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Stochastic Integral Representations for Multiparameter Random Fields with Stationary Independent Increments

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Let $\{Z_t, t \in [0,1] \times [0,1]\}$ be a two-parameter random field with stationary, independent increments. A stochastic integral representation is given for all square-integrable random variables adapted to the sigma fields of Z_t.

INTRODUCTION

It is by now well understood how to represent the martingales adapted to the sigma fields generated by a process with stationary, independent increments. Here we extend these results to two-parameter random fields with stationary, independent increments. Extensions to n-parameter fields, $n > 2$, can be achieved by the same methods: the number of types of stochastic integrals necessary, however, increases with n.

The case of the two-parameter Poisson process has been considered by Mazziotto and Szpirglas [5] and Yor [7]. While this paper was under review, a paper by Vares [6] appeared in which, independently and by different methods, results very similar to ours were obtained.

We should point out that not every proof of the one-parameter result, e.g., proofs relying on uniqueness in law, can be extended readily to the multiparameter case. Also, the multiparameter version of Ito's lemma is so cumbersome that we prefer to avoid its use here altogether.

We consider first the case where the random field has no Gaussian component. We do this in Section 1; in Section 2 we treat the general case.

1. FIELDS WITHOUT A GAUSSIAN COMPONENT

First, some notation. Let $T=[0,1]\times[0,1]$. If $s=(s_1,s_2)$, $t=(t_1,t_2)\in T$, we write $s\leqq t$ if $s_1\leqq t_1$ and $s_2\leqq t_2$, $s\vee t$ for $(s_1\vee t_1, s_2\vee t_2)$, and $](s_1,s_2),(t_1,t_2)]$ for the rectangle

$$A=\{(u_1,u_2)\in T: s_1<u_1\leqq t_1, s_2<u_2\leqq t_2\}.$$

We define lef(A), rig(A), bot(A), and top(A) to be s_1, t_1, s_2, and t_2, respectively, and we define enl$(A)=](0,0)$, $(\text{rig}(A), \text{top}(A)]-](0,0)$, $(\text{lef}(A), \text{bot}(A))]$. Let Leb$(dt)$ be usual Lebesgue measure. If $0=a_0<a_1<\cdots<a_n=1$, $0=b_0<b_1<\cdots<b_m=1$, let the partition generated by $\{a_0,\ldots,a_n,b_0,\ldots,b_m\}$ be the collection of rectangles

$$\{](a_{i-1},b_{j-1}),(a_i,b_j)]: i=1,\ldots,n, j=1,\ldots,m\}.$$

Let $(\Omega,\mathscr{F},P)$ be a probability space. If $\{X_t, t\in T\}$ is any random field and $A=](s_1,s_2),(t_1,t_2)]$ define $X(A)=X_{(t_1,t_2)}+X_{(s_1,s_2)}-X_{(s_1,t_2)}-X_{(s_2,t_1)}$. We say X is *RCLL* if, except for a set of ω's in a null set of probability 0,

$$\lim_{t\geqq s, t\to s} X_t(\omega)=X_s(\omega),$$

$$\lim_{(t_1,t_2)\to(s_1,s_2), t_1<s_1, t_2<s_2} X_{(t_1,t_2)}(\omega)$$

exists (call the limit $X_{s-}(\omega)$),

$$\lim_{(t_1,t_2)\to(s_1,s_2), t_1<s_1, t_2\geqq s_2} X_{(t_1,t_2)}(\omega)$$

exists (call the limit $X_{s-+}(\omega)$), and

$$\lim_{(t_1,t_2)\to(s_1,s_2), t_1\geqq s_1, t_2<s_2} X_{(t_1,t_2)}(\omega)$$

exists (call the limit limit $X_{s+-}(\omega)$). Let

$$\Delta X_s = X_s + X_{s-} - X_{s+-} - X_{s-+}.$$

If $\{\mathscr{F}_t, t \in T\}$ is a collection of sigma fields with $\mathscr{F}_s \subset \mathscr{F}_t$ if $s \leqq t$, a field $\{M_t, t \in T\}$ is a martingale if $E[M_t \mid \mathscr{F}_s] = M_s$, a.s. when $s \leqq t$.

By X is orthogonal to Y, we mean the usual definition $EXY=0$.

Let Z_t be a $RCLL$ field with stationary, independent increments: (i) $Z(A_1), \ldots, Z(A_n)$ are mutually independent whenever $A_1, \ldots, A_n$ are pairwise disjoint rectangles, and (ii) $E\exp(iuZ(A)) = \exp(\psi(u)$ Leb(A)), where

$$\psi(u) = iau - bu^2/2 + \int_{|x| \leqq 1} (e^{iux} - 1 - iux)m(dx) + \int_{|x|>1} (e^{iux} - 1)m(dx).$$

For the path properties of fields with independent increments, see [4].

In this section we suppose that Z has no Gaussian component, that is, $b=0$. We will also suppose that the jumps ΔZ_t are bounded, or equivalently, m has compact support. There is no real loss of generality in doing this: for if f is a bounded, continuous, strictly increasing, odd function with $f(x)=x$ if $|x| \leqq 1$, and if

$$Z'_t = Z_t - \sum_{\substack{s \leqq t \\ |\Delta Z_s| \geqq 1}} \Delta Z_s + \sum_{\substack{s \leqq t \\ |\Delta Z_s| \geqq 1}} f(\Delta Z_s),$$

then Z'_t will be a $RCLL$ field with stationary, independent increments and bounded jumps that generates the exact same sigma fields as Z_t does. In particular, it is known that Z'_t has moments of all orders.

