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OCCUPATION TIME DENSITIES FOR STABLE-LIKE PROCESSES AND OTHER PURE JUMP MARKOV PROCESSES

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Pure jump Markov processes X_t on the line associated to the operators $Af(x) = \int [f(x+h) - f(x) - f'(x)h 1_{([-1,1])}(h)]\nu(x, dh)$ are considered. Sufficient conditions for X_t to have local times that serve as occupation time densities are given. In the case where $\nu(x, dh) = |h|^{-(1+\alpha(x))}$, the stable-like case, these conditions reduce to: inf $\alpha(x) > 1$ and $\alpha(x)$ Dini continuous.

pure jump Markov processes * stable-like processes * local times * occupation times * symmetric stable processes * purely discontinuous martingales * pseudodifferential operators

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1. Introduction

Let X_t be a symmetric stable process of index $\alpha \in (0, 2)$. Then X_t has infinitesimal generator

$$A_\alpha f(x) = \theta_\alpha \int \frac{f(x+h) - f(x)}{|h|^{1+\alpha}} \, dh \tag{1.1}$$

and Lévy measure

$$\nu(dh) = \theta_\alpha |h|^{-(1+\alpha)},$$

where θ_α is a constant chosen so that $E \exp(iuX_1) = \exp(-|u|^\alpha)$. The process X_t has infinitely many jumps in finite time and has no continuous part. It is well known (see, e.g., [6]) that X_t has a local time $L_t(y)$ if and only if $\alpha > 1$. By local time we mean in this paper an occupation time density:

$$\int f(y) L_t(y) \, dy = \int_0^t f(X_s) \, ds \tag{1.2}$$

for all t and all bounded and Borel f, a.s.

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Now suppose that X_t is a pure jump Markov process on the line that is not necessarily a Lévy process. The purpose of this paper is to give sufficient conditions for X_t to have a local time that is an occupation time density.

Perhaps the simplest interesting examples to consider are what we call stable-like processes. Roughly (see Section 2 for a precise definition), a stable-like process is one with generator

$$Af(x)=\theta_{\alpha(x)}\int\frac{f(x+h)-f(x)}{|h|^{1+\alpha(x)}}\,\mathrm{d}h \tag{1.3}$$

and Lévy measure

$$\nu(x,\mathrm{d}h)=\theta_{\alpha(x)}|h|^{-(1+\alpha(x))}, \tag{1.4}$$

where $\alpha:\mathbb{R}\to(0,2)$ is a given function. Such a process may be thought of as one that at the point x behaves like the symmetric stable process of index $\alpha(x)$, but the index $\alpha(x)$ varies from point to point.

For such a process to have a local time, we would expect that we must require

$$1<\inf_x \alpha(x)\leq\sup_x \alpha(x)<2. \tag{1.5}$$

It turns out this condition is not sufficient, however, and we must require a certain amount of continuity for α. In fact, in [1] an example of a stable-like process satisfying (1.5) was constructed where the process spent positive time in a set of Lebesgue measure 0 and thus could not possibly have an occupation time density. In [3] it was shown by a rather complicated argument using the Malliavin calculus that $\alpha\in C^2$ together with (1.5) is sufficient. This raises the question, what is the right condition? In this paper we show, using an argument much simpler than that of [3], that α Dini continuous and (1.5) suffices (see Theorem 2.1). In fact (see Remark 2.3) we expect that α Dini continuous is, in a certain sense, best possible.

Of course, the stable-like processes are only a particular case of a more general situation, processes with generator

$$Af(x)=\int[f(x+h)-f(x)-f'(x)h1_{([-1,1])}(h)]\nu(x,\mathrm{d}h). \tag{1.6}$$

Our methods also give sufficient conditions for these more general pure jump Markov processes to have local times (see Section 6). What is involved are a substitute for (1.5), some mild regularity assumptions on ν, and an integral condition (Theorem 6.1(iii)) replacing the assumption of Dini continuity of $\alpha(x)$.

The general problem of when a purely discontinuous martingale has a local time was raised by Meyer [7]. Yor [10] and Yoeurp [9] both showed that Tanaka's formula was of no use whatsoever in this problem. The Lévy process case had previously been thoroughly worked out by Kesten [6] (see also Bretagnolle [5]). In [1] a rather narrow class of purely discontinuous martingales was considered, and in [3] a much larger class was dealt with by the Malliavin calculus; still in [3] considerable regularity was required.

There is an interesting connection with pseudodifferential operators that should be mentioned. A pseudodifferential operator A is associated with a symbol $\sigma(x, u)$ by

$$Af(x) = (2\pi)^{-1} \int e^{-ixu}\sigma(x, u)\hat{f}(u)\,du,$$

where $\hat{f}$ denotes the Fourier transform. When $\sigma(x, u) \equiv |u|^\alpha$, then A is a constant multiple of the generator A_α of the symmetric stable process of index α. A symbol σ is said to be of order m if

$$\sigma(x, u) = a(x)\varphi(u) + \text{lower order terms},$$

where φ is homogeneous of order m: $\varphi(bu) = b^m\varphi(u)$. Our stable-like processes correspond to operators of variable order: $\sigma(x, u) = |u|^{\alpha(x)}$; and our results can be interpreted as saying that an operator very closely related to $(\lambda - A)^{-1}$ is a bounded operator from L^1 to L^∞. So far as we know, operators of variable order have been very little studied, and few techniques have been developed with which to approach them.

Indeed, we believe this paper to be among the first to treat processes with generators A given by (1.6) without requiring considerable regularity of ν. In a companion paper [2], we build upon the estimates of Section 3 to prove uniqueness for the martingale problem for A in (1.6).

In the next section we state the precise results for stable-like processes. The main work takes place in Sections 3 and 4: in Section 3 we use some elementary Fourier analysis to obtain some estimates on A_α. In Section 4 we use these to show the expected occupation time has a density. Once we have this density, the remainder of the task of constructing local times is routine. In Section 5 we show that local times exist and that these local times are continuous in t. We also make some remarks about additive functionals. Finally, we treat the general pure jump Markov case in Section 6.

We denote the Fourier transform of f by $\hat{f}$ and the Lebesgue measure of A by $|A|$. The letter C denotes a constant whose value is unimportant and which may vary from line to line.

2. Results for stable-like processes

We now proceed to the statement of our main theorem concerning stable-like processes (see Section 6 for the general case).

Definition 2.1. *X_t is a stable-like process corresponding to the function $\alpha(x)$ if, for all $f \in C^2$,*

$$f(X_t) - f(X_0) - \int_0^t Af(X_s)\,ds$$

is a local martingale, where A is defined by (1.3).

