \(U\) with eigenvalue 1. So
\[
(A - iI)(A + iI)^{-1}z = z.
\]
Let \(v = (A + iI)^{-1}z\), or \(z = (A + iI)v\). Then
\[
z = (A - iI)(A + iI)^{-1}z = (A - iI)v,
\]
and then \(iv = -iv\), so \(v = 0\), and hence \(z = 0\), a contradiction. \(\square\)

12.3 Spectral theorem

When \(M\) is a bounded symmetric operator and \(f\) is bounded and measurable, we defined \(f(M)\) in Chapter 11. We now want to define \(f(M)\) for some unbounded functions \(f\).

**Proposition 12.8** Let \(M\) be a bounded operator and \(f\) a measurable function. Let
\[
D_f = \left\{ x : \int_{\sigma(M)} |f(\lambda)|^2 \mu_{x,x}(d\lambda) < \infty \right\}.
\]
Then
1. \(D_f\) is a dense subspace of \(H\).
2. If \(x, y \in H\),
\[
\int_{\sigma(M)} |f(\lambda)||\mu_{x,y}|(d\lambda) \leq \|y\| \left( \int_{\sigma(M)} |f(\lambda)|^2 \mu_{x,x}(d\lambda) \right)^{1/2}.
\]
3. If \(f\) is bounded and \(v = f(M)z\), then
\[
\mu_{x,v}(d\lambda) = \overline{f(\lambda)} \mu_{x,z}(d\lambda), \quad x, z \in H.
\]
Proof. (1) Let \( S \subset \sigma(M) \) and \( z = x + y \).

\[
\|E(S)z\|^2 \leq (\|E(S)x\| + \|E(S)y\|)^2 \leq 2\|E(S)x\|^2 + 2\|E(S)y\|^2.
\]

So

\[
\mu_{z,z}(S) \leq 2\mu_{x,x}(S) + 2\mu_{y,y}(S).
\]

This is true for all \( S \), so

\[
\mu_{z,z}(d\lambda) \leq 2\mu_{x,x}(d\lambda) + 2\mu_{y,y}(d\lambda).
\]

This proves that \( D_f \) is a subspace.

Let \( S_n = \{ \lambda \in \sigma(M) : |f(\lambda)| < n \} \). Then if \( x = E(S_n)z \),

\[
E(S)x = E(S)E(S_n)E(S_n)z = E(S \cap S_n)E(S_n)z = E(S \cap S_n)x,
\]

so \( \mu_{x,x}(S) = \mu_{x,x}(S \cap S_n) \). Then

\[
\int_{\sigma(M)} |f(\lambda)|^2 \mu_{x,x}(d\lambda) = \int_{S_n} |f(\lambda)|^2 \mu_{x,x}(d\lambda) \leq n^2 \|x\|^2 < \infty.
\]

To see this last line, we know it holds when \( |f|^2 \) is replaced by \( g \) and \( g \) is the characteristic function of a set. It holds for \( g \) simple by linearity, and then it holds for \( g = |f|^2 \) by monotone convergence. Therefore the range of \( E(S_n) \subset D(f) \). \( \sigma(M) = \bigcup_n S_n \), so

\[
\|E(S_n)y - y\|^2 = \|E(S_n)(y) - E(\sigma(M))(y)\|^2 = \int_{\sigma(M)} |\chi_{\sigma(M) \setminus S_n}(\lambda)|^2 \mu_{y,y}(d\lambda) \to 0
\]

by dominated convergence. Hence \( y \) is in the closure of \( D_f \).

(2) If \( x, y \in H \), \( f \) bounded,

\[
f(\lambda) \mu_{x,y}(d\lambda) \ll |f(\lambda)| \mu_{x,y}(d\lambda),
\]

so there exists \( u \) with \( |u| = 1 \) such that

\[
u(\lambda)f(\lambda) \mu_{x,y}(d\lambda) = |f(\lambda)| \mu_{x,y}(d\lambda).
\]

Hence

\[
\int_{\sigma(M)} |f(\lambda)| \mu_{x,y}(d\lambda) = (uf(M)x, y) \leq \|uf(M)x\| \|y\|.
\]
But
\[ \|u f(M)x\|^2 = \int |u f|^2 \, d\mu_{x,x} = \int |f|^2 \, d\mu_{x,x}. \]

So (2) holds for bounded \( f \). Now take a limit and use monotone convergence.

