

ADDING AND SUBTRACTING JUMPS FROM MARKOV PROCESSES

BY

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ABSTRACT. If X_t is a continuous Markov process with infinitesimal generator A , if n is a kernel satisfying certain conditions, and if B is an operator given by

$$Bg(x) = \int [g(y) - g(x)]n(x, dy),$$

then $A + B$ will be the generator of a Markov process that has Lévy system (n, dt) . Conversely, if X_t has Lévy system (n, dt) , n satisfies certain conditions, and B is defined as above, then $A - B$ will be the generator of a continuous Markov process.

1. Introduction. Suppose X_t is a Markov process with infinitesimal generator A . If we perturb A by another operator B , will $A + B$ be the generator of a Markov process? And if so, what will the new Markov process look like? In this article we show that if X_t is continuous and B is given by $Bg(x) = \int [g(y) - g(x)]n(x, dy)$ for some kernel n , then $A + B$ will be the generator of a Markov process Y_t whose jump structure is completely described by n . Conversely, if A is the generator of a Markov process X_t whose jump structure is given by n and B is as given above, then $A - B$ will be the generator of a Markov process Y_t which has no jumps; that is, all the paths are continuous.

Previous work on this problem has been done by Cook [4], in the case where $n(x, \cdot)$ is bounded in neighborhoods of x . There is a probabilistic construction of the new process Y_t due to Ikeda, Nagasawa, and Watanabe [6], Meyer [10], and Sawyer [12] (see §6) in the case $n(x, \cdot)$ is finite and sufficiently small. In this article, $n(x, \cdot)$ is allowed to be infinite (see Example 3.7). In probabilistic terms, $n(x, \cdot)$ being finite and sufficiently small means that for each path of X_t or Y_t , there will only be a finite number of jumps in each finite time interval; $n(x, \cdot)$ being infinite allows there to be infinitely many jumps in finite time intervals; in contrast to the n finite case, there may well be zero time between jumps.

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By X_t "has jump structure described by n ", we mean X_t has Lévy system (n, dt) , using the Lévy system developed by Watanabe [14] and Benveniste and Jacod [2]. Since the domains of infinitesimal generators are awkward to work with, we work instead with the resolvents R_λ and S_λ of X_t and Y_t , respectively. In §2, we give the necessary preliminaries. In §§3 and 4, we show that if BR_λ is bounded in norm, S_λ will be the resolvent of a semigroup in the cases where we are adding jumps and subtracting jumps, respectively. Example 3.7 is an example that shows n may be infinite. In §5, we show that Y_t has Lévy system (n, dt) or that Y_t is continuous depending whether we added or subtracted jumps. In §6 we describe the probabilistic construction of Y_t , when it exists.

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2. Preliminaries. We will suppose E is a compact metric space. If E is only locally compact, we can make it compact by adding the point Δ , the one point compactification. $\| \cdot \|$ will denote the usual sup norm, both for functions and operators. $f_n \rightarrow f$ will mean $\|f_n - f\| \rightarrow 0$, unless specified otherwise. Let f_n converges weakly to f mean that $\sup\|f_n\| < \infty$ and $f_n(x) \rightarrow f(x)$ for all x . We will let \mathcal{K} be the space of bounded, Borel measurable functions on E with sup norm, and \mathcal{L} will refer to a closed subspace of \mathcal{K} . We will assume throughout that all kernels m satisfy $m(x, \{x\}) = 0$ for all x . Following the notation of [3], $X_t = (\Omega, \mathcal{F}_t, X_t, \theta_t, P^x)$ will be a right continuous strong Markov process with state space E .

A Lévy system (n, dH_t) for a process X_t is a kernel $n(x, dy)$ and a perfect continuous additive functional H_t such that for all $x \in E$, for all bounded stopping times T , and all positive Borel measurable functions f on $E \times E$ that are 0 on the diagonal, the Lévy system identity holds:

$$E^x \sum_{0 < t < T} f(X_{t-}, X_t) = E^x \int_0^T \int f(X_t, y) n(X_t, dy) dH_t,$$

where both sides may be infinite.

Benveniste and Jacod [2] proved that every Hunt process has a Lévy system. We will assume throughout that $H_t(\omega) = t$ for all t and all ω . Since one can always perform a time change on X_t so that this is true (cf. [9, p. 150]), there is no real loss of generality. If X_t has Lévy system (n, dt) , and $n'(x, dy) = n(x, dy)$ for all x except for a set of potential 0, it is clear that (n', dt) satisfies the Lévy system identity. We will sometimes refer to n as the Lévy kernel for X_t .

Given any kernel m , define the Lévy operator L_m associated with m by $L_m g(x) = \int [g(y) - g(x)] m(x, dy)$ for those g 's and x 's for which the integral

is well defined. By the construction of Benveniste and Jacod, if m is a Lévy kernel for some Hunt process, $L_m g(x)$ is well defined for any g that vanishes in a neighborhood of x .

Let $m \geq n$ mean that except for a set of x 's of potential 0, $m(x, F) \geq n(x, F)$ for all Borel sets F . m is bounded if $m(x, E)$ is a bounded function of x . Let us say that m_j increases strongly to m if each m_j is bounded, and there exist Borel sets F_j contained in $E \times E$ increasing to $E \times E$ such that $m_j(x, dy) = 1_{F_j}(x, y)m(x, dy)$. We will need

PROPOSITION 2.1. *If m is the Lévy kernel of a Hunt process, there exist kernels m_j that increase strongly to m .*

PROOF. If $F = F_1 \times F_2$, $1_{F_1 \times F_2}(x, y)m(x, dy)$ is clearly a kernel that is a measure in dy and is measurable in x . By a monotone class argument $1_F(x, y)m(x, dy)$ is a kernel for all F Borel in $E \times E$.