Now suppose that x_0 is fixed and that there is one and only one probability measure P such that

$$\begin{aligned}&\text{(i) } P(X_0=x_0), \text{ and}\\ &\text{(ii) } f(X_t)-f(X_0)-\textstyle\int_0^t Af(X_s)\,ds \text{ is a } P\text{-local martingale}\\ &\quad\text{for all } f\in C^2.\end{aligned} \tag{2.1}$$

Let

$$\beta(\delta)=\sup_{|x-y|\le\delta}|\alpha(x)-\alpha(y)|,$$

$$\underline{\alpha}=\inf_x \alpha(x) \quad\text{and}\quad \bar{\alpha}=\sup_x \alpha(x).$$

One of the main results of this paper is

Theorem 2.1. *If*
(i) $1<\underline{\alpha}\le\bar{\alpha}<2$,
(ii) $\beta(\varepsilon)=o(\ln(1/\varepsilon)-1)$ as $\varepsilon\to 0$, and
(iii) $\int_0^1(\beta(x)/x)\,dx<\infty$,
then (X_t, P) *has an occupation time density.*

Remark 2.1. In the presence of (iii), condition (ii) is a very weak one. If one wanted, one could replace both conditions (ii) and (iii) by the single condition

$$\int_0^1 \frac{\beta(x)}{x^{1+\beta(x)}}\,dx<\infty. \tag{2.2}$$

As will be seen from the proof of Theorem 2.1, (2.2) is the condition that enters into the estimate of K_ε. And if (2.2) holds, then there is a sequence $\varepsilon_n\downarrow 0$ such that $\beta(\varepsilon_n)\ln(1/\varepsilon_n)\to 0$, which is all that is needed in the analysis of J_ε.

Remark 2.2. By the usual argument (cf. Section 5), it would suffice for the conditions of Theorem 2.1 to hold locally.

Remark 2.3. We conjecture that the Dini condition is best possible. By this we mean that if $\int_0^1(\beta(x)/x)\,dx=\infty$, then we expect there exists a function α with β as a modulus of continuity for which X_t does not have an occupation time density.

Remark 2.4. In a paper to appear [2] we show that (ii) and (iii) together imply that the martingale problem (2.1) has a unique solution. Hence it is superfluous to assume it.

3. Estimates for stable-like processes

In this section we obtain some estimates for approximations to the resolvents of stable-like processes. For use in Proposition 4.3 we need to estimate $-Ah_\varepsilon$ for a certain function h_ε. We write $-Ah_\varepsilon = J_\varepsilon + K_\varepsilon$, estimate J_ε in Proposition 3.2 and bound K_ε in Proposition 3.4.

Let $p_\alpha(t, x)$ be the probability density for Y_t, a symmetric stable process of index α started at 0 (so that $E \exp(iuY_t) = \exp(-t|u|^\alpha)$).

Let λ be fixed, and let

$$r_\alpha(x) = \int_0^\infty e^{-\lambda t} p_\alpha(t, x)\,dt.$$

Hence $\hat{r}_\alpha(u) = (\lambda + |u|^\alpha)^{-1}$. Let φ be an even nonnegative C^∞ function with support in $[-\frac{1}{2}, \frac{1}{2}]$ and with $\int \varphi(x)\,dx = 1$. Define $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon)$. Let $r_\alpha^\varepsilon(x) = \int r_\alpha(x-y)\varphi_\varepsilon(y)\,dy$. Then, of course, $\hat{r}_\alpha^\varepsilon(u) = \hat{\varphi}(\varepsilon u)/(\lambda + |u|^\alpha)$. The function φ will remain fixed throughout the paper and any constants C that appear may depend on the choice of φ.

We first prove

Proposition 3.1. *If $\alpha \geq \underline{\alpha} > 1$ and $\delta \leq \min((\underline{\alpha} - 1)/2, \frac{1}{4})$, then*

(i) $r_\alpha^\varepsilon(x) \leq C$, *and*

(ii) $|r_\alpha^\varepsilon(x) - r_\alpha^\varepsilon(y)| \leq C|x-y|^\delta$,

where C depends only on $\underline{\alpha}$.

Proof. Since φ_ε is trivially in the Schwartz class $\mathcal{S}$, then so is $\hat{\varphi}_\varepsilon$. We see that $r_\alpha \in L^1$ by Fubini and $\hat{r}_\alpha \in L^1$ by inspection. Hence $r_\alpha * \varphi_\varepsilon \in L^1$ (* denotes convolution) and $\hat{r}_\alpha \hat{\varphi}_\varepsilon \in L^1$; we may thus apply the Fourier inversion formula to get

$$r_\alpha^\varepsilon(x) = (2\pi)^{-1} \int e^{-ixu} \frac{\hat{\varphi}(\varepsilon u)}{\lambda + |u|^\alpha}\,du. \tag{3.1}$$

Then

$$|r_\alpha^\varepsilon(x)| \leq C \int \frac{du}{\lambda + |u|^\alpha},$$

and (i) follows.

Now $|e^{i\theta} - 1| \leq |\theta| \leq |\theta|^\delta$ for $|\theta| \leq 1$, while $|e^{i\theta} - 1| \leq 2 \leq 2|\theta|^\delta$ for $|\theta| > 1$.

Then

$$|r_\alpha^\varepsilon(x) - r_\alpha^\varepsilon(y)| \leq C \int |e^{-iu(x-y)} - 1| \frac{du}{\lambda + |u|^\alpha} \leq C|x-y|^\delta \int \frac{|u|^\delta}{\lambda + |u|^\alpha}\,du,$$

and (ii) follows. □

Let $\delta \leq \min(\frac{1}{4}, (\underline{\alpha} - 1)/2)$, and let I_0 be a fixed interval with $|I_0| \leq \delta$ and $\beta(|I_0|) \leq \delta$. Let I and I_1 be intervals such that I is contained in the interior of I_1, which is contained in the interior of I_0, and $|I_1| \leq 2|I|$. Let g be a C^∞ function such that $1_I \leq g \leq 1_{I_1}$. Let

$$h_\varepsilon(x) = \int r^\varepsilon_{\alpha(y)}(x-y)g(y)\,\mathrm{d}y. \tag{3.2}$$

An immediate consequence of Proposition 3.1 is that

$$\begin{aligned}&\text{(i)}\ |h_\varepsilon(x)| \leq C\|g\|_{L^1}, \text{ and}\\ &\text{(ii)}\ |h_\varepsilon(x)-h_\varepsilon(y)| \leq C\|g\|_{L^1}|x-y|^\delta.\end{aligned} \tag{3.3}$$

Note that, since $\varphi_\varepsilon \in C^\infty$, so is r^ε_α for each α, and hence $h_\varepsilon \in C^\infty$.