(3) Let \( g \) be continuous.
\[ \int_{\sigma(M)} g \, d\mu_{x,v} = (g(M)x, v) = (g(M)x, f(M)z) \]
\[ = ((Tg)(M)x, z) = \int g T \, d\mu_{x,z}. \]
this is true for all \( g \) continuous, so \( d\mu_{x,x} = T \, d\mu_{x,z}. \]

**Theorem 12.9** Let \( E \) be a resolution of the identity.

(1) Suppose \( f : \sigma(M) \to \mathbb{C} \) is measurable. There exists a densely defined operator \( f(M) \) with domain \( D_f \) and
\[ \langle f(M)x, y \rangle = \int_{\sigma(M)} f(\lambda) \, \mu_{x,y}(d\lambda) \]  \hspace{1cm} (12.4)
\[ \|f(M)x\| = \int_{\sigma(M)} |f(\lambda)|^2 \, \mu_{x,x}(d\lambda). \] \hspace{1cm} (12.5)

(2) If \( D_{fg} \subset D_g \), then \( f(M)g(M) = (fg)(M) \).

(3) \( f(M)^* = T(M) \) and \( f(M)f(M)^* = f(M)^*f(M) = |f|^2(M) \).

**Proof.** (1) If \( x \in D_f \), then \( \ell(y) = \int_{\sigma(M)} f \, d\mu_{x,y} \) is a bounded linear functional with norm at most \( (\int |f|^2 \, d\mu_{x,x})^{1/2} \) by (2) of the preceding proposition. Choose \( f(M)x \in H \) to satisfy (1) for all \( y \).

Let \( f_n = f \chi(\{ |f| \leq n \}) \). Then \( D_{f-f_n} = D_f \) since \( \int |f - f_n|^2 \, d\mu_{x,x} \) is finite if and only if \( \int |f|^2 \, d\mu_{x,x} \) is finite, using that \( f_n \) is bounded. By the dominated convergence theorem,
\[ \|f(M)x - f_n(M)x\|^2 \leq \int_{\sigma(M)} |f - f_n|^2 \, d\mu_{x,x} \to 0. \]
Since $f_n$ is bounded, (12.5) holds with $f_n$. Now let $n \rightarrow \infty$.

(2): Define $g_m = g\chi(|g| \leq m)$. Since $f_n$ and $g_m$ are bounded, (2) follows for $f_n, g_m$. Now let $m \rightarrow \infty$ and then $n \rightarrow \infty$.

(3) We know this holds for $f_n$ since $f_n$ is bounded. Now let $n \rightarrow \infty$. 

**Theorem 12.10** *(Change of measure principle)* Let $E$ be a resolution of the identity on $A$, $\Phi : A \rightarrow B$ one-to-one and bimeasurable. Let $E'(S') = E(\Phi^{-1}(S'))$. Then $E'$ is a resolution of the identity on $B$, and

$$\int_B f \, d\mu'_{x,y} = \int_B (f \circ \Phi) \, d\mu_{x,y}.$$  

Bimeasurable means that $\Phi$ and $\Phi^{-1}$ are both measurable. Saying that $E$ is a resolution of the identity on $A$ means that $E(C)$ is symmetric for every measurable subset $C$ of $A$, $\|E(C)\| \leq 1$, $E(\emptyset) = 0$, $E(A) = I$, $E(C \cup D) = E(C) + E(D)$ if $C$ and $D$ are disjoint, and $E(C \cap D) = E(C)E(D)$. Finally, $\langle E(C)x, y \rangle = \int \chi_C(z) \mu_{x,y}(dz)$ characterizes the measure $\mu_{x,y}$ and similarly for $\mu'_{x,y}$.