Let $G_i = \{(x, y): d(y, x) \geq 1/i\}$. Let $g_i(x) = \int 1_{G_i}(x, y)m(x, dy)$. Since m is a Lévy kernel, $g_i(x)$ is finite for all x . Also $g_i \uparrow$ as $i \rightarrow \infty$. Given a positive integer N , let $H_i = \{x: g_i(x) \leq N, g_k(x) > N \text{ for } k = i + 1, i + 2, \dots\}$. Let $F'_N = \cup_{i=1}^{\infty} (H_i \times E) \cap G_i$. $1_{F'_N}(x, y)m(x, dy)$ is bounded by N . Now let $F_j = \cup_{N=1}^j F'_N$. Then $m_j(x, E)$ is bounded by $j(j + 1)/2$, where $m_j(x, dy) = 1_{F_j}(x, y)m(x, dy)$.

Note that if m_j increases strongly to m , $g \geq 0$ and $g(x) = 0$, $L_{m_j} g(x) = m_j(x, g) \rightarrow m(x, g) = L_m g(x)$.

Recall that if X_t is a Markov process, $P_t f(x) = E^x f(X_t)$ defines a semigroup P_t on \mathcal{K} . The resolvent R_λ of P_t is given by $R_\lambda f = \int_0^\infty e^{-\lambda t} P_t f dt$. If $\lambda \neq \mu$, R_λ satisfies the resolvent identity $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu$.

Conversely, the Hille-Yosida theorem says that if $\{R_\lambda, \lambda > \lambda_0\}$ is a family of operators satisfying the resolvent identity, if $\|R_\lambda\| \leq 1/\lambda$, and if $R_\lambda(\mathcal{K})$ is dense (under the sup norm) in \mathcal{L} , a subspace of \mathcal{K} , then R_λ is the resolvent of a semigroup P_t that is strongly continuous on \mathcal{L} . An operator V is positive if $h \geq 0$ implies $Vh \geq 0$. By the construction of the Hille-Yosida theorem, if R_λ is positive, so is P_t .

If R_λ is the resolvent for a strong Markov process X_t , we have Dynkin's identity,

$$E^x R_\lambda g(X_T) - R_\lambda g(x) = E^x \int_0^T \lambda(\lambda R_\lambda - I)g(X_t) dt$$

for all $x \in E$, all $g \in \mathcal{K}$, and all bounded stopping times T , where I is the identity operator. We also know that if $g \in \mathcal{K}$, $R_\lambda g(X_t)$ is right continuous a.s.

If V is a positive linear operator and $V1 = c$, note that $\|V\| = c$, since if $-1 < g < 1$, $c - Vg = V(1 - g) \geq 0$ and $c + Vg = V(1 + g) \geq 0$.

If B is an operator such that $\|BR_\lambda\| < \delta < 1$ if $\lambda > \text{some } \lambda_0$, we can define S_λ by $S_\lambda = R_\lambda(\sum_{i=0}^{\infty} (BR_\lambda)^i)$, and $\|S_\lambda\| \leq ((1 - \delta)\lambda)^{-1}$.

$$S_\lambda = R_\lambda(I + BS_\lambda) = (I + R_\lambda B)S_\lambda.$$

We need a few basic properties of S_λ .

LEMMA 2.2. *If $R_\lambda = Q_\lambda(I + AR_\lambda)$ and $S_\lambda = R_\lambda(I + BS_\lambda)$, then $S_\lambda = Q_\lambda(I + (A + B)S_\lambda)$.*

PROOF. $S_\lambda = Q_\lambda(I + AR_\lambda)(I + BS_\lambda) = Q_\lambda(I + BS_\lambda) + Q_\lambda AR_\lambda(I + BS_\lambda) = Q_\lambda(I + BS_\lambda) + Q_\lambda AS_\lambda = Q_\lambda(I + (A + B)S_\lambda)$.

LEMMA 2.3. *If R_λ and S_λ each satisfy the resolvent identity for all μ and λ , and $S_\lambda = R_\lambda(I + BS_\lambda)$ for some λ , then $S_\mu = R_\mu(I + BS_\mu)$ for all $\mu > 0$.*

Proof is by direct computation using the resolvent identity for R_μ and S_μ .

3. Adding jumps. We assume throughout §3 that X_t is a strong Markov process with resolvent R_λ , n a kernel, B a linear operator such that $\|BR_\lambda\| < 1$ for all $\lambda > \text{some } \lambda_0$. We let $S_\lambda = R_\lambda(\sum_{i=0}^{\infty} (BR_\lambda)^i)$, and our aim is to find conditions on B so that S_λ satisfies the conditions of the Hille-Yosida theorem. Note that $S_\lambda = R_\lambda(I + BS_\lambda) = (I + S_\lambda B)R_\lambda$ if $\lambda > \lambda_0$.

LEMMA 3.1. *S_λ satisfies the resolvent identity.*

The proof is identical to that of Lemma 2.1 of Leviatan [8], a fairly straightforward calculation.

LEMMA 3.2. *$S_\lambda(\mathcal{K})$ is dense in \mathcal{L} .*

PROOF. $R_\lambda(\mathcal{K})$ is dense in \mathcal{L} . If $g = R_\lambda f$, with $f \in \mathcal{K}$, let $h = (I - BR_\lambda)f$. Then $S_\lambda h = S_\lambda(I - BR_\lambda)f = R_\lambda f$.

We first prove our result for the case where the kernel n is bounded and B is the Lévy operator of n .

THEOREM 3.3. *Suppose $n(x, E) \leq N$ for all x and B is the Lévy operator of n . Then S_λ is the resolvent of a positive contraction semigroup on \mathcal{L} .*

PROOF. Let $\lambda \geq 4N$. Since $R_\lambda 1 = 1/\lambda$ and $B1 = 0$, it follows from Lemmas 3.1, 3.2 and §2 that we need only show that if $g \geq 0$ and $g \in \mathcal{L}$, then $S_\lambda g \geq 0$. We break the proof into two steps.

