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BOOK REVIEW

ROSS G. PINSKY, *Positive Harmonic Functions and Diffusion: An Integrated Analytic and Probabilistic Approach.*

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The interplay between probability theory and partial differential equations has inspired some beautiful mathematics, including the martingale problems of Stroock and Varadhan, the Harnack inequality of Krylov and Safonov, the Malliavin calculus, and the application of Dirichlet forms to symmetric Markov processes. One of the principal ways the relationship comes about is as follows. Let $d \geq 2$ be the dimension, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ a matrix-valued function, $b\colon \mathbb{R}^d \to \mathbb{R}^d$ a vector-valued function, and W_t a d-dimensional Brownian motion. Let X_t be the solution to the stochastic differential equation

$$dX_t = \sigma(X_t)\,dW_t + b(X_t)\,dt, \tag{1}$$

which means

$$X_t = X_0 + \int_0^t \sigma(X_s)\,dW_s + \int_0^t b(X_s)\,ds,$$

where the first integral is the Itô stochastic integral. Let $a = \sigma\sigma^T$, where σ^T denotes the transpose of σ, and let $\mathscr{L}$ be the operator defined on C^2 functions by

$$\mathscr{L}f(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x_i}(x). \tag{2}$$

We illustrate the connections between X_t and $\mathscr{L}$ by three examples. Suppose that σ and b are smooth functions of x.

(1) Suppose D is an open domain in $\mathbb{R}^d$ with a smooth boundary. Let f be a continuous function on the boundary of D. A function h is harmonic in D if h is C^2 in D and $\mathscr{L}h = 0$ in D. The Dirichlet problem is to find a harmonic function h on D that is continuous on the closure of D and agrees with f on the boundary of D. If τ_D is the first time the process X_t exits D, the solution to the Dirichlet problem can be given simply by

$$h(x) = \mathbb{E}^x f(X_{\tau_D}),$$

where $\mathbb{E}^x$ denotes the expectation with respect to the law of the solution to (1) when $X_0 = x$.

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(2) Let f be a smooth bounded function on $\mathbb{R}^d$. The Cauchy problem is to find a continuous function $u(x,t)$ on $\mathbb{R}^d \times [0,\infty)$ such that $u(x,0) = f(x)$ for all x and

$$\frac{\partial u}{\partial t}(x,t) = \mathscr{L}u(x,t), \qquad x \in \mathbb{R}^d, t > 0.$$

Again there is a simple probabilistic solution, namely,

$$u(x,t) = \mathbb{E}^x f(X_t).$$

(3) If X_t is a solution to (1), then Itô's formula implies that

$$f(X_t) - f(X_0) - \int_0^t \mathscr{L}f(X_s)\,ds$$

is a martingale whenever f is a bounded C^2 function.

For more details concerning (1)–(3), see Durrett (1984) or Stroock and Varadhan (1979).

There is a great deal of more subtle information that can be extracted from the relationship between X_t and $\mathscr{L}$. The book *Positive Harmonic Functions and Diffusion: An Integrated Analytic and Probabilistic Approach* by Ross G. Pinsky is primarily concerned with what can be said about the positive harmonic functions for $\mathscr{L}$ by means of the study of X_t.

Let us look at some representative results. In what follows, we assume σ, b, and the domain D satisfy suitable regularity assumptions. Pinsky gives the precise regularity assumptions necessary in each case.

(1) Let $V\colon \mathbb{R}^d \to \mathbb{R}$ and consider the operator $\mathscr{L}_V$ defined by

$$\mathscr{L}_V f(x) = \mathscr{L}f(x) + V(x)f(x).$$

Let λ_0 be the largest eigenvalue for $\mathscr{L}_V$ in D, assuming it exists; λ_0 is called the principal eigenvalue of $\mathscr{L}_V$. There is a probabilistic characterization of λ_0.

THEOREM 1. *Let X_t be the solution to* (1). *Then for each $x \in D$,*

$$\lambda_0 = \lim_{t\to 0} \frac{1}{t} \log \mathbb{E}^x\left(\exp\left(\int_0^t V(X_s)\,ds\right); \tau_D > t\right).$$

There is also an analytic characterization of λ_0:

THEOREM 2. *λ_0 satisfies*

$$\lambda_0 = \sup_{\mu} \inf_{u} \int_D \frac{\mathscr{L}u}{u}\,d\mu,$$

where the infimum is over functions u that are positive and C^2 on the closure of D and the supremum is over probability measures μ whose density is C^1 on the closure of D and 0 *on the boundary of D.*

(2) A Green function $g\colon \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ is the fundamental solution for $\mathscr{L}_V$; this means the solution to $\mathscr{L}_V u = -f$ in D with $u = 0$ on the boundary of D is given by

$$u(x) = \int_D f(y) g(x, y)\, dy.$$

When does $\mathscr{L}_V$ have a Green function? When does $\mathscr{L}_V$ have positive harmonic functions?

THEOREM 3. *If $\lambda < \lambda_0$, where λ_0 is the principal eigenvalue, then $\mathscr{L}_V - \lambda$ has no positive harmonic functions. If $\lambda = \lambda_0$, then $\mathscr{L}_V - \lambda$ has positive harmonic functions, but does not have a Green function. If $\lambda > \lambda_0$, then $\mathscr{L}_V - \lambda$ has a Green function.*

(3) Suppose $V = 0$ and $D = \mathbb{R}^d$. The process X_t is transient if $|X_t| \to \infty$ as $t \to \infty$ and recurrent if X_t enters every open set infinitely often as $t \to \infty$. Then we have the following.

THEOREM 4. *X_t is recurrent if and only if no Green function exists.*

(4) The Martin boundary in a domain D for an operator can be characterized in terms of the sets of limit points of the functions $k(\cdot) = g(\cdot, y_n)/g(x_0, y_n)$ for sequences y_n tending to the boundary, where x_0 is a fixed point in D and g is the Green function. The Martin boundary is important because it gives a representation of all positive harmonic functions in terms of integrals with respect to measures on the Martin boundary. The Martin boundary can also be characterized by what are known as exterior harmonic measures. Let E be a subdomain of D whose closure is contained in D. Let T_E be the first hitting time of E and suppose $x \in D - \overline{E}$. Let h be a positive harmonic function on D and let $\mathscr{L}^h$ be the h-transform of $\mathscr{L}$. This means that

$$\mathscr{L}^h f(x) = \frac{\mathscr{L}(fh)(x)}{h(x)};$$

$\mathscr{L}^h$ is the operator corresponding to the process X_t conditioned by the positive harmonic function h [see Bass (1995)]. The exterior harmonic measure on the boundary of E with respect to X is the hitting distribution of X_t on the boundary of E:

$$\mu_x^h(dy) = \mathbb{P}_h^x(T_E < \tau_D, X_{T_E} \in dy).$$

Here $\mathbb{P}_h^x$ represents the law of the conditioned process. Fix a positive harmonic function h. The Martin boundary can then be characterized in terms of the set of weak limits of $\mu_{x_n}^h$ for sequences x_n tending to the boundary.

Theory such as the four examples we just gave is quite interesting. However, to appreciate the theory and truly understand it, one needs to look at lots of concrete examples. Fortunately, the book by Pinsky does just that. The details of what happens for many, many operators and diffusions are worked out carefully.

There is a huge amount of material in this book. It is very clearly written. Anyone interested in diffusions and elliptic operators should definitely purchase a copy.

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