Chapter 4

Stochastic differential equations

4.1 Poisson point processes

Poisson point processes are random measures that are related to Poisson processes. Poisson point processes are also useful in the study of excursions, even excursions of a continuous process such as Brownian motion, and they arise when studying stochastic differential equations with jumps.

Let $S$ be a metric space, $G$ the collection of Borel subsets of $S$, and $\lambda$ a measure on $(S, G)$.

**Definition 4.1** We say a map

$$N : \Omega \times [0, \infty) \times G \to \{0, 1, 2, \ldots\}$$

(writing $N_t(A)$ for $N(\omega, t, A)$) is a *Poisson point process* if

1. for each Borel subset $A$ of $S$ with $\lambda(A) < \infty$, the process $N_t(A)$ is a Poisson process with parameter $\lambda(A)$, and
2. for each $t$ and $\omega$, $N(t, \cdot)$ is a measure on $G$.

A model to keep in mind is where $S = \mathbb{R}$ and $\lambda$ is Lebesgue measure. For each $\omega$ there is a collection of points $\{(s, z)\}$ (where the collection depends on $\omega$). The number of points in this collection with $s \leq t$ and $z$ in a subset $A$ is $N_t(A)(\omega)$. Since $\lambda(\mathbb{R}) = \infty$, there are infinitely many points in every time interval.
Another example is to let $X$ be a Lévy process and let $N_t(A)$ be the number of jumps of size $A$ before time $t$. A consequence of the definition is that since $\lambda(\emptyset) = 0$, then $N_t(\emptyset)$ is a Poisson process with parameter 0; in other words, $N_t(\emptyset)$ is identically zero.

Our main result is that $N_t(A)$ and $N_t(B)$ are independent if $A$ and $B$ are disjoint.

**Theorem 4.2** Let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions. Let $S$ be a metric space furnished with a positive measure $\lambda$. Suppose that $N_t(A)$ is a Poisson point process with respect to the measure $\lambda$. If $A_1, \ldots, A_n$ are pairwise disjoint measurable subsets of $S$ with $\lambda(A_k) < \infty$ for $k = 1, \ldots, n$, then the processes $N_t(A_1), \ldots, N_t(A_n)$ are mutually independent.

This is proved exactly the same way we proved that $N_t(A)$ and $N_t(B)$ are independent in the study of Lévy processes.

We now turn to stochastic integrals with respect to Poisson point processes. In the same way that a non-decreasing function on the reals gives rise to a measure, so $N_t(A)$ gives rise to a random measure $\mu(dt, dz)$ on the product $\sigma$-field $\mathcal{B}[0, \infty) \times \mathcal{G}$, where $\mathcal{B}[0, \infty)$ is the Borel $\sigma$-field on $[0, \infty)$; $\mu$ is determined by

$$\mu([0, t] \times A)(\omega) = N_t(A)(\omega);$$

Define a non-random measure $\nu$ on $\mathcal{B}[0, \infty) \times \mathcal{G}$ by $\nu([0, t] \times A) = t\lambda(A)$ for $A \in \mathcal{G}$. If $\lambda(A) < \infty$, then $\mu([0, t] \times A) - \nu([0, t] \times A)$ is the same as a Poisson process minus its mean, hence is locally a square integrable martingale.

We can define a stochastic integral with respect to the random measure $\mu - \nu$ as follows. Suppose $H(\omega, s, z)$ is of the form

$$H(\omega, s, z) = \sum_{i=1}^{n} K_i(\omega)1_{(a_i, b_i]}(s)1_{A_i}(z),$$

where for each $i$ the random variable $K_i$ is bounded and $\mathcal{F}_{a_i}$ measurable and $A_i \in \mathcal{G}$ with $\lambda(A_i) < \infty$. For such $H$ we define

$$N_t = \int_0^t \int_H(\omega, s, z) d(\mu - \nu)(ds, dz)$$

$$= \sum_{i=1}^{n} K_i(\mu - \nu)((a_i, b_i] \cap [0, t)) \times A_i).$$

4.2. **THE LIPSCHITZ CASE**

Let us assume without loss of generality that the $A_i$ are disjoint. It is not hard to see that $N_t$ is a martingale, that $N^c = 0$, and that

$$[N]_t = \int_0^t \int H(\omega, s, z) \mu(ds, dz). \quad (4.3)$$

Since $\langle N \rangle_t$ must be predictable and all the jumps of $N$ are totally inaccessible, it follows that $\langle N \rangle_t$ is continuous. Since $[N]_t - \langle N \rangle_t$ is a martingale, we conclude