We finally suppose $EZ_t = 0$ for all t. Again, there is no loss of generality, since $Z'_t = Z_t - EZ_t$ generates the same sigma fields as Z_t.

Let $\mathscr{F}_t$ be the P-completion of $\sigma(Z_s : s \leqq t)$, $\mathscr{F}_A$ the P-completion of $\sigma(Z_B$: B a rectangle contained in $A)$. We could define $\mathscr{F}_{t-}$ (and $\mathscr{F}_{A-}$), but since $P(\Delta Z_s \neq 0$, some s in the boundary of $]0, t]) = 0$, it follows that $Z_s \in \mathscr{F}_{t-}$ if s is in the boundary of $]0, t]$, and we would then have $\mathscr{F}_{t-} = \mathscr{F}_t$.

It may be of some interest to note that the commutation hypothesis $F4$ of [2] is satisfied by the $\mathscr{F}_t$'s. That is, if $r = (r_1, r_2)$,

$s=(r_1, 1)$, and $t=(1, r_2)$, and X integrable and in $\mathscr{F}_{(1,1)}$, then

$$E[E[X\mid\mathscr{F}_s]\mid\mathscr{F}_t]=E[E[X\mid\mathscr{F}_t]\mid\mathscr{F}_s]=E[X\mid\mathscr{F}_r].$$

Let π be a partition of T into disjoint rectangles A_j such that each A_j is wholly contained in either $](0,0),r]$, $]r,(1,1)]$, $](0,r_2),s]$, or $](r_1,0),t]$. Then by independent increments and the definition of Z,

$$E\left[E\left[\prod_{A_j\in\pi}\exp(iu_jZ(A_j)-\psi(u_j)\operatorname{Leb}(A_j))\middle|\mathscr{F}_s\right]\middle|\mathscr{F}_t\right]$$

$$=E\left[\prod_{A_j\subset]0,s]}\exp(iu_jZ(A_j)-\psi(u_j)\operatorname{Leb}(A_j))\middle|\mathscr{F}_t\right]$$

$$=E\left[\prod_{A_j\subset]0,r]}\exp(iu_jZ(A_j)-\psi(u_j)\operatorname{Leb}(A_j))\middle|\mathscr{F}_r\right],$$

and similarly for $E[E[\cdot\mid\mathscr{F}_t]\mid\mathscr{F}_s]$ and $E[\cdot\mid\mathscr{F}_r]$. Thus the commutation hypothesis holds for random variables of the above product form. Taking linear combinations and limits, we get the commutation hypothesis for all integrable X in $\mathscr{F}_{(1,1)}$.

We are now ready to define stochastic integrals with respect to Z. Define (random) measures on $\Omega\times T\times\mathbb{R}$ by

$$\mu(\omega,A,I)=\sum_{s\in A}1_I(\Delta Z_s),$$

$$\nu(\omega,A,I)=m(I)\operatorname{Leb}(A),$$

where A is a rectangle in T, I an interval in $\mathbb{R}$.

Call a process U elementary if $U(\omega,r,x)=\alpha(\omega)1_A(r)1_I(x)$, where α is bounded and in $\mathscr{F}_{(\mathrm{lef}(A),\,\mathrm{bot}(A))}$, A is a rectangle, and I in an interval in $\mathbb{R}$ that is a strictly positive distance from 0.

Define $\mathrm{S}_1(U,t)=\alpha(\omega)(\mu-\nu)(\omega,A\cap]0,t],I)$.

Thus $\mathrm{S}_1(U,t)$ is just the usual Lebesgues-Stieltjes integral calculated path by path. It is immediate that $\mathrm{S}_1(U,t)$ is $RCLL$ and a martingale with respect to $\mathscr{F}_t$. Also, using independent increments and recalling that the variance of a Poisson random variable with parameter λ is just λ,

$$ES_1(U,t)^2 = E\alpha^2 E\left[\sum_{r\in A\cap]0,t]} 1_I(\Delta Z_r) - m(I)\,\mathrm{Leb}(A)\cap]0,t])\right]^2$$

$$= E\alpha^2 m(I)\,\mathrm{Leb}(A\cap]0,t])$$

$$= E\int_{]0,t]} U^2\,dv.$$

(The usual Lévy-Khintchine decomposition holds for Z. This may be seen by considering the 1-parameter process $R_s = Z_{(s^{1/2}t_1,\, s^{1/2}t_2)}$.)

