Chapter 10

Brownian motion

10.1 Definition and construction

In this section we construct Brownian motion and define Wiener measure.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \(\mathcal{B}\) be the Borel \(\sigma\)-field on \([0, \infty)\). A stochastic process, denoted \(X(t, \omega)\) or \(X_t(\omega)\) or just \(X_t\), is a map from \([0, \infty) \times \Omega\) to \(\mathbb{R}\) that is measurable with respect to the product \(\sigma\)-field of \(\mathcal{B}\) and \(\mathcal{F}\).

**Definition 10.1** A stochastic process \(X_t\) is a one-dimensional Brownian motion started at 0 if

1. \(X_0 = 0\) a.s.;
2. for all \(s \leq t\), \(X_t - X_s\) is a mean zero normal random variable with variance \(t - s\);
3. the random variables \(X_{r_i} - X_{r_{i-1}}, i = 1, \ldots, n\), are independent whenever \(0 \leq r_0 \leq r_1 \leq \cdots \leq r_n\);
4. there exists a null set \(N\) such that if \(\omega \notin N\), then the map \(t \to X_t(\omega)\) is continuous.

Brownian motion has a useful scaling property.

**Proposition 10.2** If \(X_t\) is a Brownian motion started at 0, \(a > 0\), and \(Y_t = a^{-1}X_{a^2t}\), then \(Y_t\) is a Brownian motion started at 0.
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Proof. Clearly $Y$ has continuous paths and $Y_0 = 0$. We observe that $Y_t - Y_s = a^{-1}(X_{a^2 t} - X_{a^2 s})$ is a mean zero normal with variance $t - s$. If $0 = r_0 < r_1 < \cdots < r_n$, then $a^2 r_0 \leq a^2 r_1 < \cdots < a^2 r_n$, so the $X_{a^2 r_i} - X_{a^2 r_{i-1}}$ are independent, and hence the $Y_{r_i} - Y_{r_{i-1}}$ are independent. \hfill \Box

Let us show that there exists a Brownian motion. We give the Haar function construction, which is one of the quickest ways to the construction of Brownian motion.

For $i = 1, 2, \ldots$, $j = 1, 2, \ldots, 2^i - 1$, let $\varphi_{ij}$ be the function on $[0, 1]$ defined by

$$
\varphi_{ij} = \begin{cases} 
2^{(i-1)/2}, & x \in \left[\frac{2j-2}{2^i}, \frac{2j-1}{2^i}\right); \\
-2^{(i-1)/2}, & x \in \left[\frac{2j-1}{2^i}, \frac{2j}{2^i}\right); \\
0, & \text{otherwise.}
\end{cases}
$$

Let $\varphi_{00}$ be the function that is identically 1. The $\varphi_{ij}$ are called the Haar functions. If $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2([0, 1])$, that is, $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$, note the $\varphi_{ij}$ are orthogonal and have norm 1. It is also easy to see that they form a complete orthonormal system for $L^2$: $\varphi_{00} \equiv 1$; $1_{[0,1/2)}$ and $1_{[1/2,1)}$ are both linear combinations of $\varphi_{00}$ and $\varphi_{11}$; $1_{[0,1/4)}$ and $1_{[1/4,1/2)}$ are both linear combinations of $1_{[0,1/2)}$, $\varphi_{21}$, and $\varphi_{22}$. Continuing in this way, we see that $1_{[k/2^n,(k+1)/2^n)}$ is a linear combination of the $\varphi_{ij}$ for each $n$ and each $k \leq 2^n$. Since any continuous function can be uniformly approximated by step functions whose jumps are at the dyadic rationals, linear combinations of the Haar functions are dense in the set of continuous functions, which in turn is dense in $L^2([0, 1])$.