**Proof.** Prove for $f$ the indicator of a set, use linearity, and take limits. 

**Theorem 12.11** *(Spectral theorem)* Let $A$ be a self-adjoint operator on a Hilbert space over the complex numbers. There exists a resolution of the identity $E$ such that

$$A = \int_{\sigma(A)} z \, E(dz).$$

**Proof.** Start with the unbounded operator $A$. Let $U = (A - iI)(A + iI)^{-1}$. Then $U$ is unitary with a spectrum on $\partial B(0, 1) \setminus \{1\}$. Let the resolution of the identity for $U$ be given by $\tilde{E}$.

Let $F$ be defined as in (12.3) and define $\Phi = F^{-1}$, which is a map taking $\partial B(0, 1) \setminus \{1\}$ to $\mathbb{R}$. Thus

$$\Phi(z) = \frac{i(1 + z)}{1 - z}.$$
12.3. SPECTRAL THEOREM

We check that \( A = \Phi(U) \). Since the range of \( \Phi \) is \( \mathbb{R} \), then \( \Phi(U) \) is self-adjoint by Theorem 12.9(3). Since \( \Phi(z)(1 - z) = i(1 + z) \), Theorem 12.9(2) implies that
\[
\Phi(U)(I - U) = i(I + U).
\]
In particular, the range of \( I - U \) is contained in the domain of \( \Phi(U) \). From the definition of the Cayley transform, we have
\[
A(I - U) = i(I + U)
\]
and the domain of \( A \) is equal to the range of \( I - U \). Thus \( A \subset \Phi(U) \). Since both \( A \) and \( \Phi(U) \) are self-adjoint,
\[
\Phi(U) = \Phi(U)^* \subset A^* = A \subset \Phi(U),
\]
and hence \( A = \Phi(U) \).

Let \( E(S) = \tilde{E}(\Phi^{-1}(S)) \). We have
\[
\langle Ax, y \rangle = \langle \Phi(U)x, y \rangle = \int_{\sigma(U)} \Phi(z) \langle \tilde{E}(dz)x, y \rangle.
\]
By the change of measure principle, this is equal to
\[
\int_{\sigma(A)} z \langle E(dz)x, y \rangle.
\]
\[\square\]
Chapter 13

Semigroups

13.1 Strongly continuous semigroups

Let $X$ be a Banach space over the complex numbers, $T(t) = T_t$ linear bounded operators for $t \geq 0$. $T$ is a semigroup if $T_{t+s} = T_t T_s$, $T_0 = I$.

These come up in PDE and in probability. For example, if one wants to solve the equation

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x), \quad u(0, x) = f(x),$$

where $f$ is a given function (this is the heat equation on $\mathbb{R}$), the solution is given by $u(t, x) = T_t f(x)$ for a certain semigroup $T_t$.

If $X_t$ is a Markov process, then $T_t f(x) = \mathbb{E}^x f(X_t)$ will be a semigroup, where $\mathbb{E}^x$ means expectation starting at $x$.

Here is an example: if $X$ is a Hilbert space and $\{\varphi_n\}$ is an orthonormal basis and $\lambda_j$ a sequence of real numbers increasing to infinity, let

$$T_t f = \sum_{j=1}^{\infty} e^{-\lambda_j t} \langle f, \varphi_j \rangle \varphi_j.$$

Another example is to let

$$T_t f(x) = \int f(y) \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} \, dy \quad (13.1)$$

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where \( X \) is the set of continuous functions on \( \mathbb{R} \) vanishing at infinity.

A third example is given by the next proposition.

**Proposition 13.1** Let \( A : X \to X \) be bounded. Then \( T_t = e^{tA} \) (defined as \( e^{tA} = \sum t^n A^n/n! \)) is a semigroup that is continuous in the norm topology.

**Proof.** This follows easily from the functional calculus for operators. \( \square \)

We say \( T_t \) is *strongly continuous* at \( t = 0 \) if \( \|T_t x - x\| \to 0 \) as \( t \to 0 \) for all \( x \in X \).

**Proposition 13.2** Suppose \( T_t \) is a strongly continuous semigroup at 0.

1. There exists \( b \) and \( k \) such that \( \|T_t\| \leq be^{kt} \).
2. \( T_t x \) is strongly continuous in \( t \) for all \( x \in X \).