$$\langle N \rangle_t = \int_0^t \int H(\omega, s, z) \nu(ds, dz). \quad (4.4)$$

Suppose $H(s, z)$ is predictable process in the following sense: $H$ is measurable with respect to the $\sigma$-field generated by all processes of the form (4.1). Suppose also that

$$\mathbb{E} \int_0^\infty \int_S H(s, z)^2 \nu(ds, dz) < \infty.$$

Take processes $H^n$ of the form (4.1) converging to $H$ in the space $L^2$ with norm $(\mathbb{E} \int_0^\infty \int_S H^2 \nu)^{1/2}$. The corresponding $N^n_t = \int_0^t H^n(s, z) \mu - \nu$ are easily seen to be a Cauchy sequence in $L^2$, and the limit $N_t$ we call the stochastic integral of $H$ with respect to $\mu - \nu$. As in the continuous case, we may prove that $\mathbb{E} N^n_t = \mathbb{E} [N]_t = \mathbb{E} \langle N \rangle_t$, and it follows from this, (4.3), and (4.4) that

$$[N]_t = \int_0^t \int_S H(s, z)^2 \mu(ds, dz), \quad \langle N \rangle_t = \int_0^t \int_S H(s, z)^2 \nu(ds, dz). \quad (4.5)$$

One may think of the stochastic integral as follows: if $\mu$ gives unit mass to a point at time $t$ with value $z$, then $N_t$ jumps at this time $t$ and the size of the jump is $H(t, z)$.

### 4.2 The Lipschitz case

We consider the the stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t \int F(X_{s-}, z)[\mu(dz, ds) - \nu(dz, ds)]. \quad (4.6)$$
Theorem 4.3 Suppose there is a constant $c$ such that
\[ \int \sup_x F(x, z)^2 \lambda(dz) < \infty \]
and
\[ \int |F(x, z) - F(y, z)|^2 \lambda(dz) \leq c|x - y|^2. \]
Then there exists a solution to (4.6) and the solution is pathwise unique.

Proof. Let $X_t^0 = x_0$ and define
\[ X_t^{n+1} = x_0 + \int_0^t \int F(X_s^n, z)[\mu(dz, ds) - \nu(dz, ds)]. \]
$X^n$ is a martingale, and by Doob’s inequality,
\[ \mathbb{E} \sup_{s \leq t} |X_s^{n+1} - X_s^n|^2 \leq 4\mathbb{E} |X_t^{n+1} - X_t^n|^2 \]
\[ = 4\mathbb{E} \int_0^t \int [F(X_s^n, z) - F(X_{s-}^{n-1}, z)]^2 \lambda(dz) ds \]
\[ \leq 4c \int_0^t |X_s^n - X_{s-}^{n-1}|^2 ds. \]
If we let
\[ g_n(t) = \mathbb{E} \sup_{s \leq t} |X_s^n - X_s^{n-1}|^2, \]
we then have
\[ g_n(t) \leq A \int_0^t g_{n-1}(s) ds. \]
Since $g_1$ is easily seen to be bounded, say, by $B$, we have by induction that $g_2(t) \leq ABt$, $g_3(t) \leq A^2Bt^2/2$, and so on, and therefore
\[ \sum_{n=1}^{\infty} g_n(t)^{1/2} < \infty. \]
It follows that $\sum \sup_{s \leq t} |X_s^n - X_s^{n-1}|$ converges in $L^2$, and it is routine to see that therefore $X_t = \lim X_t^n$ exists and is the solution to (4.6).
4.3. ANALOGUE OF YAMADA-WATANABE THEOREM

If $X$ and $Y$ are two solutions and $g(t) = \mathbb{E} \sup_{s \leq t} |X_s - Y_s|^2$, we obtain similarly that

$$g(t) \leq A \int_0^t g(s) \, ds.$$ 

We may also assume that $g$ is bounded by $B$ for $t \leq t_0$. We then obtain $g(t) \leq ABt$, $g(t) \leq A^2 B t^2 / 2!$, and so on, and therefore $g$ must be identically zero, or we have pathwise uniqueness.