(1) Let $m = \inf_{x \in E} S_\lambda g(x)$. Let $\varepsilon > 0$. Since $S_\lambda g = R_\lambda(I + BS_\lambda)g$, $S_\lambda g(X_t)$ is right continuous. Let $T = \min(U, 1)$ where $U = \inf\{t > 0: S_\lambda g(X_t) - S_\lambda g(X_0) \geq 2\varepsilon\}$. We have $S_\lambda g(X_U) \geq S_\lambda g(X_0) + 2\varepsilon$ provided $U < \infty$. Suppose for the remainder of the proof that $S_\lambda g(x) < m + \varepsilon$.

If $P^x(T = U) > \frac{2}{3}$, then

$$\begin{aligned} \frac{\varepsilon}{3} &\leq E^x S_\lambda g(X_T) - S_\lambda g(x) \\ &= E^x \int_0^T (\lambda R_\lambda - I)(I + BS_\lambda)g(X_t) dt \leq 6\|g\|E^x T. \end{aligned}$$

It follows that whether or not $P^x(T = U) > \frac{2}{3}$, we have $E^x T > \min(\frac{1}{3}, \varepsilon/(18\|g\|))$.

(2) If $S_\lambda g(y) < m + 2\varepsilon$, $-BS_\lambda g(y) \leq 2\varepsilon N$. We then have

$$\begin{aligned} m - S_\lambda g(x) &\leq E^x Sg(X_T) - S_\lambda g(x) = E^x \int_0^T (\lambda S_\lambda - I - BS_\lambda)g(X_t) dt \\ &\leq E^x \int_0^T \lambda S_\lambda g(X_t) dt - E^x \int_0^T BS_\lambda g(X_t) dt \\ &\leq \lambda(m + 2\varepsilon)E^x T + 2\varepsilon NE^x T. \end{aligned}$$

Now if \bar{m} were less than 0, we could pick ε small enough so that $\lambda(m + 2\varepsilon) + 2\varepsilon N < \lambda\bar{m}/2$. We would then have that if $S_\lambda g(x) < m + \varepsilon$, $S_\lambda g(x) \geq m + \lambda(E^x T)|\bar{m}|/2$, a contradiction to the definition of m since $E^x T$ is bounded away from 0 by step (1).

Comment. By the Hille-Yosida theorem there is a semigroup P_t such that

$$S_\lambda f = \int e^{-dt} P_t f dt, \quad \lambda > 4N, \quad f \in \mathcal{L}.$$

We can use this equation to define $S_\lambda f$ for all λ . By Fubini's theorem and the fact that P_t is a semigroup, we have that S_λ satisfies the resolvent identity for all λ and $\mu > 0$. By Lemma 2.3, $S_\lambda = R_\lambda(I + BS_\lambda)$ for all λ , not just $\lambda > 4N$.

We now allow B to be unbounded.

THEOREM 3.4. *Suppose there exists a sequence of bounded kernels n_j , with Lévy operators L_{n_j} , and $L_{n_j} S_\lambda g \rightarrow BS_\lambda g$ for all $g \in \mathcal{L}$. Then S_λ is the resolvent of a positive contraction semigroup on \mathcal{L} .*

PROOF. Let $\lambda \geq \lambda_0$. Again, as in Theorem 3.3, we need only show $S_\lambda g \geq 0$ if $g \in \mathcal{L}$ and $g \geq 0$. Let $S_j^\mu g = \sum_{i=0}^\infty R_\mu (L_{n_j} R_\mu)^i$ for $\mu \geq 2\|L_{n_j}\|$. By (3.3), S_j^μ is a resolvent for a semigroup, and $S_j^\mu = R_\mu + S_j^\mu L_{n_j} R_\mu$ for all μ . Since $S_\lambda = R_\lambda + R_\lambda BS_\lambda$,

$$\begin{aligned} S_j^\lambda (g - L_{n_j} S_\lambda g + BS_\lambda g) &= R_\lambda g + R_\lambda BS_\lambda g \\ &\quad + S_j^\lambda L_{n_j} (R_\lambda g + R_\lambda BS_\lambda g) - S_j^\lambda L_{n_j} S_\lambda g \\ &= S_\lambda g. \end{aligned}$$

Then $\|S_j^\lambda g - S_\lambda g\| \leq \|L_{n_j} S_\lambda g - BS_\lambda g\|/\lambda \rightarrow 0$ as $j \rightarrow \infty$. Since $S_j^\lambda g \geq 0$, our result follows.

COROLLARY 3.5. *Suppose B is the infinitesimal generator of a Markov process such that $BR_\lambda(\mathcal{L}) \subset \mathcal{L}$. Then S_λ is the resolvent of a contraction semigroup on \mathcal{L} .*

PROOF. First of all, it is clear that $BS_\lambda = \sum_{i=0}^{\infty} (BR_\lambda)^i$ maps \mathcal{L} into \mathcal{L} . If T_λ is the resolvent of the Markov process generated by B , let $L_{n_j} = j(jT_j - I)$:

$$L_{n_j}g(x) = \int [g(y) - g(x)]j^2T_j(x, dy).$$

Since $L_{n_j}S_\lambda g = j(jT_j - I)S_\lambda g = jT_jBS_\lambda g \rightarrow BS_\lambda g$ as $j \rightarrow \infty$ if $g \in \mathcal{L}$, the result follows by Theorem 3.4.

Corollary 3.5 is also proved in [5] and [7]. As in the following example, Corollary 3.5 shows that in some cases perturbing by a drift term may be viewed as the limit of perturbation by jumps.

