4.4 Doob’s inequalities

The first interesting consequences of the optional stopping theorems are Doob’s inequalities. If $M_n$ is a martingale, denote $M^*_n = \max_{i \leq n} |M_i|$.

**Theorem 4.8** If $M_n$ is a martingale or a positive submartingale,

$$\mathbb{P}(M^*_n \geq a) \leq \mathbb{E}[|M_n|; M^*_n \geq a]/a \leq \mathbb{E}|M_n|/a.$$  

**Proof.** Set $M_{n+1} = M_n$. Let $N = \min\{j : |M_j| \geq a\} \wedge (n + 1)$. Since $|\cdot|$ is convex, $|M_n|$ is a submartingale. If $A = (M^*_n \geq a)$, then $A \in \mathcal{F}_N$ because

$$A \cap (N \leq j) = (N \leq n) \cap (N \leq j) = (N \leq j) \in \mathcal{F}_j.$$  

By Corollary 4.4

$$\mathbb{P}(M^*_n \geq a) \leq \mathbb{E}\left[\frac{M^*_n}{a}; M^*_n \geq a\right] \leq \frac{1}{a} \mathbb{E}[|M_N|; A] \leq \frac{1}{a} \mathbb{E}[|M_n|; A] \leq \frac{1}{a} \mathbb{E}|M_n|.$$  

For $p > 1$, we have the following inequality.

**Theorem 4.9** If $p > 1$ and $\mathbb{E}|M_i|^p < \infty$ for $i \leq n$, then

$$\mathbb{E}(M^*_n)^p \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_n|^p.$$  

**Proof.** Note $M^*_n \leq \sum_{i=1}^n |M_i|$, hence $M^*_n \in L^p$. We write, using Theorem 4.8 for the first inequality,

$$\mathbb{E}(M^*_n)^p = \int_0^\infty pa^{p-1}\mathbb{P}(M^*_n > a) \, da \leq \int_0^\infty pa^{p-1}\mathbb{E}[|M_n| 1_{(M^*_n \geq a)}/a] \, da$$

$$= \mathbb{E} \int_0^{M^*_n} pa^{p-2}|M_n| \, da = \frac{p}{p-1}\mathbb{E}[(M^*_n)^{p-1}|M_n|]$$

$$\leq \frac{p}{p-1}(\mathbb{E}(M^*_n)^p)^{(p-1)/p}(\mathbb{E}|M_n|^p)^{1/p}.$$  

The last inequality follows by Hölder’s inequality. Now divide both sides by the quantity $(\mathbb{E}(M^*_n)^p)^{(p-1)/p}$.  

$\blacksquare$
4.5 Martingale convergence theorems

The martingale convergence theorems are another set of important consequences of optional stopping. The main step is the upcrossing lemma. The number of upcrossings of an interval \([a, b]\) is the number of times a process crosses from below \(a\) to above \(b\).

To be more exact, let
\[
S_1 = \min\{k : X_k \leq a\}, \quad T_1 = \min\{k > S_1 : X_k \geq b\},
\]
and
\[
S_{i+1} = \min\{k > T_i : X_k \leq a\}, \quad T_{i+1} = \min\{k > S_{i+1} : X_k \geq b\}.
\]
The number of upcrossings \(U_n\) before time \(n\) is
\[
U_n = \max\{j : T_j \leq n\}.
\]

**Theorem 4.10 (Upcrossing lemma)** If \(X_k\) is a submartingale,
\[
\mathbb{E} U_n \leq (b - a)^{-1} \mathbb{E} [(X_n - a)^+]\]

**Proof.** The number of upcrossings of \([a, b]\) by \(X_k\) is the same as the number of upcrossings of \([0, b - a]\) by \(Y_k = (X_k - a)^+\). Moreover \(Y_k\) is still a submartingale. If we obtain the inequality for the the number of upcrossings of the interval \([0, b - a]\) by the process \(Y_k\), we will have the desired inequality for upcrossings of \(X\).

So we may assume \(a = 0\). Fix \(n\) and define \(Y_{n+1} = Y_n\). This will still be a submartingale. Define the \(S_i, T_i\) as above, and let \(S'_i = S_i \wedge (n + 1), T'_i = T_i \wedge (n + 1)\). Since \(T_{i+1} > S_{i+1} > T_i\), then \(T'_{n+1} = n + 1\).

We write
\[
\mathbb{E} Y_{n+1} = \mathbb{E} Y_{S'_i} + \sum_{i=0}^{n+1} \mathbb{E} [Y_{T'_i} - Y_{S'_i}] + \sum_{i=0}^{n+1} \mathbb{E} [Y_{T'_{i+1}} - Y_{T'_i}].
\]
All the summands in the third term on the right are nonnegative since \(Y_k\) is a submartingale. For the \(j\)th upcrossing, \(Y_{T'_j} - Y_{S'_j} \geq b - a\), while \(Y_{T'_j} - Y_{S'_j}\) is always greater than or equal to 0. So
\[
\sum_{i=0}^{\infty} (Y_{T'_i} - Y_{S'_i}) \geq (b - a) U_n.
\]
So
\[ \mathbb{E} U_n \leq \mathbb{E} Y_{n+1}/(b - a). \] (4.2)

This leads to the martingale convergence theorem.

