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OCCUPATION TIMES OF d-DIMENSIONAL SEMIMARTINGALES

by

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1.0 Introduction

If X_t is a stochastic process, f a nonnegative function, define $A_t(f) = \int_0^t f(X_s)ds$. When X_t is a Markov process, $A_t(f)$ is an example of what are called additive functionals. Over the past 20 years, an extensive theory has been developed about additive functionals of Markov processes. One of the major results is that, given a measure μ, under quite general conditions on X and μ, one can construct an additive functional $A_t(\mu)$ as the limit of additive functionals $A_t(f_n)$ for some sequence of functions f_n. When μ is point mass at a point x, $A_t(\mu)$ is called the local time at x. More generally, when μ is concentrated on a set C, $A_t(\mu)$ is called (one of) the local times at the set C. Local times have turned out to be among the most useful examples of additive functionals.

Suppose now that X_t is a d-dimensional semimartingale: an $\mathbb{R}^d$-valued process, each of whose coordinates is a semimartingale. Let us call processes $A_t(f)$ defined above, as well as uniform limits of such processes, occupation times. (The reason for the change of name is that there will, in general, be processes that satisfy the abstract definition of additive functional but are not occupation times.)

Our goals in this paper are

(1.1) to construct an occupation time $A_t(\mu)$ corresponding to a measure μ;

(1.2) to give conditions under which $A_t(\mu_a)$ is jointly continuous in a and t;

(1.3) to give integral representations of $A_t(\mu)$ and $A_t(f)$ in terms of $A_t(\lambda(s,v))$, where $\lambda(s,v)$ is $d-1$ dimensional Lebesgue measure on the hyperplane $\{y: y*v = s\}$, v on the unit ball, $y*v$ the inner product of y and v.

Of these, (1.3) is the generalization to the d-dimensional semi-martingale case of the results obtained in [1] for Brownian motion.

The precise results are given in section 2 as theorems T1, T2, T3 and T4. Let us briefly describe the hypotheses needed on X_t and μ (given more precisely in section 2 as A2 and M1). The process X_t must satisfy $dX_t = \sigma_t(X)dW_t + \eta_t\, dt$. Here η_t need only be bounded and predictable; $\sigma_t(X)$ is bounded, strictly elliptic, continuous in t, and for each t, $\sigma_t(X)$ is a functional on the path $\{X_s, s \le t\}$. Note that $\sigma_t(X)$ is a functional of the entire past of X, not just the current value X_t as is the case when X is Markov. As a functional, we would like σ_t to be twice Fréchet differentiable; in fact, slightly less will do.

The conditions we put on μ are that $\mu(\mathbb{R}^d)$ be finite (this can be weakened: see 6.1) and that $\mu(B(r,x)) \le cr^{d-2+\varepsilon}$, where B(r,x) is the ball of radius r centered at x and c,ε are constants > 0 independent of r and x. If μ has a bounded density, we could let $\varepsilon = 2$; if μ is concentrated on a surface in a nice way, we could let $\varepsilon = 1$. Of course, μ could be much wilder than either of these (cf. 8.3).

Looking at the particular case when X_t is Markov, (1.1) is, as we stated above, known; (1.3) is new, while (1.2), at least in the generality in which it is stated, may perhaps be new. At the heart of our method are estimates on the size and smoothness of $\rho(z)$, the density of $\lambda(C) = E\int_0^t 1_C(X_s)ds$ with respect to Lebesgue measure. Bounds on $\rho(z)$ are well-known when σ_t is only strictly elliptic and Hölder continuous. However, to get smoothness of $\rho(z)$, one needs that the coefficients of the adjoint operator for X be Hölder continuous, which translates to requiring that σ be in $C^{2+\alpha}$ for some α (see [3, Chapter 6]). Thus, even in the Markov case, the classical results on parabolic partial differential equations do not give an improvement on our results for (1.2) and (1.3).

Returning to the general case, Maisonneuve [7] has a way of constructing an increasing process which has support $\{(t,\omega): X_t(\omega) \in C\}$. Such a process need not be an occupation time, however. Krylov [6] has bounds on the size of the density $\rho(z)$, but his bounds are not good enough for our purposes and his methods give no information about the smoothness of ρ. If one did not want (1.1), (1.2), or (1.3), but only wanted to construct some occupation time on a very smooth surface C, there is an easy way to proceed. One maps $\mathbb{R}^d$ into $\mathbb{R}^d$ smoothly so that C gets mapped into a subset of $\mathbb{R}^{d-1}\times\{0\}$, and one now has what is essentially a one-dimensional problem. This procedure is carried out in section 8, but it works only when C is quite smooth.

Since our argument is rather long, we will give a brief outline. In section 2 we introduce some notation and give statements of the results. First proving our results under the restrictions that σ_t is close to the identity matrix I, that $\eta_t = 0$, and that the dimension $d \geq 3$ (see R1,2,3 of section 2), we begin in section 3 with an integration by parts formula. We take a formula of Haussmann [4], and following the approach used by Bismut [2] to prove Malliavin's results, we apply it to the functional $L(X) = \int_0^\tau g(X_s)h_s ds$, where τ is a fixed time, g a differentiable function, and h_s functionals to be specified later. After some calculations, we get that

$$E \int_0^\tau Dg(X_s)G_s h_s ds$$

is equal to an expression involving $g(X_s)$, h_s, and some other terms (see 3.4), where G_s is a certain auxiliary process and Dg is the gradient of g. We would like to let $h_s = G_s^{-1}$ and then use Hölder's inequality. But, easy examples show that G_s^{-1} need not be integrable except in the Markov case. This is not just a technical point: it turns out that G_s is intimately related to the Fréchet derivative of the mapping $W_t \to X_t$. The singularity of G has implications to the existence of densities for X_t. The device we use to get around the singularity of G is to stop at a randomized stopping time; the randomization is necessary to preserve Fréchet differentiability at appropriate places.

After using Hölder's inequality, we are faced with bounding expressions of the form $E\int_0^\tau |g|^p(X_s)s^{-\beta}ds$, $0 < \beta < \frac{1}{2}$. We do this in section 4 by comparing the radial process $|X_s - w|$ to a Bessel process of appropriate index ν. (This is why we need $\sigma_t \approx I$.) Letting g equal successively the first order partial derivatives of the Newtonian potential of the indicator of $B(r,w)$, we achieve the necessary estimates for the density of the potential of the stopped process.

In section 5 we use these estimates to get estimates of $|E \int_0^{S\wedge\tau}(\mu * \phi_b - \mu * \phi_a)(X_s)ds|$, S our stopping time, ϕ_b an approximation to the identity. We do an iteration scheme to suppress S from this expression, and then, following roughly along the lines of the corresponding proof for Markov additive functionals, we estimate $E|\int_0^\tau(\mu * \phi_b - \mu * \phi_a)(X_s)ds|^p$.

Using Kolmogorov's theorem and techniques from [1], we prove our results in section 6, subject to the restrictions R1,2,3. Finally, in section 7, we eliminate these restrictions: $\sigma_t \approx I$ by an iteration procedure, no drift by a Girsanov transformation, $d \geq 3$ by a projection argument.

2.0 Notation and statement of results

If $x \in \mathbb{R}^d$, $|x|$ is the Euclidean norm of x. If K is a $d \times d$ matrix, $|K| = \sup\{|Kx| : |x| \leq 1\}$ and K^* is the transpose of K. e_j will denote the j^{th} standard basis vector for $\mathbb{R}^d$. Let $B(r,x) = \{y \in \mathbb{R}^d : |y - x| < r\}$.

If f is an $\mathbb{R}$ or $\mathbb{R}^d$-valued function, $\|f\| = \sup\{|f(x)|\}$; if the domain of f is $[0,\infty)$, $\|f\|_\tau = \sup\{|f(t)| : t \leq \tau\}$; let

$$\|f\|_{C^\alpha} = \|f\| + \sup_{y \neq x} |f(y) - f(x)|/|y - x|^\alpha.$$

If μ is a measure, $\|\mu\|_V$ is the total variation of μ and $\|\mu\|_{W-\alpha} = \sup\{|\int f d\mu| : \|f\|_{C^\alpha} \leq 1\}$. If μ is a measure on $[0,\infty)$, we will let $[\mu,f]$ denote $\int \mu(ds)f(s)$; we will frequently write integrals in this order since we identify elements of $\mathbb{R}^d$ with $d \times 1$ matrices, with the exception that we will consider $\mathbb{R}^d$-valued measures as $1 \times d$ matrices.

If L is a real-valued functional on $C[0,\infty)$, the continuous $\mathbb{R}^d$-valued functions, the Fréchet derivative of L at f is an $\mathbb{R}^d$-valued measure $L'(f)$ such that

$$|L(f + g) - L(f) - [L'(f),g]| = o(\|g\|).$$

The letter c with a subscript will denote constants; $c_{2.1}(\alpha,n)$, for example, is a constant whose value depends on α and n.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. When necessary, we will assume $\Omega = C[0,\infty)$; let $\mathcal{F}_t$ be the right continuous, completed filtration generated by W_t, a d-dimensional Brownian motion on Ω. Let W_t^j be the j^{th} coordinate of W_t. If M is a martingale, let $\langle M,M\rangle$ be the quadratic variation of M (see [8] for details about stochastic integration). Let $\phi_b(y) = (2\pi b)^{-d/2}\, e^{-|y|^2/2b}$.

Suppose X_t satisfies

$$X_0 = x_0, \quad dX_t = \sigma_t dW_t + \eta_t dt, \tag{2.1}$$

where $\sigma_t \in \mathbb{R}^{d\times d}$, $\eta_t \in \mathbb{R}^d$, respectively, and are both bounded and predictable. We will refer to σ_t and η_t as the diffusion and drift coefficients of X_t respectively. We will suppose $\sigma_t(\omega) = \hat{\sigma}_t(X(\omega))$, a.s., where $\hat{\sigma}_t$ is a $d \times d$ matrix valued functional on $C[0,t]$. Since no confusion should result, we will drop the ^'s on the $\hat{\sigma}$ and write σ_t, $\sigma_t(X)$, $\sigma(t)$, or $\sigma(t,X)$ as needed.

We will make the following assumptions on σ and η.

A1 (i) $|\eta_t|$ *is bounded by* $c_{2.1}$, *independent of* t; η_t *is predictable;*

(ii) $|\sigma_t|$ *is bounded by* $c_{2.2}$, *independent of* t; σ_t *is predictable;*

(iii) σ_t *is uniformly strictly elliptic: there exists* $c_{2.3}$ *independent of* t *such that* $x^*\sigma_t x \geq c_{2.3}|x|^2$ *for all* x;

(iv) *for each* i,j, $\sigma_{ij}(t)$ *is a functional on* $C[0,t]$ *which is Fréchet differentiable;* $\sup_{t\leq\tau}\big|\,\|\sigma'_{ij}(t,f)\|_V\big| \leq c_{2.4}(\tau)$, $c_{2.4}$ *independent of* f, i, *and* j;

(v) $\sigma'_{ij}(t,f)$ *is itself Fréchet differentiable: there exists*

$\sigma''_{ij}(t)$, *a measure-valued linear functional on* $C[0,t]$ *such that* $|\| \sigma'_{ij}(t, f+g) - \sigma'_{ij}(t,f) - \sigma''_{ij}(t,f)(g)\|_V| = o(\|g\|_t)$; *moreover*, $\sup_{t\le\tau} |\| \sigma''_{ij}(t,f)(g)\|_V| \le c_{2.5}(\tau)\|g\|_\tau$, $c_{2.5}$ *independent of* f, i, *and* j.

In section 5, we will weaken A1 to:

A2 (i) A1(i) *still holds;*

(ii) *there exists a sequence of functionals* σ^n *satisfying* A1(ii)-(v), *the constants* $c_{2.2}$, $c_{2.3}$, $c_{2.4}$, $c_{2.5}$ *independent of* n, *such that for each* $f \in C[0,\tau]$, $\sup_{t\le\tau}|\sigma_t^n(f) - \sigma_t(f)| \to 0$.

We impose the following condition on measures μ (and μ_a and λ, where appropriate)

M1 (i) $\mu(\mathbb{R}^d) \le c_{2.6} < \infty$;

(ii) *for all* x *and* r, $\mu(B(r,x)) \le c_{2.7} r^{d-2+c_{2.8}}$, *where* $c_{2.7}$, $c_{2.8}$ *are independent of* x,r *and are* > 0.