EXAMPLE 3.6. Suppose R_λ is the resolvent of Brownian motion on the real line, $B = d/dx$. Here \mathcal{L} is \mathcal{C}_0 , the continuous functions that vanish at infinity. If $g \in \mathcal{C}_0$, $R_\lambda g$ is twice differentiable, hence $BR_\lambda g$ is continuous, and it is easily checked that $BR_\lambda g \in \mathcal{C}_0$. Also, since

$$2\|g\| \geq \|\lambda R_\lambda g - g\| = \|AR_\lambda g\| = \|\frac{1}{2}(R_\lambda g)''\|,$$

where A is the generator of Brownian motion, the identity $\|f'\|^2 < 4\|f\| \|f''\|$ gives $\|BR_\lambda g\|^2 < 16\|R_\lambda g\| \|g\|$, or $\|BR_\lambda g\| \leq 4\|g\|/\lambda^{1/2}$. It follows by Corollary 3.5 that S_λ is the resolvent of a semigroup on \mathcal{C}_0 ; it is clear that the generator of this semigroup is $\frac{1}{2}d^2/dx^2 + d/dx$.

EXAMPLE 3.7. Suppose that R_λ is the resolvent of Brownian motion on the real line,

$$Bg(x) = \int [g(y) - g(x)]n(x, dy),$$

where $\int_{|y-x|<\epsilon} |y-x|n(x, dy) \rightarrow 0$ uniformly in x as $\epsilon \rightarrow 0$ where $N = \int \min[|y-x|, 1]n(x, dy)$ is bounded. Note that it is possible here for $n(x, E)$ to be infinite for all x . Define n_j by $n_j(x, dy) = 1_{D_j}(x, y)n(x, dy)$ where $D_j = \{(x, y): d(x, y) \geq 1/j\}$. As in Proposition 2.1, n_j is a kernel bounded by jN . As in Example 3.6 $(R_\lambda g)'$ is bounded in norm if $g \in \mathcal{C}_0$, or $|R_\lambda g(y) - R_\lambda g(x)| \leq c|y-x|$, where $c = 4\|g\|/\lambda^{1/2}$. Since $R_\lambda g_i \rightarrow R_\lambda g$ weakly if $g_i \rightarrow g$ weakly, a monotone class argument gives the above inequality for all g , not just those in \mathcal{C}_0 . Since $S_\lambda g = R_\lambda(I + BS_\lambda)g$, $|S_\lambda g(y) - S_\lambda g(x)| \leq c'|y-x|$, where $c' = 8\|g\|$, if λ is large enough.

$$\begin{aligned} \|BS_\lambda g - L_{n_j}S_\lambda g\| &\leq \int_{|y-x|<1/j} |S_\lambda g(y) - S_\lambda g(x)| n(x, dy) \\ &\leq 8\|g\| \int_{|y-x|<1/j} |y-x|n(x, dy) \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

A similar argument shows that $\|BR_\lambda\| \leq 2N\|g\|/\lambda + 4N\|g\|/\lambda^{1/2}$; hence $\|BS_\lambda\| \rightarrow 0$ as $\lambda \rightarrow \infty$, and clearly $\|BS_\lambda\| < 1$ if λ is large enough. It follows by Theorem 3.4 that S_λ is the resolvent of a contraction semigroup on \mathcal{C}_0 . We will show in §5 that the associated Markov process has Lévy kernel n .

4. Subtracting jumps. We assume throughout §4 that X_t is a Hunt process with resolvent R_λ and Lévy system (n, dt) , that B is a linear operator, and that $\|BR_\lambda\| < 1$ for all $\lambda >$ some λ_0 . We let $S_\lambda = R_\lambda(\sum_{i=0}^\infty (-BR_\lambda)^i)$, and again our aim is to show that S_λ satisfies the conditions of the Hille-Yosida theorem.

THEOREM 4.1. *Suppose n is bounded by N and B is the Lévy operator of n . Then S_λ is the resolvent of a positive contraction semigroup on \mathcal{L} .*

PROOF. Let $\lambda \geq 4N$. As in Lemmas 3.1 and 3.2, S_λ satisfies the resolvent identity and $S_\lambda(\mathcal{K})$ is dense in \mathcal{L} . So we need only show that if $g \geq 0, g \in \mathcal{L}$, then $S_\lambda g \geq 0$. $S_\lambda g(X_t) = R_\lambda(I - BS_\lambda)g(X_t)$ is right continuous. Let $m = \inf_{x \in E} S_\lambda g(x)$. Let $\varepsilon > 0$. Let $T = \min(U, V, 1)$, where $U = \inf\{t > 0: S_\lambda g(X_t) - S_\lambda g(X_0) \geq 2\varepsilon\}$ and $V = \inf\{t > 0: X_t \neq X_{t-}\}$. As in Theorem 3.3, if $U < \infty$ we have $S_\lambda g(X_U) \geq S_\lambda g(X_0) + 2\varepsilon$. We break the remainder of the proof into two steps. Let us suppose throughout that $S_\lambda g(x) < m + \varepsilon$.

(1) First we show $E^x T$ is bounded away from 0. We have that if $P^x(T = U) \geq \frac{7}{12}$,

$$\begin{aligned} \varepsilon/3 &\leq 2\varepsilon P^x(T = U) - 2\varepsilon P^x(T < U) \\ &\leq E^x(S_\lambda g(X_T) - S_\lambda g(x); T = U) + E^x(S_\lambda g(X_T) - S_\lambda g(x); T < U) \\ &= E^x R_\lambda(I - BS_\lambda)g(X_T) - R_\lambda(I - BS_\lambda)g(x) \\ &= E^x \int_0^T (\lambda R_\lambda - I)(I - BS_\lambda)g(X_s) ds \leq 6\|g\| E^x T. \end{aligned}$$

If $P^x(T = V) \geq \frac{1}{3}$, let $t_0 = 1/(6N)$. Then

$$P^x(V < t_0) \leq E^x \sum_{0 < t < t_0} 1_{(X_t \neq X_{t-})} = E^x \int_0^{t_0} n(X_t, E) dt \leq N t_0 = \frac{1}{6}.$$

In this case, $P^x(T \geq t_0) \geq \frac{1}{6}$, or $E^x T \geq 1/(36N)$.

