Chapter 2

Lévy processes

A Lévy process is a process with stationary and independent increments whose paths are right continuous with left limits. Having stationary increments means that the law of $X_t - X_s$ is the same as the law of $X_{t-s} - X_0$ whenever $s < t$. Saying that $X$ has independent increments means that $X_t - X_s$ is independent of $\sigma(X_r; r \leq s)$ whenever $s < t$.

We want to examine the structure of Lévy processes. We know three examples: the Poisson process, Brownian motion, and the deterministic process $X_t = t$. It turns out all Lévy processes can be built up out of these as building blocks. We will show how to construct Lévy processes and we will give a representation of an arbitrary Lévy process.

Recall that we use $X_{t-} = \lim_{s \downarrow t, s \to t} X_s$ and $\Delta X_t = X_t - X_{t-}$.

2.1 Examples

Let us begin at looking at some simple Lévy processes. Let $P^j_t$, $j = 1, \ldots, J$, be a sequence of independent Poisson processes with parameters $\lambda_j$, resp. Each $P^j_t$ is a Lévy process and the formula for the characteristic function of a Poisson random variable shows that the characteristic function of $P^j_t$ is

$$\mathbb{E} e^{iuP^j_t} = \exp(t\lambda_j(e^{iu} - 1)).$$

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Therefore the characteristic function of $a_j P_t^j$ is
\[ \mathbb{E} e^{iua_j P_t^j} = \exp(t\lambda_j(e^{iua_j} - 1)) \]
and the characteristic function of $a_j P_t^j - a_j \lambda_j t$ is
\[ \mathbb{E} e^{iua_j P_t^j - a_j \lambda_j t} = \exp(t\lambda_j(e^{iua_j} - 1 - iua_j)). \]
If we let $m_j$ be the measure on $\mathbb{R}$ defined by $m_j(dx) = \lambda_j \delta_{a_j}(dx)$, where $\delta_{a_j}(dx)$ is point mass at $a_j$, then the characteristic function for $a_j P_t^j$ can be written as
\[ \exp \left( t \int_{\mathbb{R}} [e^{iux} - 1] m_j(dx) \right) \] (2.1)
and the one for $a_j P_t^j - a_j \lambda_j t$ as
\[ \exp \left( t \int_{\mathbb{R}} [e^{iux} - 1 - iux] m_j(dx) \right). \] (2.2)

Now let
\[ X_t = \sum_{j=1}^J a_j P_t^j. \]
It is clear that the paths of $X_t$ are right continuous with left limits, and the fact that $X$ has stationary and independent increments follows from the corresponding property of the $P^j$’s. Moreover the characteristic function of a sum of independent random variables is the product of the characteristic functions, so the characteristic function of $X_t$ is given by
\[ \mathbb{E} e^{iuX_t} = \exp \left( t \int_{\mathbb{R}} [e^{iux} - 1] m(dx) \right) \] (2.3)
with $m(dx) = \sum_{j=1}^J \lambda_j \delta_{a_j}(dx)$.

The process $Y_t = X_t - t \sum_{j=1}^J a_j \lambda_j$ is also a Lévy process and its characteristic function is
\[ \mathbb{E} e^{iuY_t} = \exp \left( t \int_{\mathbb{R}} [e^{iux} - 1 - iux] m(dx) \right), \] (2.4)
again with $m(dx) = \sum_{j=1}^J \lambda_j \delta_{a_j}(dx)$. 
Remark 2.1 Recall that if \( \varphi \) is the characteristic function of a random variable \( Z \), then \( \varphi'(0) = i \mathbb{E} Z \) and \( \varphi''(0) = -\mathbb{E} Z^2 \). If \( Y_t \) is as in the paragraph above, then clearly \( \mathbb{E} Y_t = 0 \), and calculating the second derivative of \( \mathbb{E} e^{iuY_t} \) at 0, we obtain
\[
\mathbb{E} Y_t^2 = t \int x^2 m(dx).
\]

The following lemma is a restatement of Corollary 1.8.

Lemma 2.2 If \( X_t \) is a Lévy process and \( T \) is a finite stopping time, then \( X_{T+t} - X_T \) is a Lévy process with the same law as \( X_t - X_0 \) and independent of \( \mathcal{F}_T \).