4.3 Analogue of Yamada-Watanabe theorem

Our main result in this section is the analogue of the Yamada-Watanabe condition for diffusions. We suppose $X$ is a symmetric stable process of index $\alpha \in (1, 2)$ and look at the equation

$$dV_t = F(V_{t-}) \, dX_t. \tag{4.7}$$

**Theorem 4.4** Suppose $\alpha \in (1, 2)$, suppose $F$ is bounded and continuous, and suppose $\rho$ is a nondecreasing continuous function on $[0, \infty)$ with $\rho(0) = 0$ and $|F(x) - F(y)| \leq \rho(|x - y|)$ for all $x, y \in \mathbb{R}$. If

$$\int_{0+} \frac{1}{\rho(x)^\alpha} \, dx = \infty, \tag{4.8}$$

then the solution to the SDE (4.7) is pathwise unique.

We normalize our symmetric stable processes so that $\sum_{s \leq t} 1_{\{\Delta X_s \in A\}}$ is a Poisson process with parameter $\int_A |y|^{-1-\alpha} \, dy$.

Recall (2.8), (2.9), and (2.10).

Suppose $X_t$ is a symmetric stable process of index $\alpha \in (1, 2)$. We define the Poisson point process $\mu$ by

$$\mu(A \times [0, t]) = \sum_{s \leq t} 1_A(\Delta X_s),$$
the number of times before time $t$ that $X_t$ has jumps whose size lies in the set $A$. We define the compensating measure $\nu$ by

$$\nu(A) = \mathbb{E} \mu(A \times [0,1]) = \int_A \frac{1}{|x|^{1+\alpha}} dx.$$  

Set

$$\mathcal{L} f(x) = \int [f(x + w) - f(x) - f'(x)w] |w|^{-\alpha} dw$$  

for $C^2_b$ functions $f$, where $C^2_b$ is the set of $C^2$ functions $f$ such that $f$, $f'$, and $f''$ are bounded. There is convergence of the integral for large $w$ since $\alpha > 1$. There is convergence for small $w$ by using Taylor’s theorem and the fact that $\alpha < 2$.

For $C^2_b$ functions $\mathcal{L}$ coincides with the infinitesimal generator of $X$. Let us explain this further.

If $X_t$ is a Lévy process with Lévy measure $m$, then

$$\mathbb{E} e^{iu(X_t+x)} - e^{iux} = e^{iux} \left( \mathbb{E} e^{iuX_t} - 1 \right) = e^{iux} \left( e^{tf |u|^{\alpha} - 1} - iuh 1_{|h| \leq 1} m(h) - 1 \right).$$

Dividing by $t$ and letting $t \to 0$,

$$\lim_{t \to 0} \frac{\mathbb{E} e^{iu(X_t+x)} - e^{iux}}{t} = e^{iux} \int [e^{iuh} - 1 - iuh 1_{|h| \leq 1}] m(h).$$

Replacing $u$ by $-u$, multiplying by $\frac{1}{2\pi} \hat{f}(u)$, and integrating $u$ over $\mathbb{R}$, we get

$$\lim_{t \to 0} \frac{\mathbb{E} f(X_t + x) - f(x)}{t} = \int [f(x + h) - f(x) - hf'(x) 1_{|h| \leq 1}] m(h) = \mathcal{L} f(x),$$

or

$$\lim_{t \to 0} \frac{P_t f(x) - f(x)}{t} = \mathcal{L} f(x),$$

provided $f \in C^2_b$, the collection of $C^2$ functions such that $f$, $f'$, and $f''$ are bounded and provided one shows that it is valid to interchange the limit with the integral in two places (it is).

There is a slight discrepancy here in the definition of $\mathcal{L}$. We use that for $\alpha \in (1,2)$ and $m(dw) = |w|^{-\alpha} dw$, we have that $\int w1_{|w|>1} m(dw)$ is integrable and equals 0.
Let
\[ G_f(x) = \int_0^\infty e^{-\lambda s}P_s f(x) \, ds \]
where \( P_s f(x) = \mathbb{E} f(X_s + x) \). We have
\[
P_t G_f(x) = \int_0^\infty e^{-\lambda s}P_{s+t} f(x) \, ds = e^{\lambda t} \int_0^\infty e^{-\lambda(s+t)}P_s f(x) \, ds
\]
\[
= e^{\lambda t} \int_t^\infty e^{-\lambda s}P_s f(x) \, ds
\]
\[
= (e^{\lambda t} - 1) \int_t^\infty e^{-\lambda s}P_s f(x) \, ds + \int_t^\infty e^{-\lambda s}P_s f(x) \, ds.
\]
So
\[
\frac{P_t G_f(x) - G_f(x)}{t} = \frac{e^{\lambda t} - 1}{t} \int_t^\infty e^{-\lambda s}P_s f(x) \, ds - \int_0^t e^{-\lambda s}P_s f(x) \, ds.
\]
Since \( P_s f(x) = \mathbb{E} f(X_s + x) \to f(x) \) as \( s \to 0 \), we obtain
\[
\mathcal{L} G_f(x) = \lambda G_f(x) - f(x).
\]