We now want to study

$$J_\varepsilon(x) = \int_{x-\varepsilon}^{x+\varepsilon} -A_{\alpha(x)} r^\varepsilon_{\alpha(y)}(x-y)g(y)\,\mathrm{d}y. \tag{3.4}$$

Proposition 3.2. *Under the hypotheses of Theorem* 2.1, $J_\varepsilon(x) \to g(x)$ *uniformly in* x *as* $\varepsilon \to 0$.

Proof. We may suppose $\varepsilon \leq \frac{1}{2}$. For the moment suppose $1 < \underline{\alpha} \leq \alpha_1, \alpha_2$. Recalling that φ_ε has support in $[-1, 1]$, we see that if $|z| \geq 2$, then

$$A_{\alpha_1}\varphi_\varepsilon(z) = \theta_{\alpha_1}\int \frac{\varphi_\varepsilon(z+h)-\varphi_\varepsilon(z)}{|h|^{1+\alpha_1}}\,\mathrm{d}h \leq \frac{C\|\varphi_\varepsilon\|_{L^\infty}}{(|z|-1)^{1+\alpha_1}}.$$

Since $A_{\alpha_1} r^\varepsilon_{\alpha_2}(z) = r_{\alpha_2} * A_{\alpha_1}\varphi_\varepsilon(z)$, it follows that $A_{\alpha_1} r^\varepsilon_{\alpha_2} \in L^1$. Since $\hat{\varphi}$ is in the Schwartz class, then $(A_{\alpha_1} r^\varepsilon_{\alpha_2})^\wedge$ is also in L^1. (Recall that $\hat{A}_{\alpha_1}(u) = -|u|^{\alpha_1}$, where $\hat{A}_{\alpha_1}$ denotes the Fourier transform of the operator A_{α_1}.) We may then apply the Fourier inversion formula to get

$$J_\varepsilon(x) = (2\pi)^{-1}\int_{x-\varepsilon}^{x+\varepsilon}\int \mathrm{e}^{-\mathrm{i}u(x-y)}\frac{|u|^{\alpha(x)}}{\lambda+|u|^{\alpha(y)}}\hat{\varphi}(\varepsilon u)\,\mathrm{d}u\,g(y)\,\mathrm{d}y. \tag{3.5}$$

A change of variables gives us

$$\begin{aligned}J_\varepsilon(x) &= (2\pi)^{-1}\int_{-1}^{1}\int \mathrm{e}^{-\mathrm{i}uy}\frac{|u/\varepsilon|^{\alpha(x)}}{\lambda+|u/\varepsilon|^{\alpha(x+\varepsilon y)}}\hat{\varphi}(u)\,\mathrm{d}u\,g(x+\varepsilon y)\,\mathrm{d}y\\ &= (2\pi)^{-1}\int_{-1}^{1}\int \mathrm{e}^{-\mathrm{i}uy}\hat{\varphi}(u)\,\mathrm{d}u\,g(x+\varepsilon y)\,\mathrm{d}y\\ &\quad + (2\pi)^{-1}\int_{-1}^{1}\int \mathrm{e}^{-\mathrm{i}uy}[\Psi_\varepsilon(x,y,u)-1]\hat{\varphi}(u)\,\mathrm{d}u\,g(x+\varepsilon y)\,\mathrm{d}y,\end{aligned} \tag{3.6}$$

where

$$\Psi_\varepsilon(x,y,u) = \frac{|u/\varepsilon|^{\alpha(x)}}{\lambda+|u/\varepsilon|^{\alpha(x+\varepsilon y)}}.$$

By the Fourier inversion formula, the first term on the last line of (3.6) is equal to $\int_{-1}^{1}\varphi(y)g(x+\varepsilon y)\,dy$, which converges uniformly to $g(x)$ as $\varepsilon\to 0$ since $g\in C^{\infty}$ with compact support and φ has support in $[-\frac{1}{2},\frac{1}{2}]$. We will see that the second term on the last line of (3.6) converges uniformly to 0 as $\varepsilon\to 0$ once we show

$$\int|\Psi_\varepsilon(x,y,u)-1||\hat\varphi(u)|\,du\to 0\quad\text{as } \varepsilon\to 0 \text{ uniformly in } x \text{ and } y. \tag{3.7}$$

Now

$$|\Psi_\varepsilon(x,y,u)|\le|u/\varepsilon|^{\alpha(x)-\alpha(x+\varepsilon y)}\le|u/\varepsilon|^{\delta}+|u/\varepsilon|^{-\delta}$$

since $x,y\in I_0$ and $\beta(|I_0|)\le\delta$. Then, recalling that $\hat\varphi\in\mathscr{S}$,

$$\int_{|u|\ge\varepsilon^{-1}}|\Psi_\varepsilon(x,y,u)-1||\hat\varphi(u)|\,du\le C\varepsilon^{-\delta}\int_{|u|\ge\varepsilon^{-1}}(1+|u|^{\delta})u^{-3}\,du\to 0. \tag{3.8}$$

Next,

$$\int_{-\varepsilon^{1/2}}^{\varepsilon^{1/2}}|\Psi_\varepsilon(x,y,u)-1||\hat\varphi(u)|\,du\le C\varepsilon^{-\delta}\int_{-\varepsilon^{1/2}}^{\varepsilon^{1/2}}|u|^{-\delta}\,du\to 0. \tag{3.9}$$

Finally, by Hypothesis (ii) of Theorem 2.1,

$$|\Psi_\varepsilon(x,y,u)-|u/\varepsilon|^{\alpha(x)-\alpha(x+\varepsilon y)}|\to 0$$

and

$$||u/\varepsilon|^{\alpha(x)-\alpha(x+\varepsilon y)}-1|\to 0$$

as $\varepsilon\to 0$, uniformly for $|u|\in[\varepsilon^{1/2},\varepsilon^{-1}]$. This, with the fact that $|\hat\varphi(u)|$ is integrable, (3.8), and (3.9) proves (3.7). □

Now let

$$K_\varepsilon(x)=\int_{[x-\varepsilon,x+\varepsilon]^c}-A_{\alpha(x)}r^{\varepsilon}_{\alpha(y)}(x-y)g(y)\,dy. \tag{3.10}$$

Clearly,

$$|K_\varepsilon(x)|\le\int M(x,y)g(y)\,dy,$$

where

$$M(x,y)=\sup_{0<\varepsilon\le|x-y|}|A_{\alpha(x)}r^{\varepsilon}_{\alpha(y)}(x-y)|. \tag{3.11}$$

So we need to study $M(x,y)$.