We will need the following lemma:

Lemma 2.3 Suppose \( X_1, \ldots, X_n \) are independent exponential random variables with parameters \( a_1, \ldots, a_n \), resp.
(1) Then \( \min(X_1, \ldots, X_n) \) is an exponential random variable with parameter \( a_1 + \cdots + a_n \).
(2) The probability that \( X_i \) is the smallest of the \( n \) exponentials is
\[
\frac{a_i}{a_1 + \cdots + a_n}.
\]

Proof. (1) Write
\[
\mathbb{P}(\min(X_1, \ldots, X_n) > t) = \mathbb{P}(X_1 > t, \ldots, X_n > t) = \mathbb{P}(X_1 > t) \cdots \mathbb{P}(X_n > t) = e^{-a_1 t} \cdots e^{-a_n t} = e^{-(a_1 + \cdots + a_n)t}.
\]

(2) Without loss of generality we may suppose \( i = 1 \). Let’s first do the case \( n = 2 \). The joint density of \( (X_1, X_2) \) is \( a_1 e^{-a_1 x} a_2 e^{-a_2 y} \) and we want to integrate this over \( x < y \). Doing the integration yields \( a_1/(a_1 + a_2) \). For the general case of \( n > 2 \), apply the above to \( X_1 \) and \( \min(X_2, \ldots, X_n) \). \( \square \)

If \( P_1, \ldots, P_n \) are independent Poisson processes with parameters \( \lambda_1, \ldots, \lambda_n \) resp., let \( X_t = \sum_{i=1}^n b_i P_i(t) \). By the above lemma, the times between jumps
of $X$ are independent exponentials with parameter $\lambda_1 + \cdots + \lambda_n$. At each
jump, $X$ jumps $b_i$ with probability $\lambda_i/(\lambda_1 + \cdots + \lambda_n)$.

Thus another way to construct $X$ is to let $U_1, U_2, \ldots$ be independent exponentials with parameter $\lambda_1 + \cdots + \lambda_n$ and let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables independent of the $U_i$’s such that $\mathbb{P}(Y_k = b_j) = \lambda_j/(\lambda_1 + \cdots + \lambda_n)$. We then let $X_0$ be 0, let $X_t$ be piecewise constant, and at time $\sum_{i=1}^m U_i$ we let $X$ jump by the amount $Y_m$.

2.2 Construction of Lévy processes

A process $X$ has bounded jumps if there exists a real number $K > 0$ such that $\sup_t |\Delta X_t| \leq K$, a.s.

Lemma 2.4 If $X_t$ is a Lévy process with bounded jumps and with $X_0 = 0$, then $X_t$ has moments of all orders, that is, $\mathbb{E}|X_t|^p < \infty$ for all positive integers $p$.

Proof. Suppose the jumps of $X_t$ are bounded in absolute value by $K$. Since $X_t$ is right continuous with left limits, there exists $M > K$ such that $\mathbb{P}(\sup_{s \leq t} |X_s| \geq M) \leq 1/2$.

Let $T_1 = \inf\{t : |X_t| \geq M\}$ and $T_{i+1} = \inf\{t > T_i : |X_t - X_{T_i}| > M\}$. For $s < T_1$, $|X_s| \leq M$, and then $|X_{T_1}| \leq |X_{T_1}| + |\Delta X_{T_1}| \leq M + K \leq 2M$. We have

$$
\mathbb{P}(T_{i+1} < t) \leq \mathbb{P}(T_i < t, T_{i+1} - T_i < t)
$$

$$
= \mathbb{P}(T_i < t) \mathbb{P}(T_{i+1} - T_i < t)
$$

$$
= \mathbb{P}(T_i < t) \mathbb{P}(T_i < t),
$$

using Lemma 2.2. Now

$$
\mathbb{P}(T_1 < t) \leq \mathbb{P}(\sup_{s \leq t} |X_s| \geq M) \leq \frac{1}{2},
$$

so $\mathbb{P}(T_{i+1} < t) \leq \frac{1}{2} \mathbb{P}(T_i < t)$, and then by induction, $\mathbb{P}(T_i < t) \leq 2^{-i}$. Therefore

$$
\mathbb{P}(\sup_{s \leq t} |X_s| \geq 2(i + 1)M) \leq \mathbb{P}(T_i < t) \leq 2^{-i}
$$
and the lemma now follows immediately.

A key lemma is the following.

**Lemma 2.5** Suppose $I$ is a finite interval of the form $(a,b)$, $[a,b)$, $(a,b]$, or $[a,b]$ with $a > 0$ and $m$ is a finite measure on $\mathbb{R}$ giving no mass to $I^c$. Then there exists a Lévy process $X_t$ satisfying (2.3)

**Proof.** First let us consider the case where $I = [a,b)$. We approximate $m$ be a discrete measure. If $n \geq 1$, let $z_j = a + j(b-a)/2^n$, $j = 0, \ldots, 2^n - 1$, and let

$$m_n(dx) = \sum_{j=0}^{2^n-1} m([z_j, z_{j+1})) \delta_{z_j}(dx),$$

where $\delta_{z_j}$ is point mass at $z_j$. The measures $m_n$ converge weakly to $m$ as $n \to \infty$ in the sense that

$$\int f(x) m_n(dx) \to \int f(x) dx$$

whenever $f$ is a bounded continuous function on $\mathbb{R}$.