Finally, if $P^x(T = U) < \frac{7}{12}$ and $P^x(T = V) < \frac{1}{3}$, we must have $P^x(T = 1) \geq \frac{1}{12}$. In any case, we have $E^x T \geq c = \min(\frac{1}{12}, 1/(36N)\varepsilon/(18\|g\|))$.

(2) We now show $m \geq 0$. Note

$$S_\lambda g(X_T) - S_\lambda g(x) - (S_\lambda g(X_T) - S_\lambda g(X_{T-})) \geq m - S_\lambda g(x).$$

If $t < T$, $S_\lambda g(X_t) = S_\lambda g(X_{t-})$, and so

$$S_\lambda g(X_T) - S_\lambda g(X_{T-}) = \sum_{0 < t < T} (S_\lambda g(X_t) - S_\lambda g(X_{t-})).$$

Then by the Lévy system identity and Dynkin's identity,

$$\begin{aligned} m - S_\lambda g(x) &\leq E^x S_\lambda g(X_T) - S_\lambda g(x) - E^x \sum_{0 < t < T} (S_\lambda g(X_t) - S_\lambda g(X_{t-})) \\ &= E^x \int_0^T (\lambda R_\lambda - I)(I - BS_\lambda)g(X_t) dt - E^x \int_0^T BS_\lambda g(X_t) dt \\ &\leq E^x \int_0^T \lambda S_\lambda g(X_t) dt \leq \lambda(m + 2\varepsilon)E^x T. \end{aligned}$$

But if m were less than 0, we could let $\varepsilon = |m|/4$, and by selecting x so that $S_\lambda g(x)$ were close enough to m , we would have a contradiction.

THEOREM 4.2. *Suppose there exists a sequence of bounded kernels n_j with Lévy operators L_{n_j} such that (i) n_j strongly increases to n and (ii) $L_{n_j} S_\lambda g \rightarrow BS_\lambda g$ for all $g \in \mathcal{L}$. Then S_λ is the resolvent of a positive contraction semigroup on \mathcal{L} .*

PROOF. Suppose $g \geq 0$, $g \in \mathcal{L}$. Let $S_j^\mu g = \sum_{i=0}^\infty R_\mu (-L_{n_j} R_\mu)^i$ for $\mu > 2\|L_{n_j}\|$. The proof goes exactly as for Theorem 3.4 provided we show $S_j^\mu g \geq 0$. But the proof of Theorem 4.1 goes through exactly as before if we redefine V to be $\inf\{t > 0: (X_{t-}, X_t) \in F_j\}$ and we consider $\sum (S_\lambda g(X_t) - S_\lambda g(X_{t-}))1_{((X_{t-}, X_t) \in F_j)}$ instead of $\sum (S_\lambda g(X_t) - S_\lambda g(X_{t-}))$, where for each j , F_j is a set such that $n_j(x, dy) = 1_{F_j}(x, n) n(x, dy)$.

EXAMPLE 4.3. Consider Example 3.7. From the fact that if $S_\lambda = R_\lambda + S_\lambda BR_\lambda$, $R_\lambda = S_\lambda(I - BR_\lambda)$, it should be clear that if the process constructed in Example 3.7 does have Lévy system (n, dt) , a consequence of §5, then the semigroup constructed as a result of Theorem 4.2 will just be that of Brownian motion.

5. Lévy system. At this point we make three assumptions. Suppose X_t is a Hunt process. We know by §§3 and 4 that S_λ is the resolvent of a semigroup Q_t on \mathcal{L} .

I. *We assume Q_t can be extended to a semigroup Q_t on \mathcal{K} .*

Let Y_t be a Markov process that has Q_t as its semigroup.

II. *We assume there is a version of Y_t which is a Hunt process.*

By [2], Y_t has a Lévy system (m, dH_t) .

III. *We assume that we can suppose that H_t is absolutely continuous with respect to Lebesgue measure for all ω .*

As a consequence of III, we may suppose that Y_t has a Lévy system of the form (m, dt) (see proof of Proposition 5.1 below).

I, II, and III can be shown to hold under fairly general conditions, in particular when B is bounded, but the proofs are long. See [1]. In some cases, however, it is easy to verify I, II, and III. For example, it is well known that if \mathcal{L} is the collection of bounded continuous functions, I and II must hold. A

simple condition that guarantees III, much stronger than is necessary, is the following.

PROPOSITION 5.1. *Suppose that given any pair of disjoint compact sets F_1 and F_2 , there exists an $f \in \mathcal{K}$ (depending on F_1 and F_2) such that $R_\lambda f > 0$ and $R_\lambda f$ is 0 on F_1 and bounded away from 0 on F_2 . Then III holds.*

PROOF. If F_1 and F_2 are two disjoint compact sets, let $f \in \mathcal{K}$ such that $R_\lambda f > 0$ and $R_\lambda f$ is 0 on F_1 and bounded away from 0 on F_2 , say by δ .