We can now state our main theorems: T1,2,3 and 4. In each, we suppose X is given by 2.1 and we suppose A2 holds. Let $A_t(f) = \int_0^t f(X_s)ds$. Let $\mu * \phi_b(x)$ be the density of the measure $\mu * \phi_b(dx)$.

T1 THEOREM. *Suppose* μ *satisfies* M1. *Then there exists an increasing, continuous process* $A_t(\mu)$ *such that for all* τ,

$$\sup_{t\le\tau}|A_t(\mu * \phi_b) - A_t(\mu)| \to 0, \text{ a.s. } as\ b \to 0.$$

T2 THEOREM. *Suppose* $\{\mu_a,\ 0 \le a \le 1\}$ *satisfies* M1 *for each* a, $c_{2.6}$, $c_{2.7}$, $c_{2.8}$ *independent of* a. *Suppose for some* α,

$$\| \mu_a - \mu_b \|_{W-\alpha} \le c_{2.9}(\alpha)|b - a|^{c_{2.10}(\alpha)}.$$

Then there exist versions of $A_t(\mu_a)$ *such that*

$$\sup_{t \le \tau} \sup_{|a-b| < \delta} |A_t(\mu_a) - A_t(\mu_b)| \to 0, \quad \text{a.s., } as\ \delta \to 0.$$

Let

(2.2) $$I_b(y) = (2\pi)^{-d} \int_0^\infty \cos(qy) q^{d-1} e^{-bq^2/2}\, dq,$$

and

(2.3) $$\hat{B}_t(\mu,b) = \int \int_{(|v|=1)} \int_{-\infty}^{\infty} I_b(s - y^* v) A_t(\lambda(s,v)) ds\, dv\, \mu(dy).$$

Here the integral with respect to dv is with respect to surface measure on the unit ball, and $\lambda(s,v)$ is $d-1$ dimensional Lebesgue measure on the hyperplane $\{z\colon z^* v = s\}$.

T3 THEOREM. *Suppose* μ *satisfies* M1. *Then*

$$\sup_{t \le \tau} |A_t(\mu) - \hat{B}_t(\mu,b)| \to 0, \quad \text{a.s. } as\ b \to 0.$$

If f is a continuous function on $\mathbb{R}^d$, let

$$g_v(s) = \int_{(y^* v = s)} f(y) d\lambda(s,v).$$

T4 THEOREM. *Suppose* μ *satisfies* M1. *Suppose* d *is odd and* $D_{d-1} g_v(s) = \partial^{d-1} g_v(s)/\partial s^{d-1}$ *is in* $L_1(-\infty,\infty)$. *Then*

$$A_t(f) = \tfrac{1}{2}(-1)^{(d-1)/2} \int_{(|v|=1)} \int_{-\infty}^{\infty} D_{d-1} g_v(s) A_t(\lambda(s,v)) ds dv, \text{ a.s.}$$

T1,2,3, and 4 will be first proved under some restrictions; the restrictions will be removed in section 7.

R1 *There exists a constant* $c_{2.11} < (d+1)^{-1}(c_{2.8} \wedge 1)/16$ *such that* $|\sigma_t - I| \le c_{2.11}$ *for all* t.

R2 $\eta_t \equiv 0$.

R3 $d \ge 3$.

Finally, we need to state Haussmann's formula. Since we will be considering functionals that will be defined in terms of stochastic integrals, we need to modify slightly what we mean by the Fréchet derivative of such a functional.

Let S be the set of $C[0,\infty)$-valued processes adapted to F_t, identifying two processes that are equal, P-a.s. If L is a functional defined on S, mapping Z to an r.v. L(Z), we will say L is P-Fréchet differentiable if L satisfies

(2.4) (i) $E|L(Z)|^q \le c_{2.12}(q,Z) < \infty$ for all q;

(ii) $L(Z) \in F_\tau$ for some τ independent of Z;

(iii) if Q is any probability equivalent to P, if $Z, Y \in S$, and if the P-distribution of Z is equal to the Q-distribution of Y, then the P-distribution of L(Z) is equal to the Q-distribution of L(Y);

(iv) there exists a functional L' mapping $Z \in S$ to L'(Z), a random measure (i.e., for each ω, L'(Z) is a measure on $C[0,\tau]$) such that $E|L(Z+Y) - L(Z) - [L'(Z),Y]|^q = o(E\|Y\|^q)$ for all q;

(v) $E|\|L'(Z)\|_v|^q \le c_{2.12}(q) < \infty$.

We define the auxiliary $\mathbb{R}^{d\times d}$-valued process ψ_t^s by

(2.5) $$\psi_t^s = 0 \quad \text{if } t < s; \quad \psi_t^s e_j = Ie_j + \int_s^t a(s,r)dW_r \quad \text{if } t \geq s,\ 1 \leq j \leq d,$$

where $a(s,r)$ is $\mathbb{R}^{d\times d}$-valued and $a(s,r)_{ik} = [\sigma_{ik}'(r),\ \psi_\cdot^s e_j]$.

We then have, upon making trivial modifications to his proof, Haussmann's formula

(2.6) THEOREM (Haussmann [4]). *If* L *satisfies* (2.4), σ *satisfies* A1, X *is given by* (2.1), *and* R2 *holds, then*

$$L(X) = EL(X) + \int_0^\tau E([L'(X),\psi_\cdot^t]|F_t)\sigma_t dW_t.$$

3.0 Integration by parts

Throughout this section we assume that σ satisfies A1 and that R1,2,3 hold. We need to introduce some auxiliary processes.

Let G_t, J_t be $\mathbb{R}^{d\times d}$-valued processes given by

$$G_t = \int_0^t \psi_t^s ds, \qquad J_t = t^{-1}G_t,$$

where ψ_t^s is given by (2.5).

We need the following facts about ψ, G, and J.

(3.1) LEMMA. (i) $E \sup_{s \leq r \leq t} |\psi_r^s|^q \leq c_{3.1}(q,t)$;

(ii) $E \sup_{s \leq t} |G_s - sI|^q \leq c_{3.2}(q,t)$;

(iii) $J_t \to I$, a.s., *as* $t \to 0$; $E \sup_{s \leq t} |J_s - I|^q \leq c_{3.3}(q,t)$;

(iv) *If* $\theta > 1/8$ *and* $S_\theta = \inf\{t: |J_t - I| > \theta\}$, *then*
$$E\, e^{-S_\theta} < c_{3.4}(\theta) < 1.$$

PROOF. We will omit the proof (i) since it is similar to (ii), but easier.

(ii) Let $b_{ik}(r) = \int_0^r a_{ik}(s,r)ds = \int_0^r \int_0^r \sigma'_{ik}(r)(du)\psi_u^s e_j ds$

$$= [\sigma'_{ik}(r), G e_j],$$

using Fubini and the fact that $\psi_u^s = 0$ if $u < s$. It follows then that

$$|b_{ik}(r)| \le \|\sigma'_{ik}(r)\|_V \sup_{u \le r}|G_u e_j| \le c_{3.5} \sup_{u \le r}|G_u e_j|, \quad c_{3.5} = c_{2.4}(r).$$

Suppose $H(s,r) = 1_{[s_1,s_2]}(s)1_F(\omega)1_{[r_1,r_2]}(r)$, where $s_1 \le s_2 \le r_1 \le r_2$, $F \in F_{r_1}$. Direct calculation shows that

$$\int_0^t \int_0^t H(s,r)dW_r ds = \int_0^t \int_0^t H(s,r)ds dW_r. \tag{3.2}$$

By linearity and taking limits, (3.2) holds for $H(s,r) = a_{ik}(s,r)1_{(s \le r)}$, and so we get

$$G_t e_j = \int_0^t \psi_t^s e_j ds = te_j + \int_0^t \int_s^t a(s,r)dW_r ds = te_j + \int_0^t b_r dW_r.$$

Each component of $G_t e_j - te_j$ is a martingale, and so, using Burkholder's inequality and taking $q \ge 2$,

$$E \sup_{u \le t}|G_u e_j - ue_j|^q \le c_{3.6}(q)E\Big(\int_0^t \operatorname{trace}(b_r^* b_r)dr\Big)^{q/2}$$

$$\le c_{3.7}(q)E\Big(\int_0^t \sup_{u \le r}|G_u e_j|^2 dr\Big)^{q/2}$$

$$\le c_{3.8}(q)t^{q/2-1}\, E\int_0^t \Big(\sup_{u \le r}|G_u e_j - ue_j|^q + r^q\Big)dr \qquad \text{(Hölder)}.$$

By Gronwall's inequality, (ii) follows with $c_{3.2}(q,t) = c_{3.9}(q)t^{3q/2}$ for $t \le 1$.

$$\text{(iii)}\quad E \sup_{u \le 2^{-p}} |J_u - I|^q \le E \sum_{i=p}^{\infty} \sup_{2^{-(i+1)} \le u \le 2^{-i}} |J_u - I|^q$$

$$\le E \sum_{i=p}^{\infty} \sup_{2^{-(i+1)} \le u \le 2^{-i}} |G_u - uI|^q\, 2^{(i+1)q}$$

$$\le c_{3.9}(q) \sum_{i=p}^{\infty} 2^{(i+1)q}(2^{-i})^{3q/2} \le c_{3.10}(q)2^{-p},$$

if $q \ge 2$. This proves that $J_t \to I$ a.s. as $t \to 0$, and the fact that $|J_u - I| \le 2|G_u - uI|$ if $u \ge \frac{1}{2}$ gives the remainder of (iii).

$$\text{(iv)}\quad P(S_\theta \le t) = P(\sup_{u \le t}|J_u - I| \ge \theta) \le \theta^{-2}E(\sup_{u \le t}|J_u - I|^2)$$

$$\le c_{3.10}(2)\theta^{-2}2^{-p}$$

if $t = 2^{-p}$ for some p; here $c_{3.10}$ depends only on $c_{2.4}(1)$. Since $\theta > 1/8$, take p sufficiently large so that $P(S_\theta \le 2^{-p}) \le \frac{1}{2}$. Then $Ee^{-S_\theta} \le \frac{1}{2} + \frac{1}{2}e^{-2^{-p}} = c_{3.11} < 1$. □

(3.3) LEMMA. ψ_t^s, G_t, J_t, $t \le \tau$ *each satisfy* (2.4 i - v); *in fact, the constant* $c_{2.13}$ *in* (2.4v) *may be chosen so that*

$$E \sup_{t \le \tau}|\,\|\psi_t^{s\prime}\|_V|^q,\ E \sup_{t \le \tau}|\,\|G_t'\|_V|^q,\ \textit{and}\ E \sup_{t \le \tau}|\,\|J_t'\|_V|^q\ \textit{are all} \le c_{2.13}.$$

PROOF. The proof for G_t is similar to that for ψ_t^s, while the results for J_t follow from those for G_t as in (3.1). Thus we will prove only the results for ψ_t^s. (i) and (ii) follow from (3.1) and the definitions.

(iii) $$d\psi_t^s e_j = a(s,t)dW_t = a(s,t)\sigma_t^{-1}dX_t,$$

where $a(s,t)$ depends on ψ_t^s and X. This stochastic differential equation may be solved by Picard iteration, from which (iii) follows.

(iv) Let $\psi_t^s(X)' = 0$ if $t < s$, and solve, for each j and each continuous process Y,

$$[\psi_t^s(X)'e_j, Y] = \int_s^t a'(s,r,X,Y)dW_r \quad \text{if } t \geq s,$$

where

$$a'_{ik}(s,r,X,Y) = [\sigma'_{ik}(r,X),[\psi_\cdot^s(X)'e_j,Y]] + [\sigma''_{ik}(r,X)(Y),\psi_\cdot^s e_j].$$

We claim that $\psi_t^s(X)'$ is the required random measure of (2.4iv). The proof is so similar to steps in the proof of Haussmann's formula (cf. [4, p.23] and also the proof (3.1ii) above, essentially applications of Burkholder's and Gronwall's inequalities, that we omit the details.