**Proposition 4.5** Suppose \( \alpha \in (1, 2) \), \( f \) is in \( C_b^2 \), and
\[
Z_t = \int_0^t H_s \, dX_s,
\]
where \( H_t \) is a bounded predictable process. Then
\[
f(Z_t) = f(Z_0) + M_t + \int_0^t |H_s|^\alpha \mathcal{L} f(Z_{s-}) \, ds,
\]
where \( M_t \) is a martingale.

**Proof.** Let \( X_t^n = \sum_{s \leq t} \Delta X_s 1_{(|\Delta X_s| \leq n)} \) and \( Y_t^n = X_t - X_t^n \). Then \( X_t^n \) is a Lévy process with symmetric Lévy measure which is equal to \( \nu \) on \( [-n, n] \) and 0 outside this interval. Hence \( X_t^n \) is a square integrable martingale and so \( \int_0^t H_s \, dX_s^n \) is also a square integrable martingale since \( H \) is bounded. On the other hand
\[
\mathbb{E} \left| \int_0^t H_s \, dY_s^n \right| \leq \|H\|_\infty \mathbb{E} \sum_{s \leq t} |\Delta X_s| 1_{(|\Delta X_s| > n)} < \infty
\]
because \( \alpha \in (1, 2) \). The right hand side tends to 0 as \( n \to \infty \) by dominated convergence. Therefore \( Z_t \) is the \( L^1 \) limit of the square integrable martingales \( \int_0^t H_s \, dX_s^n \), and it follows that \( Z_t \) is a martingale.

Write \( K(s, y) \) for \([f(Z_s + H_s y) - f(Z_s) - f'(Z_s)H_s y] \). Note that \( \Delta Z_s = H_s \Delta X_s \). Note also that \( |K(s, y)| \) is bounded by a constant times \((|y| \wedge y^2)\).

If \( f \in C_b^2 \), we have by Ito’s formula that

\[
f(Z_t) = f(Z_0) + \int_0^t f'(Z_s) \, dZ_s + \sum_{s \leq t} [f(Z_s) - f(Z_{s_-}) - f'(Z_{s_-}) \Delta Z_s]
\]

\[
= f(Z_0) + \int_0^t f'(Z_s) \, dZ_s + \int_0^t \int K(s, y) \mu(dy, ds)
\]

\[
= f(Z_0) + M_t + \int_0^t \int K(s, y) \nu(dy) ds,
\]

where

\[
M_t = \int_0^t f'(Z_s) \, dZ_s + \int_0^t \int K(s, y) (\mu(dy, ds) - \nu(dy) ds).
\]

The first term on the right is a martingale by the argument of the first paragraph of this proof. For each \( m \) we have then that \( \int_{|y| \leq m} K(s, y)^2 \nu(dy) \) is bounded, and so for each \( m \)

\[
W^m_t = \int_0^t \int_{|y| \leq m} K(s, y) (\mu(dy, ds) - \nu(dy) ds)
\]

is a martingale. Since \( W^k_t - W^m_t \) is a martingale for each \( k \), then

\[
\mathbb{E} \int_0^t \int_{m < |y| \leq k} |K(s, y)| (\mu(dy, ds) + \nu(dy) ds) \leq c_1 \int_0^t \int_{m < |y| \leq k} |y| \nu(dy) ds
\]

\[
\leq c_2 m^{1-\alpha},
\]

where \( c_1 \) and \( c_2 \) are positive finite constants not depending on \( m \) or \( k \). Letting \( k \to \infty \), we see that

\[
\mathbb{E} \int_0^t \int_{m < |y|} |K(s, y)| (\mu(dy, ds) + \nu(dy) ds) \leq c_2 m^{1-\alpha}.
\]

Therefore \( M_t \) is the limit in \( L^1 \) of the martingales \( \int_0^t f(Z_s) \, dZ_s + W^m_t \), and hence is itself a martingale.
4.3. ANALOGUE OF YAMADA-WATANABE THEOREM

We make the change of variable \( w = H_s y \). Since \( y \to H_s y \) is monotone if \( H_s \neq 0 \) we have that the integral with respect to \( \nu(dy) \) is

\[
\int [f(Z_{s-} + H_s y) - f(Z_{s-}) - f'(Z_{s-}) H_s y] \frac{dy}{|y|^{1+\alpha}}
\]

\[
= \int [f(Z_{s-} + w) - f(Z_{s-}) - f'(Z_{s-}) w] H_s |w|^{-\alpha} dw,
\]

\[
= |H_s|^\alpha Lf(Z_{s-})
\]

if \( H_s \neq 0 \). This equality clearly also holds when \( H_s = 0 \). We therefore arrive at (4.10).