We let $U_1, U_2, \ldots$ be independent exponential random variables with parameter $m(I)$. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables independent of the $U_i$’s with $\mathbb{P}(Y_i \in dx) = m(dx)/m(I)$. We let $X_t$ start at 0 and be piecewise constant with jumps of size $Y_j$ at times $\sum_{i=1}^j U_i$.

If we define $X^n_t$ is the exact same way, except that we replace $m$ by $m_n$ and we let $Y^n_i = z_j$ if $Y_i \in [z_j, z_{j+1})$, then we know from the previous section that $X^n_t$ is a Lévy process with Lévy measure $m_n$. Moreover each $Y^n_i$ differs from $Y_i$ by at most $(b-a)2^{-n}$, so

$$\sup_{s \leq t} |X^n_s - X_s| \leq (b-a)2^n N,$$

where $N$ is the number of jumps of these processes before time $t$. $N$ is a Poisson random variable with parameter $m(I)$, so has moments of all orders. It follows that $X^n_t$ converges uniformly to $X_t$ almost surely on each finite interval, and the difference goes to 0 in $L^p$ for each $p$. 
We conclude that the law of $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same law as that of $X_{t-s}$ because these hold for each $X^n$.

Since $x \to e^{iux}$ is a bounded continuous function and $m_n$ converges weakly to $m$, starting with

$$\mathbb{E} \exp(iuX^n_t) = \exp \left( t \int [e^{iux} - 1] m_n(dx) \right),$$

and passing to the limit, we obtain that the characteristic function of $X$ under $\mathbb{P}$ is given by (2.3).

If now the interval $I$ contains the point $b$, we follow the above proof, except we let $P^{2n-1}_t$ be a Poisson random variable with parameter $m([z_{n-1}, b])$. Similarly, if $I$ does not contain the point $a$, we change $P^0_t$ to be a Poisson random variable with parameter $m((a, z_1))$. With these changes, the proof works for intervals $I$, whether or not they contain either of their endpoints. \qed

**Remark 2.6** If $X$ is the Lévy process constructed in Lemma 2.5, then $Y_t = X_t - \mathbb{E}X_t$ will be a Lévy process satisfying (2.4).

Here is the main theorem of this section.

**Theorem 2.7** Suppose $m$ is a measure on $\mathbb{R}$ with $m(\{0\}) = 0$ and

$$\int (1 \wedge x^2)m(dx) < \infty.$$

Suppose $b \in \mathbb{R}$ and $\sigma \geq 0$. There exists a Lévy process $X_t$ such that

$$\mathbb{E} e^{iuX_t} = \exp \left( t \left\{ iub - \sigma^2 u^2 / 2 + \int_{\mathbb{R}} [e^{iux} - 1 - iux1_{|x| \leq 1}] m(dx) \right\} \right). \quad (2.5)$$

The above equation is called the Lévy-Khintchine formula. The measure $m$ is called the Lévy measure. If we let

$$m(dx) = \frac{1 + x^2}{x^2} m'(dx)$$
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and
\[ b = b' + \int_{|x| \leq 1} \frac{x^3}{1 + x^2} m(dx) - \int_{|x| > 1} \frac{x}{1 + x^2} m(dx), \]
then we can also write
\[ \mathbb{E} e^{iuX_t} = \exp \left( t \left\{ iub' - \sigma^2 u^2 / 2 + \int_{\mathbb{R}} \left[ e^{ix} - 1 - \frac{ix}{1 + x^2} \right] \frac{1 + x^2}{x^2} m'(dx) \right\} \right). \]

Both expressions for the Lévy-Khintchine formula are in common use.

**Proof.** Let \( m(dx) \) be a measure supported on \((0, 1]\) with \( \int x^2 m(dx) < \infty \). Let \( m_n(dx) \) be the measure \( m \) restricted to \((2^{-n}, 2^{-n+1}]\). Let \( Y^n_t \) be independent Lévy processes whose characteristic functions are given by (2.4) with \( m \) replaced by \( m_n \); see Remark 2.6. Note \( \mathbb{E} Y^n_t = 0 \) for all \( n \) by Remark 2.1. By the independence of the \( Y^n_t \)'s, if \( M < N \),
\[ \mathbb{E} \left( \sum_{n=M}^{N} Y^n_t \right)^2 = \sum_{n=M}^{N} \mathbb{E} (Y^n_t)^2 = \sum_{n=M}^{N} t \int x^2 m_n(dx) = t \int_{2^{-N}}^{2^{-M}} x^2 m(dx). \]

By our assumption on \( m \), this goes to 0 as \( M, N \to \infty \), and we conclude that \( \sum_{n=0}^{N} Y^n_t \) converges in \( L^2 \) for each \( t \). Call the limit \( Y_t \). It is routine to check that \( Y_t \) has independent and stationary increments. Each \( Y^n_t \) has independent increments and is mean 0, so
\[ \mathbb{E} [Y^n_t - Y^n_s | \mathcal{F}_s] = \mathbb{E} [Y^n_t - Y^n_s] = 0, \]
or \( Y^n \) is a martingale. By Doob's inequalities and the \( L^2 \) convergence,
\[ \mathbb{E} \sup_{s \leq t} \left| \sum_{n=M}^{N} Y^n_s \right|^2 \to 0 \]
as \( M, N \to \infty \), and hence there exists a subsequence \( M_k \) such that \( \sum_{n=1}^{M_k} Y^n_s \) converges uniformly over \( s \leq t \), a.s. Therefore the limit \( Y_t \) will have paths that are right continuous with left limits.