Let $\psi_{ij}(t) = \int_0^t \varphi_{ij}(r)dr$. Let $Y_{ij}$ be a sequence of independent identically distributed standard normal random variables. Set

$$
V_0(t) = Y_{00}\psi_{00}(t), \quad V_i(t) = \sum_{j=1}^{2^{i-1}} Y_{ij}\psi_{ij}(t), \quad i \geq 1.
$$

If $\{e_i\}, i = 1, \ldots, N$ is a finite orthonormal set and $f = \sum_{i=1}^N a_i e_i$, then

$$
\langle f, e_j \rangle = \sum_{i=1}^N a_i \langle e_i, e_j \rangle = a_j.
$$
or
\[ f = \sum_{i=1}^{N} (f, e_i) e_i. \]

If \( f = \sum_{i=1}^{N} a_i e_i \) and \( g = \sum_{j=1}^{N} b_j e_j \), then
\[ \langle f, g \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i b_j \langle e_i, e_j \rangle = \sum_{i=1}^{N} a_i b_i, \]
or
\[ \langle f, g \rangle = \sum_{i=1}^{N} \langle f, e_i \rangle \langle g, e_i \rangle. \]

We need one more preliminary. If \( Z \) is a standard normal random variable, then \( Z \) has density \((2\pi)^{-1/2}e^{-x^2/2}\). Since
\[ \int x^4e^{-x^2/2}dx < \infty, \]
then \( E Z^4 < \infty \). We then have
\[ \mathbb{P}(|Z| > \lambda) = \mathbb{P}(Z^4 > \lambda^4) \leq \frac{E Z^4}{\lambda^4}. \quad (10.1) \]

**Theorem 10.3** \( \sum_{i=0}^{\infty} V_i(t) \) converges uniformly in \( t \) a.s. If we call the sum \( X_t \), then \( X_t \) is a Brownian motion started at 0.

**Proof.** *Step 1.* We first prove convergence of the series. Let
\[ A_i = (|V_i(t)| > i^{-2} \text{ for some } t \in [0, 1]). \]

We will show \( \sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty \). Then by the Borel–Cantelli lemma, except for \( \omega \) in a null set, there exists \( i_0(\omega) \) such that if \( i \geq i_0(\omega) \), we have \( \sup_t |V_i(t)(\omega)| \leq i^{-2} \). This will show \( \sum_{i=0}^{I} V_i(t)(\omega) \) converges as \( I \to \infty \), uniformly over \( t \in [0, 1] \). Moreover, since each \( \psi_{ij}(t) \) is continuous in \( t \), then so is each \( V_i(t)(\omega) \), and we thus deduce that \( X_t(\omega) \) is continuous in \( t \).
Now for \( i \geq 1 \) and \( j_1 \neq j_2 \), for each \( t \) at least one of \( \psi_{ij_1}(t) \) and \( \psi_{ij_2}(t) \) is zero. Also, the maximum value of \( \psi_{ij} \) is \( 2^{-(i+1)/2} \). Hence

\[
\mathbb{P}(|V_i(t)| > i^{-2} \text{ for some } t \in [0, 1]) \\
\leq \mathbb{P}(|Y_{ij}| \psi_{ij}(t) > i^{-2} \text{ for some } t \in [0, 1], \text{ some } 0 \leq j \leq 2^{i-1}) \\
\leq \mathbb{P}(|Y_{ij}| 2^{-(i+1)/2} > i^{-2} \text{ for some } 0 \leq j \leq 2^{i-1}) \\
\leq \sum_{j=0}^{2^{i-1}-1} \mathbb{P}(|Y_{ij}| 2^{-(i+1)/2} > i^{-2}) \\
= (2^{i-1} + 1) \mathbb{P}(|Z| > 2^{(i+1)/2} i^{-2})
\]

where \( Z \) is a standard normal random variable. Using (10.1), we conclude \( \mathbb{P}(A_i) \) is summable in \( i \).

**Step 2.** Next we show that the limit, \( X_t \), satisfies the definition of Brownian motion. It is obvious that each \( X_t \) has mean zero and that \( X_0 = 0 \).