**Proof.** We claim \( \|T_t\| \) is bounded near 0. If not, there exists \( t_j \to 0 \) such that \( \|T_{t_j}\| \to \infty \). By the uniform boundedness principle, \( T_{t_j} x \) cannot converge to \( x \) for all \( x \), a contradiction to strong continuity. So there exists \( a, b \) such that \( \|T_t\| \leq b \) for \( t \leq a \).

Write \( t = na + r \). \( T_t = T_a^n T_r \), so

\[
\|T_t\| \leq \|T_a\|^n \|T_r\| \leq b^{n+1} \leq be^{kt}
\]

with \( k = \frac{1}{a} \log b \).

2. \( T_t x - T_s x = T_s [T_{t-s} x - x] \), so

\[
\|T_t x - T_s x\| \leq \|T_s\| \|T_{t-s} x - x\| \to 0.
\]

\( \square \)

Suppose \( D \) is dense in \( X \) and \( A : D \to X \) is closed. \( z \in \rho(A) \), the resolvent set, if \( z - A \) maps \( D = D(A) \) one-to-one onto \( X \). Thus \( \rho(A) = \sigma(A)^c \). Write \( R(z) = R_z = (zI - A)^{-1} \).
Since $A$ is closed, then $R_z$ is closed. To see this, suppose $x_n \to x$ and $y_n = R_z x_n \to y$. Then

$$Ay_n = zy_n - (z - A)y_n = zy_n - x_n \to zy - x.$$  

Since $A$ is closed, $y \in D(A)$ and $Ay = zy - x$, or $(z - A)y = x$. So $y = R_z x$, which proves $R_z$ is closed.

$R_z$ is defined on all of $X$, so by the closed graph theorem, $R_z$ is a bounded operator.

Let $T$ be a strongly continuous one parameter semigroup. The infinitesimal generator $A$ is defined by

$$Ax = \lim_{h \to 0} \frac{T_h x - x}{h},$$

where we mean that the difference of the two sides goes to 0 in norm. The domain of $A$ consists of those $x$ for which the strong limit exists.

As an example, with $T_t$ defined by (13.1), if $f \in C^2$ vanishes at infinity, then using Taylor’s theorem,

$$\frac{T_h f(x) - f(x)}{h} = \frac{1}{h} \int [f(y) - f(x)] \frac{1}{\sqrt{2\pi h}} e^{-(y-x)^2/2h} dy + f'(x) \int (y - x) \frac{1}{\sqrt{2\pi h}} e^{-(y-x)^2/2h} dy + \frac{1}{2} f''(x) \int (y - x)^2 \frac{1}{\sqrt{2\pi h}} e^{-(y-x)^2/2h} dy + \int E(h) \frac{1}{\sqrt{2\pi h}} e^{-(y-x)^2/2h} dy = \frac{1}{2} f''(x) + E(h)/h \to \frac{1}{2} f''(x),$$

where $E(h)$ is a remainder term that goes to 0 faster than $h$; we used standard facts about the Gaussian density. One can improve the above to show that the convergence is uniform, and we can then conclude that $C^2 \subset D(A)$ and $Af = \frac{1}{2} f''$.

**Proposition 13.3** (1) A commutes with $T_t$ in the sense that if $x \in D(A)$, then $T_t x \in D(A)$ and $AT_t x = T_t Ax$. 


(2) $D(A)$ is dense in $X$.

(3) $D(A^n)$ is dense.

(4) $A$ is closed.

(5) If $\|T_t\| \leq be^{kt}$ and Re $z > k$, then $z \in \rho(A)$. The resolvent of $A$ is the Laplace transform of $T_1$.

**Proof.**

(1) 

$$\frac{T_{t+h} - T_t}{h} x = T_{t} \frac{T_h - I}{h} x = \frac{T_h - I}{h} T_t x.$$ 

If $x \in D(A)$, the middle term converges to $T_t Ax$. So the limit exists in the third term, and therefore $T_t x \in D(A)$. Moreover $\frac{d}{dt} T_t x = T_t Ax = AT_t x$.