(v) $$|a'_{ik}(s,r,X,Y)| \leq \|\sigma'_{ik}(r,X)\|_V \sup_{u\leq r}|[\psi_u^s(X)'e_j,Y]|$$
$$+ \|\sigma''_{ik}(r,X)(Y)\|_V \sup_{u\leq r}|\psi_u^s e_j|.$$

By Burkholder's inequality, if $\|Y\|_\tau \leq 1$, a.s.,

$$E \sup_{u\leq t} [\psi_u^s(X)'e_j,Y]|^q \leq c_{3.12}(q,\tau)E\Big(\int_0^t \text{trace}(a'(s,r,X,Y)^* a'(s,r,X,Y))dr\Big)^{\frac{q}{2}}$$

$$\leq c_{3.13}(q,\tau)E\int_0^t \sup_{u\leq r}|[\psi_u^s(X)'e_j,Y]|^q dr + c_{3.14}(q,\tau)E\int_0^t \sup_{u\leq r}|\psi_u^s|^q dr\,\|Y\|_\tau^q.$$

Using 3.1 and Gronwall completes the proof. □

The first step in our integration by parts is

(3.4) PROPOSITION. *Suppose* g *is a bounded differentiable function whose gradient* Dg *is also bounded. Suppose for each* $s \le \tau$, h_s *is a functional satisfying* (2.4) *with* $c_{2.12}$ *and* $c_{2.13}$ *independent of* s, *such that* $h_s(X)$ *is right continuous in* s, a.s. *Then*

$$E\int_0^\tau Dg(X_s)^* G_s h_s(X) e_k ds = E\int_0^\tau g(X_s)h_s(X)M_s ds - E\int_0^\tau g(X_s)[h_s'(X),G]e_k ds,$$

where $M_t = \int_0^t \sigma_s^{-1} e_k dW_s$.

PROOF. Multiplying both sides of (2.6) by M_τ and taking expectations, we get

$$(3.5) \quad EL(X)M_\tau = E\int_0^\tau E([L',\psi_\cdot^t]|F_t)\sigma_t\sigma_t^{-1}e_k dt = E\int_0^\tau [L',\psi_\cdot^t]e_k dt.$$

Now let $L(X) = \int_0^\tau g(X_s)h_s(X)ds$. $[L',Y]$ is easily seen to be

$$\int_0^\tau Dg(X_s)^* Y_s h_s(X)ds + \int_0^\tau g(X_s)[h_s'(X),Y]ds.$$

Substituting in (3.5),

$$EL(X)M_\tau = E\int_0^\tau\int_0^\tau Dg(X_s)^*\psi_s^t h_s(X)ds e_k dt + E\int_0^\tau\int_0^\tau g(X_s)[h_s'(X),\psi_\cdot^t]ds e_k dt$$

$$= E\int_0^\tau Dg(X_s)^* G_s h_s(X)e_k ds + E\int_0^\tau g(X_s)[h_s'(X),G]e_k ds,$$

using Fubini and recalling that $\psi_s^t = 0$ if $s < t$, hence $\int_0^\tau \psi_s^t dt = G_s$.

Finally, let $L_t = \int_0^t g(X_s)h_s(X)ds$. L_t is a process of bounded variation. Integrating by parts ([8, p. 285]),

$$EL_\tau M_\tau = E\int_0^\tau L_s dM_s + E\int_0^\tau M_s dL_s + E\langle M,L\rangle_\tau = E\int_0^\tau g(X_s)h_s(X)M_s ds. \qquad \square$$

We next show that we can weaken the assumptions on h.

(3.6) PROPOSITION. *Suppose for each* n, $L_n(X) = \int_0^\tau g(X_s)h_n(s,X)ds$, *where*

(i) *for each* n, $h_n(s)$ *is a functional satisfying* (2.4) *with* $c_{2.12}$ *and* $c_{2.13}$ *independent of* s *(but not necessarily of* n*), such that* $h_n(s,X)$ *is right continuous in* s, a.s.;

(ii) *there is a constant* $c_{3.15}$ *such that* $sh_n(s,X) \le c_{3.15}$, a.s. *for all* n *for all* $s \le \tau$; *and* $sh_n(s,X)$ *converges to a functional* $sh_s(X)$, a.s. *for almost all* $s \le \tau$;

(iii) $c_{3.16}(q) = \sup_{s\le\tau}\sup_n |E\,[h_n'(s),G]|^q < \infty$.

Let $\varepsilon > 0$, $1 < p < 2$, *and* $\beta = 1 + \varepsilon - p/2$. *Then*

$$\left|E\int_0^\tau Dg(X_s)^*G_sh_s(X)e_kds\right| \le c_{3.17}(\tau,\varepsilon,p)\left(E\int_0^\tau |g|^p(X_s)s^{-\beta}ds\right)^{1/p}.$$

PROOF. First of all, in (3.4) replace h_s by $h_n(s)$. Since $sh_n(s,X)$ converges boundedly to $sh_s(X)$ and $|s^{-1}G_s|^q = |J_s|^q$ has finite expectation,

$$E\int_0^\tau Dg(X_s)^*G_sh_n(s,X)e_kds \to E\int_0^\tau Dg(X_s)^*G_sh_s(X)e_kds.$$

Secondly, using Hölder and (iii),

$$E\int_0^\tau g(X_s)[h_n'(s,X),G]e_kds \le c_{3.18}\left(E\int_0^\tau |g|^p(X_s)ds\right)^{1/p}$$

$$\le c_{3.18}\tau^{\beta/p}\left(E\int_0^\tau |g|^p(X_s)s^{-\beta}ds\right)^{1/p}.$$

Thirdly,

$$E\int_0^\tau g(X_s)h_n(s,X)M_sds = E\int_0^\tau g(X_s)s^{-\beta/p}(sh_n(s,X))M_ss^{\beta/p-1}ds$$

$$\leq c_{3.19}\Big(E\int_0^\tau |g|^p(X_s)s^{-\beta}ds\Big)^{1/p}\Big(E\int_0^\tau |M_s|^q s^{(\beta/p-1)q}ds\Big)^{1/q},$$

using Hölder and (ii).

By Burkholder's inequality, $E|M_s|^q \leq c_{3.20}(q)s^{q/2}$. Our main result follows since $\int_0^\tau s^{q/2}s^{(\beta/p-1)q}ds \leq c_{3.20}(\varepsilon,p,\tau) < \infty$, noting that $q/2 + (\beta/p - 1)q = -1 + q\varepsilon/p > -1$. □

One would like to let $h_s(X) = G_s^{-1}$, but G_s^{-1} need not, except in the Markov case, be integrable. So we must construct a different h_s. The idea behind what follows is to stop the process X before $|G_s^{-1}|$ gets too large. Stopping at an ordinary stopping time would destroy the Fréchet differentiability of either L or σ; indeed, we stop at a randomized stopping time.

Let Ψ be a C^∞ function on $[0,\infty)$ such that $\Psi(x) = 1$ if $x \leq 2/3$, $\Psi(x) = 2x$ if $x \geq 3/4$, and $\Psi(x) \geq 1$ for all x. For any $d\times d$ matrix K, let $\Gamma(K) = I + (K-I)/\Psi(|K-I|)$.

Since $|\Gamma(K)-I| \leq 3/4$, $\Gamma(K)$ is invertible with inverse $\sum_{n=0}^\infty (I - \Gamma(K))^n$ and $|\Gamma(K)^{-1}| \leq 4$. Furthermore, if $|K-I| < 2/3$, $\Gamma(K) = K$.

Let $m_s = \sup_{r\leq s}|J_r - I|$. Let $\theta_0 = 1/4$, $\theta_1 = 1/2$, hence $(\theta_1 - \theta_0)^{-1} = 4$. Let $S_\theta = \inf\{t\colon m_t \geq \theta\}$. Observe that $m_s < \theta$ if and only if $s < S_\theta$, and if $s < S_\theta$, $\theta \leq \theta_1$, $\Gamma(J_s) = J_s$.

Define $h_s(X) = 4(m_s \vee \theta_1 - m_s \vee \theta_0)(\Gamma(J_s)^{-1})_{ki}s^{-1}$, k,i fixed. Note $|sh_s| \leq 16\theta_1$.

The main result of this section is

(3.7) THEOREM. *Let* $\varepsilon > 0$, $1 < p < 2$, $\beta = 1 + \varepsilon - p/2$. *Then*

$$\Big|E(\theta_1 - \theta_0)^{-1}\int_{\theta_0}^{\theta_1}\int_0^{S_\theta\wedge\tau} D_i g(X_s)ds\,d\theta\Big| \leq c_{3.21}(p,\tau,\varepsilon)\Big(E\int_0^\tau|g|^p(X_s)s^{-\beta}ds\Big)^{1/p}.$$

The value of $c_{3.21}$ *depends on* X *and* σ *only through* $c_{2.2}$, $c_{2.3}$, $c_{2.4}$, *and* $c_{2.5}$ *and may be chosen so as to be an increasing function of* τ.

PROOF. Let

$$\begin{aligned}
I_k &= \int_0^\tau Dg(X_s)^* G_s h_s(X) e_k ds \\
&= 4 \int_0^\tau Dg(X_s)^* G_s s^{-1} \Gamma(J_s)^{-1}_{ki} \Big(\int_{\theta_0}^{\theta_1} 1_{(\theta > m_s)} d\theta\Big) e_k ds \\
&= 4 \int_{\theta_0}^{\theta_1} \int_0^\tau Dg(X_s)^* G_s s^{-1} \Gamma(J_s)^{-1}_{ki} 1_{(s < S_\theta)} ds\ e_k d\theta \\
&= 4 \int_{\theta_0}^{\theta_1} \int_0^{\tau \wedge S_\theta} Dg(X_s)^* G_s (G_s^{-1})_{ki}\ e_k ds\ d\theta \\
&= (\theta_1 - \theta_0)^{-1} \int_{\theta_0}^{\theta_1} \int_0^{\tau \wedge S_\theta} (G_s^{-1})_{ki} \sum_{j=1}^d (G_s)_{jk} D_j g(X_s) ds\ d\theta,
\end{aligned}$$

using the fact that $\Gamma(J_s) = J_s$ if $s < S_{\theta_1}$. Here $D_j g(X_s)$ is the j^{th} coordinate of $Dg(X_s)$. If we sum over k, $|E \sum_{k=1}^d I_k|$ will be the desired left-hand side, since

$$\sum_j \sum_k (G_s)_{jk} (G_s^{-1})_{ki} D_j g(X_s) = D_i g(X_s).$$

It only remains to construct the appropriate sequence h_n and to apply (3.6). For fixed n, let $\Lambda_{n,\delta}\colon (\mathbb{R}^{d\times d})^n \to \mathbb{R}$ be continuously differentiable such that for all δ, $(y_1,\ldots,y_n)$, and $(z_1,\ldots,z_n)$,

$$|\Lambda_{n,\delta}(y_1,\ldots,y_n) - \Lambda_{n,\delta}(z_1,\ldots,z_n)| \le 2 \max(|y_1 - z_1|,\ldots,|y_n - z_n|)$$

and such that

$$\Lambda_{n,\delta}(y_1,\ldots,y_n) \to \max(|y_1|,\ldots,|y_n|) \quad \text{as } \delta \to 0$$

uniformly on compact sets. Let

$$m_s^n = \Lambda_{n,\delta_n}(J_{(\tau/n)\wedge s} - I, J_{(2\tau/n)\wedge s} - I, \dots, J_{(n\tau/n)\wedge s} - I),$$

where $\delta_n \to 0$ fast enough so that for almost all s, $m_s^n \to m_s$, a.s. Let $h_n(s,X) = (m_s^n \wedge \theta_1 - m_s^n \wedge \theta_0)(\Gamma(J_s)^{-1})_{ki}(s^{-1} \wedge n)$.

Clearly (ii) of (3.6) is satisfied. Since J_s is P-Fréchet differentiable, so is m_s^n. $\| m_s^{n\prime}\|_V \le c_{3.22} \sup_{s\le\tau} |\| J_s'\|_V|$. Since Ψ is smooth, $\Gamma(J_s)$ is P-Fréchet differentiable, $(\Gamma(J_s)^{-1})' = \sum_{n=0}^{\infty} n(I - \Gamma(J_s))^{n-1} \cdot(\Gamma(J_s))'$. Hence $|\|(\Gamma(J_s)^{-1})'\|_V| \le c_{3.23} |\| J_s'\|_V|$. Thus (3.6i) is satisfied.

Finally,

$$\begin{aligned} E|[h_n'(s,X), G]|^q &\le E(\sup_{s\le\tau}|s\|h_n'(s,X)\|_V| \sup_{s\le\tau}|J_s|)^q \\ &\le (E \sup_{s\le\tau}|s\|h_n'(s,X)\|_V|^{2q})^{\frac12}(E \sup_{s\le\tau}|J_s|^{2q})^{\frac12} \\ &\le c_{3.24} < \infty \end{aligned}$$

by (3.1) and (3.3).