Let \( D = \{ k/2^n : n \geq 0, 0 \leq k \leq 2^n \} \), the dyadic rationals. If \( s, t \in D \) and \( s < t \), then there exists \( N \) such that \( s = k/2^N, t = \ell/2^N \), and then

\[
\mathbb{E}[X_sX_t] = \sum_{i=0}^{N} \sum_{j} \sum_{k=0}^{N} \sum_{\ell} \mathbb{E}[Y_{ij}Y_{k\ell}]\psi_{ij}(s)\psi_{k\ell}(t) \\
= \sum_{i=0}^{N} \sum_{j} \psi_{ij}(s)\psi_{ij}(t) \\
= \sum_{i=0}^{N} \sum_{j} \langle 1_{[0,s]}, \varphi_{ij}(s) \rangle \langle 1_{[0,t]}, \varphi_{ij}(t) \rangle \\
= \langle 1_{[0,s]}, 1_{[0,t]} \rangle = s.
\]

Thus \( \text{Cov}(X_s, X_t) = s \). Applying this also with \( s \) replaced by \( t \) and \( t \) replaced by \( s \), we see that \( \text{Var} X_t = t \) and \( \text{Var} X_s = s \), and therefore

\[
\text{Var}(X_t - X_s) = t - 2s + s = t - s.
\]

Clearly \( X_t - X_s \) is normal, hence

\[
\mathbb{E}e^{iu(X_t - X_s)} = e^{-u^2(t-s)/2}.
\]
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For general $s, t \in [0, 1]$, choose $s_m < t_m$ in $D$ such that $s_m \to s$ and $t_m \to t$. Using dominated convergence and the fact that $X$ has continuous paths,

$$
\mathbb{E} e^{i(u_t - u_s)} = \lim_{m \to \infty} \mathbb{E} e^{i(u_{t_m} - u_{s_m})} = \lim_{m \to \infty} e^{-u^2(t_m - s_m)/2} = e^{-u^2(t - s)/2}.
$$

This proves (2).

We can work backwards from this to see that if $s < t$, then

$$
t - s = \text{Var} (X_t - X_s) = \text{Var} X_t - 2 \text{Cov} (X_s, X_t) + \text{Var} X_s
$$

or

$$
\text{Cov} (X_s, X_t) = s.
$$

Suppose $0 \leq r_1 < \cdots < r_n$ are in $D$. There exists $N$ such that each $r_i = m_i/2^N$. So $X_{r_k} = \sum_{i=0}^{N} \sum_{j} Y_{ij} \psi_{ij}(r_k)$, and hence the $X_{r_k}$ are jointly normal.

If $r_i < r_j$, then

$$
\text{Cov} (X_{r_i} - X_{r_{i-1}}, X_{r_j} - X_{r_{j-1}}) = (r_i - r_i) - (r_{i-1} - r_{i-1}) = 0,
$$

and hence the increments are independent. We therefore have

$$
\mathbb{E} e^{i \sum_{k=1}^{n} u_k (X_{r_k} - X_{r_{k-1}})} = \prod_{k=1}^{n} \mathbb{E} e^{i u_k (X_{r_k} - X_{r_{k-1}})}. \tag{10.2}
$$

Now suppose the $r_k$ are in $[0, 1]$, take $r_k^m$ in $D$ converging to $r_k$ with $r_1^k < \cdots < r_n^k$. Replacing $r_k$ by $r_k^m$, letting $m \to \infty$, and using dominated convergence, we have (10.2). This proves independent increments.

The stochastic process $X_t$ induces a measure on $C([0, 1])$. We say $A \subset C([0, 1])$ is a cylindrical set if

$$
A = \{ f \in C([0, 1]) : (f(r_1), \ldots, f(r_n)) \in B \}
$$

for some $n \geq 1$, $r_1 \leq \cdots \leq r_n$, and $B$ a Borel subset of $\mathbb{R}^n$. For $A$ a cylindrical set, define $\mu(A) = \mathbb{P}(\{X(\omega) \in A\})$, where $X$ is a Brownian motion and $X(\omega)$ is the function $t \to X_t(\omega)$. We extend $\mu$ to the $\sigma$-field generated
by the cylindrical sets. If $B$ is in this $\sigma$-field, then $\mu(B) = \mathbb{P}(X \in B)$. The probability measure $\mu$ is called Wiener measure.