(2) We claim

$$T_t x - x = A \int_0^t T_s x ds.$$ 

To see this, $T_s x$ is a continuous function of $s$. Using a Riemann sum approximation,

$$\frac{T_h - I}{h} \int_0^t T_s x ds = \frac{1}{h} \int_0^t [T_{s+h} x - T_s x] ds = \frac{1}{h} \int_t^{t+h} T_s x ds - \frac{1}{h} \int_0^h T_s x ds 
\rightarrow T_t x - x.$$ 

So $\int_0^t T_s x ds \in D(A)$. But $\frac{1}{h} \int_0^t T_s x ds \rightarrow x$.

(3) Let $\phi$ be $C^{\infty}$ and supported in $(0, 1)$. Let

$$x_\phi = \int_0^1 \phi(s) T_s x ds.$$ 

Then

$$Ax_\phi = \int_0^1 \phi(s) AT_s x ds = \int_0^1 \phi(s) \frac{\partial}{\partial s} T_s x ds = - \int_0^1 \phi'(s) T_s x ds$$

by integration by parts. Repeating, $x_\phi \in D(A^n)$. Now take $\phi_j$ approximating the identity.
13.2. GENERATION OF SEMIGROUPS

(4) \( T_t x - x = \int_0^t T_s Ax \, ds \): To see this, both are 0 at 0. The derivative on the left is \( T_t Ax \), which is the same as the derivative on the right. Let \( x_n \in D(A) \), \( x_n \to x \), \( Ax_n \to y \). Then

\[
T_t x_n - x_n = \int_0^t T_s Ax_n \, ds \to \int_0^t T_s y \, ds.
\]

The left hand term converges to \( T_t x - x \). Divide by \( t \) and let \( t \to 0 \). The right hand side converges to \( y \). Therefore \( x \in D(A) \) and \( Ax = y \).

(5) Let

\[
L(z)x = \int_0^\infty e^{-zs} T_s x \, ds.
\]

The Riemann integral converges when \( \Re z > k \).

\[
\|L(z)x\| \leq \int_0^\infty be^{(k-\Re z)s} \|x\| \, ds \leq \frac{b}{\Re z - k} \|x\|.
\]

We claim \( L(z) = R_z \). Check that \( e^{-zt}T_t \) is also a semigroup with infinitesimal generator \( A - zI \).

Hence

\[
e^{-zt}T_t - x = (A - zI) \int_0^t e^{-zs} T_s x \, ds.
\]

As \( t \to \infty \), the left hand side tends to \(-x\) and the right hand side tends to \((A - zI) L(z)x\). Since \( A \) is closed, \( x = (zI - A) L(z)x \). So \( L(z) \) is the right inverse of \((zI - A)\). Similarly, we see that it is also the left inverse.

\[
13.2\; \text{Generation of semigroups}
\]

**Proposition 13.4** A strongly continuous semigroup of operators is uniquely defined by its infinitesimal generator.

**Proof.** If \( S, T \) have the same generator, let \( x \in D(A) \) and

\[
\frac{d}{dt} S_t T_{s-t} x = S(t)AT_{s-t}x - S_t AT_{s-t}x = 0.
\]
Therefore
\[ 0 = \int_0^s \frac{d}{dr} S_r T_{s-r} x \, dr = S_s T_0 x - S_0 T_s x, \]
or \( S_s x = T_s x. \) Now use the fact that \( D(A) \) is dense.

\( T_t \) is a contraction if \( \|T_t\| \leq 1 \) for all \( t. \)

**Proposition 13.5** The infinitesimal generator of a strongly continuous semigroup of contractions has \((0, \infty) \subseteq \rho(A)\) and
\[ \|R_\lambda\| = \| (\lambda I - A)^{-1} \| \leq \frac{1}{\lambda}. \] (13.2)

**Proof.** We already did this: this is the case \( b = 1, k = 0. \) We have
\[ \|L(z)x\| \leq \frac{1}{|\text{Re} z - k|} \|x\|. \]

**Proposition 13.6** Suppose \( B \) is an extension of \( A \) and there exists \( \lambda \in \rho(A) \cap \rho(B). \) Then \( A = B. \)

**Proof.** Suppose \( x \in D(B) \setminus D(A). \) We know \((\lambda - B)x \in X, \) so
\[ (\lambda - A)^{-1}(\lambda - B)x \in D(A) \subseteq D(B). \]
Then
\[ (\lambda - B)(\lambda - A)^{-1}(\lambda - B)x = (\lambda - A)(\lambda - A)^{-1}(\lambda - B)x = (\lambda - B)x. \]
Hit both sides with \((\lambda - B)^{-1}\) to obtain \((\lambda - A)^{-1}(\lambda - B)x = x. \) So \( x \in D(A), \) a contradiction.