The assertion about $c_{3.21}$ follows by showing the corresponding assertions for $c_{3.1}, c_{3.2}, \dots$, noting that $|\sigma_t^{-1}| \le c_{2.3}^{-1}$. □

4.0 Densities of potentials

Throughout this section we assume σ satisfies A1 and that R1,2,3 hold. We begin by proving an elementary lemma that will be needed to handle some technical points later on. This lemma is an immediate corollary of Krylov's results on the existence of densities, but nothing so powerful is needed.

(4.1) *LEMMA.* (i) *For all* x, $E\int_0^\tau 1_{B(\varepsilon,x)}(X_s)ds \to 0$ *as* $\varepsilon \to 0$;

(ii) *For all* $r > 0$ *and all* x,

$$E\int_0^\tau 1_{[B(r+\varepsilon,x)-B(r,x)]}(X_s)ds \to 0 \quad as \quad \varepsilon \to 0.$$

PROOF. Just for the duration of this proof, let us assume without loss of generality that $X_0 = x_0 = 0$. Let X_t^i be the i^{th} coordinate of X_t.

(i) Let $Y_t = X_t^1$. Y_t is a martingale whose diffusion coefficient $d\langle Y,Y\rangle_t/dt = \sigma_{11} > c_{4.1} > 0$. The quantity in question is less than or equal to $E\int_0^\tau 1_{[y-\varepsilon,y+\varepsilon]}(Y_s)ds$, where y is the first coordinate of x.

Let f be a function such that $f(0) = f'(0) = 0$, f'' exists and is continuous, and $1_{[y-\varepsilon,y+\varepsilon]} \le f'' \le 1_{[y-2\varepsilon,y+2\varepsilon]}$. Ito's lemma gives

$$\tfrac{1}{2}c_{4.1}E\int_0^\tau 1_{[y-\varepsilon,y+\varepsilon]}(Y_s)ds \le \tfrac{1}{2}E\int_0^\tau f''(Y_s)d\langle Y,Y\rangle_s = Ef(Y_\tau) - Ef(Y_0).$$

Since $\|f'\| \le 4\varepsilon$, $Ef(Y_\tau) - Ef(Y_0) \le \|f'\| E|Y_\tau| \to 0$ as $\varepsilon \to 0$.

(ii) Let $Y_t = |X_t - x|^2$. Using Ito's lemma, we get that Y_t is a semimartingale with drift $\sum_i 2\langle X^i - x_i, X^i - x_i\rangle_t \le c_{4.2}t$ and

$$\langle Y,Y\rangle_t = \sum_{i,j}\int_0^\tau (X_s^i - x_i)(X_s^j - x_j)\sigma_{ij}ds$$

$$\ge c_{4.3}\int_0^\tau |X_s - x|^2 ds = c_{4.3}\int_0^\tau Y_s ds,$$

using the strict ellipticity of σ. (Here x_i is the i^{th} coordinate of x.)

Let f be a function such that $f(0) = f'(0) = 0$, f'' exists and is continuous, and $1_{[r-\varepsilon,r+\varepsilon]} \le f'' \le 1_{[r-2\varepsilon,r+2\varepsilon]}$. Applying Ito's lemma to $f(Y_t)$,

$$Ef(Y_\tau) - Ef(Y_0) \ge -E\int_0^\tau f'(Y_s)c_{4.2}ds + \tfrac{1}{2}c_{4.3}E\int_0^\tau 1_{[r-\varepsilon,r+\varepsilon]}(Y_s)(r-\varepsilon)ds.$$

Since $\|f'\| \le 4\varepsilon$ and $E|Y_\tau| < \infty$, we get the result by letting $\varepsilon \to 0$. □

Let $p_t(x,y)$ be the transition density (with respect to Lebesgue measure) for a Bessel process of index ν (recall that such a process Z_t^ν satisfies $dZ_t^\nu = dW_t + (\nu - 1)/(2Z_t)dt)$. We need an estimate for $U_{\tau,\beta,\nu}(x,y) = \int_0^\tau t^{-\beta} p_t(x,y)dt,\ 0 < \beta < 1/2,\ \nu > 2$.

(4.2) *PROPOSITION. Let* $0 \le \lambda \le 1,\ 0 < \beta < 1/2,\ \nu > 2$. *Then*

$$U_{\tau,\beta,\nu}(x,y) \le c_{4.4}(\nu,\tau,\beta)[y^{\nu-1-2(1-\lambda)\beta} 1_{(y\le 1)} + y 1_{(y>1)}][1 + x^{2-\nu-2\lambda\beta}].$$

PROOF. We have that

$$p_t(x,y) = t^{-1} e^{-(x^2 + y^2/2t)} (xy)^{1-\nu/2}\ I_{\nu/2-1}(xy/t) y^{\nu-1} dt, \tag{4.3}$$

where $I_p(u)$ is the modified Bessel function [5, p. 225].

It is well known that

$$I_p(u) \sim c_{4.5}(p)u^p,\ u \to 0; \quad I_p(u) \sim c_{4.6}(p)e^u/u^{\frac{1}{2}},\ \ u \to \infty.$$

Thus, there exists $c_{4.7}(p)$ such that if $u \le 1$, $I_p(u) \le c_{4.7}(p)u^p$, while if $u \ge 1$, $I_p(u) \le c_{4.7}(p)e^u/u^{\frac{1}{2}}$. Applying this to (4.3), we get

$$\begin{aligned} U_{\tau,\beta,\nu}(x,y) &\le c_{4.7}(\nu/2 - 1)y^{\nu-1} \int_{xy\wedge\tau}^{\tau} t^{-\nu/2-\beta} e^{-(x^2+y^2)/2t} dt \\ &\quad + c_{4.7}(\nu/2 - 1)(y/x)^{(\nu-1)/2} \int_0^{xy\wedge\tau} t^{-\frac{1}{2}-\beta} e^{-(x-y)^2/2t} dt \\ &= J_1 + J_2. \end{aligned}$$

$$\begin{aligned} J_1 &\le c_{4.7}(\nu/2 - 1)y^{\nu-1} \int_0^\infty t^{-\nu/2-\beta} e^{-(x^2+y^2)/2t} dt \\ &\le c_{4.8}(\nu,\beta)y^{\nu-1}(x^2 + y^2)^{(2-\nu-2\beta)/2}. \end{aligned}$$

If $y \geq 1$, this last expression is

$$\leq c_{4.8}(\nu,\beta)y^{\nu-1}y^{2-\nu-2\beta} \leq c_{4.8}(\nu,\beta)y.$$

If $y \leq 1$,

$$J_1 \leq c_{4.8}(\nu,\beta)y^{\nu-1-2(1-\lambda)\beta}x^{2-\nu-2\lambda\beta}, \quad 0 \leq \lambda \leq 1.$$

To investigate J_2, note that

$$e^{-(x-y)^2/2t} \leq e^{-(x-y)^2/2xy} = e^2 e^{-\frac{1}{2}(x/y+y/x)},$$

since $t \leq xy$. Note also that

$$e^{-\frac{1}{2}(x/y+y/x)}(x/y)^p \leq c_{4.9}(\nu) < \infty \quad \text{for } 0 < x,y \text{ and } -\nu-3 \leq p \leq \nu+3.$$

When $y \geq 1$,

$$J_2 \leq c_{4.10}(\nu)\int_0^{xy\wedge\tau} t^{-\frac{1}{2}-\beta}dt = c_{4.11}(\nu,\tau,\beta).$$

When $y \leq 1$,

$$J_2 \leq c_{4.12}(\nu)y^{\nu-1}x^{2-\nu}(xy)^{-\frac{1}{2}}(x/y)^{(1-2\lambda)\beta}$$

$$\cdot\int_0^{xy\wedge\tau} t^{-\frac{1}{2}-\beta}e^{-(x-y)^2/2t}(x/y)^{(\nu-2)/2-(1-2\lambda)\beta}dt$$

$$\leq c_{4.13}(\nu,\beta)y^{\nu-1}x^{2-\nu}(xy)^{-\frac{1}{2}}(x/y)^{(1-2\lambda)\beta}\int_0^{xy} t^{-\frac{1}{2}-\beta}dt$$

$$\leq c_{4.14}(\nu,\beta)y^{\nu-1-2(1-\lambda)\beta}x^{2-\nu-2\lambda\beta}.$$

Summing, we get our result. □

We next derive a connection between the right-hand side of (3.7) and Bessel processes. Recall $X_0 = x_0$, a.s.

(4.4) *THEOREM. Let* $w = (w_1,\ldots,w_d)$ *be fixed and suppose* $w \neq x_0$. *Suppose* $\hat{g}(y) = \sup_{|z-w|=y} |g(z)|$ *is nonincreasing in* y. *Suppose* $0 < \beta < 1/2$ *and*

$$2 < \nu < 1 + (1 + c_{2.11})^{-2}[(d-1) - (d+1)c_{2.11}].$$

Then,

$$E \int_0^\tau |g|^p(X_t)t^{-\beta}dt \leq c_{4.15} \int_0^\infty |\hat{g}|^p(y)U_{2\tau,\beta,\nu}(|x_0 - w|,y)dy.$$

PROOF. Let $Y_t = |X_t - w|$. Using Ito's lemma for $t \leq \inf\{t:|X_t - w| < 1/n\}$, we get that Y_t satisfies $Y_0 = |x_0 - w|$ and

$$dY_t = \sum_{i,j} \frac{X_t^i - w_i}{|X_t - w|} \sigma_{ij} dW_t^j + \frac{1}{2} \sum_{i,j} \frac{1_{(i=j)}|X_t - w|^2 - (X_t^i - w_i)(X_t^j - w_j)}{|X_t - w|^3} \sigma_{ij} dt.$$

Thus Y_t is a semimartingale with diffusion coefficient

$$a_t = d\langle Y,Y\rangle/dt = (X_t - w)^*\sigma\sigma^*(X_t - w)/|X_t - w|^2$$

and drift coefficient $b_t/(2Y_t)$, where

$$b_t = \text{trace } \sigma - ((X_t - w)^*\sigma(X_t - w)/|X_t - w|^2).$$

Using R1, $b_t/a_t > \nu - 1$.

We now time change Y_t. Let $C_t = \langle Y,Y\rangle_t^{-1}$. Note $\frac{1}{2} \leq dC_t/dt \leq 2$. Let $Z_t = Y_{C_t}$. Z_t is a semimartingale adapted to $G_t = \sigma(X_{C_s};\ s \leq t)$.

Checking that

$$Z_t - \frac{1}{2} \int_0^t b_{C_s}/(a_{C_s} Z_s)ds = Y_{C_t} - \frac{1}{2} \int_0^{C_t} b_r/Y_r dr$$

is a G_t-martingale with $\langle Z,Z\rangle_t = t$, we see that Z_t is equal to a one-dimensional Brownian motion $\hat{W}_t$ plus $\frac{1}{2}\int_0^t b_{C_s}/(a_{C_s}Z_s)ds$. By a comparison theorem for stochastic differential equations (for example, see [5, p. 352]), we conclude that $Z_t \geq Z_t^\nu$ for all $t \leq \inf\{t: |X_t - w| < 1/n\}$, where Z_t^ν is a Bessel process of index ν defined in terms of $\hat{W}_t$. Since $\nu > 2$, Z_t^ν never hits 0, hence Z_t never does either, hence $Z_t \geq Z_t^\nu$ for all t.