We defined Brownian motion for $t \in [0, 1]$. To define Brownian motion for $t \in [0, \infty)$, take a sequence $\{X^n_t\}$ of independent Brownian motions on $[0, 1]$ and piece them together as follows. Define $X_t = X^1_t$ for $0 \leq t \leq 1$. For $1 < t \leq 2$, define $X_t = X_1 + X^2_{t-1}$. For $2 < t \leq 3$, let $X_t = X_2 + X^3_{t-2}$, and so on.

### 10.2 Nowhere differentiability

We have the following result, which says that except for a set of $\omega$’s that form a null set, $t \to X_t(\omega)$ is a function that does not have a derivative at any time $t \in [0, 1]$.

**Theorem 10.4** With probability one, the paths of Brownian motion are nowhere differentiable.

**Proof.** Note that if $Z$ is a normal random variable with mean 0 and variance 1, then

$$\mathbb{P}(|Z| \leq r) = \frac{1}{\sqrt{2\pi}} \int_{-r}^{r} e^{-x^2/2} \, dx \leq 2r. \tag{10.3}$$

Let $M, h > 0$ and let

$$A_{M, h} = \{s \in [0, 1] : |X_t - X_s| \leq M|t - s| \text{ if } |t - s| \leq h\},$$

$$B_n = \{k \leq 2n : |X_{k/n} - X_{(k-1)/n}| \leq 4M/n, \quad |X_{(k+1)/n} - X_{k/n}| \leq 4M/n, \quad |X_{(k+2)/n} - X_{(k+1)/n}| \leq 4M/n\}.$$

We check that $A_{M, h} \subset B_n$ if $n \geq 2/h$. To see this, if $\omega \in A_{M, h}$, there exists an $s$ such that $|X_t - X_s| \leq M|t - s|$ if $|t - s| \leq 2/n$; let $k/n$ be the largest multiple of $1/n$ less than or equal to $s$. Then

$$|(k + 2)/n - s| \leq 2/n \quad \text{and} \quad |(k + 1)/n - s| \leq 2/n,$$

and therefore

$$|X_{(k+2)/n} - X_{(k+1)/n}| \leq |X_{(k+2)/n} - X_s| + |X_s - X_{(k+1)/n}|$$

$$\leq 2M/n + 2M/n < 4M/n.$$
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Similarly \( |X_{(k+1)/n} - X_{k/n}| \) and \( |X_{k/n} - X_{(k-1)/n}| \) are less than \( 4M/n \).

Using the independent increments property, the stationary increments property, and (10.3),

\[
\mathbb{P}(B_n) \leq 2n \sup_{k \leq 2n} \mathbb{P}( |X_{k/n} - X_{(k-1)/n}| < 4M/n, |X_{(k+1)/n} - X_{k/n}| < 4M/n, \\
|X_{(k+2)/n} - X_{(k+1)/n}| < 4M/n ) \\
\leq 2n \mathbb{P}( |X_{1/n}| < 4M/n, |X_{2/n} - X_{1/n}| < 4M/n, \\
|X_{3/n} - X_{2/n}| < 4M/n ) \\
= 2n \mathbb{P}( |X_{1/n}| < 4M/n) \mathbb{P}( |X_{2/n} - X_{1/n}| < 4M/n) \\
\times \mathbb{P}( |X_{3/n} - X_{2/n}| < 4M/n ) \\
= 2n( \mathbb{P}( |X_{1/n}| < 4M/n))^3 \\
\leq cn \left( \frac{4M}{\sqrt{n}} \right)^3,
\]

which tends to 0 as \( n \to \infty \). Hence for each \( M \) and \( h \),

\[
\mathbb{P}(A_{M,h}) \leq \limsup_{n \to \infty} \mathbb{P}(B_n) = 0.
\]

This implies that the probability that there exists \( s \leq 1 \) such that

\[
\limsup_{h \to 0} \frac{|X_{s+h} - X_s|}{|h|} \leq M
\]

is zero. Since \( M \) is arbitrary, this proves the theorem. \( \square \)