To do this, we define for $\varepsilon\in(0,1)$

$$\Gamma_\varepsilon(u)=\int_0^u e^{-it}\hat\varphi(\varepsilon t)\,dt \tag{3.12}$$

$$\Lambda_\varepsilon(u)=\int_0^u\Gamma_\varepsilon(t)\,dt, \tag{3.13}$$

and

$$H_u(x)=\frac{\sin ux}{x}. \tag{3.14}$$

Proposition 3.3

(i) $\Gamma_\varepsilon(u)=-H_u * \varphi_\varepsilon(1)$;

(ii) $|\Gamma_\varepsilon(u)|\leq C(|u|\wedge 1)$, *C independent of* ε,

(iii) $|A_\varepsilon(u)|\leq C(u^2\wedge 1)$, *C independent of* ε; *and*

(iv) *for each* ε, $|\Gamma_\varepsilon(u)|\leq C_\varepsilon(1\wedge|u|^{-2})$.

Proof. The convolution in (i) is well-defined since $H_u\in L^2$. Fix u. Let g_n be C^∞ functions with support in $[-2u, 2u]$ and range in $[0, 1]$; let G_n be their inverse Fourier transforms. Then

$$\begin{aligned}(2\pi)^{-1}\int e^{-it}g_n(t)\hat{\varphi}(\varepsilon t)\,dt &= G_n * \varphi_\varepsilon(1)=\int G_n(y)\varphi_\varepsilon(1-y)\,dy\\ &=\int_{|1-y|\leq\varepsilon/2} G_n(y)\varphi_\varepsilon(1-y)\,dy,\end{aligned} \tag{3.15}$$

and

$$G_n(y)=(2\pi)^{-1}\int e^{-iyt}g_n(t)\,dt. \tag{3.16}$$

Now suppose g_n converges boundedly and pointwise to $1_{[-u,u]}(t)$ as $n\to\infty$. The left side of (3.15) converges to $(2\pi)^{-1}\int_{-u}^{u} e^{-it}\hat{\varphi}(\varepsilon t)\,dt=\pi^{-1}\Gamma_\varepsilon(u)$ by dominated convergence. And since all the g_n have support in $[-2u, 2u]$, and are uniformly bounded, $G_n(y)\to(2\pi)^{-1}\int_{-u}^{u} e^{-ity}\,dt$ boundedly and pointwise. Direct evaluation shows that this last integral is $-\pi^{-1}H_u(y)$. So taking the limit in (3.15) as $n\to\infty$ and recalling the support of φ_ε is $[-\varepsilon/2, \varepsilon/2]$, we get

$$\pi^{-1}\Gamma_\varepsilon(u)=\int_{|1-y|\leq\varepsilon/2}-\pi^{-1}H_u(y)\varphi_\varepsilon(1-y)\,dy=-\pi^{-1}\int H_u(y)\varphi_\varepsilon(1-y)\,dy,$$

which proves (i).

To prove (ii), observe that

$$|\Gamma_\varepsilon(u)|\leq\int_0^u \|\hat{\varphi}\|_{L^\infty}\,dt\leq Cu. \tag{3.17}$$

And from (i),

$$\begin{aligned}|\Gamma_\varepsilon(u)| &= \left|\int\frac{\sin u(1-y)}{1-y}\varphi_\varepsilon(y)\,dy\right|\\ &\leq \sup_{|y|\leq\varepsilon/2}\left|\frac{\sin u(1-y)}{1-y}\right|\int\varphi_\varepsilon(y)\,dy\leq 2.\end{aligned}$$

From (3.17), we see $|\Lambda_\varepsilon(u)|\leq Cu^2$. And

$$|\Lambda_\varepsilon(u)|=\left|\int_0^u\int_{|1-y|\leq\varepsilon/2}\frac{\sin ty}{y}\varphi_\varepsilon(1-y)\,\mathrm{d}y\,\mathrm{d}t\right|$$
$$=\left|\int_{|1-y|\leq\varepsilon/2}\frac{1-\cos uy}{y^2}\varphi_\varepsilon(1-y)\,\mathrm{d}y\right|$$
$$\leq\sup_{|1-y|\leq\varepsilon/2}\left|\frac{1-\cos uy}{y^2}\right|\int\varphi_\varepsilon(1-y)\,\mathrm{d}y\leq 8,$$

which proves (iii).

Finally, for fixed $\varepsilon\leq 1$,

$$\int_0^\infty \mathrm{e}^{-it}\hat{\varphi}(\varepsilon t)\,\mathrm{d}t=\pi\varphi_\varepsilon(1)=0,$$

and so

$$|\Gamma_\varepsilon(u)|=|-\int_u^\infty \mathrm{e}^{-it}\hat{\varphi}|\varepsilon t|\,\mathrm{d}t|\leq\int_u^\infty|\hat{\varphi}(\varepsilon t)|\,\mathrm{d}t\leq C_\varepsilon u^{-2}$$

since $\hat{\varphi}_\varepsilon\in\mathcal{S}$. $\square$

Recall the definition of M in (3.11). We now can prove the following result.

Proposition 3.4. *Under the hypotheses of Theorem* 2.1,

$$M(x,y)\leq C\left(1+\frac{\beta(|x-y|)}{|x-y|}\right).$$

Proof. As in the proof of Proposition 3.1,

$$-A_{\alpha(x)}r^\varepsilon_{\alpha(y)}(z)=(2\pi)^{-1}\int \mathrm{e}^{-iuz}\frac{|u|^{\alpha(x)}}{\lambda+|u|^{\alpha(y)}}\hat{\varphi}(\varepsilon u)\,\mathrm{d}u,$$

where $z=x-y$. Since

$$\left|\frac{|u|^{\alpha(x)}}{\lambda+|u|^{\alpha(y)}}-\frac{|u|^{\alpha(x)}}{|u|^{\alpha(y)}}\right|=\frac{\lambda}{\lambda+|u|^{\alpha(y)}}|u|^{\alpha(x)-\alpha(y)}$$
$$\leq\frac{C\lambda(|u|^\delta+|u|^{-\delta})}{\lambda+|u|^\alpha},$$

which is integrable, and $\hat{\varphi}$ is bounded, it suffices to show

$$B_\varepsilon(x,y)\leq\frac{C\beta(|x-y|)}{|x-y|},\tag{3.18}$$

where

$$B_\varepsilon(x,y)=(2\pi)^{-1}\int e^{-iuz}|u|^\gamma\hat{\varphi}(\varepsilon u)\,du,$$