If U is a finite linear combination of elementary processes, define $S_1(U,t)$ by linearity. Again $S_1(U,t)$ is a martingale, and using the fact that $\Sigma 1_I(\Delta Z_r)$ and $\Sigma 1_J(\Delta Z_r)$ are independent if $I\cap J=\emptyset$,

$$ES_1(U,t)^2 = E\int_{]0,t]} U^2\,dv.$$

Let $\mathscr{K}_1$ be the L_2 closure of the set of finite linear combinations of elementary processes with respect to the measure $E\int_T \cdot\, dv$. For $U\in\mathscr{K}_1$ define $S_1(U,(1,1))$ as an $L_2(P)$ limit, and $S_1(U,t)$ by $E[S_1(U,(1,1))\,|\,\mathscr{F}_t]$. By the maximal inequality of Cairoli [1] $S_1(U,t)$ can be taken to be *RCLL*. Again, $S_1(U,t)$ will be a martingale with second moment $E\int_{]0,t]} U^2\,dv$.

We also need a stochastic integral corresponding to the Type 2 integrals of [3]. Call $U(\omega,r,x,s,y)$ elementary if

$$U(\omega,r,x,s,y) = \alpha(\omega)1_A(r)1_I(x)1_B(s)1_J(y),$$

where α is bounded and in $\mathscr{F}_{(\mathrm{lef}(B),\,\mathrm{bot}(A))}$, I and J are intervals in $\mathbb{R}$ that are both a positive distance from 0, and A and B are rectangles such that $\mathrm{bot}(A)\geqq \mathrm{top}(B)$ and $\mathrm{rig}(A)\leqq \mathrm{lef}(B)$.

Define $S_2(U,t) = \alpha(\omega)(\mu-\nu)(\omega, A\cap]0,t], I)(\mu-\nu)(\omega, B\cap]0,t], J)$. Using independent increments, $S_2(U,t)$ can be shown to be a martingale and $ES_2(U,t)^2 = E\int_{]0,t]} U^2\,dv\,dv$ (see the analogous statement in [3].)

Extend the definition of S_2 to linear combination of elementary processes by linearity, and by an $L_2(P)$ limit to processes in $\mathscr{K}_2$, the

L_2 closure of the set of finite linear combinations of elementary processes with respect to the measure $E\int_T \cdot\, dv\, dv$. Note that if $U \in \mathscr{K}_2$, $U(\omega, r, x, s, y) = 0$ if $r \leqq s$ or $s \leqq r$, since the same is true for every elementary process.

If $U_1 \in \mathscr{K}_1, U_2 \in \mathscr{K}_2$, then $\mathrm{S}_1(U_1, t_1)$ is orthogonal to $\mathrm{S}_2(U_2, t_2)$. This may be proved by first demonstrating orthogonality when U_1 and U_2 are elementary, and then taking limits.

We will write $\mathrm{S}_1(U)$ for $\mathrm{S}_1(U, 1, 1))$ and $\mathrm{S}_2(U)$ for $\mathrm{S}_2(U, (1, 1))$.

THEOREM 1 *Suppose Z_t is a RCLL field with stationary, independent increments, mean 0, bounded jumps, and no Gaussian component. If $X \in \mathscr{F}_{(1,1)}$ and $EX^2 < \infty$, then $X = EX + \mathrm{S}_1(U_1) + \mathrm{S}_2(U_2)$, a.s. for some $U_1 \in \mathscr{K}_1, U_2 \in \mathscr{K}_2$.*

Before proving Theorem 1, we first need some lemmas. A *RCLL* martingale M_t is continuous if $P(\Delta M_t \neq 0$, some $t \in T) = 0$. M_t has paths of bounded variation if $\sup_\pi(\sum_i |M(A_i)|: A_i \in$ the partition π of $T) < \infty$, the sup being over all partitions of T.

LEMMA 1 *If M_t is a continuous martingale with paths of bounded variation and $M_0 = 0$, then $M_t = 0$ a.s.*

Proof Fix $t = (t_1, t_2) \in T$, and define the 1-parameter martingale $\{N_s, 0 \leqq s \leqq 1\}$ by $N_s = M_{(st_1, st_2)}$. N is easily seen to be continuous.

We next show that N_s has paths of bounded variation. Let

$$0 = s_0 < s_1 < \cdots < s_n = 1.$$

Let

$$A_{i1} =](0, s_i t_2), (s_i t_1, s_{i+1} t_2)],\ A_{i2} =](s_i t_1, s_i t_2), (s_{i+1} t_1, s_{i+1} t_2)],$$

and

$$A_{i3} =](s_i t_1, 0), (s_{i+1} t_1, s_i t_2)].$$

Then

$$|N_{s_{i+1}} - N_{s_i}| \leqq |M(A_{i1})| + |M(A_{i2})| + |M(A_{i3})|$$

The A_{i1}'s, A_{i2}'s, A_{i3}'s, $i = 0, \ldots, n-1$ are all mutually disjoint; let π^0 be the smallest partition of T into rectangles that contain all of the

A_{i1}'s, A_{i2}'s, and A_{i3}'s. Then

$$\sum_{i=0}^{n-1} |N_{s_{i+1}} - N_{s_i}| \leq \sum_{B_j \in \pi^0} |M(B_j)|.$$

Finally, since N is a continuous martingale with paths of bounded variation, by the 1-parameter theorem, $N_1 = 0$. □

Let

$$Y_t^\varepsilon = \sum_{\substack{r \in]0,t] \\ |\Delta Z_r| \geq \varepsilon}} \Delta Z_r - \int_{(-\varepsilon,\varepsilon)^c} x m(dx) \operatorname{Leb}(]0,t])$$