**Theorem 13.7** (Hille-Yosida theorem) Let \( A \) be a densely defined unbounded operator such that \((0, \infty) \subseteq \rho(A)\) and
\[ \|R_\lambda\| = \| (\lambda I - A)^{-1} \| \leq \frac{1}{\lambda}. \] (13.3)
Then \( A \) is the infinitesimal generator of a strongly continuous semigroup of contractions.
13.2. GENERATION OF SEMIGROUPS

Note that saying \((0, \infty) \subset \rho(A)\) implies that \(\lambda - A\) is one-to-one and onto from the domain of \(A\) to the Banach space, which means the range of \(\lambda - A\) is all of the Banach space.

**Proof.** Note \(nR_n - I = R_nA\) since \(R_n(nI - A) = I\). Let \(A_n = nAR_n\). Then \(A_n = n^2R_n - nI\), so \(A_n\) is a bounded operator. Define \(T_n(t) = e^{tA_n}\).

**Step 1.** We show \(nR_nx \to x\) for all \(x\).

To prove this,

\[
\|nR_nx - x\| = \|R_nA(x)\| \leq \frac{1}{n}\|Ax\|,
\]

so the claim is true for \(x \in D(A)\). Since \(\|nR_n\| \leq 1\) and \(D(A)\) is dense in \(X\), this proves the claim.

**Step 2.** We show that if \(x \in D(A)\), then \(A_n(x) \to A(x)\):

\[
A_nx = nAR_nx = nR_nAx \to Ax.
\]

**Step 3.** We show that \(T_n(s)x\) converges for all \(x\).

We have

\[
T_n(t) = e^{tA_n} = e^{-nt}e^{n^2R_nt} = e^{-nt}\sum \frac{(n^2t)^m}{m!}(R_n)^m,
\]

so \(\|T_n(t)\| \leq e^{nt}e^{nt} = 1\).

\(A_n\) and \(A_m\) commute with \(T_n\) and \(T_m\).

\[
\frac{d}{dt}T_n(s - t)T_m(t)x = T_n(s - t)T_m(t)[A_m - A_n]x.
\]

The norm of the right hand side is bounded by \(\|A_nx - A_mx\|\). So

\[
\|T_n(s)x - T_m(s)x\| \leq s\|A_nx - A_mx\| \to 0
\]

as \(n, m \to \infty\). Therefore \(T_n(s)x\) converges, say, to \(T_x\), uniformly in \(s\). \(D(A)\) is dense so this holds for all \(x\).

\(T_n(s)\) is a strongly continuous semigroup of contractions, so the same holds for \(T_s\).
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Step 4. It remains to show that $A$ is the infinitesimal generator of $T$. We have

$$T_n(t)x - x = \int_0^t T_n(s)A_n x \, ds.$$  

If $x \in D(A)$, we can let $n \to \infty$ to get

$$T_t x - x = \int_0^t T_s Ax \, ds.$$  

If $B$ is the generator of $T$, dividing by $t$ and letting $t \to 0$, we get $D(A) \subset D(B)$ and $B = A$ on $D(A)$. So $B$ is an extension of $A$. If $\lambda > 0$, then $\lambda \in \rho(A), \rho(B)$, which implies $B$ cannot be a proper extension by Proposition 13.6.

\[\Box\]

13.3 Perturbation of semigroups

Lemma 13.8 (Lumer-Phillips) Let $A$ be densely defined in a Hilbert space $B$ and suppose $(0, \infty) \subset \rho(A)$. Then $\| R_\lambda \| \leq 1/\lambda$ if and only if $\Re \langle x, Ax \rangle \leq 0$ for all $x \in D(A)$.