By Lemma 3.2 or its counterpart from the subtracting jumps case, $R_\lambda f = S_\lambda h$ for some $h \in \mathcal{K}$. Let $T = \inf\{t > 0: Y_{t-} \in F_1, Y_t \in F_2\}$. Let $U = \inf\{t > 0: Y_t \in F_1\}$. By Dynkin's identity,

$$\begin{aligned} \delta P^x(T \leq t) &\leq E^x S_\lambda h(Y_{T \wedge t}) - E^x S_\lambda h(Y_{U \wedge t}) \\ &= E^x \int_{U \wedge t}^{T \wedge t} \lambda(\lambda S_\lambda - I)h(Y_s) ds \leq 2\lambda \|h\| t. \end{aligned}$$

Let $T_0 = 0, T_{i+1} = T_i + T \circ \theta_{T_i}$, the $(i + 1)$ st time Y_t jumps from F_1 to F_2 . An induction argument and the strong Markov property gives $P^x(T_i < t) \leq (ct)^i$, where $c = 2\lambda \|h\|/\delta$. Hence if $t < 1/(2c)$,

$$E^x \sum_{s < t} 1_{(Y_{s-} \in F_1, Y_s \in F_2)} \leq \sum_{i=0}^{\infty} iP^x(T_i < t) \leq 4ct.$$

Repeating the construction of Benveniste and Jacod, one can see that one can take their additive functional H_t so that $E^x H_t \leq c't$ for some constant c' . By [11, Lemma 6.6], $H_t \leq c't$ a.s.; hence $H_t = \int_0^t h(Y_u) du$ for some h , bounded by c' by Proposition 5.3 below. If we let $m'(x, dy) = h(x)m(x, dy)$, it follows that (m', dt) is a Lévy system for Y_t .

Since it should be clear from the context which we mean, we are using P^x, E^x to refer to probabilities and expectations for both X_t and Y_t .

Note that in Example 3.7, $R_\lambda(K)$ contains the collection of twice continuously differentiable functions with compact support, and hence satisfies Proposition 5.1.

We now want to show that if X_t is continuous and we added jumps, Y_t has Lévy system (n, dt) and that if X_t has Lévy system (n, dt) and we subtracted jumps, Y_t is continuous. We first prove some lemmas. Let us say $L_n \ll L_m$, where L_n and L_m are Lévy operators if: whenever G is an open set, g a continuous bounded function ≥ 0 with support contained in G^c , $1_G(x)L_n g(x) \leq 1_G(x)L_m g(x)$ for all x except possibly for a set of potential 0.

LEMMA 5.2. *A set D has potential 0 with respect to X_t if and only if it has potential 0 with respect to Y_t .*

PROOF. We have $R_\lambda 1_D = 0$. Hence $S_\lambda 1_D = R_\lambda 1_D \pm S_\lambda B R_\lambda 1_D = 0$. The "if" part follows by symmetry.

PROPOSITION 5.3. *If Z_t is any Hunt process, $g \geq 0$, either $f \geq 0$ or f bounded, and*

$$E^x \int_0^t f(Z_u) du \leq E^x \int_0^t g(Z_u) du < \infty$$

for all x and t , then $f \leq g$ except for a set of Z -potential 0.

PROOF. By Proposition 2.8 of [11],

$$E^x \int_T^U f(Z_u) du \leq E^x \int_T^U g(Z_u) du$$

for all bounded stopping times $T \leq U$. Let $\varepsilon > 0$, K any compact contained in $\{x: f(x) > g(x) + \varepsilon\}$, F_i compact sets increasing to K^c such that F_i is disjoint from K . Let

$$\begin{aligned} T_{i1} &= \inf\{t: Z_t \in K\}, & U_{i1} &= \inf\{t > T_{i1}: Z_t \in F_i\}, \\ T_{i2} &= \inf\{t > U_{i1}: Z_t \in K\}, & U_{i2} &= \inf\{t > T_{i2}: Z_t \in F_i\}, \end{aligned}$$

etc. By quasi-left continuity, $T_{ij} \rightarrow \infty$, $U_{ij} \rightarrow \infty$ as $j \rightarrow \infty$. If L is any real,

$$\begin{aligned} \sum_{j=1}^{\infty} E^x \int_{T_{ij} \wedge L}^{U_{ij} \wedge L} f(Z_u) du &= E^x \int_0^L f(Z_u) 1_{(U \in [T_{ij}, U_{ij}] \text{ for some } j)} du \\ &\rightarrow E^x \int_0^L f(Z_u) 1_K(Z_u) du \end{aligned}$$

as $i \rightarrow \infty$ by dominated convergence.

Combining with a similar equation for g , we have

$$E^x \int_0^L f(Z_u) 1_K(Z_u) du \leq E^x \int_0^L g(Z_u) 1_K(Z_u) du.$$

We must have that K has potential 0, and since L was arbitrary, the result follows.

LEMMA 5.4. *Suppose Z_t is a Hunt process with Lévy system (m, dt) , Lévy operator L_m , and resolvent V_λ . Suppose G is an open set, g is a nonnegative, bounded continuous function with support in G^c . Suppose $\lambda V_\lambda g \rightarrow g$ weakly as $\lambda \rightarrow \infty$. Suppose $\{g_\lambda\}$ is a collection of functions that converge weakly to g as $\lambda \rightarrow \infty$ and that D is a bounded operator such that $Dg_\lambda \rightarrow Dg$ weakly. Suppose T and U are two bounded stopping times such that $T(\omega) < t < U(\omega)$ implies $Z_t(\omega) \in G$. Then*

$$E^x \int_T^U \lambda^2 V_\lambda g(Z_t) dt \rightarrow E^x \int_T^U L_m g(Z_t) dt < \infty; \quad (1)$$

$$E^x \int_T^U \lambda V_\lambda Dg_\lambda(Z_t) dt \rightarrow E^x \int_T^U Dg(Z_t) dt. \quad (2)$$

PROOF. (1) By the hypotheses on g , T , and U ,

$$\begin{aligned} E^x \int_T^U \lambda^2 V_\lambda g(Z_t) dt &= E^x \int_T^U \lambda(\lambda V_\lambda - I)g(Z_t) dt \\ &= E^x \lambda V_\lambda g(Z_U) - E^x \lambda V_\lambda g(Z_T) \\ &\rightarrow E^x g(Z_U) - E^x g(Z_T) = E^x \sum_{T < t < U} [g(Z_t) - g(Z_{t-})] \\ &= E^x \int_T^U L_m g(Z_t) dt. \end{aligned}$$