LEMMA 2 *Let t be fixed. Then $\exp(iuY_t^\varepsilon) = c + \mathrm{S}_1(U,t)$, a.s., where c is a constant, $U \in \mathcal{K}_1$, $U(\cdot,s,x) \in \mathcal{F}_t$, and $U(\omega,\cdot,x)$ is supported on $]0,t]$ (i.e., $U(\omega,s,x)=0$ if $s \notin]0,t]$).*

Proof We will write Y instead of Y^ε. Let

$$V(\omega,r,x) = \exp(-\gamma \operatorname{Leb}(]0,r] + iuY_{r-})(e^{iux}-1)1_{(\varepsilon,\infty)}(|x|),$$

where $\gamma = \int_{(-\varepsilon,\varepsilon)^c}(e^{iux} - 1 - iux)m(dx)$. Since $V(\omega,r,x)=0$ if $|x|<\varepsilon$ and Z has only finitely many jumps greater than ε, a.s., it is easy to check that $N_s = \mathrm{S}_1(V,s)$ is equal, in this case, to a Lebesgues-Stieltjes integral, and that

$$N_s = \sum_{r \in]0,s]} \exp(-\gamma \operatorname{Leb}(]0,r]) + iuY_{r-})(e^{iu\Delta Y_r} - 1)$$

$$- \int_{]0,s]} \exp(-\gamma \operatorname{Leb}(]0,r]) + iuY_{r-}) \int_{(-\varepsilon,\varepsilon)^c} (e^{iux}-1)m(dx) \operatorname{Leb}(dr).$$

N_s has paths of bounded variation since Y only has finitely many jumps, a.s.

Let $M_s = \exp(iuY_s - \gamma \operatorname{Leb}(]0,s]) - 1$. M_s will be a martingale since Y has independent increments, and M_s is *RCLL* and of bounded variation since Y is.

Since the locations of the jumps of Y form a Poisson process, it is easy to see that P(for some $s, \Delta Y_s \neq 0$ but $Y_{s+-} \neq Y_{s-}$ or $Y_{s-+} \neq Y_{s-}$) $=0$. Then $\Delta Y_s = (Y_s - Y_{s-})1_{(\Delta Y_s \neq 0)}$, a.s., and hence $\Delta N_s = \Delta M_s$. Then

$M_s - N_s$ is continuous, and hence by Lemma 1, $M_t - N_t = 0$. Then

$$\exp(iuY_t) = \exp(\gamma \operatorname{Leb}(]0,t])(1+N_t),$$

and the result follows by the linearity of S_1. □

Comment By stationarity, the same argument shows that $\exp(iuY^\varepsilon(A)) = c + S_1(U)$, where A is a rectangle, $U \in \mathscr{F}_A$, and $U(\omega,\cdot,x)$ is supported on A.

LEMMA 3 *Suppose* $X_n = c_n + S_1(U_n) + S_2(V_n)$, a.s., for $c_n \in \mathbb{R}$, $U_n \in \mathscr{K}_1$, $V_n \in \mathscr{K}_2$. *Suppose* $E|X_n - X|^2 \to 0$. *Then*

$$X = c + S_1(U) + S_2(V), \text{ a.s. } \textit{for some } U \in \mathscr{K}_1, V \in \mathscr{K}_2.$$

Proof By orthogonality of S_1 integrals to S_2 integrals and the fact that both types of stochastic integrals have 0 expectation,

$$E|X_n - X_m|^2 = (c_n - c_m)^2 + E\int (U_n - U_m)^2\, dv + E\int (V_n - V_m)^2\, dv\, dv.$$

By the completeness of L_2, there is a process $U \in \mathscr{K}_1$ such that $E\int (U_n - U)^2\, dv \to 0$. Similarly we find $V \in \mathscr{K}_2$ and $c \in \mathbb{R}$.

$$E|X - (c + S_1(U) + S_2(V))|^2 \leq 2E|X - X_n|^2$$
$$+ 2((c_n - c)^2 + E\int (U_n - U)^2\, dv + E\int (V_n - V)^2\, dv\, dv),$$

from which we get our result. □

LEMMA 4 *Suppose* G, H *are disjoint rectangles,*

$$M = c + S_1(U_1) + S_2(U_2), \text{ a.s.},$$

$$N = d + S_1(V_1) + S_2(V_2), \text{ a.s.}$$

where $U_1, V_1 \in \mathscr{K}_1$; $U_2, V_2 \in \mathscr{K}_2$; $U_1, U_2 \in \mathscr{F}_G$; $V_1, V_2 \in \mathscr{F}_H$; *and* $U_1(\omega,\cdot,x)$, $V_1(\omega,\cdot,x)$, $U_2(\omega,\cdot,x,\cdot,y)$, $V_2(\omega,\cdot,x,\cdot,y)$ *are supported on* G, H, $G \times G$, $H \times H$, *respectively. Then*

$$MN = cd + S_1(W_1) + S_2(W_2), \text{ a.s.}$$

for some

$$W_1 \in \mathscr{K}_1, W_2 \in \mathscr{K}_2; W_1, W_2 \in \mathscr{F}_{G \cup H};$$

W_1 *supported on* $G \cup H$; *and* W_2 *supported on* $(G \cup H) \times (G \cup H)$.