If \( m \) is a measure supported in \((1, \infty)\) with \( m(\mathbb{R}) < \infty \), we do a similar procedure starting with Lévy processes whose characteristic functions are of the form (2.3). We let \( m_n(dx) \) be the restriction of \( m \) to \((2^n, 2^{n+1}]\), let \( X^n_t \) be independent Lévy processes corresponding to \( m_n \), and form \( X_t = \sum_{n=0}^{\infty} X^n_t \).
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Since $m(\mathbb{R}) < \infty$, for each $t_0$, the number of times $t$ less than $t_0$ at which any one of the $X_t^n$ jumps is finite. This shows $X_t$ has paths that are right continuous with left limits, and it is easy to then see that $X_t$ is a Lévy process.

Finally, suppose $\int x^2 \wedge 1 \, m(dx) < \infty$. Let $X_t^1$, $X_t^2$ be Lévy processes with characteristic functions given by (2.3) with $m$ replaced by the restriction of $m$ to $(1, \infty)$ and $(-\infty, -1)$, resp., let $X_t^3$, $X_t^4$ be Lévy processes with characteristic functions given by (2.4) with $m$ replaced by the restriction of $m$ to $(0, 1]$ and $[-1, 0)$, resp., let $X_t^5 = bt$, and let $X_t^6 = \sigma t$ be $\sigma$ times a Brownian motion. Suppose the $X_t^n$’s are all independent. Then their sum will be a Lévy process whose characteristic function is given by (2.5).

A key step in the construction was the centering of the Poisson processes to get Lévy processes with characteristic functions given by (2.4). Without the centering one is forced to work only with characteristic functions given by (2.3).

2.3 Representation of Lévy processes

We now work towards showing that every Lévy process has a characteristic function of the form given by (2.5).

Lemma 2.8 If $X_t$ is a Lévy process and $A$ is a Borel subset of $\mathbb{R}$ that is a positive distance from 0, then

\[ N_t(A) = \sum_{s \leq t} 1_A(\Delta X_s) \]

is a Poisson process.

Saying that $A$ is a positive distance from 0 means that $\inf\{|x| : x \in A\} > 0$.

Proof. Since $X_t$ has paths that are right continuous with left limits and $A$ is a positive distance from 0, then there can only be finitely many jumps of $X$ that lie in $A$ in any finite time interval, and so $N_t(A)$ is finite and has paths that are right continuous with left limits. It follows from the fact that $X_t$ has stationary and independent increments that $N_t(A)$ also has stationary and independent increments. We now apply Proposition 1.10.
2.3. REPRESENTATION OF LÉVY PROCESSES

Our main result is that \( N_t(A) \) and \( N_t(B) \) are independent if \( A \) and \( B \) are disjoint.

**Theorem 2.9** Let \( \{ \mathcal{F}_t \} \) be a filtration satisfying the usual conditions. Suppose that \( N_t(A) \) is a Poisson point process with respect to the measure \( \lambda \). If \( A_1, \ldots, A_n \) are pairwise disjoint measurable subsets of \( \mathbb{R} \) with \( \mathbb{E} N_t(A_k) < \infty \) for \( k = 1, \ldots, n \), then the processes \( N_t(A_1), \ldots, N_t(A_n) \) are mutually independent.

**Proof.** Define \( \lambda(A) = \mathbb{E} N_t(A) \). The previous lemma shows that if \( \lambda(A) < \infty \), then \( N_t(A) \) is a Poisson process, and clearly its parameter is \( \lambda(A) \). We first make the observation that because \( A_1, A_2, \ldots, A_n \) are disjoint, then no two of the \( N_t(A_k) \) have jumps at the same time.

To prove the theorem, it suffices to let \( 0 = r_0 < r_1 < \cdots < r_m \) and show that the random variables

\[
\{ N_{r_j}(A_k) - N_{r_{j-1}}(A_k) : 1 \leq j \leq m, 1 \leq k \leq n \}
\]

are independent. Since for each \( j \) and each \( k \), \( N_{r_j}(A_k) - N_{r_{j-1}}(A_k) \) is independent of \( \mathcal{F}_{r_{j-1}} \), it suffices to show that for each \( j \leq m \), the random variables

\[
\{ N_{r_j}(A_k) - N_{r_{j-1}}(A_k) : 1 \leq k \leq n \}
\]

are independent. We will do the case \( j = m = 1 \) and write \( r \) for \( r_j \) for simplicity; the case when \( j, m > 1 \) differs only in notation.