If the last property holds, we say $A$ is dissipative. An example is the Laplacian:

$$\langle f, Af \rangle = \int f(x) \Delta f(x) \, dx = -\int |\nabla f(x)|^2 \, dx \leq 0$$

by integration by parts. Another example is if

$$Af(x) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(\cdot) \frac{\partial f}{\partial x_j}(\cdot) \right)(x).$$

To verify this, we again use integration by parts.

**Proof.** Suppose

$$\|(\lambda I - A)^{-1}u\|^2 \leq \frac{1}{\lambda^2} \|u\|^2.$$
Let \( x = (\lambda I - A)^{-1} u \). So
\[
\langle x, x \rangle \leq \frac{1}{\lambda^2} \langle \lambda x - Ax, \lambda x - Ax \rangle.
\]
This becomes
\[
2\text{Re} \langle x, Ax \rangle = \langle x, Ax \rangle + \langle Ax, x \rangle \leq \frac{1}{\lambda} \| Ax \|^2.
\]
This is true for all \( \lambda \), so let \( \lambda \to \infty \).

For the converse,
\[
\langle x, Ax \rangle + \langle Ax, x \rangle = 2\text{Re} \langle x, Ax \rangle \leq 0 \leq \frac{1}{\lambda} \| Ax \|^2
\]
for all \( \lambda > 0 \). Now reverse the above steps. \( \square \)

**Theorem 13.9** (Trotter) Suppose \( A \) is the infinitesimal generator of a semigroup of contractions in a Hilbert space. Let \( B \) be a densely defined dissipative operator such that \( D(A) \subset D(B) \) and there exist \( b > 0 \) and \( a \in (0, 1) \) such that
\[
\| Bx \| \leq a \| Ax \| + b \| x \|, \quad x \in D(A).
\]
Then \( A + B \) (defined on \( D(A) \)) is the generator of a contraction semigroup.

**Proof.** First, \( A + B \) is closed: Let \( x_n \to x \) and \( y_n = (A + B)x_n \to y \). So
\[
A(x_n - x_m) = y_n - y_m - B(x_n - x_m),
\]
and
\[
\| A(x_n - x_m) \| \leq \| y_n - y_m \| + a \| A(x_n - x_m) \| + b \| x_n - x_m \|.
\]
Since \( a < 1 \), then \( Ax_n \) converges. Therefore \( Bx_n \) converges. \( A \) is closed, so \( Ax_n \to Ax \). If \( x \in D(A) \subset D(B) \),
\[
\| Bx_n - Bx \| \leq a \| A_n x - Ax \| + b \| x_n - x \| \to 0.
\]
Then \( (A + B)x_n \to (A + B)x \).
Next, $\lambda \in \rho(A + B)$: By the Lumer-Phillips lemma, $A$ is dissipative. $B$ is also. So $A + B$ is dissipative. By Lumer-Phillips,

$$\|x\| \leq \frac{1}{\lambda} \|(\lambda I - (A + B))x\|.$$  

One immediate consequence of this is that the operator $\lambda - (A + B)$ is one-to-one. Another is that the range of $\lambda - (A + B)$ is closed, because if $y_n \to y$, then $y_n = (\lambda - (A + B))x_n$ for some $x_n$. The inequality shows that $\|x_n - x_m\| \to 0$. If $x_n \to x$, then $y = (\lambda - (A + B))x$, since $A + B$ is a closed operator. Therefore the range of $(A + B) - \lambda I$ is closed.

The range is $X$: if not, there exists $v \neq 0$ perpendicular to the range. $A - \lambda I$ is invertible, so there exists $x \in D(A)$ such that $(A - \lambda I)x = v$. Then $v + Bx$ is in the range, or $\langle v + Bx, v \rangle = 0$. So $\|v\|^2 + \langle Bx, v \rangle = 0$, or

$$\|v\|^2 \leq \|Bx\| \|v\|,$$

and so $\|v\| \leq \|Bx\|$. Then

$$\|Ax - \lambda x\| \leq \|Bx\| \leq a\|Ax\| + b\|x\|.$$  

Squaring and use the fact that $A$ is dissipative,

$$\|Ax\|^2 + \lambda^2\|x\|^2 \leq a^2\|Ax\|^2 + 2ab\|Ax\| \|x\| + b^2\|x\|^2.$$  

This holds for all $\lambda > 0$, so for $\lambda$ large enough, $\|x\| = 0$. So $x = 0$ and the range is the whole space.