Finally, using $\frac{1}{2} \leq dC_s/ds \leq 2$ and $|\hat{g}|$ nonincreasing,

$$\begin{aligned}
E\int_0^\tau |g|^p(X_t)t^{-\beta}dt &\leq E\int_0^\tau |\hat{g}|^p(Y_t)t^{-\beta}dt \\
&= E\int_0^{\langle Y,Y\rangle_\tau} |\hat{g}|^p(Z_t)C_t^{-\beta}dC_t \\
&\leq c_{4.16}(\beta)E\int_0^{2\tau}|\hat{g}|^p(Z_t)t^{-\beta}dt \\
&\leq c_{4.16}(\beta)E\int_0^{2\tau}|\hat{g}|^p(Z_t^\nu)t^{-\beta}dt \\
&\leq c_{4.16}(\beta)\int_0^\infty|\hat{g}|^p(y)U_{2\tau,\beta,\nu}(|x_0 - w|,y)dy. \qquad \square
\end{aligned}$$

We come now to the main result of this section. Let S_θ, θ_0, θ_1 be as in Section 3. Define

$$\lambda(F) = E(\theta_1 - \theta_0)^{-1}\int_{\theta_0}^{\theta_1}\int_0^{\tau\wedge S_\tau} 1_F(X_s)ds\, d\tau.$$

(4.5) *THEOREM. Suppose* $1 < p < 2$, $\varepsilon < 0$, $0 < \lambda < 1$, $d-1 < \nu < d$, *and* ν *satisfies the hypothesis of* (4.4). *Let*

$$\beta = 1 + \varepsilon - p/2, \quad \gamma = (1-d)p + \nu - 2(1-\lambda)\beta,$$

and suppose $\beta < \frac{1}{2}$, $\gamma > 0$.

Then, $\lambda(F)$ *has a density* $\rho(z)$ *with respect to Lebesgue measure which satisfies*

(i) $|\rho(z)| \le c_{4.17}(1 + |z - x_0|^{(2-\nu-2\lambda\beta)/p})$;

and

(ii) $|\rho(z) - \rho(z')| \le c_{4.18}\zeta^{-c_{4.19}}|z - z'|^{\gamma/(p+\lambda)}$,

where $\zeta = \min(|z - x_0|, |z' - x_0|, 1)$.

COMMENT. In (5.2) we show how to select p, ε, λ, and ν to satisfy the constraints we have put on them.

PROOF. Let $\rho_r(z) = r^{-d}\lambda(B(r,z))$. The first step is to show that $\rho_r(z)$ converges as $r \to 0$. Fix z. Let

$$g_{ir}(y) = (y_i - z_i)/|y - z|^d \quad \text{if } |y - z| > r,$$

$$= (y_i - z_i)/r^d \quad \text{if } |y - z| \le r.$$

Note that $|g_{ir}(y)| \le |y - z|^{1-d}$ and that $\sum_i D_i g_{ir}(y) = dr^{-d}$ if $|y - z| < r$, 0 if $|y - z| > r$.

(3.7) is not valid for g_{ir} since g_{ir} is not differentiable at $|y - z| = r$. However, g_{ir} is uniformly Lipschitz; an easy approximation argument together with (4.1ii) shows that (3.7) may be applied to g_{ir}. We do so, and summing over i, we get

$$(4.6) \qquad |d\rho_r(z)| \le c_{4.20}(p,\tau,\varepsilon)\sum_{i=1}^{d}\Big(E\int_0^\tau |g_{ir}|^p(X_t)t^{-\beta}dt\Big)^{1/p}$$

$$\le c_{4.21}\Big(E\int_0^\tau |X_t - z|^{(1-d)p}\, t^{-\beta}dt\Big)^{1/p}$$

$$\le c_{4.22}\Big(\int_0^\infty y^{(1-d)p}\, U_{2\tau,\beta,\nu}(|x_0 - z|, y)dy\Big)^{1/p}$$

$$\le c_{4.23}\big(1 + |x_0 - z|^{2-\nu-2\lambda\beta}\big)^{1/p},$$

using (4.2), R3, and the hypothesis that $\gamma > 0$.

Now applying (3.7) to $g_{ir} - g_{is}$ and summing over i, we get

$$(4.7)\quad d|\rho_r(z) - \rho_s(z)| \le c_{4.24}\sum_{i=1}^{d}\Big(E\int_0^\tau |g_{ir} - g_{is}|^p(X_t)t^{-\beta}dt\Big)^{1/p}$$

$$\le c_{4.25}\Big(E\int_0^\tau |X_t - z|^{p(1-d)}1_{[0,r\vee s]}(|X_t - z|)t^{-\beta}dt\Big)^{1/p}$$

$$\le c_{4.26}\Big(\int_0^{r\vee s} y^{(1-d)p}\, U_{2\tau,\beta,\nu}(|x_0 - z|,y)dy\Big)^{1/p}$$

$$\le c_{4.27}\Big(\big(|x_0 - z|^{2-\nu-2\lambda\beta} + 1\big)(r\vee s)^\gamma\Big)^{1/p}.$$

Here we use the fact that $g_{ir} - g_{is}$ is 0 if $|y - z| \ge r\vee s$.

Thus, $\rho_r(z)$ converges, say to $\rho(z)$. Taking the limit in (4.6) gives (i), while taking the limit in (4.7) gives

$$(4.8)\qquad |\rho_r(z) - \rho(z)| \le c_{4.28}\big(|x_0 - z|^{(2-\nu-2\lambda\beta)/p} + 1\big)r^{\gamma/p}.$$

The next step is to show $\rho(z)$ is continuous in z.

$$\lambda(B(r,z') - B(r,z)) \le \lambda(B(r + |z - z'|,z) - B(r,z)) \to 0$$

as $|z - z'| \to 0$ by (4.1ii). But

$$|\rho_r(z) - \rho_r(z')| \le r^{-d}\lambda(B(r,z)\,\Delta B(r,z')),$$

and so $\rho_r(z)$ is continuous in z. (Δ denotes symmetric difference.) By (4.8), $\rho(z)$ is the uniform limit of $\rho_r(z)$ in regions bounded away from x_0, and hence ρ is continuous in such a region.

It follows (cf. [9, Ch. 8]) that ρ is a density for λ in any region bounded away from x_0. Since (4.1i) shows that λ assigns no mass to $\{x_0\}$, ρ is a density for λ over all of $\mathbb{R}^d$.

We now establish (ii). Let $\zeta = \min(|x_0 - z|,|x_0 - z'|,1)$, $\delta = |z - z'|$, and suppose $r,\delta < \zeta/6$. Since $B(r,z') - B(r,z) \subseteq B(r+\delta,z) - B(r,z)$, it follows that the Lebesgue measure of $B(r,z')\,\Delta B(r,z)$ is $\le c_{4.29}\,\delta r^{d-1}$ as long as $\delta < r/2$.

Then $|\rho_r(z) - \rho_r(z')| \le r^{-d}\lambda(B(r,z)\,\Delta B(r,z'))$

$$= r^{-d}\int_{B(r,z)\Delta B(r,z')}\rho(w)dw$$

$$\le c_{4.30}r^{-d}\delta r^{d-1}\,\zeta^{(2-\nu-2\lambda\beta)/p}$$

$$\le c_{4.31}r^{-1}\,\delta\zeta^{-c_{4.32}},$$

since $\rho(w)$ is bounded as long as $|w - x_0| > \zeta/3$.

Together with (4.8),

$$|\rho(z) - \rho(z')| \le c_{4.33}\zeta^{-c_{4.32}}(\delta r^{-1} + r^{\gamma/p}).$$

Letting $r = \delta^{p/(p+\gamma)}$, $\delta = r^{(p+\gamma)/p} < r/2$ if δ is small enough, and then

$$|\rho(z) - \rho(z')| \le c_{4.34}\zeta^{-c_{4.32}}\,\zeta^{\gamma/(p+\gamma)}.$$

Since $\gamma/(p+\gamma) > 0$, this proves (ii). □

5.0 Potentials of occupation times

In this section we will assume that R1,2,3 hold. We will also assume that σ satisfies A1 until (5.4), after which we will allow σ to satisfy A2 instead.

We state some elementary results. Recall that ϕ_b is the density of a normal random variable with mean 0 and variance b.

(5.1) *PROPOSITION. Let* λ, μ *be measures satisfying* M1:

$$\lambda(\mathbb{R}^d),\ \mu(\mathbb{R}^d) \le c_{2.6}$$

$$\lambda(B(r,x)),\mu(B(r,x)) \le c_{2.7} r^{d-2+c_{2.8}} \quad \textit{for all } x,r.$$

Then, (i) $\mu * \phi_b(dx)$ *has density* $\int \phi_b(x-y)\mu(dy)$, *which is bounded and uniformly continuous in* x;

(ii) $\mu * \phi_b$, $\lambda * \phi_b$ *satisfy* M1 *with the same constants* $c_{2.6}$, $c_{2.7}$, *and* $c_{2.8}$;

(iii) $\|\mu * \phi_a - \mu * \phi_b\|_{W-\alpha} \le c_{5.1}|b-a|^\alpha$, *where* $c_{5.1}$ *depends only on* α, $c_{2.6}$, $c_{2.7}$, *and* $c_{2.8}$;

(iv) *If* $\|\mu - \lambda\|_{W-\alpha} \le c_{5.2}$, *then* $\|\mu * \phi_b - \lambda * \phi_b\|_{W-\alpha} \le c_{5.2}$;

Let $p,\varepsilon,\nu,\beta,\gamma,\rho(z)$ *be as in* (4.3). *Suppose*

$$(\nu - 2 + 2\lambda\beta)/p < d - 2 + c_{2.8}.$$

Then

(v) $\int \rho(z)\lambda(dz) \le c_{5.3}$, *where the value of* $c_{5.3}$ *depends on* λ *only through* $c_{2.6}$, $c_{2.7}$, *and* $c_{2.8}$;

(vi) $\int \rho(z)(\lambda-\mu)(dz) \le c_{5.4}\|\lambda-\mu\|^{c_{3.5}}_{W-\gamma/(p+\gamma)}$, *where the value of* $c_{5.4}$ *depends on* λ *and* μ *only through* $c_{2.6}$, $c_{2.7}$, *and* $c_{2.8}$.

PROOF. Very similar results are proved in section 3 of [1]. □

It is not clear that $\nu,\beta,\lambda,\varepsilon,p$ may be selected to satisfy all the constraints we have placed on them. We pause to verify that we can do so.

(5.2) *PROPOSITION. If* R1 *holds*, ε,p,ν, *and* λ *may be selected so that*

(i) $\beta < 1/2$;

(ii) $\gamma > 0$;

(iii) $(\nu - 2 + 2\lambda\beta)/p < d - 2 + c_{2.8}$, where β and γ were defined in (4.5).

PROOF. Under R1, we may select ν in (4.4) so that

$$d - 1/3(c_{2.8} \wedge 1) < \nu < d + 1/3(c_{2.8} \wedge 1)$$

by taking ν less than but close to $1 + (1 + c_{2.11})^{-2}[(d-1) - (d+1)c_{2.11}]$.

Let $\lambda = \frac{1}{2}(c_{2.8} \wedge 1)$. Take ε small enough so that $\nu + \lambda - 2\varepsilon(1-\lambda) > d$, choose p close to 1 so that

$$1 < p < (\nu - (1-\lambda)(2 + 2\varepsilon))/(d + \lambda - 2),$$

and then, if necessary, choose ε even smaller so that $1 + \varepsilon - p/2 < 1/2$. (i), (ii), and (iii) now follow. □

If we apply (5.1v,vi) to $\mu * \phi_b, \mu * \phi_a$, we get

$$\left|E(\theta_1 - \theta_0)^{-1} \int_{\theta_0}^{\theta_1} \int_0^{S_\theta \wedge \tau} \mu * \phi_b(X_s) ds\, d\theta\right| \le c_{5.3} \quad \text{and}$$

$$\left|E(\theta_1 - \theta_0)^{-1} \int_{\theta_0}^{\theta_1} \int_0^{S_\theta \wedge \tau} (\mu * \phi_b - \mu * \phi_a)(X_s) ds\, d\theta\right| \le c_{5.6}|b - a|^{c_{5.7}}.$$

Here $\mu * \phi_b(x)$ is the density of $\mu * \phi_b$ evaluated at x.

Let $\tilde{\theta}_0$ be a random variable that is uniform on $[\theta_0, \theta_1]$ and independent of W_t. In the usual way, we enlarge the probability space and redefine the probability P and the sigma fields F_t to ensure that this is possible. If $S^0 = \inf\{t: m_t \ge \tilde{\theta}_0\}$, what we have proved is that

$$\left|E \int_0^{S^0 \wedge \tau} \mu * \phi_b(X_s) ds\right| \le c_{5.3}$$

$$\left|E \int_0^{S_0 \wedge \tau} (\mu * \phi_b - \mu * \phi_a)(X_s) ds\right| \le c_{5.6}|b - a|^{c_{5.7}}.$$

We now attempt to eliminate the S^0 from the above two expressions.