We will prove this using induction. We start with the case \( n = 2 \) and show the independence of \( N_r(A_1) \) and \( N_r(A_2) \). Each \( N_t(A_k) \) is a Poisson process, and so \( N_t(A_k) \) has moments of all orders. Let \( u_1, u_2 \in \mathbb{R} \) and set

\[
\phi_k = \lambda(A_k)(e^{iu_k} - 1), \quad k = 1, 2.
\]

Let

\[
M^k_t = e^{iu_kN_t(A_k) - t\phi_k}.
\]

We see that \( M^k_t \) is a martingale because \( \mathbb{E} e^{iu_kN_t(A_k)} = e^{t\phi_k} \), and therefore

\[
\mathbb{E} [M^k_t | \mathcal{F}_s] = M^k_s\mathbb{E} [e^{iu(N_t(A_k) - N_s(A_k)) - (t-s)\phi_k} | \mathcal{F}_s] = M^k_s e^{-(t-s)\phi_k} \mathbb{E} [e^{iu(N_t(A_k) - N_s(A_k))}] = M^k_s,
\]

for \( t \geq s \).
using the independence and stationarity of the increments of a Poisson process.

Now we can write
\[ \mathbb{E}[M_t^1 M_t^2] = \mathbb{E}[M_t^1] + \mathbb{E} \int_0^t M_s^1 dM_s^2 \]
\[ = 1 + \mathbb{E} \int_0^t M_s^1 dM_s^2, \]
using that \( M_0^2 = 1 \), \( M^1 \) is a martingale, and Proposition 1.17. (Here \( M^2 \) is the difference of two increasing processes; the adjustments needed are easy.)

Since we have argued that no two of the \( N_t(A_k) \) jump at the same time, the same is true for the \( M_t^k \) and so the above is equal to
\[ 1 + \mathbb{E} \int_0^t M_s^1 dM_s^2. \]

It therefore remains to prove that the above integral is equal to 0.

If \( H_s \) is a process of the form
\[ H_s(\omega) = K(\omega)1_{(a,b)}(s) \]
where \( K \) is \( \mathcal{F}_a \) measurable, then
\[ \int_0^t H_s dM_s^2 = K(M_{t\wedge b}^2 - M_{t\wedge a}^2), \]
and conditioning on \( \mathcal{F}_a \), the expectation is zero:
\[ \mathbb{E}[K(M_{t\wedge b}^2 - M_{t\wedge a}^2)] = \mathbb{E}[\mathbb{E}[M_{t\wedge b}^2 - M_{t\wedge a}^2 | \mathcal{F}_a]] = 0, \]
using that \( M^2 \) is a martingale. We are doing Lebesgue-Stieltjes integrals here, but the argument is similar to one used with stochastic integrals. The expectation is also 0 for linear combinations of such \( H \)’s. Since \( M_{s-}^1 \) is left continuous, we can approximate it by such \( H \)’s, and therefore the integral is 0 as required.

We thus have
\[ \mathbb{E} M_t^1 M_t^2 = 1. \]
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This implies
\[ \mathbb{E}
\left[
\exp\left(i(u_1 N_r(A_1) + u_2 N_r(A_2))\right)
\right] = \exp\left(iu_1 N_r(A_1) + u_2 N_r(A_2)\right) \cdot \mathbb{E}\left[e^{iu_1 N_r(A_1)}\right] \cdot \mathbb{E}\left[e^{iu_2 N_r(A_2)}\right]. \]

Since this holds for all \( u_1, u_2 \), then \( N_r(A_1) \) and \( N_r(A_2) \) are independent. We conclude that the processes \( N_t(A_1) \) and \( N_t(A_2) \) are independent.

To handle the case \( n = 3 \), we first show that \( M^1_t M^2_t \) is a martingale. We write
\[
\begin{align*}
\mathbb{E}[M^1_t M^2_t | F_s] &= M^1_s M^2_s e^{-(t-s)(\phi_1 + \phi_2)} \mathbb{E}[e^{i(u_1 (N_t(A_1) - N_s(A_1)) + u_2 (N_t(A_2) - N_s(A_2)))} | F_s] \\
&= M^1_s M^2_s e^{-(t-s)(\phi_1 + \phi_2)} \mathbb{E}[e^{i(u_1 (N_t(A_1) - N_s(A_1)) + u_2 (N_t(A_2) - N_s(A_2)))}]
\end{align*}
\]

using the fact that \( N_t(A_1) \) and \( N_t(A_2) \) are independent of each other and each have stationary and independent increments.