Proof If (i) $\mathsf{S}_1(U_1)\mathsf{S}_1(V_1)$; (ii) $\mathsf{S}_1(U_1)\mathsf{S}_2(V_2)$; (iii) $\mathsf{S}_2(U_2)\mathsf{S}_1(V_1)$; and (iv) $\mathsf{S}_2(U_2)\mathsf{S}_2(V_2)$ can all be represented in the desired form, then by linearity of S_1 and S_2 we will have our result. We will do (iv), the other cases being easier. We write U for U_2, V for V_2.

Suppose first that U and V are elementary: $U = \alpha 1_A(s) I_I(x) 1_B(t) 1_J(y)$, $V = \beta 1_C(s) 1_K(x) 1_D(t) 1_L(y)$, α, β bounded, $\text{bot}(A) \geqq \text{top}(B)$, $\text{rig}(A) \leqq \text{lef}(B)$, $\text{bot}(C) \geqq \text{top}(D)$, $\text{rig}(C) \leqq \text{lef}(D)$. Let $\{F_i\}$ be a partition of T into rectangles fine enough so that each of A, B, C, D, G, H may be written as a union of rectangles in $\{F_i\}$. Let δ be the mesh size of the partition.

If $M_t = E[\mathsf{S}_2(U) \mid \mathscr{F}_t] = \mathsf{S}_2(U, t)$, a calculation shows that

$$M(F_i) = \alpha(\mu - \nu)(\omega, A \cap \text{enl}(F_i), I)(\mu - \nu)(\omega, B \cap \text{enl}(F_i), J)$$

and similarly for $N(F_i)$. Then

$$\mathsf{S}_2(U)\mathsf{S}_2(V) = \Big(\sum_i M(F_i)\Big)\Big(\sum_j N(F_j)\Big)$$

$$= \sum_1 M(F_i)N(F_j) + \sum_2 M(F_i)N(F_j) + \sum_3 M(F_i)N(F_j)$$

$$+ \sum_4 M(F_i)N(F_j) + \sum_5 M(F_i)N(F_j) + \sum_6 M(F_i)N(F_j)$$

where $\sum_1$ is the sum over pairs i, j such that every point in F_i is $\leqq$ every point in F_j; $\sum_2$: every point in F_j is $\leqq$ every point in F_i; $\sum_3$: for every $s \in F_i, t \in F_j$, neither $s \leqq t$ nor $t \leqq s$; $\sum_4$: $i = j$; $\sum_5$ $\text{top}(F_i) = \text{top}(F_j)$, $\text{bot}(F_i) = \text{bot}(F_j)$, $\text{rig}(F_i) \leqq \text{lef}(F_j)$; and $\sum_6$: $\text{lef}(F_i) = \text{lef}(F_j)$, $\text{rig}(F_i) = \text{rig}(F_j)$, $\text{top}(F_i) \leqq \text{bot}(F_j)$.

If $M(F_i)\neq 0$, we have $F_i\subset G$, or $F_i\cap H=\emptyset$, or $N(F_i)=0$. So $\sum_4=0$.

$$\sum_1=\sum_j\Big(\sum_{i\in\Gamma(j)} M(F_i)N(F_j)\Big)$$

$$=\Big(\beta\sum_{i\in\Gamma(j)} M(F_i)\Big)(\mu-\nu)(\omega, C\cap \mathrm{enl}(F_j), K)(\mu-\nu)(\omega, D\cap \mathrm{enl}(F_j), L),$$

a stochastic integral of type S_2; here $\sum_{i\in\Gamma(j)}$ is the sum over all i such that every point of F_i is $\leqq$ every point of F_j. $\sum_2$ is similar.

Now, consider a term in $\sum_3$. Suppose that $\mathrm{bot}(F_i)\geqq \mathrm{top}(F_j)$, $\mathrm{rig}(F_i)\leqq \mathrm{lef}(F_j)$, the other case being treated similarly. If $A\cap \mathrm{enl}(F_i)=\emptyset$, $M(F_i)=0$, and the term is 0. The same is true if $D\cap \mathrm{enl}(F_j)=\emptyset$.

$$\mathrm{bot}(A\cap \mathrm{enl}(F_i))=\mathrm{bot}(F_i)\geqq \mathrm{top}(F_j)\geqq \mathrm{top}(D\cap \mathrm{enl}(F_j)),$$

and

$$\mathrm{rig}(A\cap \mathrm{enl}(F_i))\leqq \mathrm{rig}(F_i)\leqq \mathrm{lef}(F_j)=\mathrm{lef}(D\cap \mathrm{enl}(F_j)).$$

Similar calculations show that $B\cap \mathrm{enl}(F_i)$ and $C\cap \mathrm{enl}(F_j)$ are contained in $]0,(\mathrm{lef}(D\cap \mathrm{enl}(F_j)), \mathrm{bot}(A\cap \mathrm{enl}(F_i)))]$.

So,

$$M(F_i)N(F_j)=\gamma_{ij}(\mu-\nu)(\omega, A\cap \mathrm{enl}(F_i), I)(\mu-\nu)(\omega, D\cap \mathrm{enl}(F_j), L),$$

where

$$\gamma_{ij}=\alpha\beta(\mu-\nu)(\omega, B\cap \mathrm{enl}(F_i), J)(\mu-\nu)(\omega, C\cap \mathrm{enl}(F_j), K),$$

or $M(F_i)N(F_j)$ is a stochastic integral of type S_2. It is routine to check that γ_{ij} has its support in the appropriate set.