(5.3) *PROPOSITION*. (i) $\left|E \int_0^{\tau} \mu * \phi_b(X_s)ds\right| \le c_{5.8}$

(ii) $\left|E \int_0^{\tau} (\mu * \phi_b - \mu * \phi_a)(X_s)ds\right| \le c_{5.9}|b - a|^{c_{5.7}}$, *where* $c_{5.7}$, $c_{5.8}$, *and* $c_{5.9}$ *depend on* X, σ, *and* μ *only through* $c_{2.2}$ - $c_{2.8}$, $c_{2.11}$.

PROOF. We prove (i), (ii) being similar. Let $Q_{\omega}^0(\cdot)$ be a regular conditional probability distribution (r.c.p.d) for $E(\cdot|F_{S^0})$. Since X_{S^0} and S^0 are F_{S^0}-measurable, we claim that for each ω', $\hat{X}_t(\omega) = X_{S^0+t}(\omega)$ is a d-dimensional martingale under $Q_{\omega'}^0$, and the diffusion coefficients $\hat{\sigma}$ will still satisfy A1 and R1. In fact, under $Q_{\omega'}^0$, $\hat{\sigma}_t(\omega) = \sigma_{t+S^0(\omega')}(Y)$ where $Y_s(\omega) = X(\omega')$ if $s \le S^0(\omega')$, $Y_s(\omega) = \hat{X}_{s-S^0(\omega')}(\omega)$ if $s > S^0(\omega')$.

Now let $S^1 = \inf\{t: \hat{m}_t \ge \tilde{\theta}_1\}$, where $\tilde{\theta}_1$ is an r.v. that is uniform on $[\theta_0,\theta_1]$, and, under Q_{ω}^0, independent of $\tilde{\theta}_0, W_t$; $\hat{m}_t$ is defined in a manner analogous to m_t. We then get, as before,

$$\left|Q_{\omega}^0 \int_0^{S^1\wedge(\tau-S^0)} \mu * \phi_b(X_s)ds\right| \le c_{5.3}.$$

Then

$$\left|E \int_0^{(S^0+S^1)\wedge\tau} \mu * \phi_b(X_s)ds\right| \le \left|E \int_0^{S^0\wedge\tau} \mu * \phi_b(X_s)ds\right|$$

$$+ \left|E\left(\left|Q_{\omega}^0 \int_0^{S^1\wedge(\tau-S^0)} \mu * \phi_b(\hat{X}_s)ds\right|; \ S^0 \le \tau\right)\right|$$

$$\le c_{5.3} + c_{5.3}\, P(S^0 \le \tau).$$

Repeat, letting Q_{ω}^1 be an r.c.p.d. for $E(\cdot|F_{S^0+S^1})$, etc. By induction, we get

$$\left|E \int_0^{(S^0+\cdots+S^n)\wedge\tau} \mu * \phi_b(X_s)ds\right| \le c_{5.3}(1 +\cdots+ P(S^0 +\cdots+ S^n \le \tau)).$$

To complete the proof, it suffices to show that

(a) $S^0 + \cdots + S^n \to \infty$, a.s., and

(b) $\sum_{n=0}^{\infty} P(S^0 + \cdots + S^n \le \tau) \le c_{5.10} < \infty.$

By (3.1iv),

$$Ee^{-(S^0+S^1)} = Ee^{-S^0}(Q_\omega^0 e^{-S^1}) \le c_{3.4} Ee^{-S^0} \le c_{3.4}{}^2.$$

By induction, $Ee^{-(S^0+\cdots+S^n)} \le c_{3.4}{}^n \to 0$, which proves (a).

$$P(S^0 + \cdots + S^n \le \tau) \le Ee^{-(S^0+\cdots+S^n)}/e^{-\tau} \le e^\tau c_{3.4}{}^n;$$

using Chebyshev, and summing over n gives (b).

(5.4) *PROPOSITION.* (5.3) *holds if* σ *only satisfies* A2.

PROOF. Take a sequence of σ_n's converging to σ as in A2. Let X^n be the solution to $X_0^n = x_0$, $dX_t^n = \sigma_t^n(X^n)dW_t$, and let P^n be the law induced on $C[0,\infty)$ by X^n. Let P^0 be the law induced on $C[0,\infty)$ by X. Since X is the unique pathwise solution of a stochastic differential equation, P^0 satisfies a uniqueness in law property [10], and it is not hard to show that P^n converges weakly to P^0.

Since $\mu * \phi_b$ is continuous, $L(f) = \int_0^\tau \mu * \phi_b(f(s))ds$ is a continuous functional on $C[0,\tau]$. But

$$\left|E\int_0^\tau \mu * \phi_b(X_s)ds\right| = \left|P^0 L(X)\right| = \lim_{n\to\infty}\left|P^n L(X)\right|$$

$$= \lim_{n\to\infty}\left|E\int_0^\tau \mu * \phi_b(X_s^n)ds\right| \le c_{5.8},$$

independent of n.

The proof of (5.3ii) is similar. □

We also need (5.4) for conditional probabilities.

(5.5) *THEOREM. If* $t \le \tau$,

(i) $\left|E\left(\int_t^\tau \mu * \phi_b(X_s)ds \middle| F_t\right)\right| \le c_{5.8}$, a.s.;

(ii) $\left|E\left(\int_t^\tau (\mu * \phi_b - \mu * \phi_a)(X_s)ds \middle| F_t\right)\right| \le c_{5.9}|b - a|^{c_{5\ 7}}$, a.s.

PROOF. Let Q_ω be an r.c.p.d. for $E(\cdot|F_t)$. Letting $\hat{X}_s = X_{t+s} = X_t + \int_t^{t+s} \sigma(X_r)dW_r$, one checks as above that $(\hat{X}_s, Q_\omega)$ satisfies A2 and R1,2,3, and hence

$$|Q_\omega \int_0^{t-\tau} \mu * \phi_b(\hat{X}_s)ds| \le c_{5.11}(\tau - t). \tag{5.6}$$

$c_{5.11}$ may be chosen to be an increasing function of $\tau - t$, and hence is $\le c_{5.12}(\tau)$, independent of t. But this is just what we needed.

(ii) is similar. □

Our final step is the following.

(5.7) *THEOREM. Suppose* A *and* B *are two increasing processes,* $C = A - B$. *Suppose* $|E(A_\tau - A_t|F_t) + E(B_\tau - B_t|F_t)| \le N$ *for all* $t \le \tau$ *and that* $|U_t| \le \varepsilon < 1$ *for all* $t \le \tau$, *where* $U_t = E(C_\tau - C_t|F_t)$. *Then*

(i) $E(\sup_{t \le \tau}|C_t|)^p \le c_{5.13}(N,p)\varepsilon^{p/2}$ *and*

(ii) $E(\sup_{t \le \tau}|A_t|^p) \le c_{5.14}N^p$.

PROOF. First of all,

$$(C_\tau - C_t)^2 = 2\int_t^\tau (C_\tau - C_s)dC_s,$$

and so,

$$E((C_\tau - C_t)^2|F_t) = 2E\left(\int_t^\tau E(C_\tau - C_s|F_s)dC_s|F_t\right)$$

$$\leq 2\varepsilon E\left(\int_t^\tau d|C_s|\ |F_t\right)$$

$$\leq 2\varepsilon N.$$

Secondly, if $M_t = E(C_\tau|F_t)$, $U_t = M_t - C_t$, and if $t \leq \tau$,

$$E(\langle M,M\rangle_\tau - \langle M,M\rangle_t|F_t) = E((M_\tau - M_t)^2|F_t)$$

$$\leq 2U_t^2 + 2E((C_\tau - C_t)^2|F_t)$$

$$\leq 2\varepsilon^2 + 4\varepsilon N \leq c_{5.15}(N)\varepsilon.$$

Integrating by parts gives

$$\langle M,M\rangle_\tau^n = n\int_0^\tau (\langle M,M\rangle_\tau - \langle M,M\rangle_t)d\langle M,M\rangle_t^{n-1},$$

and

$$E\langle M,M\rangle_\tau^n = nE\int_0^\tau E(\langle M,M\rangle_\tau - \langle M,M\rangle_t|F_t)d\langle M,M\rangle_t^{n-1}$$

$$\leq n\ c_{5.15}(N)\varepsilon\ E\int_0^\tau d\langle M,M\rangle_t^{n-1}.$$

By induction,

$$E\langle M,M\rangle_\tau^p \leq c_{5.15}{}^p(N)\varepsilon^p\ p!.$$

Finally, since $C_t = M_t - U_t$,

$$E\sup_{t\leq\tau}|C_t|^p \leq c_{5.16}(p)E\sup_{t\leq\tau}|M_t|^p + c_{5.16}(p)E\sup_{t\leq\tau}|U_t|^p$$

$$\leq c_{5.17}(p)E\langle M,M\rangle_\tau^{p/2} + c_{5.16}(p)\varepsilon^p \quad \text{(Burkholder)}$$

$$\leq c_{5.18}\varepsilon^{p/2}.$$

(ii) is similar. □

6.0 Proof of theorems

In this section we prove T1,2,3,4 under the assumptions that A2 and R1,2,3 hold.

PROOF of T1. Abbreviate $A_t(\mu * \phi_b)$ by A_t^b. Combining (5.5) and (5.7),

$$E \sup_{t \le \tau} |A_t^b - A_t^a|^p \le c_{6.1}(p)|b - a|^{c_{6.2}p}.$$

Take p large enough so that $c_{6.2}p > 1$. Kolmogorov's theorem then implies that there exist versions of A_t^b such that A_t^b is uniformly jointly continuous, a.s., $0 \le t \le \tau$, $0 < b \le 1$. Since the density of $\mu * \phi_b$ is uniformly continuous in b for b in a closed interval not containing 0, there is, in fact, no need to take versions. It is then immediate that there exists a process $A_t(\mu)$ that is the uniform limit of $A_t(\mu * \phi_b)$, a.s. □

If we replaced $\mu * \phi_b(y)dy$ by some other set of measures converging appropriately to μ, one would like to know that one gets the same limit process $A_t(\mu)$. This will follow from the proof of T2 below.

(6.1) *COROLLARY. Suppose* μ *satisfies* M1(ii), *but we replace* M1(i) *by* $\int_{\mathbb{R}^d} e^{-|y|^2} \mu(dy) < \infty$.

Then there exists a process $A_t(\mu)$ *such that* $\sup_{t \le \tau} |A_t(\mu * \phi_b) - A_t(\mu)| \to 0$.

PROOF. If $T = \inf\{t: |X_t| \ge M\}$, it suffices to show $A_t(\mu * \phi_b)$ converges uniformly for $t \le \tau \wedge T$, and then let $M \to \infty$. But if $\mu_0 = \mu|_{B(2M,0)}$ (the restriction of μ to $B(2M,0)$) and $\mu_1 = \mu|_{B^c(2M,0)}$, then

$$A_t(\mu * \phi_b) = A_t(\mu_0 * \phi_b) + A_t(\mu_1 * \phi_b).$$

μ_0 satisfies M1, and so the first term on the right converges.

$$\mu_1 * \phi_b(x) = \int_{B^c(2M,0)} \phi_b(x-y)\mu_1(dy) \to 0$$

uniformly as $b \to 0$ for $|x| \le M$, and so

$$A_{\tau \wedge T}(\mu_1 * \phi_b) = \int_0^{\tau\wedge T} \mu_1 * \phi_b(X_s) \to 0 \quad \text{a.s.}$$

as $b \to 0$. □

If we replace ϕ_b by an approximation to the identity that has compact support, we can define $A_t(\mu)$ for any μ locally bounded and satisfying M1(ii).

Before proving T2, we need the following technical lemma.

(6.2) *Suppose* $0 < \alpha, \beta \le 1$, μ *a signed measure*, $\|\mu\|_V \le c_{6.3}$, *and* $\|\mu\|_{W-\alpha} \le \delta$. *Then* $\|\mu\|_{W-\beta} \le c_{6.4}\, \delta^{c_{6.5}}$, $c_{6.4}$, $c_{6.5}$ *depending only on* α, β, *and* $c_{6.3}$.