$z=x-y$, $\gamma=\alpha(x)-\alpha(y)$, and $|z|\geq\varepsilon$. By a change of variables,

$$B_\varepsilon(x,y)=\pi^{-1}|z|^{-(1+\gamma)}\int_0^\infty e^{-iu}u^\gamma(\tilde{\varepsilon}u)\,du, \tag{3.19}$$

where $\tilde{\varepsilon}=\varepsilon/|z|\leq 1$. We bound (3.19) by integrating by parts twice (look at the real and imaginary parts separately):

$$\int_\eta^N e^{-iu}u^\gamma\hat{\varphi}(\tilde{\varepsilon}u)\,du=\Gamma_{\tilde{\varepsilon}}(u)u^\gamma]_\eta^N-\gamma\Lambda_{\tilde{\varepsilon}}(u)u^{\gamma-1}]_\eta^N$$
$$+\gamma(\gamma-1)\int_\eta^N \Lambda_{\tilde{\varepsilon}}(u)u^{\gamma-2}\,du. \tag{3.20}$$

Recalling that $|\gamma|\leq\delta\leq\frac{1}{4}$ and using Proposition 3.3,

$$\left|\int_0^\infty e^{-iu}u^\gamma\hat{\varphi}(\tilde{\varepsilon}u)\,du\right|\leq|\gamma(\gamma-1)|\int_0^\infty|\Lambda_{\tilde{\varepsilon}}(u)|u^{\gamma-2}\,du\leq C\gamma.$$

Hence

$$B_\varepsilon(x,y)\leq\frac{C\gamma}{|z|^{1+\gamma}}\leq\frac{C\beta(|x-y|)}{|z|^{1+\gamma}}.$$

The proof of (3.18) and thus of the proposition is completed by realizing that Hypothesis (ii) of Theorem 2.1 implies that $|z|^\gamma$ is bounded independently of x and y, provided $x,y\in I_0$. □

4. Expected occupation times

Throughout this section we assume the hypotheses of Theorem 2.1 hold. By Hypothesis (iii) of Theorem 2.1, we see that

$$\limsup_{|I_0|\downarrow 0}\sup_y\int_{I_0}M(x,y)\,dx=0.$$

So let $\delta=\min(\frac{1}{4},(\underline{\alpha}-1)/2)$, and suppose I_0 is an open interval satisfying $|I_0|\leq\delta$, $\beta(|I_0|)\leq\delta$, and $\sup_y\int_{I_0}M(x,y)\,dx\leq\frac{1}{4}$. Let $T=\inf\{t\colon X_t\notin I_0\}$, and define a measure

$$S(A)=E\int_0^T 1_A(X_s)\,ds. \tag{4.1}$$

The goal of this section is to show $S(A)\leq C|A|$, C depending only on β.

We need a couple of preliminary propositions. Let $T_\eta=\inf\{t\colon|X_t-x_0|\geq\eta\}$.

Proposition 4.1. *If $\eta < 1$, then $P(T_\eta \le r) \le Cr/\eta^2$.*

Proof. Without loss of generality, assume $x_0 = 0$ and $I_0 \subseteq [-1, 1]$. Let f be a bounded C^2 function with bounded first and second derivatives such that $f(x) = x^2$ if $|x| \le 2$ and $f(x) \ge 4$ if $|x| \ge 2$. We have

$$Af(y) = \theta_{\alpha(y)} \int \frac{f(y+h) - f(y)}{|h|^{1+\alpha(y)}}\, dh = \theta_{\alpha(y)} \int_0^\infty \frac{f(y+h) + f(y-h) - 2f(y)}{h^{1+\alpha(y)}}\, dh.$$

For $h \ge 1$, the integrand is bounded above by $C\|f\|_{L^\infty} h^{-(1+\underline{\alpha})}$, which is integrable on $[1, \infty)$. For $h \le 1$, the integrand is bounded by $C\|f''\|_{L^\infty} h^2/h^{1+\alpha(y)} \le ch^{1-\bar\alpha}$, which is integrable on $[0, 1]$. Hence Af is bounded above. Then

$$\eta^2 P(T_\eta \le r) \le Ef(X_{T_\eta \wedge r}) - Ef(X_0) = E\int_0^{T_\eta \wedge r} Af(X_s)\, ds$$

$$\le r\|Af\|_{L^\infty}. \quad \square$$

We also have the following result.

Proposition 4.2. *$ET \le C < \infty$, where C depends only on $\underline{\alpha}$ and $\bar\alpha$.*

Proof. Without loss of generality, assume $X_0 = 0$ and $I_0 \subseteq [-1, 1]$.

Suppose $z \ge 4$, and let g be a C^2 function with range $[0, 1]$ such that $g(x) = 0$ if $|x| \le z/2 + 1$, $g(x) = 1$ if $|x| \ge z$, and $\|g''\|_{L^\infty} \le 4$. For $x \in I_0$.

$$Ag(x) = \theta_{\alpha(x)} \int \frac{g(x+h) - g(x)}{|h|^{1+\alpha(x)}}\, dh \le 2C \int_{z/2}^\infty \frac{dh}{h^{1+\underline{\alpha}}} = Cz^{-\underline{\alpha}}.$$

Then

$$P(|X_{T\wedge t}| \ge z) \le Eg(X_{T\wedge t}) - g(0) = E\int_0^{T\wedge t} Ag(X_s)\, ds$$

$$\le Cz^{-\underline{\alpha}} E(T \wedge t). \tag{4.2}$$

Multiplying (4.2) by $(1+\delta)z^\delta$ and integrating from 4 to ∞, we then get

$$E|X_{T\wedge t}|^{1+\delta} \le C(1 + E(T\wedge t)). \tag{4.3}$$

Next let f be a convex C^2 function ≥ 0 such that $f(x) = x^2$ for $|x| \le 2$ and $f(x) = C|x|$ for $|x| \ge 3$. Note for $y \in I_0$ and $h \ge 1$,

$$\frac{f(y+h) + f(y-h) - 2f(y)}{h^{1+\alpha(y)}} \le \frac{C\|f'\|_{L^\infty} h}{h^{1+\underline{\alpha}}},$$

which is integrable on $[1, \infty]$.