(2)

$$\begin{aligned} E^x \int_U^T \lambda V_\lambda Dg_\lambda(Z_t) dt &= E^x V_\lambda Dg_\lambda(Z_u) - E^x V_\lambda Dg_\lambda(Z_t) + E^x \int_T^U Dg_\lambda(Z_t) dt \\ &\rightarrow E^x \int_T^U Dg(Z_t) dt. \end{aligned}$$

LEMMA 5.5. Suppose Z_t is a Hunt process with resolvent V_λ , Lévy system (m, dt) and Lévy operator L_m . Let D be a bounded operator such that whenever $\{f_n\}$ converges weakly to f , Df_n converges weakly to Df . Suppose that W_λ is a collection of positive linear operators such that $W_\lambda = V_\lambda(I - DW_\lambda)$, $\|\lambda W_\lambda\| < 1$, and $\lambda W_\lambda g$ converges weakly to g as $\lambda \rightarrow \infty$ whenever g is bounded and continuous. Then $L_m \gg D$.

PROOF. Let G be open, g nonnegative, bounded, and continuous with support in G^c . Suppose T and U are stopping times such that if $T(\omega) < t < U(\omega)$, $Z_t(\omega) \in G$. Since $W_\lambda g \geq 0$,

$$E^x \int_T^U \lambda^2 V_\lambda g(Z_t) dt \geq E^x \int_T^U \lambda V_\lambda D(\lambda W_\lambda g)(Z_t) dt.$$

Letting $\lambda \rightarrow \infty$, by Lemma 5.4, we get

$$\infty > E^x \int_T^U L_m g(Z_t) dt \geq E^x \int_T^U Dg(Z_t) dt.$$

Since $\|Dg\| < \infty$, the result now follows by an argument very similar to Lemma 5.3.

LEMMA 5.6. Suppose m and n are two kernels, L_m and L_n their Lévy operators, and $L_m \gg L_n$. Then $m \gg n$.

PROOF. Suppose $\{G_i\}$ is a countable open basis for the topology of E . For each G_i , select a countable dense subset $\{g_{ij}\}$ of the positive continuous functions with support in G_i^c . If

$$N = \{x: 1_{G_i}(x)L_m g_{ij}(x) < 1_{G_i}(x)L_n g_{ij}(x) \text{ for some } i, j\},$$

N has potential 0. Note that if $x \in G_i$, $L_m g_{ij}(x) = m(x, g_{ij})$ and similarly for

L_n . A monotone class argument shows that if F is any Borel set and $x \notin N$, $m(x, F \cap G_i^c) > n(x, F \cap G_i^c)$, for each i . It follows that $m \geq n$.

THEOREM 5.7. *Suppose X_t, R_λ, n, B , and S_λ are as in §4. Suppose B is the Lévy operator of n . Suppose n_j is a sequence of bounded kernels that increase strongly to n such that if L_{n_j} is the Lévy operator of n_j , $L_{n_j}R_\lambda g \rightarrow BR_\lambda g$ for all $g \in \mathcal{K}$. Then Y_t is continuous a.s.*

PROOF. Suppose Y_t has Lévy kernel $m > 0$ with Lévy operator L_m . Let m_k be bounded kernels strongly increasing to m , L_{m_k} the Lévy operators. If $V_\lambda = S_\lambda(\sum_{i=0}^{\infty} (-L_{m_k} S_\lambda)^i)$, V_λ is positive by Theorem 4.1 and the proof of Theorem 4.2. Now let $W_\lambda = V_\lambda(\sum_{i=0}^{\infty} ((B - L_{n_j}) V_\lambda)^i)$. Since $V_\lambda = S_\lambda(I - L_{m_k} V_\lambda)$, BV_λ will have norm < 1 if λ is large enough; hence for large enough λ so will $(B - L_{n_j})V_\lambda$. Since for all $f \in \mathcal{K}$, $W_\lambda f = R_\lambda h$ for some h (Lemma 2.2), $(L_{n_j} - L_{n_j})W_\lambda f \rightarrow (B - L_{n_j})W_\lambda f$. Hence by Theorem 3.4, W_λ is positive. By I and II (valid in this case by [1]), W_λ is the resolvent of a Hunt process, hence $\lambda W_\lambda g \rightarrow g$ weakly whenever g is continuous.

By Lemma 2.2, $W_\lambda = R_\lambda(I - (L_{n_j} + L_{m_k})W_\lambda)$. By Lemma 5.5, $B \gg L_{n_j} + L_{m_k}$. Since if $g(x) = 0$, $g \geq 0$, $L_{n_j}g(x) \uparrow n_j(x, g) \uparrow n(x, g) = Bg(x)$, letting $j \rightarrow \infty$ gives $B \gg B + L_{m_k}$. Similarly, letting $k \rightarrow \infty$, we get $0 \gg L_m$, or $m = 0$ by Lemma 5.6. Hence the expected number of jumps in finite time is 0 by the Lévy system identity, i.e., Y_t is continuous a.s.

Finally we show that if X_t is continuous and we added jumps, Y_t has Lévy system (n, dt) .