PROOF. Suppose $\|f\|_{C^\beta} \le 1$.

$$|\int f\, d\mu| \le |\int (f - f*\phi_b) d\mu| + |\int f*\phi_b\, d\mu|. \tag{6.3}$$

Since $\|f\|_{C^\beta} \le 1$, $\|f - f*\phi_b\| \le c_{6.6} b^\beta$, and so the first term on the right of (6.3) is $\le c_{6.3}\, c_{6.6}\, b^\beta$. If $b \le 1$, $\|f*\phi_b\|_{C^\alpha} \le c_{6.7}\, b^{-c_{6.8}}$ and so the second term on the right of (6.3) is $\le c_{6.7} \delta b^{-c_{6.8}}$.

Summing, and then letting $b = \delta^{1/(\beta + c_{6.8})}$ gives

$$|\int f\, d\mu| \le c_{6.9}\, \delta^{\beta/(\beta+c_{6.8})}.$$ □

PROOF of T2. In view of (6.2), we may suppose that the α in the statement of T2 is equal to $\gamma/(\beta+\gamma)$, γ as in (4.5). By (5.1) and the fact that if $b < b'$, $\phi_{b'} = \phi_b * \phi_{b'-b}$,

$$\| \mu_a * \phi_b - \mu_{a'} * \phi_{b'} \|_{W-\alpha} \le c_{6.10}(|a-a'|^{c_{6.11}} + |b-b'|^{c_{6.11}}).$$

Applying (5.1v,vi), repeating the arguments of (5.3), (5.4) and (5.5), and then applying (5.7),

$$E \sup_{t \le \tau} |A_t^{b,a} - A_t^{b',a'}|^p \le c_{6.12}(p)|(a,b)-(a',b')|^{c_{6.11}p},$$

where $A_t^{b,a} = \int_0^t \mu_a * \phi_b(X_s)ds$.

If p is chosen large enough so that $c_{6.11}p > 2$, then by the two parameter version of Kolmogorov's theorem, there exist versions of $A_t^{b,a}$ that are jointly uniformly continuous, a.s., $t \le \tau$, $0 \le a \le 1$, $0 < b \le 1$. Again, it is immediate that there exist processes A^a such that A^a is the uniform limit of $A^{b,a}$, $b \to 0$, and A^a is jointly continuous in t and a. □

Note that

$$E \sup_{t \le \tau} |A_t(\mu_a) - A_t(\mu_b)|^p \le c_{6.12}|b-a|^{c_{6.11}p}$$

Using the generalization of the lemma of Garsia, Rodemich, and Rumsey [10, p. 60], one gets that there exists a $\delta > 0$ and a random variable $H(\omega) < \infty$, a.s. such that

$$\sup_{t \le \tau} |A_t(\mu_a)(\omega) - A_t(\mu_b)(\omega)| \le H(\omega)|b-a|^{\delta} \text{ for all } a,b.$$

We now prove T3 and T4. The proofs are those of [1]; we here mainly point out the necessary modifications.

PROOF of T3. Let T be a stopping time $\le \inf\{t: |X_t| \ge M\} \wedge \tau$. Let $K: [0,\infty) \to \mathbb{R}$ such that K has support in $[0,1]$, K is continuous, and $\int_{\mathbb{R}^d} K(|y|)dy = 1$. Let $K_a(y) = a^{-d}K(|y|/a)$, $1 > a > 0$.

Let $\lambda(s,v)$ be $d-1$ dimensional Lebesgue measure on the hyperplane $\{y: v*y = s\}$, where $s \in \mathbb{R}$, $v \in \mathbb{R}^d$, and $|v| = 1$.

First of all, $A_t(\lambda(s,v))$ is jointly continuous in t, s, v, a.s. This may be proved as in the proof of T2 and [1, section 2].

Secondly, if $a \le 1$, $A_T(\lambda(s,v) * K_a) = A_T(\lambda(s,v)|_{B(M+1,0)} * K_a)$. In particular, $A_T(\lambda(s,v) * K_a) = 0$ if $|s| \ge M+1$. Then $A_t(\lambda(s,v) * K_a) \to A_t(\lambda(s,v))$ uniformly in s, v, $t \le T$ by T2. We also get

$$E\, A_T^2(\lambda(s,v) * K_a) \le E\, A_T^2(\lambda(s,v)|_{B(M+1,0)} * K_a) \le c_{6.13},$$

$c_{6.13}$ independent of s, v, a, and therefore

$$E\, A_T(\lambda(s,v) * K_a) \to E\, A_T(\lambda(s,v)).$$

Now if $a \le 1$, let

$$f_a(x) = E \int_0^T K_a(X_s - x)ds.$$

f_a is bounded and has support in $B(M+1,0)$. Apply the Radon transform formula [1, section 4] to f_a to get

$$(6.4)\quad f_a * \phi_b(y) =$$

$$(2\pi)^{-d} \int_{(|v|=1)} \int_0^\infty \int_{-\infty}^\infty e^{iq(s-v*y)} q^{d-1} e^{-bq^2/2} [\textstyle\int f_a(z)\lambda(s,v)(dz)]ds\,dq\,dv.$$

Integrate both sides of (6.4) with respect to $\mu(dy)$. The left-hand side is

$$\int f_a * \phi_b(y)\mu(dy) = E \int_0^T [\textstyle\int K_a(X_t - y)\mu * \phi_b(y)dy]dt$$

$$\to E \int_0^T \mu * \phi_b(X_t) \quad \text{as} \quad a \to 0$$

since $\mu * \phi_b$ is bounded and uniformly continuous, hence $K_a * (\mu * \phi_b) \to \mu * \phi_b$ uniformly.

On the right-hand side of (6.4),

$$\int f_a(z)\lambda(s,v)(dz) = E \int \int_0^T K_a(X_t - z)dt\ \lambda(s,v)(dz)$$

$$= E\ A_T(\lambda(s,v) * K_a)$$

$$\to E\ A_T(\lambda(s,v)) \quad \text{as} \quad a \to 0.$$

Using dominated convergence and Fubini, we get

$$E\ A_T(\mu * \phi_b) = E\ \hat{B}_T(\mu,b).$$

The same argument shows that

$$E(A_T(\mu * \phi_b) - A_t(\mu * \phi_b) | \mathcal{F}_t) = E(\hat{B}_T(\mu,b) - \hat{B}_t(\mu,b) | \mathcal{F}_t) \quad \text{on } t \le T;$$

therefore $A_t(\mu * \phi_b) - \hat{B}_t(\mu,b)$, $t \le T$, is a continuous martingale with paths of bounded variation, hence 0, a.s. Finally, $A_t(\mu * \phi_b) \to A_t(\mu)$, $t \le T$ by T1.

To finish the proof, it suffices to take a sequence of such stopping times T increasing to τ. Under R1,2,3, we may use the sequence $T_M = \inf\{t: |X_t| \ge M\} \wedge \tau$. □

PROOF of T4. By a method identical to that of [1], we get

(6.5) $E(A_T(\mu * \phi_b)) =$

$$E\tfrac{1}{2}(-1)^{(d-1)/2} \int_{(|v|=1)} \int_{-\infty}^{\infty} D_{d-1} g_v(s)[A_T(\lambda(s,v))]ds\ dv + J_b,$$

where T is a stopping time $\le \inf\{t: |X_t| \ge M\} \wedge \tau$ and J_b is a term that $\to 0$ as $b \to 0$.

$$E\, A_T^2(\mu * \phi_b) \le E\, A_\tau^2(\mu * \phi_b) \le c_{6.14}.$$

So the left-hand side of (6.5) approaches $E\, A_T(\mu)$ as $b \to 0$.

We get a similar equation for $E(A_T(\mu) - A_t(\mu)|F_t)$, $t \le T$, and arguing as in the proof of T3, we see that our result holds for $t \le T$. Then take the appropriate sequence of T's. □

One could, of course, weaken the hypothesis that $\mu(\mathbb{R}^d) < \infty$ as in (6.1).

7.0 Removal of Restrictions

In this section we show that T1,2,3, and 4 still hold when restrictions R1,2, and 3 are removed.

REMOVAL OF R1. Given a measure(s) μ (μ_a) satisfying M1, let ε be chosen so that

$$2\varepsilon < (d+1)^{-1}(c_{2.8} \wedge 1)/16.$$

Define a C^∞ function $\Gamma: \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ such that $|\Gamma(K) - I| < 2\varepsilon$ for all K and $\Gamma(K) = K$ if $|K - I| < \varepsilon$. (Cf. construction following (3.6).)

Consider the process $\hat{X}_t$ that solves $\hat{X}_t = x_0$, $d\hat{X}_t = \Gamma(\sigma(\hat{X}))dW_t$. $\Gamma(\sigma)$ satisfies A2 and R1. Clearly $X_t = \hat{X}_t$ up to time $T_1 = \inf\{t: |\sigma_t - I| > \varepsilon\} \wedge \tau$. Applying T1 to the process $\hat{X}_t$, we get that $A_t(\mu * \phi_b)$ converges uniformly to a process $A_t(\mu)$, $t \le T_1$. Also, if $t \le T_1$,

$$E(|A_{T_1}(\mu * \phi_b) - A_t(\mu * \phi_b)|^p|F_t) \le E\Big(\Big|\int_t^\tau \mu * \phi_b(\hat{X}_s)ds\Big|^p\Big|F_t\Big)$$

$$\le c_{7.1}(p).$$

Let C_1 be a square root of $\sigma^{-1}_{T_1}$, let Q_ω be an r.c.p.d. for $E(\cdot|F_{T_1})$, let $X^{(1)}_t = C_1 X_{t+T_1}$, and let $\sigma^{(1)}_t$ be the diffusion coefficient of $X^{(1)}_t$. Using Ito's formula, we check that $\sigma^{(1)}_t = C_1 \sigma_{t+T_1} C_1$. Also $\sigma^{(1)}_0 = I$.

Let $f_b(y) = \mu * \phi_b(C_1 y)$. Applying T2 to the process $\hat{X}^{(1)}_t$ corresponding to $\Gamma(\sigma^{(1)}_t)$, $\int_0^t f_b(X^{(1)}_s)ds$ converges uniformly for $t \leq T_2 = \inf\{t: |\sigma^{(1)}_t - I| > \varepsilon\} \wedge (\tau - T_1)$, a.s. (dQ_ω).

Note again that

$$Q_\omega(|A_{T_1+T_2}(\mu * \phi_b) - A_{T_1}(\mu * \phi_b)|^p) = Q_\omega\Big(\Big|\int_0^{T_2} f_b(\hat{X}^{(1)}_s)ds\Big|^p\Big)$$

$$\leq c_{7.1}(p),$$

and similarly for conditional expectations.

Thus, $A_t(\mu * \phi_b)$ converges uniformly for $t \leq T_1 + T_2$ and

$$E\, A^p_{T_1+T_2}(\mu * \phi_b) \leq c_{7.2}(p).$$

Continue by induction: $X^{(2)}_t = C_2 X_{t+T_1+T_2}$, where C_2 is a square root of $\sigma^{-1}_{T_1+T_2}, \ldots$. Letting $V_n = T_1 + \cdots + T_n$, we get

(i) $A_t(\mu * \phi_b)$ converges uniformly for $t \leq V_n$, and

(ii) $E\, A^p_{V_n}(\mu * \phi_b)) \leq c_{7.3}(p,n)$ (and similarly for conditional expectations).

Using the first of these two facts, we have T1 for $t \leq V_n$. If we let the stopping time T in the proofs of T3, T4 be $V_n \wedge \inf\{t: |X_t| \geq M\}$ and also use (ii), we have the results of T3, T4 for $t \leq V_n$. We can get T2 for $t \leq V_n$ in an exactly similar fashion. To finish the proof, it remains to show $V_n(\omega) = \tau$ for some n on, a.s.

Recall that $T_{n+1} = \inf\{t: |\sigma_t^{(n)} - I| > \varepsilon\} \wedge (\tau - V_n)$, and $\sigma_t^{(n)} = C_n\sigma_{t+V_n}C_n$, C_n a square root of $\sigma_{V_n}^{-1}$. We then get $|C_n\sigma_{V_{n+1}}C_n - I| = \varepsilon$ on the set $V_{n+1} < \tau$.

Let $V = \sup_n V_n$. Recall $\operatorname{trace}(K_1K_2) = \operatorname{trace}(K_2K_1)$. By the continuity of σ, $\sigma_{V_n} \to \sigma_V$, and so

$$\operatorname{trace}(C_n\sigma_{V_{n+1}}C_n - I)^2 = \operatorname{trace}(C_n^4\sigma_{V_{n+1}}^2) - 2\operatorname{trace}(C_n^2\sigma_{V_{n+1}}) + d$$

$$\to \operatorname{trace}((\sigma_V^{-1})^2\sigma_V^2) - 2\operatorname{trace}(\sigma_V^{-1}\sigma_V) + d = 0.$$

We must therefore have $V_{n+1}(\omega) = \tau$ for some n on, as required. □

Next we allow there to be drift.