We then conclude $Af(y)$ is bounded for $y \in I_0$ (cf. proof of Proposition 4.1). Also, since f is convex, if $y \in I_0$, then

$$Af(y) \ge \theta_{\alpha(y)} \int_0^1 \frac{f(y+h) + f(y-h) - 2f(y)}{h^{1+\alpha(y)}}\, dh \ge C\int_0^1 h^{1-\underline{\alpha}}\, dh = C.$$

Approximating f from below by bounded C^2 functions which have uniformly bounded first and second derivatives and which equal f on $|x|\leq 2$, it is easy to see that

$$Ef(X_{T\wedge t})-f(0)=E\int_0^{T\wedge t} Af(X_s)\,ds\geq CE(T\wedge t). \tag{4.4}$$

Since $Ef(X_{T\wedge t})\leq C(1+E|X_{T\wedge t}|)$, we then get

$$\begin{aligned} ET\wedge t&\leq C(1+E|X_{T\wedge t}|)\leq C(1+(E|X_{T\wedge t}|^{1+\delta})^{1/(1+\delta)})\\ &\leq C(1+(E(T\wedge t))^{1/(1+\delta)}). \end{aligned} \tag{4.5}$$

But this implies $ET\wedge t\leq C$ with C independent of t. Now use monotone convergence to let $t\uparrow\infty$. □

Proposition 4.3. *Suppose $\alpha(x)\in C^1$, and let $L=\|\alpha'\|_{L^\infty}<\infty$. Suppose $\underline{\alpha}\leq\alpha(x)\leq\bar{\alpha}$. Then $E\int_0^{T\wedge t}1_A(X_s)\,ds\leq C(t^{\delta/3}\wedge 1)|A|$ for all Borel sets $A\subseteq I_0$ with C depending only on $\underline{\alpha}$ and $\bar{\alpha}$ and independent of t and L.*

Proof. Since $-Ah_\varepsilon(x)=J_\varepsilon(x)+K_\varepsilon(x)$, we have from Propositions 3.2 and 3.4 that

$$-Ah_\varepsilon(x)\geq g(x)-v_\varepsilon(x)-Mg(x) \tag{4.6}$$

for $y\in I_0$, where $v_\varepsilon(x)\to 0$ uniformly as $\varepsilon\to 0$ and $Mg(x)=\int_{I_0}M(x,y)g(y)\,dy$.

Proposition 3.4 tells us that

$$M(x,y)\leq C\left(1+\frac{\beta(|x-y|)}{|x-y|}\right)\leq C(1+L),$$

and hence

$$Mg(x)\leq C(1+L)\|g\|_{L^1}. \tag{4.7}$$

By Proposition 3.1(i), $|Eh_\varepsilon(X_0)-Eh_\varepsilon(X_{T\wedge t})|\leq C\|g\|_{L^1}$. But we also have, taking $\eta=t^{1/3}$ in Proposition 4.1,

$$\begin{aligned} |Eh_\varepsilon(X_0)-Eh_\varepsilon(X_{T\wedge t})|&\leq\|h_\varepsilon\|_{L^\infty}P(|X_{T\wedge t}-X_0|\geq\eta)+C\|g\|_{L^1}\eta^\delta\\ &\leq C\|g\|_{L^1}(P(T_\eta\leq t)+t^{\delta/3})\\ &\leq C\|g\|_{L^1}(t^{1/3}+t^{\delta/3}). \end{aligned}$$

Therefore,

$$|Eh_\varepsilon(X_0)-Eh_\varepsilon(X_{T\wedge t})|\leq C\|g\|_{L^1}(1\wedge t^{\delta/3}). \tag{4.8}$$

Choose ε small enough so that

$$\|v_\varepsilon\|_{L^\infty} \leq \|g\|_{L^1}(1 \wedge t^{\delta/3}). \tag{4.9}$$

From

$$\begin{aligned} Eh_\varepsilon(X_0) - Eh_\varepsilon(X_{T\wedge t}) &= E\int_0^{T\wedge t} -Ah_\varepsilon(X_s)\,\mathrm{d}s \\ &\geq E\int_0^{T\wedge t} g(X_s)\,\mathrm{d}s - E\int_0^{T\wedge t} v_\varepsilon(X_s)\,\mathrm{d}s \\ &\quad - E\int_0^{T\wedge t} Mg(X_s)\,\mathrm{d}s, \end{aligned}$$

and Proposition 4.1, (4.7), (4.8), and (4.9),

$$\begin{aligned} E\int_0^{T\wedge t} g(X_s)\,\mathrm{d}s &\leq C\|g\|_{L^1}(1 \wedge t^{\delta/3} + (1+L)ET) \\ &\leq [C_1(1\wedge t^{\delta/3}) + C_2(1+L)]\|g\|_{L^1}, \end{aligned} \tag{4.10}$$

with C_1 and C_2 independent of t and L. Since $1_I \leq g$, and $\|g\|_{L^1} \leq |I_1| \leq 2|I|$, we get

$$E\int_0^{T\wedge t} 1_I(X_s)\,\mathrm{d}s \leq [C_1(1\wedge t^{\delta/3}) + C_2(1+L)]|I|. \tag{4.11}$$

Since the interval I was arbitrary, the measure $S(t, A) = E\int_0^{T\wedge t} 1_A(X_s)\,\mathrm{d}s$ has a density $s_t(x)$ with respect to Lebesgue measure that is bounded by $C_1(1\wedge t^{\delta/3}) + C_2(1+L)$.

But we can use this fact to get a better estimate for the Mg term.

$$\begin{aligned} E\int_0^{T\wedge t} Mg(X_s)\,\mathrm{d}s &= \int Mg(x)s_t(x)\,\mathrm{d}x = \int\int_{I_0} M(x,y)s_t(x)g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \tfrac{1}{4}\|s_t\|_{L^\infty}\|g\|_{L^1}. \end{aligned} \tag{4.12}$$

Using (4.11) and also (4.8) and (4.9) in (4.10) yields

$$E\int_0^{T\wedge t} g(X_s)\,\mathrm{d}s \leq C\|g\|_{L^1}(1\wedge t^{\delta/3}) + \tfrac{1}{4}\|s_t\|_{L^\infty}\|g\|_{L^1} \tag{4.13}$$

As above, this implies that the density $s_t(x)$ of $S(t, A)$ is bounded:

$$\|s_t\|_{L^\infty} \leq C(1\wedge t^{\delta/3}) + \tfrac{1}{4}\|s_t\|_{L^\infty}$$

or since $\|s_t\|_{L^\infty}$ is finite, we have finally $\|s_t\|_{L^\infty} \leq C(1 \wedge t^{\delta/3})$, C independent of t and L. □

The key estimate can now be proven.

Theorem 4.4

(i) $E \int_0^T 1_A(X_s)\,ds \leq C|A|$, *and*

(ii) $E \int_0^T e^{-\lambda s} 1_A(X_s)\,ds \leq C\lambda^{-\delta/3}|A|$.

Proof. Let α_n be a sequence of C^1 functions converging uniformly to α such that $\underline{\alpha} \leq \inf_{n,x} \alpha_n(x) \leq \sup_{n,x} \alpha_n(x) \leq \bar{\alpha}$, $\|\alpha_n'\|_{L^\infty}$ is finite for each n, and $\sup_{n,|x-y|\leq\delta} |\alpha_n(x)-\alpha_n(y)| \leq \beta(\delta)$. Let P_n be the solution to the martingale problem (2.1) with α replaced by α_n.