Note that \( M^3_t = e^{iu_3 N_t(A_3)} \) has no jumps in common with \( M^1_t \) or \( M^2_t \). Therefore if \( \overline{M}^3_t = M^3_{t\wedge T} \), then
\[ \mathbb{E} \left[ \overline{M}^3_\infty (\overline{M}^1_\infty \overline{M}^2_\infty) \right] = 0, \]

and as before this leads to
\[ \mathbb{E} \left[ M^3_r (M^1_r M^2_r) \right] = 1. \]

As above this implies that \( N_r(A_1) \), \( N_r(A_2) \), and \( N_r(A_3) \) are independent. To prove the general induction step is similar.

We will also need the following corollary.

**Corollary 2.10** Let \( F_t \) and \( N_t(A_k) \) be as in Theorem 4.2. Suppose \( Y_t \) is a process with paths that are right continuous with left limits such that \( Y_t - Y_s \) is independent of \( F_s \) whenever \( s < t \) and \( Y_t - Y_s \) has the same law as \( Y_{t-s} \) for each \( s < t \). Suppose moreover that \( Y \) has no jumps in common with any of the \( N_t(A_k) \). Then the processes \( N_t(A_1), \ldots, N_t(A_n) \), and \( Y_t \) are independent.
Proof. The law of $Y_0$ is the same as that of $Y_t - Y_s$, so $Y_0 = 0$, a.s. By the fact that $Y$ has stationary and independent increments,

$\mathbb{E} e^{iuY_{s+t}} = \mathbb{E} e^{iuY_s} \mathbb{E} e^{iu(Y_{s+t} - Y_s)} = \mathbb{E} e^{iuY_s} \mathbb{E} e^{iuY_t}$,

which implies that the characteristic function of $Y$ is of the form $\mathbb{E} e^{iuY_t} = e^{t\psi(u)}$ for some function $\psi(u)$.

We fix $u \in \mathbb{R}$ and define

$$M_t^Y = e^{iuY_t - t\psi(u)}.$$ 

As in the proof of Theorem 4.2, we see that $M_t^Y$ is a martingale. Since $M_t^Y$ has no jumps in common with any of the $M_t^k$, if $\overline{M}_t^Y = M_t^{Y^r}$, we see as above that

$$\mathbb{E} [\overline{M}_{\infty}^Y (\overline{M}_{\infty}^1 \cdots \overline{M}_{\infty}^n)] = 1,$$

or

$$\mathbb{E} [M_t^Y M_t^1 \cdots M_t^r] = 1.$$

This leads as above to the independence of $Y$ from all the $N_t(A_k)$’s. $\square$

Here is the representation theorem for Lévy processes.

**Theorem 2.11** Suppose $X_t$ is a Lévy process with $X_0 = 0$. Then there exists a measure $m$ on $\mathbb{R} - \{0\}$ with

$$\int (1 \land x^2) \, m(dx) < \infty$$

and real numbers $b$ and $\sigma$ such that the characteristic function of $X_t$ is given by (2.5).

**Proof.** Define $m(A) = \mathbb{E} N_1(A)$ if $A$ is a bounded Borel subset of $(0, \infty)$ that is a positive distance from 0. Since $N_1(\bigcup_{k=1}^\infty A_k) = \sum_{k=1}^\infty N_1(A_k)$ if the $A_k$ are pairwise disjoint and each is a positive distance from 0, we see that $m$ is a measure on $[a, b]$ for each $0 < a < b < \infty$, and $m$ extends uniquely to a measure on $(0, \infty)$. 


2.3. REPRESENTATION OF LÉVY PROCESSES

First we want to show that $\sum_{s \leq t} \Delta X_s \mathbb{1}_{(\Delta X_s > 1)}$ is a Lévy process with characteristic function

$$\exp \left( t \int_1^\infty [e^{inx} - 1] m(dx) \right).$$

Since the characteristic function of the sum of independent random variables is equal to the product of the characteristic functions, it suffices to suppose $0 < a < b$ and to show that

$$\mathbb{E} e^{iuZ_t} = \exp \left( t \int_{(a,b]} [e^{inx} - 1] m(dx) \right),$$

where

$$Z_t = \sum_{s \leq t} \Delta X_s \mathbb{1}_{(a,b]}(\Delta X_s).$$

Let $n > 1$ and $z_j = a + j(b - a)/n$. By Lemma 2.8, $N_t((z_j, z_{j+1}])$ is a Poisson process with parameter

$$\ell_j = \mathbb{E} N_1((z_{j-1}, z_j]) = m((z_j, z_{j+1}]).$$

Thus $\sum_{j=0}^{n-1} z_j N_t((z_j, z_{j+1}])$ has characteristic function

$$\prod_{j=0}^{n-1} \exp(t\ell_j(e^{inz_j} - 1)) = \exp \left( t \sum_{j=0}^{n-1} (e^{inz_j} - 1) \ell_j \right),$$

which is equal to

$$\exp \left( t \int (e^{inx} - 1) m_n(dx) \right),$$

where $m_n(dx) = \sum_{j=0}^{n-1} \ell_j \delta_{z_j}(dx)$. Since $Z^n_t$ converges to $Z_t$ as $n \to \infty$, passing to the limit shows that $Z_t$ has a characteristic function of the form (2.5).

