The supremum of Brownian local times on Hölder curves, II

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Abstract

Abstract: For $f : [0,1] \to \mathbb{R}$, we consider L_t^f, the local time of space-time Brownian motion on the curve f. Let $\mathcal{S}_\alpha$ be the class of all functions whose Hölder norm of order α is less than or equal to 1. We show that the supremum of L_1^f over f in $\mathcal{S}_\alpha$ is finite if $\alpha > \frac{1}{2}$.

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1 Introduction

The main claim of [1] was that the supremum of Brownian local times over all α-Hölder curves is finite if $\alpha > 1/2$ (see Theorem 1.1 below for the precise statement). An error in the proof was pointed out to us by A. Vatamanelu; however we were able to establish the claim for $\alpha > 5/6$ in [2]. The purpose of this note is to prove the original claim from [1], that finiteness of the supremum indeed holds for all $\alpha \in (1/2, 1]$. We also showed in [1] that $\alpha = 1/2$ is the critical value; see Theorem 3.8 of that paper for the precise statement.

Let W_t be one-dimensional Brownian motion and let $f : [0,1] \to \mathbb{R}$ be a Hölder continuous function. There are a number of equivalent ways to define

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L_t^f, the local time of W_t along the curve f, one of which is as the limit in probability of

$$\frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{(f(s)-\varepsilon, f(s)+\varepsilon)}(W_s)\, ds$$

as $\varepsilon \to 0$. See [1, Sect. 2] for a discussion of other ways of defining L_t^f. Let

$$\mathcal{S}_\alpha = \{f : \sup_{0\le t\le 1} |f(t)| \le 1, |f(s)-f(t)| \le |s-t|^\alpha \text{ if } s,t \le 1\}.$$

Our main result in this paper is the following.

Theorem 1.1. *For any $\alpha \in (1/2, 1]$, there exists $\widetilde{L}_t^f$ such that*

(i) for each $f \in \mathcal{S}_\alpha$, we have $\widetilde{L}_t^f = L_t^f$ for all t, a.s.,

(ii) with probability one, $f \to \widetilde{L}_1^f$ is a continuous map on $\mathcal{S}_\alpha$ with respect to the supremum norm, and

(iii) with probability one, $\sup_{f\in\mathcal{S}_\alpha} \widetilde{L}_1^f < \infty$.

The interest in Theorem 1.1 has several sources. One is that the metric entropy of $\mathcal{S}_\alpha$ is far too large for chaining arguments to work; nevertheless the supremum is finite a.s. Another is the work of Holden and Sheffield [3] on scaling limits of the Schelling model, where they used some of the techniques in [1] to analyze local times of random fields over Lipschitz surfaces.

In the interests of space we present only the changes needed to [1] to prove our result and refer to the original paper for the unchanged part of the proof.

2 The finiteness of the supremum

Let W_t be a Brownian motion. A key ingredient in our proof is Lemma 3.1 of [1]. The proof there is correct; the error in [1] was in how this lemma was applied further on.

We replace Propositions 3.2 and 3.3 in [1] by the following.

Consider an integer $N > 0$. For $0 \le \ell \le N, -N^\alpha - 1 \le m \le N^\alpha$, let $R_{\ell m}$ be the rectangle defined by

$$R_{\ell m} = [\ell/N, (\ell+1)/N] \times [m/N^\alpha, (m+1)/N^\alpha].$$

Proposition 2.1. *Let $\alpha \in (1/2, 1]$ and $\varepsilon \in (0, 1/16)$. There exist c_1, c_2, and c_3 such that:*

(i) there exists a set D_N with $\mathbb{P}(D_N) \le c_1 \exp(-c_2 N^{\varepsilon/2})$;

(ii) if $\omega \notin D_N$ and $f \in \mathcal{S}_\alpha$, then there are at most $c_3 N^{(1/2)+2\varepsilon}$ rectangles $R_{\ell m}$ in $[0,1] \times [-1,1]$ which contain both a point of the graph of f and a point of the graph of $W_t(\omega)$.

Proof. Let $M = \lfloor N^\varepsilon \rfloor$ and set

$$Q_{ik} = [i/M, (i+1)/M] \times [k/M^{1/2}, (k+1)/M^{1/2}],$$

for $0 \le i \le M$ and $-M^{1/2} - 1 \le k \le M^{1/2}$. Let $J = \lceil N/M \rceil$.

Let

$$I_{ikj} = \{\exists t \in [i/M + (j-1)/N, i/M + j/N] : \\ k/M^{1/2} \le W_t \le (k+1)/M^{1/2}\},$$

$$A_{ik} = \sum_{j=1}^{J} \mathbf{1}_{I_{ikj}},$$

and

$$C_{ik} = \{A_{ik} \ge J^{(1/2)+\varepsilon}\}.$$

By Lemma 3.1 of [1] with $\lambda = J^\varepsilon$ and the Markov property applied at i/M we have $\mathbb{P}(C_{ik}) \le c_4 \exp(-c_5 J^\varepsilon)$.

There are at most $c_6 M^{3/2}$ rectangles Q_{ik}, so if $D_N = \cup_{i,k} C_{ik}$, where $0 \le i \le M$ and $-M^{1/2} - 1 \le k \le M^{1/2}$, then

$$\mathbb{P}(D_N) \le c_7 M^{3/2} \exp(-c_5 J^\varepsilon) \le c_7 \exp(-c_8 N^{\varepsilon/2}).$$

Let f be any function in $\mathcal{S}_\alpha$. If f intersects Q_{ik} for some i and k, then f might intersect $Q_{i,k-1}$ and $Q_{i,k+1}$. But because $f \in \mathcal{S}_\alpha$ and $\alpha > 1/2$, it cannot intersect Q_{ir} for any r such that $|r - k| > 1$. Therefore f can intersect at most $3(M+1)$ of the Q_{ik}.

Now suppose $\omega \notin D_N$. Look at any one of the Q_{ik} that f intersects. Since $\omega \notin D_N$, then there are at most $J^{(1/2)+\varepsilon}$ integers j that are less than J and for which the path of $W_t(\omega)$ intersects $([i/M+(j-1)/N, i/M+j/N]\times[-1,1])\cap Q_{ik}$. If f intersects a rectangle $R_{\ell m}$, then it can intersect a rectangle $R_{\ell r}$ only if $|r-m|\le 1$, since $f\in\mathcal{S}_\alpha$. Therefore there are at most $3J^{(1/2)+\varepsilon}$ rectangles $R_{\ell m}$ contained in Q_{ik} which contain both a point of the graph of f and a point of the graph of $W_t(\omega)$.

Since there are at most $3(M+1)$ rectangles Q_{ik} which contain a point of the graph of f, there are therefore at most

$$3(M+1)3J^{(1/2)+\varepsilon}\le c_9N^{(1/2)+2\varepsilon}$$

rectangles $R_{\ell m}$ that contain both a point of the graph of f and a point of the graph of $W_t(\omega)$. □

Our present Proposition 2.1 is almost identical to Proposition 3.3 in [1], so the latter can be omitted. With this change, the remainder of [1], beyond Proposition 3.3, can proceed as in the original.

References

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