Now use the Hille-Yosida theorem.  

\[ \square \]

### 13.4 Groups of unitary operators

We prove Stone’s theorem.

**Theorem 13.10** (1) Suppose $A$ is self-adjoint and $H$ is a Hilbert space. There exists a strongly continuous group $U(t)$ of unitary operators with infinitesimal generator $iA$.

(2) Given a strongly continuous group of unitary operators, the generator is of the form $iA$ where $A$ is self-adjoint.
Proof. (1) We saw in our proof that the spectrum of an unbounded self-adjoint operator is real that \( \| (z - A)^{-1} \| \leq 1/|\text{Im} \, z| \). So if \( \lambda > 0 \) and \( z = -i\lambda \), then
\[
\| (\lambda - iA)^{-1} \| = \| (iz - iA)^{-1} \| = \| (z - A)^{-1} \| \leq \frac{1}{|\text{Im} \, iz|} = \frac{1}{\lambda}.
\]
The resolvent set of \( iA \) contains the positive reals. So \( iA \) and \( -iA \) satisfy the Hille-Yosida theorem. Let \( U(t), V(t) \) be the respective semigroups.

\( V \) and \( U \) are inverses:
\[
\frac{d}{dt} U(t)V(t) = U(t)iAV(t)x - U(t)iAV(t)x = 0.
\]
So \( U(t)V(t)x \) is independent of \( t \). When \( t = 0 \), we get \( x \). So \( U(t)V(t)x = x \) if \( x \in D(A) \). But \( D(A) \) is dense.

Both \( U \) and \( V \) are contractions. Since \( U(t)V(t) = I \), they must be norm preserving. This is because
\[
\|x\| = \|U(t)V(t)x\| \leq \|V(t)x\| \leq \|x\|,
\]
so \( \|x\| = \|V(t)x\| \) and similarly with \( U \). Since they are invertible, they are unitary. Define \( U(t) = V(-t) \) for \( t < 0 \).

(2) Let \( V(t) = U(-t) \). Then \( U(t) \) and \( V(t) \) are strongly continuous semigroups of contractions, and the infinitesimal generators are additive inverses. So the generators are \( B, -B \).

Since both \( B, -B \) are infinitesimal generators, all real numbers except 0 are in the resolvent set of \( B \). Take \( x \in D(B) \).
\[
\|U(t)x\|^2 = (U(t)x, U(t)x) = \|x\|^2.
\]
Take the derivative with respect to \( t \):
\[
(Bx, x) + (x, Bx) = 0.
\]
Letting \( A = -iB \) so that \( B = iA \), we see that
\[
\langle Ax, x \rangle = \langle x, Ax \rangle \tag{13.4}
\]
for all \( x \in D(A) \). Using (13.4) with \( x \) replaced by \( x + y \) and with \( x \) replaced by \( y \), we obtain
\[
\langle Ax, y \rangle + \langle Ay, x \rangle = \langle x, Ay \rangle + \langle y, Ax \rangle. \tag{13.5}
\]
Replacing $y$ by $iy$ in (13.5),

$$-i\langle Ax, y \rangle + i\langle Ay, x \rangle = -i\langle x, Ay \rangle + i\langle y, Ax \rangle.$$  

Dividing this by $i$ and subtracting from (13.5) we have

$$\langle Ax, y \rangle = \langle x, Ay \rangle.$$  

Therefore $A$ is symmetric and $A^*$ is an extension of $A$. It follows that $B^*$ is an extension of $-B$. We showed in the previous chapter that the adjoint of $(\lambda - B)^{-1}$ was $(\bar{\lambda} - B^*)^{-1}$, and it follows that $\rho(B^*) = \overline{\rho(B)}$. If $z \neq 0$ and $z \in \mathbb{R}$, then $z \in \rho(B)$, so $z \in \rho(B^*)$. Also $z \in \rho(-B)$. By Proposition 13.6 $B^*$ cannot be a proper extension of $-B$, hence $B^* = -B$, and so $A^* = A$, or $A$ is self-adjoint. \qed
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