We note for future reference that we have shown (take \( H_s = 1 \) a.s.) that

\[ f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds \]

is a martingale if \( f \in C_b^2 \).

We now prove Theorem 4.4.

**Proof of Theorem 4.4.** Let \( Y^1 \) and \( Y^2 \) be any two solutions to (4.7), let \( Z_t = Y^1_t - Y^2_t \), and let \( H_t = F(Y^1_t) - F(Y^2_t) \). Then \( Z_t = \int_0^t H_s dX_s \).

Let \( a_n \) be numbers decreasing to 0 so that \( \int_{a_{n+1}}^{a_{n}} \rho(x)^{-\alpha} dx = n \). For each \( n \) let \( h_n \) be a nonnegative \( C^2 \) function with support in \( [a_{n+1}, a_n] \) whose integral is 1, and with \( h_n(x) \leq 2/(n \rho(x)^\alpha) \). This is possible since

\[
\int_{a_{n+1}}^{a_{n}} 1/(n \rho(x)^\alpha) dx = 1.
\]

Fix \( \lambda > 0 \), let \( g_\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(x,0) dt \), where \( p_t(x,y) \) is the transition density for \( X_t \), and let \( G_\lambda f(x) = \int f(y) g_\lambda(x-y) dy \). We have shown that that \( g_\lambda(x) \) is bounded, and is continuous in \( x \). Furthermore, \( g_\lambda(x) < g_\lambda(0) \) if \( x \neq 0 \). Let \( f_n(x) = G_\lambda h_n(x) \). By interchanging differentiation and integration and using translation invariance, \( f_n \) is in \( C_b^2 \) since \( h_n \) is \( C_b^2 \).

Define \( A_t = \int_0^t |H_s|^\alpha ds \). By Ito’s product formula,

\[
\mathbb{E} e^{-\lambda A_t} f_n(Z_t) - f_n(0) = \mathbb{E} \int_t^0 e^{-\lambda A_s} d[f_n(Z_s)] - \mathbb{E} \int_0^t e^{-\lambda A_s} \lambda |H_s|^\alpha f_n(Z_{s-}) ds
\]

\[
= \mathbb{E} \int_0^t e^{-\lambda A_s} |H_s|^\alpha Lf_n(Z_{s-}) ds - \mathbb{E} \int_0^t e^{-\lambda A_s} \lambda |H_s|^\alpha f_n(Z_{s-}) ds.
\]
Since $\mathcal{L}_f = \mathcal{L}G_\lambda h_n = \lambda G_\lambda h_n - h_n = \lambda f_n - h_n$, we have

$$f_n(0) - \mathbb{E} e^{-\lambda A_t} f_n(Z_t) = \mathbb{E} \int_0^t e^{-\lambda A_s} |H_s| h_n(Z_{s-}) ds.$$ 

Note $|H_s| \leq \rho(|Z_{s-}|)$, so using our bound for $h_n$, the right hand side is less than $2t/n$ in absolute value, which tends to 0 as $n \to \infty$. The measures $h_n(y)dy$ all have mass 1 and they tend weakly to point mass at 0. Since $g_\lambda$ is continuous in $x$, then $f_n(x) \to g_\lambda(x)$ as $n \to \infty$. We conclude

$$g_\lambda(0) - \mathbb{E} e^{-\lambda A_t} g_\lambda(Z_t) = 0.$$ 

We noted above that $g_\lambda(x) < g_\lambda(0)$ if $x \neq 0$, while clearly $A_t < \infty$ since $F$ is bounded. We deduce $\mathbb{P}(Z_t = 0) = 1$. This holds for each $t$, and we conclude that $Z$ is identically 0. \hfill \qed

**Remark 4.6** The above proof breaks down for $\alpha = 1$ since $g_\lambda$ is no longer a bounded function.