Next we show that $m(1, \infty) < \infty$. (We write $m(1, \infty)$ instead of $m((1, \infty))$ for esthetic reasons.) If not, $m(1, K) \to \infty$ as $K \to \infty$. Then for each fixed $L$ and each fixed $t$,

$$\limsup_{K \to \infty} \mathbb{P}(N_t(1, K) \leq L) = \limsup_{K \to \infty} \sum_{j=0}^L e^{-tm(1,K)} \frac{m(1,K)^j}{j!} = 0.$$

This implies that $N_t(1, \infty) = \infty$ for each $t$. However, this contradicts the fact that $X_t$ has paths that are right continuous with left limits.
We define \( m \) on \((-\infty, 0)\) similarly.

We now look at

\[
Y_t = X_t - \sum_{s \leq t} \Delta X_s 1_{(|\Delta X_s| > 1)}.
\]

This is again a Lévy process, and we need to examine its structure. This process has bounded jumps, hence has moments of all orders. By subtracting \( c_1 t \) for an appropriate constant \( c_1 \), we suppose \( Y_t \) has mean 0. Let

\[
I_1, I_2, \ldots
\]

be an ordering of the intervals \([2^{-(m+1)}, 2^{-m}), (-2^{-m}, -2^{-(m+1)}] : m \geq 0\). Let

\[
\tilde{X}_t^k = \sum_{s \leq t} \Delta X_s 1_{(\Delta X_s \in I_k)}
\]

and let \( X_t^k = \tilde{X}_t^k - \mathbb{E} \tilde{X}_t^k \). By the fact that all the \( X^k \) have mean 0 and are independent,

\[
\sum_{k=1}^{\infty} \mathbb{E} (X_t^k)^2 \leq \mathbb{E} \left[ \left( Y_t - \sum_{k=1}^{\infty} X_t^k \right)^2 \right] + \mathbb{E} \left[ \left( \sum_{k=1}^{\infty} X_t^k \right)^2 \right] = \mathbb{E} (Y_t)^2 < \infty.
\]

Hence

\[
\mathbb{E} \left[ \sum_{k=M}^{N} X_t^k \right]^2 = \sum_{k=M}^{N} \mathbb{E} (X_t^k)^2
\]

tends to 0 as \( M, N \to \infty \), and thus \( X_t - \sum_{k=1}^{N} X_t^k \) converges in \( L^2 \). The limit, \( X^c_t \), say, will be a Lévy process independent of all the \( X_t^k \). Moreover, \( X^c \) has no jumps, i.e., it is continuous. Since all the \( X^k \) have mean 0, then \( \mathbb{E} X_t^c = 0 \). By the independence of the increments,

\[
\mathbb{E} [X_t^c - X_s^c | \mathcal{F}_s] = \mathbb{E} [X_t^c - X_s^c] = 0,
\]

and we see \( X^c \) is a continuous martingale. Using the stationarity and independence of the increments,

\[
\mathbb{E} [(X_{s+t}^c)^2] = \mathbb{E} [(X_s^c)^2] + 2 \mathbb{E} [X_s^c (X_{s+t}^c - X_s^c)] + \mathbb{E} [(X_{s+t}^c - X_s^c)^2]
\]
\[
= \mathbb{E} [(X_s^c)^2] + \mathbb{E} [(X_t^c)^2],
\]
which implies that there exists a constant $c_2$ such that $\mathbb{E}(X_t^c)^2 = c_2 t$. We then have

\[
\mathbb{E}[(X_t^c)^2 - c_2 t \mid \mathcal{F}_s] = (X_s^c)^2 - c_2 s + \mathbb{E}[(X_t^c - X_s^c)^2 \mid \mathcal{F}_s] - c_2 (t - s) \\
= (X_s^c)^2 - c_2 s + \mathbb{E}[(X_t^c - X_s^c)^2] - c_2 (t - s) \\
= (X_s^c)^2 - c_2 s.
\]

The quadratic variation process of $X^c$ is therefore $c_2 t$, and by Lévy’s theorem, $X_t^c/\sqrt{t}$ is a constant multiple of Brownian motion.