THEOREM 5.8. *Suppose X_t, n, R_λ, B , and S_λ are as in §3. Suppose B is the Lévy operator for n . Suppose there exist bounded kernels n_j strongly increasing to n with Lévy operator L_{n_j} such that $L_{n_j}S_\lambda g \rightarrow BS_\lambda g$ for all $g \in \mathcal{K}$, then Y_t has Lévy system (n, dt) .*

Suppose Y_t has Lévy system m , Lévy operator L_m . We show $m \geq n$. Let

$$V_\lambda = R_\lambda \left(\sum_{i=0}^{\infty} ((B - L_{n_j})R_\lambda)^i \right).$$

As in Theorem 5.7, V_λ is positive. By Lemma 2.2, $V_\lambda = S_\lambda(I - L_{n_j}V_\lambda)$. Hence $L_m \gg L_{n_j}$. Since $L_{n_j}g(x) \uparrow Bg(x)$ if $g(x) = 0$, letting $j \rightarrow \infty$ gives $L_m \gg B$. By Lemma 5.6, $m \geq n$.

Next we must show $m \leq n$. Let m_k be bounded kernels strongly increasing to m , L_{m_k} the Lévy operators. Let j be fixed. Again letting $V_\lambda = R_\lambda(\sum_{i=0}^{\infty} ((B - L_{n_j})R_\lambda)^i)$, we know V_λ is positive. Let p be the Lévy kernel for the process with resolvent V_λ , L_p the Lévy operator. We first show $L_p = L_m - L_{n_j}$.

If $T_\lambda = S_\lambda(\sum_{i=0}^{\infty} (-L_{m_k} S_\lambda)^i)$, T_λ is positive since Y_t has Lévy kernel $m > m_k$.

$$T_\lambda = R_\lambda(I + (B - L_{m_k})T_\lambda) = R_\lambda(I + ((B - L_{n_j}) + (L_{n_j} - L_{m_k}))T_\lambda),$$

from which it follows that $T_\lambda = V_\lambda((I + L_{n_j} - L_{m_k})T_\lambda)$. By Lemma 5.5, $L_p \gg L_{m_k} - L_{n_j}$. Letting $k \rightarrow \infty$, $L_p \gg L_m - L_{n_j}$.

On the other hand, let p_r be a sequence of kernels increasing strongly to p , with Lévy operators L_{p_r} . If $U_\lambda = V_\lambda(\sum_{i=0}^\infty (-L_{p_r} V_\lambda)^i)$, U_λ is positive, and

$$U_\lambda = V_\lambda(I - L_{p_r} U_\lambda) = S_\lambda(I - (L_{p_r} + L_{n_j})U_\lambda).$$

Hence $L_m \gg L_{p_r} + L_{n_j}$. Letting $r \rightarrow \infty$, $L_p \ll L_m - L_{n_j}$. Hence $p = m - n_j$.

We know $n_j(x, dy) = 1_{H_j}(x, y)n(x, dy)$ for sets H_j , $m_k(x, dy) = 1_{F_k}(x, y)m(x, dy)$ for sets F_k . Let

$$q_{jk}(x, dy) = 1_{F_k}(x, y)m(x, dy) - 1_{F_k}(x, y)n_j(x, dy),$$

$s_{jk}(x, dy) = 1_{F_k \cap H_j^c}(x, y)n(x, dy)$, $L_{q_{jk}}$, $L_{s_{jk}}$ the associated Lévy operators. Since $n_j \leq n \leq m$, $q_{jk} \geq 0$. But $q_{jk} \leq m_k$, hence q_{jk} is bounded. $s_{jk}(x, dy) \leq 1_{F_k}(x, dy)m(x, dy)$, hence $s_{jk} \leq m_k$, and s_{jk} is also bounded. Note also that $(q_{jk} - s_{jk})(x, dy) = 1_{F_k}(x, y)(m - n)(x, dy)$, which is independent of j .

Since $q_{jk} \leq m - n_j$, the Lévy kernel for the process with resolvent V_λ ,

$$W_\lambda = V_\lambda\left(\sum_{i=0}^\infty (-L_{q_{jk}} V_\lambda)^i\right)$$

is positive by §4. Then $Z_\lambda^j = W_\lambda(\sum_{i=0}^\infty (L_{s_{jk}} W_\lambda)^i)$ is positive by Theorem 3.4. If we denote $L_{q_{jk}} - L_{s_{jk}}$ simply by $L_{(k)}$, $Z_\lambda^j = V_\lambda(I - L_{(k)}Z_\lambda^j)$, hence

$$Z_\lambda^j = R_\lambda(I + (B - L_{n_j} - L_{(k)})Z_\lambda^j).$$

Letting $j \rightarrow \infty$, we get $Z_\lambda = R_\lambda(\sum_{i=0}^\infty (-L_{(k)}R_\lambda)^i)$ is positive. $Z_\lambda = R_\lambda(I - L_{(k)}Z)$. Since X_t is continuous, the Lévy operator of X_t is 0, and by Lemma 5.5, $0 \gg L_{(k)}$. Letting $k \rightarrow \infty$, by Lemma 5.6 we conclude that $m - n \leq 0$.

6. A probabilistic construction. It would be nice to have a probabilistic construction of Y_t . If $n(x, E)$ is finite and small enough, one exists. In general, as in Example 3.7, none is known.

Suppose X_t has finitely many jumps in finite time. Kill X_t at the time T of the first jump, then “restart” it with distribution X_{T-} . Proceed until the first jump of the new process and kill again, etc. “Restarting” can be made precise through the work of Ikeda, Nagasawa, and Watanabe [6] or Meyer [10]. See also Stroock [13] in the case n is bounded. That this pieced together process is the same as the process constructed by the methods of this paper follows readily from [12]. If X_t has infinitely many jumps in finite time, as when $n(x, E)$ is infinite, the time of the first jump will, in general, be identically equal to 0, and this method will fail.

To add jumps, kill the process according to the multiplicative functional $\exp(-n(X_t, E))$. At the time of death T , “restart” it with distribution $n(X_{T-}, dy)/n(X_T, E)$. Again it is not too difficult to check that this new process is the Y_t one would have constructed through the methods of this

paper. Again, however, if $n(x, E)$ is infinite, this method will fail since $\exp(-n(X, E))$ would be identically 0.

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