Next consider $\sum_5$. Suppose $\mathrm{top}(F_i)=\mathrm{top}(F_j)$, $\mathrm{bot}(F_i)=\mathrm{bot}(F_j)$, $\mathrm{rig}(F_i)\leqq \mathrm{lef}(F_j)$, $\mathrm{top}(F_l)=\mathrm{top}(F_n)$, $\mathrm{bot}(F_l)=\mathrm{bot}(F_n)$, and $\mathrm{rig}(F_l)\leqq \mathrm{lef}(F_n)$. First conditioning on $\mathscr{F}_{(1,\mathrm{bot}(F_j)\vee \mathrm{bot}(F_n))}$, and then using independent increments, $EM(F_i)N(F_j)M(F_l)N(F_n)=0$ unless $i=l$ and $j=n$. When $i=l$ and $j=n$, we have, using independent increments again,

$$EM(F_i)^2N(F_j)^2 \leqq \alpha^2\beta^2 E[(\mu-\nu)(A\cap \mathrm{enl}(F_i), I)(\mu-\nu)(B\cap \mathrm{enl}(F_i), J)$$
$$\times(\mu-\nu)(C\cap \mathrm{enl}(F_j), K)(\mu-\nu)(D\cap \mathrm{enl}(F_j), L)]^2$$
$$\leqq k^4\alpha^2\beta^2 \mathrm{Leb}(A\cap \mathrm{enl}(F_i))\mathrm{Leb}(B\cap \mathrm{enl}(F_i))$$
$$\times \mathrm{Leb}(C\cap \mathrm{enl}(F_j))\,\mathrm{Leb}(D\cap \mathrm{enl}(F_j)),$$
$$\leqq k^4\alpha^2\beta^2 \mathrm{Leb}(F_i)\,\mathrm{Leb}(F_j),$$

where k is a bound for $m(I\cup J\cup K\cup L)$.

For fixed j,

$$\sum_{\{i: i,j\in \sum_5\}} \mathrm{Leb}(F_i) \leqq \mathrm{Leb}(]0, \mathrm{bot}(F_j)), (1, \mathrm{top}(F_j)])$$
$$= \mathrm{top}(F_j) - \mathrm{bot}(F_j) \leqq \delta.$$

Then

$$E\left(\sum_5 M(F_i)N(F_j)\right)^2 = E\left(\sum_{i,j,l,n} M(F_i)N(F_j)M(F_l)N(F_n)\right)$$
$$= E\sum_5 M(F_i)^2N(F_j)^2$$
$$\leqq \sum_5 k^4\alpha^2\beta^2 \mathrm{Leb}(F_i)\,\mathrm{Leb}(F_j)$$
$$\leqq k^4\alpha^2\beta^2 \sum_j \mathrm{Leb}(F_j) \sum_{\{i: i,j\in\sum_5\}} \mathrm{Leb}(F_i)$$
$$\leqq k^4\alpha^2\beta^2\delta \sum_j \mathrm{Leb}(F_j)$$
$$\leqq k^4\alpha^2\beta^2\delta.$$

$\sum_6$ may be treated similarly. Since $E(\sum_5)^2$ and $E(\sum_6)^2 \to 0$ as the partition becomes finer,

$$S_2^*(U)S_2(V)-\sum_5-\sum_6\to S_2(U)S_2(V)$$

in L_2 as the partition becomes finer. But

$$S_2(U)S_2(V)-\sum_5-\sum_6=\sum_1+\sum_2+\sum_3+\sum_4,$$

which, we have shown, can be represented in the desired form. Using Lemma 3, we get that $S_2(U)S_2(V)$ can be represented in the desired form.

Now suppose U, V are not necessarily elementary. By the definition of $\mathscr{K}_2$, $ES_2(U)^2, ES_2(V)^2<\infty$. By the independence of $\mathscr{F}_G$ and $\mathscr{F}_H$,

$$E(S_2(U)S_2(V))^2=ES_2(U)^2ES_2(V)^2<\infty.$$

Approximating U and V by finite linear combinations of elementary processes gives our result by Lemma 3. □

Proof of Theorem 1 By Lemma 3, it suffices to show $\prod_j \exp(iu_jZ(F_j))$ can be represented in the desired form for each partition $\{F_j\}$ of T. Since $Y^\varepsilon(F_j)\to Z(F_j)$ in probability as $\varepsilon\to 0$ for each rectangle F_j, it suffices to show $\prod_j \exp(iu_jY^\varepsilon(F_j))$ can be represented in the desired form.