To complete the proof, it remains to show that $\int_{-1}^1 x^2 m(dx) < \infty$. But by Remark 2.1,

\[
\int_{I_k} x^2 m(dx) = \mathbb{E}(X^k_t)^2,
\]

and we have seen that

\[
\sum_k \mathbb{E}(X^k_t)^2 \leq \mathbb{E}Y_1^2 < \infty.
\]

Combining gives the finiteness of $\int_{-1}^1 x^2 m(dx)$.

\[\square\]

### 2.4 Symmetric stable processes

Let $\alpha \in (0, 2)$. If

\[
m(dx) = \frac{c}{|x|^{1+\alpha}} \, dx,
\]

we have what is called a symmetric stable process of index $\alpha$. We see that $\int 1 \wedge x^2 m(dx)$ is finite.

Because $|x|^{-1-\alpha}$ is symmetric, in the Lévy-Khintchine formula we can take $iux 1_{(|x|<a)}$ for any $a$ instead of $iux 1_{(|x|<1)}$. Then

\[
\int \left[ e^{iux} - 1 - iux 1_{(|x|<1)} \right] \frac{c}{|x|^{1+\alpha}} \, dx = \int \left[ e^{iux} - 1 - iux 1_{(|x|<1/|u|)} \right] \frac{c}{|x|^{1+\alpha}} \, dx \\
= \int \left[ e^{iy} - 1 - iy 1_{(|y|<1)} \right] |u|^{1+\alpha} \frac{dy}{|u|} \\
= -c'|u|^\alpha.
\]
In the last line we have the negative sign because the imaginary part of 
\(e^{iy} - 1 - iy1_{|y|<1}\) is zero and the real part is negative since \(|\cos y| \leq 1\). Therefore is \(X_t\) is a symmetric stable process of index \(\alpha\),

\[\mathbb{E}e^{iuX_t} = e^{-c't|u|^\alpha}.\]

An exercise is to show that if \(a > 0\) and \(X_t\) is a symmetric stable process of index \(\alpha\), then \(X_{at}\) and \(a^{1/\alpha}X_t\) have the same law.

By Exercise 6.7.4 of Chung’s book,

\[\mathbb{P}(X_1 > A) \sim cA^{-\alpha}, \quad A \to \infty,\] (2.7)

where \(f \sim g\) means the ratio of the two sides goes to 0.

Since \(e^{-c't|u|^\alpha}\) is integrable, \(X_t\) has a continuous density function \(p_t(x)\). We have

\[p_t(0) = \frac{1}{2\pi} \int e^{-c't|u|^\alpha} \, du,\] (2.8)

and by a change of variables,

\[p_t(0) = ct^{-1/\alpha}.\] (2.9)

If \(x \neq 0\), then

\[p_t(x) = \frac{1}{2\pi} \int e^{-iux}e^{-c't|u|^\alpha} \, du = \frac{1}{2\pi} \int (\cos ux - i \sin ux)e^{-c't|u|^\alpha} \, du.\]

Since \(\sin ux\) is an odd function of \(u\), the imaginary term is 0. Since \(\cos ux \leq 1\) and in fact is strictly less than 1 except at countably many values of \(u\), we see that

\[p_t(x) < p_t(0).\] (2.10)

If \(\beta < 1\), we can take \(m(dx) = c/|x|^{1+\beta}\) for \(x > 0\) and 0 for \(x < 0\). We can also take the Lévy-Khintchine exponent to be just \([e^{iux} - 1]\) if we take the drift term to cancel out the \(\int iux1_{(|x|<1)}\) term. This reflects that here we do not need to subtract the mean to get convergence of the compound Poisson processes. In this case we get the one-sided stable processes of index \(\beta\). The paths of such a process only increase.

There is a notion of subordination which is very curious. Suppose that \(T_t\) is a one-sided stable process of index \(\beta\) with \(\beta < 1\). Let \(W_t\) be a Brownian
motion independent of $T_t$. Then $Y_t = W_{T_t}$ is a symmetric stable process of index $2\beta$. Let’s see why that is so.

That $Y$ is a Lévy process is not hard to see. We must therefore calculate the Lévy measure $m$. If $P_t$ is a Poisson process with parameter $\lambda$, then

$$\mathbb{E} e^{uP_t} = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} e^{uk} = e^{-\lambda t} e^{u\lambda t} = e^{\lambda t(e^u-1)}.$$ 

Using that the moment generating function of independent random variables is the product of the moment generating functions and taking limits, we see that

$$\mathbb{E} e^{-uT_t} = e^{-cu^{\beta}}.$$ 

Then

$$\mathbb{E} e^{iuY_t} = \mathbb{E} \int e^{iuW_s} \mathbb{P}(T_t \in ds)$$

$$= \int e^{-u^2 s/2} \mathbb{P}(T_t \in ds)$$

$$= \mathbb{E} e^{-u^2 T_t/2}$$

$$= e^{-ct(u^2/2)^\beta}$$

$$= e^{-c'|u|^{2\beta}}.$$