REMOVAL OF R2. Define a probability Q on $C[0,\infty)$ by

$$(dQ/dP)|_{F_t} = \exp(M_t - \tfrac{1}{2}\langle M,M\rangle_t),$$

where $M_t = -\int_0^t (\sigma_t^{-1}\eta_t)^* dW_t$.

If $\hat{W}_t = \int_0^t \sigma_t^{-1}dX_t$, then under Q, $\hat{W}_t$ will be a Brownian motion. Under Q, (X,σ) satisfy A2 and R2, hence $\sup_{t\le\tau}|A_t(\mu * \phi_b) - A_t(\mu)| \to 0$, a.s. (dQ). Since P is absolutely continuous with respect to Q, this limit is 0, a.s. (dP). This proves T1, and T2 follows analogously.

T3 and T4 would follow as in section 6, provided we had bounds on $E_P(|A_T(\mu * \phi_b) - A_t(\mu * \phi_b)|^p | F_t)$, $t \le T$, T a stopping time as in the proofs of T3, T4. Such bounds are obtained by applying the following lemma, letting $R = P_\omega$, an r.c.p.d. for $E_P(\cdot|F_t)$ and $L = A_T(\mu * \phi_b) - A_{t\wedge T}(\mu * \phi_b)$.

(7.1) *LEMMA. Let* R *be a probability,* W_t *a* d-*dimensional Brownian motion,* $a_t(\cdot)$ *an* $\mathbb{R}^{d\times d}$ *valued functional on* $C[0,t]$, b_t *a predictable* *process. Suppose* $|a_t^{-1}| \le c_{7.4}$, $|b_t| \le c_{7.5}$.

Suppose Y_t *is a solution to* $Y_0 = y_0$, $dY_t = a_t(Y)dW_t + b_t dt$.

Suppose Z_t *is the unique (in law) solution to* $Z_0 = y_0$, $dZ_t = a_t(Z)dW_t$. *Let* L *be a functional on* $C[0,\tau]$. *Then* $E_R|L(Y)|^p \le c_{7.6}(E_R|L(Z)|^{2p})^{\frac{1}{2}}$, $c_{7.6}$ *depending only on* p, $c_{7.4}$, $c_{7.5}$, *and* τ.

PROOF. Define a probability Q on $C[0,\infty)$ by

$$(dQ/dR)|_{F_t} = N_t = \exp(M_t - \tfrac{1}{2}\langle M,M\rangle_t),$$

where $M_t = -\int_0^t (a_t^{-1}(Y)b_t)^* dW_t$. Under Q, $\hat{W}_t = \int_0^t a_t^{-1}(Y)dY_t$ is a Brownian motion, or $dY_t = \int_0^t a_t(Y)d\hat{W}_t$. By the uniqueness in law of Z,

$$E_Q|L(Y)|^{2p} = E_R|L(Z)|^{2p}.$$

Then

$$E_R|L(Y)|^p = E_Q(|L(Y)|^p(dR/dQ))$$

$$\le E_Q(|L(Y)|^{2p})^{\frac{1}{2}}(E_Q N_\tau^{-2})^{\frac{1}{2}}.$$

The result follows from standard estimates on $E_Q(N_\tau^{-2}) = E_R(N_\tau^{-1})$. □

Finally, we remove the restriction $d \ge 3$.

REMOVAL OF R3. Given a measure μ on $\mathbb{R}^d$, $d \le 2$, define $\hat{\mu}(C \times D) = \mu(C)\lambda|_{B(M,0)}(D)$, where $C \subseteq \mathbb{R}^d$, $D \subseteq \mathbb{R}^2$, and λ is Lebesgue measure on $\mathbb{R}^2$. Let $\hat{X}_t = (X_t, \hat{W}_t)$, where $\hat{W}_t$ is a 2-dimensional Brownian motion independent of the Brownian motion in terms of which X is defined.

If $t \le T = \inf\{t: |X_t| \ge M\}$, $X_t \in C$ if and only if $\hat{X}_t \in C \times B(M,0)$. It is not hard to see that $A_t((\mu * \phi_b)\hat{})$ has a uniform limit, $t \le T$, $b \to 0$ using T2. Here ϕ_b is the density of a d-dimensional normal random variable. Call this limit $A_t(\mu)$. T2 may be proved in the same fashion.

$E\, A_t^p((\mu * \phi_b)^\wedge) \le c_{7.7}(p)$ because $(\mu * \phi_b)^\wedge$ satisfies M1. Using this, together with the analogous statement for conditional expectations, we can prove T3 and T4 as in section 6. □

8.0 An Alternate Approach

If one merely wants to construct an occupation time whose support is a given smooth surface, there is a simpler way to proceed. Of course, the results obtained are much weaker also.

Let C be all or part of a C_2 surface: suppose there exists $f: \mathbb{R}^d \to \mathbb{R}^d$ such that $f \in C^2$, the Jacobian of f is bounded away from 0, and $f(C) = \bar{B} \times \{0\}$, B an open (possible unbounded) subset of $\mathbb{R}^{d-1}$, $\bar{B}$ = closure of B.

Suppose X_t is an $\mathbb{R}^d$-valued process, $dX_t = \sigma_t dW_t + \eta_t dt$, where σ_t, η_t are bounded and predictable, σ_t is uniformly strictly elliptic, and σ_t is continuous in t.

(8.1) *THEOREM. There exists a continuous increasing process* $\hat{A}_t$ *such that*

(i) *the support of* $\hat{A}_t = C$, *i.e.,* C *is the smallest closed set for which* $\int 1_{C^c}(X_s)d\hat{A}_s = 0$;

(ii) *there exist nonnegative functions* g_n *such that* $\hat{A}_t$ *is the uniform limit of* $\int_0^t g_n(X_s)ds$, $t \le \tau$, a.s.

It will be apparent from the proof that the conditions on X and C can be slightly weakened by a localization argument. Also, if one has a family of curves C_a that vary smoothly in a: $f(C_a) = \bar{B} \times \{a\}$, then $\hat{A}_t^a$

may be chosen to vary smoothly in a.

First we consider the one-dimensional case.

(8.2) *LEMMA. Suppose* $M_t = \int_0^t \sigma_s dW_s$, $d = 1$, σ_s *continuous in* s, $\sigma_s \geq c_{8.1}$ *for all* s, a.s. *Then there exist versions of* $A_t^\varepsilon = (2\varepsilon)^{-1} \int_0^t 1_{[-\varepsilon,\varepsilon]}(M_s)ds$ *that converge uniformly,* $t \leq \tau$, a.s. *to an increasing process whose support is* $\{0\}$.

PROOF. First of all, if g_ε is given by $g_\varepsilon(0) = 0$, $g_\varepsilon'(x) = (\min(1,|x|/\varepsilon))\mathrm{sgn}(x)$, Ito's formula gives

$$L_t^\varepsilon = (2\varepsilon)^{-1} \int_0^t 1_{[-\varepsilon,\varepsilon]}(M_s)d\langle M,M\rangle_s = g_\varepsilon(M_t) - g_\varepsilon(M_0) - \int_0^t g_\varepsilon'(M_s)dM_s.$$

By a familiar argument, L_t^ε converges uniformly in $t \leq \tau$, a.s. to L_t, the local time of M at 0. (One estimates $E|\int_0^t (g_\varepsilon' - g_\delta')(M_s)dM_s|^p$ by Burkholder's inequality, and then uses Kolmogorov's theorem.)

If H_s is continuous and bounded, $\int_0^t H_s dL_s^\varepsilon$ converges uniformly, $t \leq \tau$, to $\int_0^t H_s dL_s$, a.s. This may be proved by fixing ω, and then approximating $H_s(\omega)$ by a step function $H_s^n = \sum_{i=1}^n k_i 1_{[s_i,s_{i+1})}(s)$ so that $\sup_{s \leq \tau}|H_s^n - H_s| < \delta$.

Now let $H_s = \sigma_s^{-1}$, let $A_t = \int_0^t H_s dL_s$, and observe that $A_t^\varepsilon = \int_0^t H_s dL_s^\varepsilon \to A_t$. □

It is known that versions of L_t^x, the local time at x, can be chosen that are jointly continuous in x and t. It follows that $A_t^x = \int_0^t H_s dL_s^x$ can be taken to be jointly continuous in x and t.

PROOF of (8.1). By localization, let us suppose $\|D_{ij}f\|$, $\|D_i f\|$, $\|f\|$ are all bounded. Let $Y_t = f(X_t)$. Using Ito's lemma, Y_t is a d-dimensional semimartingale satisfying

$$dY_t^k = \sum_i D_i f^k(X_t)dX_t^i + \frac{1}{2} \sum_{i,j} D_{ij} f^k(X_t)d\langle X^i,X^j\rangle_t.$$

Here f^k is the k^{th} coordinate of f.

Let $J_{ik}(t) = D_i f^k(X_t)$. Note that the drift coefficients of Y, $\frac{1}{2} \sum D_{ij} f^k(X_t)\sigma_{ij}(t)$, are bounded.

The diffusion coefficients of Y_t are given by

$$d\langle Y^k,Y^\ell\rangle_t = \langle \sum_i J_{ik}(t)dX_t^i, \sum_j J_{j\ell}dX_t^j\rangle = (J_t^*\sigma_t J_t)_{k\ell}dt.$$

Since σ is strictly elliptic and symmetric, $J_t^*\sigma_t J_t$ is nonnegative definite and will be strictly elliptic provided $\det(J^*\sigma J) > c_{8.2} > 0$. This follows from the hypothesis that the Jacobian of f is bounded away from 0.

Thus the problem is reduced to the case where $C = \bar{B} \times \{0\}$, B an open subset of $\mathbb{R}^{d-1}$ and Y_t satisfies the same conditions at X_t. Arguing as in section 7, we may assume that Y_t has 0 drift.

By (8.2),

$$A_t^\varepsilon = \int_0^t (2\varepsilon)^{-1} 1_{\mathbb{R}^{d-1}\times[-\varepsilon,\varepsilon]}(Y_s)ds$$

$$= \int_0^t (2\varepsilon)^{-1} 1_{[-\varepsilon,\varepsilon]}(M_s)ds \to A_t,$$

where M_s is the d^{th} coordinate of Y_s.

Let $h: \mathbb{R}^d \to \mathbb{R}$ be a continuous bounded function whose support is exactly $\bar{B} \times \mathbb{R}$ and such that $h(y_1,\ldots,y_d)$ does not depend on y_d. Then

$$\hat{A}_t^\varepsilon = \int_0^t (2\varepsilon)^{-1} 1_{\mathbb{R}^{d-1}\times[-\varepsilon,\varepsilon]}(Y_s)h(Y_s)ds$$

$$= \int_0^t h(Y_s)dA_s^\varepsilon \to \int_0^t h(Y_s)dA_s = \hat{A}_t,$$

as in this proof of (8.2).

To finish the proof, we must show that f(C) is the support of $\hat{A}$. By the definition of $\hat{A}$, support ($\hat{A}$) = support (h) ∩ support (A). Since $d\langle M,M\rangle_t/dt$ is bounded above and below away from 0, support (A) = support (L) = $\mathbb{R}^{d-1} \times \{0\}$. And $(\bar{B} \times \mathbb{R}) \cap (\mathbb{R}^{d-1} \times \{0\}) = \bar{B} \times \{0\} = f(C)$. □

(8.3) *EXAMPLE.* Let $C_u = \{(t, f(t) + u): 0 \le t \le 1\}$, where f(t) is a continuous, nowhere differentiable function (e.g., a typical Brownian path). Obviously (8.1) does not apply to C_u.

Let $\mu_u(B)$ = (one-dimensional) Lebesgue measure of $\{t: (t, f(t) + u) \in B\}$. Clearly $\mu_u(B(r,x)) \le 2r$.

By T1, $A\,(\mu_u)$ exists. It is conceivable that $A_1(\mu_u)$ is 0 for some u. (This would depend on the process X_s; it would be impossible for it to occur for a two-dimensional Brownian motion, for example.) However, for almost all u, $A_1(\mu_u)$ must be nonzero.

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