There is no difficulty constructing P_n: let $\sigma_n(x,z) = \theta_{\alpha(x)}|z|^{-1/\alpha(x)}$, and construct the stochastic differential equation

$$dX_t^n = \int \sigma_n(X_{s-}^n, z)\mu(dz, ds), \quad X_0^n = x_0, \tag{4.14}$$

where μ is a Poisson point process with compensator $dz\,dt$ (cf. [3]). It is easy to check that such a σ_n satisfies the conditions of [8, Ch. 3], so a pathwise unique solution to (4.14) exists. Let P_n be the probability measure induced on $D[0,\infty)$ by X_t^n.

Standard techniques (using, e.g., Proposition 4.1) show that the sequence is tight, and moreover it is easy to see that any subsequential limit point solves the martingale problem for α. Since we are assuming uniqueness for this martingale problem, $P_n \to^w P$.

Take g as in Proposition 4.3. If $I_0 = (a,b)$, $M_s = \sup_{r\leq s} X_r$, and $m_s = \inf_{r\leq s} X_r$, then since $P_n \to^w P$,

$$Eg(X_s)1_{(M_s<b)}1_{(m_s>a)} \leq \limsup_n E_n g(X_s)1_{(M_s<b)}1_{(m_s>a)}.$$

The estimate in Proposition 4.3 depends only on $\underline{\alpha}$, $\bar{\alpha}$, and β, and so applies uniformly for all the P_n's. Therefore,

$$E\int_0^{T\wedge t} g(X_s)\,ds = E\int_0^t g(X_s)1_{(M_s<b)}1_{(m_s>a)}\,ds$$

$$\leq \limsup_n E_n \int_0^{T\wedge t} g(X_s)\,ds \leq C\|g\|_{L^1}(1\wedge t^{\delta/3}). \tag{4.15}$$

Letting $t\uparrow\infty$ proves (i).

To prove (ii), write

$$E\int_0^T e^{-\lambda s}g(X_s)\,ds = E\int_0^\infty\int_s^\infty \lambda\, e^{-\lambda r}\,dr\,g(X_s)1_{(s<T)}\,ds$$
$$= \int_0^\infty \lambda\, e^{-\lambda r}E\int_0^{T\wedge t} g(X_s)\,ds\,dr$$
$$\leq C\|g\|_{L^1}\int_0^\infty \lambda\, e^{-\lambda r}r^{\delta/3}\,dr. \quad \square \tag{4.16}$$

5. Local times

With the estimates of Theorem 4.4 we can now finish the proof of Theorem 2.1. Let $Q_t(\omega, d\omega')$ be a regular conditional probability for $E(\cdot\,|\mathcal{F}_t)$:

$$Q_t(\omega, A) = E(A|\mathcal{F}_t)(\omega) \quad \text{a.s.}$$

By the arguments of [1, Section 4] we see that (X_{t+s}, Q_s) is a solution to the martingale problem for α, but with starting point $X_s(\omega)$ instead of x_0. If $B=\{y$: there is more than one solution to the martingale problem for α starting at $y\}$, it is not hard to see that $P(X_t$ ever hits $B)=0$; consequently a.s. (X_{t+s}, Q_s) is the unique solution to the martingale problem for α starting at $X_s(\omega)$. We may then apply Theorem 4.4 to (X_{t+s}, Q_s) to conclude (cf. [1])

$$E\left[\int_s^{T\vee s} e^{-\lambda r}g(X_r)\,dr\,\Big|\,\mathcal{F}_s\right] \leq C\|g\|_{L^1}(\lambda^{-\delta/3}\wedge 1). \tag{5.1}$$

We now apply, nearly verbatim, the arguments of [1, Section 4] to see that there exist local times $L_t(y)$ that are an occupation time density for X_t up to time T; by [1, Section 5], these $L_t(y)$ are continuous in t. By looking at (X_{T+t}, Q_T), we can repeat the procedure (cf. [1, Section 6]) to construct continuous $L_t(y)$, $t\leq T_2$, where $T_2=\inf\{t: |X_t - X_{T_1}|\geq \frac{1}{2}|I_0|\}$ and $T_1 = T$. Continue by iteration to get $L_t(y)$ for $t\leq T_n$. Proposition 4.1 implies that there exists a $\gamma<1$ such that $Q_{T_i}\,e^{-T}\leq\gamma$, a.s., for all i. Then

$$E\,e^{-T_2} = E\,e^{-T_1}Q_{T_1}\,e^{-T}\leq \gamma E\,e^{-T_1}\leq\gamma^2.$$

By induction, $E\,e^{-T_n}\leq\gamma^n\to 0$, hence $T_n\to\infty$, hence $L_t(y)$ can be defined for all t. Theorem 2.1 is proved.

We defined our processes in terms of a martingale problem, partly to accommodate the approximation of α by α_n and partly to be able to use the machinery developed in [1]. But we could also have used the machinery of additive functionals and potential theory (see, e.g., [4]). Suppose that for every starting point x the martingale problem for α has a unique solution P^x. Consider the process X_t killed at time T. We will show below that $U(x, A) = E^x\int_0^T 1_A(X_s)\,ds$ has a density $\bar{s}(x, y)$ with respect to Lebesgue measure that is excessive in x for every y. One would then let $L_t(y)$ be the additive functional whose potential is $\bar{s}(x, y)$. An additional argument (that would use Theorem 4.4(ii)) is necessary to show $L_t(y)$ is a continuous additive functional. That additional argument, as well as details involving measurability, are left to the reader.

Proposition 5.1. *There exists a function $\bar{s}(x, y)$ that is excessive in x for every y such that $U(x, A) = \int_A \bar{s}(x, y)\,\mathrm{d}y$ for all x and A.*

Proof. Let $U^\alpha f(x) = E^x \int_0^\infty \mathrm{e}^{-\alpha t} f(X_s)\,\mathrm{d}s$. Let $s(x, y)$ be a jointly measurable density for $U(x, \mathrm{d}y)$. If $f \geq 0$ is bounded and continuous, $Uf(x) = \int s(x, y) f(y)\,\mathrm{d}y = E^x \int_0^T f(X_t)\,\mathrm{d}t$ is excessive and, by [4, II. 2.3],

$$\alpha U^\alpha Uf(x) \leq \beta U^\beta Uf(x) \leq Uf(x) \text{ if } \alpha \leq \beta.$$

It follows that for every x, α, and β with $\alpha < \beta$ that

$$\alpha U^\alpha S(x, y) \leq \beta U^\beta s(x, y) \leq s(x, y) \quad \text{for almost all } y.$$

So by Fubini we can find a null set N such that if $y \notin N$, then

$$\alpha U^\alpha s(x, y) \leq \beta U^\beta s(x, y) \leq s(x, y) \quad \text{for almost all } \alpha < \beta, x.$$

If $y \in N$, let $\bar{s}(x, y) = 0$. This is trivially excessive in x.