**Theorem 4.11** If \( X_n \) is a submartingale such that \( \sup_n \mathbb{E} X_n^+ < \infty \), then \( X_n \) converges a.s. as \( n \to \infty \).

**Proof.** Let \( U(a, b) = \lim_{n \to \infty} U_n \). For each \( a, b \) rational, by monotone convergence,
\[ \mathbb{E} U(a, b) \leq c(b - a)^{-1} \mathbb{E} (X_n - a)^+ < \infty. \]

So \( U(a, b) < \infty \), a.s. Taking the union over all pairs of rationals \( a, b \), we see that a.s. the sequence \( X_n(\omega) \) cannot have \( \lim \sup X_n > \lim \inf X_n \). Therefore \( X_n \) converges a.s., although we still have to rule out the possibility of the limit being infinite. Since \( X_n \) is a submartingale, \( \mathbb{E} X_n \geq \mathbb{E} X_0 \), and thus
\[ \mathbb{E} |X_n| = \mathbb{E} X_n^+ + \mathbb{E} X_n^- = 2 \mathbb{E} X_n^+ - \mathbb{E} X_n \leq 2 \mathbb{E} X_n^+ - \mathbb{E} X_0. \]

By Fatou’s lemma, \( \mathbb{E} \lim_n |X_n| \leq \sup_n \mathbb{E} |X_n| < \infty \), or \( X_n \) converges a.s. to a finite limit.

**Corollary 4.12** If \( X_n \) is a positive supermartingale or a martingale bounded above or below, \( X_n \) converges a.s.

**Proof.** If \( X_n \) is a positive supermartingale, \( -X_n \) is a submartingale bounded above by 0. Now apply Theorem 4.11.

If \( X_n \) is a martingale bounded above, by considering \( -X_n \), we may assume \( X_n \) is bounded below. Looking at \( X_n + M \) for fixed \( M \) will not affect the convergence, so we may assume \( X_n \) is bounded below by 0. Now apply the first assertion of the corollary.
4.6 APPLICATIONS OF MARTINGALES

Proposition 4.13 If $X_n$ is a martingale with $\sup_n \mathbb{E}|X_n|^p < \infty$ for some $p > 1$, then the convergence is in $L^p$ as well as a.s. This is also true when $X_n$ is a submartingale. If $X_n$ is a uniformly integrable martingale, then the convergence is in $L^1$. If $X_n \to X_\infty$ in $L^1$, then $X_n = \mathbb{E}[X_\infty | \mathcal{F}_n]$.

$X_n$ is a uniformly integrable martingale if the collection of random variables $X_n$ is uniformly integrable.

Proof. The $L^p$ convergence assertion follows by using Doob’s inequality (Theorem 4.9) and dominated convergence. The $L^1$ convergence assertion follows since a.s. convergence together with uniform integrability implies $L^1$ convergence. Finally, if $j < n$, we have $X_j = \mathbb{E}[X_n | \mathcal{F}_j]$. If $A \in \mathcal{F}_j$,

$$\mathbb{E}[X_j; A] = \mathbb{E}[X_n; A] \to \mathbb{E}[X_\infty; A]$$

by the $L^1$ convergence of $X_n$ to $X_\infty$. Since this is true for all $A \in \mathcal{F}_j$, $X_j = \mathbb{E}[X_\infty | \mathcal{F}_j]$. \hfill \Box

4.6 Applications of martingales

One application of martingale techniques is Wald’s identities.

Proposition 4.14 Suppose the $Y_i$ are i.i.d. with $\mathbb{E}|Y_1| < \infty$, $N$ is a stopping time with $\mathbb{E}N < \infty$, and $N$ is independent of the $Y_i$. Then $\mathbb{E}S_N = (\mathbb{E}N)(\mathbb{E}Y_1)$, where the $S_n$ are the partial sums of the $Y_i$.

Proof. $S_n - n(\mathbb{E}Y_1)$ is a martingale, so $\mathbb{E}S_{n \wedge N} = \mathbb{E}(n \wedge N)\mathbb{E}Y_1$ by optional stopping. The right hand side tends to $(\mathbb{E}N)(\mathbb{E}Y_1)$ by monotone convergence. $S_{n \wedge N}$ converges almost surely to $S_N$, and we need to show the expected values converge.

Note

$$|S_{n \wedge N}| = \sum_{k=0}^\infty |S_{n \wedge k}| 1_{(N=k)} \leq \sum_{k=0}^\infty \sum_{j=0}^{n \wedge k} |Y_j| 1_{(N=k)}$$

$$= \sum_{j=0}^n \sum_{k>j} |Y_j| 1_{(N=k)} = \sum_{j=0}^n |Y_j| 1_{(N \geq j)} \leq \sum_{j=0}^\infty |Y_j| 1_{(N \geq j)}.$$
The last expression, using the independence, has expected value
\[
\sum_{j=0}^{\infty} (\mathbb{E}|Y_j|) \mathbb{P}(N \geq j) \leq (\mathbb{E}|Y_1|)(1 + \mathbb{E}N) < \infty.
\]
So by dominated convergence, we have \(\mathbb{E}S_{n\wedge N} \to \mathbb{E}S_N\).

Wald’s second identity is a similar expression for the variance of \(S_N\).