By Lemma 4 and an induction argument, it suffices to show $\exp(iu_jY^\varepsilon(F_j))$ is of the desired form for each j. But this follows by Lemma 2 and the comment immediately following it. □

2. FIELDS WITH A GAUSSIAN COMPONENT

Suppose cW_t+Z_t is a *RCLL* field with stationary, independent increments, where W_t is the Wiener field: a continuous Gaussian random field with stationary, independent increments, $EW(A)=0$, and $\mathrm{Cov}(W(A), W(B))=\mathrm{Leb}(A\cap B)$; W_t is independent of Z_t; and Z_t is a *RCLL* field with stationary, independent increments with no Gaussian component. If $\mathscr{G}_t$ is the completion of $\sigma(cW_s+Z_s; s\leq t)$, we will represent all $\mathscr{G}_{(1,1)}$-measurable square-integrable random variables as stochastic integrals. There is no loss of generality in assuming $c=1$, and as in Section 1, we may assume Z_t has bounded jumps and mean 0.

To state our representation, we need three more types of stochastic integrals. (Types 3 and 4 are merely the type 1 and type 2 integrals of [3].) If $U(\omega, s)=\alpha(\omega)1_A(s)$, α bounded and in $\mathscr{G}_{(\text{lef}(A),\,\text{bot}(A))}$, call U elementary, and let $S_3(U,t)=\alpha(\omega)W(A\cap]0,t])$. Extend this definition by linearity and L_2 limits to $\mathscr{K}_3$, the L_2 closure of the collection of finite linear combinations of such elementary processes with respect to the measure $\int\cdot\, d\text{Leb}$.

If $U(\omega,r,s)=\alpha(\omega)1_A(r)1_B(s)$, where α is bounded and in $\mathscr{G}_{(\text{lef}(B),\,\text{bot}(A))}$, $\text{bot}(A)\geqq\text{top}(B)$, and $\text{rig}(A)\leqq\text{lef}(B)$, call U elementary, and define $S_4(U,t)=\alpha(\omega)W(A\cap]0,t])W(B\cap]0,t])$. Extend by linearity and L_2 limits to $\mathscr{K}_4$, the L_2 closure of the collection of finite linear combinations of elementary processes with respect to the measure $E\int\cdot\, d\text{Leb}\, d\text{Leb}$.

Finally, if $U(\omega,r,s,x)=\alpha(\omega)1_A(r)1_B(s)1_I(x)$, where I is an interval in $\mathbb{R}$ that is a positive distance from 0, for any $r\in A$, $s\in B$, neither $r\leqq s$ nor $s\leqq r$, and α is bounded and in $\mathscr{G}_q$, where $q=(\text{lef}(A)\vee\text{lef}(B),\text{bot}(A)\vee\text{bot}(B))$, call U elementary, and define $S_5(U,t)=\alpha(\omega)W(A\cap]0,t])(\mu-\nu)(\omega,B\cap]0,t],I)$. Extend by linearity and L_2 limits to $\mathscr{K}_5$, the L_2 closure of the collection of finite linear combinations of such elementary processes with respect to the measures $E\int\cdot\, d\text{Leb}\, d\nu$.

By using independent increments and the independence of W_t and Z_t, all five types of stochastic integrals are martingales and are mutually orthogonal if the integrands are elementary. By taking limits, the same is true if the integrands are in $\mathscr{K}_i$, $i=1,\ldots,5$. We also have

$$ES_i(U,t)^2=E\int_{]0,t]}U^2\,d\text{Leb},\ E\int_{]0,t]}U^2\,d\text{Leb}\,d\text{Leb},$$

or

$$E\int_{]0,t]}U^2\,d\text{Leb}\,d\nu,\quad \text{if}\quad i=3,4,\text{ or }5,$$

respectively. Again we write $S_i(U)$ for $S_i(U,(1,1))$, $i=1,\ldots,5$.

The proof of the following is exactly analogous to the proof of Lemma 3.

LEMMA 5 *Suppose*

$$X_n = c_n + \sum_{i=1}^{5} S_i(U_i^n), \text{a.s.}, \ U_i^n \in \mathscr{K}_i.$$

If $E|X_n - X|^2 \to 0$, *then there exists* c *and* $U_i \in \mathscr{K}_i$, $i = 1, \ldots, 5$ *such that*

$$X = c + \sum_{i=1}^{5} S_i(U_i), \text{ a.s.}$$

THEOREM 2 *If* $X \in \mathscr{G}_{(1,1)}, EX^2 < \infty$, *then* $X = EX + \sum_{i=1}^{5} S_i(U_i)$ *for some* $U_i \in \mathscr{K}_i$, $i = 1, \ldots, 5$.

Proof By Lemma 5 it suffices to prove the result for

$$X = \left[\prod_j \exp(iu_j Y^\varepsilon(F_j))\right]\left[\prod_j \exp(iv_j W(F_j))\right],$$

where $\{F_j\}$ is a partition of T and Y^ε is defined as in Lemma 2. By Section 6 of [3],

$$\prod_j \exp(iv_j W(F_j)) = c + S_3(U_3) + S_4(U_4), U_3, U_4 \in \mathscr{K}_3, \mathscr{K}_4,$$

respectively, and both U_3, U_4 in the P-completion of $\sigma(W_s; s \in T)$. Using Theorem 1, our result will then follow by the following lemma.