Now fix $y \notin N$. Fix $\alpha < \beta$, and pick $\gamma \in (\alpha, \beta)$ such that $\gamma U^\gamma s(x, y) \leq s(x, y)$ for almost all x. Since X_t spends 0 time in the set $\{x: \gamma U^\gamma s(x, y) > s(x, y)\}$ by Theorem 4.4, then

$$\begin{aligned} U^\beta \gamma U^\gamma s(x, y) &= E^x \int_0^\infty \mathrm{e}^{-\beta t} \gamma U^\gamma s(X_t, y)\,\mathrm{d}t \\ &\leq E^x \int_0^\infty \mathrm{e}^{-\beta t} s(X_t, y)\,\mathrm{d}t = U^\beta s(x, y). \end{aligned}$$

Using the resolvent identity,

$$\begin{aligned} \gamma U^\gamma s(x, y) &= \gamma U^\beta s(x, y) + (\beta - \gamma) U^\beta \gamma U^\gamma s(x, y) \\ &\leq \gamma U^\beta s(x, y) + (\beta - \gamma) U^\beta s(x, y) = \beta U^\beta s(x, y). \end{aligned}$$

A similar argument shows that $\alpha U^\alpha s(x, y) \leq \gamma U^\gamma s(x, y)$. We thus have that if $y \notin N$, $\beta U^\beta s(x, y)$ increases as $\beta \to \infty$. Call the limit $\bar{s}(x, y)$.

We have for f bounded and continuous

$$\begin{aligned} \int \bar{s}(x, y) f(y)\,\mathrm{d}y &= \lim_\beta \int \beta U^\beta s(x, y) f(y)\,\mathrm{d}y = \lim_\beta \beta U^\beta Uf(x) \\ &= Uf(x). \end{aligned}$$

So $\bar{s}(x, y)$ is a density for $U(x, \mathrm{d}y)$.

Since $\|\gamma U^{\gamma}s\|_{L^\infty}\le \|s\|_{L^\infty}<\infty$,

$$\beta U^{\beta}\bar{s}(x,y)=\lim_{\gamma}\beta U^{\beta}\gamma U^{\gamma}s(x,y)=\lim_{\gamma}\frac{\beta\gamma}{\beta-\gamma}[U^{\gamma}s(x,y)-U^{\beta}s(x,y)]$$
$$=\beta U^{\beta}s(x,y)\le \bar{s}(x,y).$$

And this same calculation shows $\beta U^{\beta}\bar{s}(x,y)=\beta U^{\beta}s(x,y)$, which increases to $\bar{s}(x,y)$ as $\beta\to\infty$. So by [4, II. 2.3], $\bar{s}(x,y)$ is excessive in x. □

6. Pure jump Markov processes

In this section we consider the general case where we have Lévy measure $\nu(x,\mathrm{d}h)$ and

$$Af(x)=\int [f(x+h)-f(x)-f'(x)h1_{[-1,1]}(h)]\nu(x,\mathrm{d}h) \tag{6.1}$$

for $f\in C^2$. We suppose there is one and only one probability measure P for which $P(X_0=x_0)=1$ and $f(X_t)-f(X_0)-\int_0^t Af(X_s)\,\mathrm{d}s$ is a P-local martingale for each $f\in C^2$.

Let

$$\Phi(x,u)=\int [\mathrm{e}^{\mathrm{i}uh}-1-\mathrm{i}uh1_{([-1,1])}(h)]\nu(x,\mathrm{d}h), \tag{6.2}$$

and let

$$W(x,y,u)=\Phi(x,u)/\Phi(y,u).$$

We make the mild regularity assumptions:
There exists $\delta\in(0,1)$ such that

$$\begin{aligned}
&\text{(i)} && \inf_x \Phi(x,u)\ge C|u|^{1+\delta}\ \text{for}\ |u|\ge 1;\\
&\text{(ii)} && \sup_x\int (h^2\wedge 1)\nu(x,\mathrm{d}h)<\infty;\ \text{and}\\
&\text{(iii)} && u^2\left|\frac{\partial^2}{\partial u^2}W(x,y,u)\right|\le C_{x,y}(|u|^{-\delta}+|u|^{\delta}).
\end{aligned} \tag{6.3}$$

Define

$$R(\varepsilon)=\sup_{|u|\in[\varepsilon^{1/2},\varepsilon^{-1}]}\ \sup_{x\in\mathbb{R},|y|\le 1}\left|W\left(x,x+\varepsilon y,\frac{u}{\varepsilon}\right)-1\right|, \tag{6.4}$$

and

$$M(x,y)=|x-y|^{-3}\int_{-\infty}^{\infty}(u^2\wedge 1)\left|\frac{\partial^2}{\partial u^2}W\left(x,y,\frac{u}{|x-y|}\right)\right|\mathrm{d}u. \tag{6.5}$$

If $\nu_n(x, dh)$ is another Lévy measure, define A_n, P_n, Φ_n, W_n, R_n, and M_n analogously. We suppose that there exists a sequence of measures $\nu_n(x, dh)$ such that (6.3) is satisfied for each n (the constants may depend on n), and $P_n \to^w P$.

We then have the following result.

Theorem 6.1. *If*

(i) *for each n,* $R_n(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

(ii) *for each n,* $\sup_{x,y} M_n(x, y) < \infty$; *and*

(iii) $$\lim_{\delta \to 0} \sup_n \sup \int_0^\delta M_n(x, y)\, dx = 0,$$

then (X_t, P) *has an occupation time density* $L_t(y)$ *that is continuous in t for each y.*

Remark 6.1. The assumptions (6.3) are really quite mild. For example, in the Lévy process case one needs $\int [\lambda + \Phi(u)]^{-1}\, du < \infty$ to have a local time, and so (i) is only slightly more restrictive. The assumptions could undoubtedly be weakened slightly.

Remark 6.2. The assumption of a sequence of measures ν_n is somewhat inelegant. For any particular example of a ν, it is clear how the ν_n should be defined, but it is difficult to give a recipe that works in general.

Remark 6.3. Many of the assumptions and definitions are given in terms of $\Phi(x, u)$ rather than $\nu(x, dh)$ directly. That should not be unexpected: even in the Lévy process case most results about local times are stated in terms of the Lévy-Khintchine exponent.

Proof. The hypotheses of the theorem are set up so that the proof of Theorem 2.1 goes through step by step; only two comments are required. The first is that Hypothesis (ii) is used to get the analogue of Proposition 4.1. The second is that instead of constructing local times up to time $T = \inf\{t: X_t \notin I_0\}$ and then iterating, construct local times up to time $T \wedge 1$, and then iterate. This makes the analogue of Proposition 4.2 unnecessary. □

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