LEMMA 6 *Suppose* $M = c + S_1(U_1) + S_2(U_2)$, a.s., $N = d + S_3(U_3) + S_4(U_4)$ a.s., $U_i \in \mathscr{K}_i$, $i = 1, \ldots, 4, U_1, U_2 \in \mathscr{F}_{(1,1)}, U_3, U_4 \in$ *the* P-*completion of* $\sigma(W_s, s \in T)$. *Then*

$$MN = cd + \sum_{i=1}^{5} S_i(V_i), \text{ a.s.}, \ V_i \in \mathscr{K}_i.$$

Proof As in the proof of Lemma 4, it suffices to consider a number of simpler cases. We will consider the case $M = S_2(U_2)$,

$N = S_4(U_4)$. By the independence of W and Z, $EM^2N^2 = EM^2EN^2 < \infty$. Using Lemma 5 and the last paragraph of the proof of Lemma 4, it suffices to consider the case where U_2 and U_4 are elementary. So suppose $U_2 = \alpha 1_A(r)1_I(x)1_B(s)1_J(y)$, $U_4 = \beta 1_C(r)1_D(s)$.

Let $M_t = S_2(U_2, t)$, $N_t = S_4(U_4, t)$. As in the proof of Lemma 4, let $\{F_i\}$ be a partition of T such that $A, B, C\,D$ are each unions of rectangles in $\{F_i\}$, and write

$$MN = \left(\sum_i M(F_i)\right)\left(\sum_j N(F_j)\right) = \sum_1 + \sum_2 + \sum_3 + \sum_4 + \sum_5 + \sum_6,$$

the sums being defined as before.

Exactly as in the proof of Lemma 4, we may show that $E(\sum_5)^2, E(\sum_6)^2 \to 0$ as the partition becomes finer.

Suppose for the moment that we have shown $EM(F_i)M(F_j) = 0$ whenever $i \neq j$. Then,

$$EM^2 = E\left(\sum_i M(F_i)\right)^2 = \sum_{i \neq j} EM(F_i)M(F_j) + \sum_i EM(F_i)^2 = \sum_i EM(F_i)^2.$$

Secondly, by the independence of M_t and N_t,

$$E\left(\sum_4\right)^2 = E\left(\sum_i M(F_i)N(F_i)\right)^2 = \sum_{i,j} E(M(F_i)M(F_j))E(N(F_i)N(F_j))$$

$$= \sum_i EM(F_i)^2 EN(F_i)^2 \leqq \sup_i EN(F_i)^2 \sum_i EM(F_i)^2$$

$$= \sup_i EN(F_i)^2 EM^2.$$

But if k is a bound for β^2,

$$EN(F_i)^2 = E\int \beta^2 1_{C \cap \mathrm{enl}(F_i)}(r) 1_{D \cap \mathrm{enl}(F_i)}(s)\, \mathrm{Leb}(dr)\, \mathrm{Leb}(ds)$$

$$\leqq k\, \mathrm{Leb}(C \cap \mathrm{enl}(F_i))\, \mathrm{Leb}(D \cap \mathrm{enl}(F_i))$$

$$\leqq k\, \mathrm{Leb}(F_i),$$

which $\to 0$ as the mesh size of the partition tends to 0.

Thus

$$E\left(\sum_4\right)^2 \to 0,$$

or

$$MN - \sum_4 - \sum_5 - \sum_6 \to MN$$

in L_2 as the partition becomes finer. But

$$MN - \sum_4 - \sum_5 - \sum_6 = \sum_1 + \sum_2 + \sum_3,$$

which, as the argument in Lemma 4 shows, can be represented in the desired form.

Lemma 5 will then give us our desired result, provided we show $EM(F_i)M(F_j)=0$ if $i \neq j$.

If $F_i \neq F_j$, either $\mathrm{rig}(F_i) \leqq \mathrm{lef}(F_j)$, $\mathrm{rig}(F_j) \leqq \mathrm{lef}(F_i)$, $\mathrm{top}(F_i) \leqq \mathrm{bot}(F_j)$, or $\mathrm{top}(F_j) \leqq \mathrm{bot}(F_i)$. We will do the case $\mathrm{rig}(F_i) \leqq \mathrm{lef}(F_j)$, the other cases being similar.

$$M(F_j) = \alpha(\omega)(\mu - v)(\omega, A \cap \mathrm{enl}(F_j), I)(\mu - v)(\omega, B \cap \mathrm{enl}(F_j), J).$$

If either $A \cap \mathrm{enl}(F_j) = \emptyset$ or $B \cap \mathrm{enl}(F_j) = \emptyset$, $M(F_j) = 0$.

Otherwise,

$$\mathrm{rig}(A \cap \mathrm{enl}(F_j)) = \mathrm{rig}(A) \leqq \mathrm{lef}(B) \leqq \mathrm{lef}(B \cap \mathrm{enl}(F_j)) = \mathrm{lef}(F_j).$$

$$\alpha \in \mathscr{F}_{(\mathrm{lef}(B),\, \mathrm{bot}(A))} \subset \mathscr{F}_{(\mathrm{lef}(F_j),\, 1)}.$$

Then

$$\begin{aligned} EM(F_i)M(F_j) &= E[M(F_i)\alpha(\mu - v)(\omega, A \cap \mathrm{enl}(F_j), I) \\ &\qquad \times E[(\mu - v)(\omega, B \cap \mathrm{enl}(F_j), J) \mid \mathscr{F}_{(\mathrm{lef}(F_j),\, 1)}]] \\ &= 0, \end{aligned}$$

since the conditional expectation is identically 0 by independent increments. □

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