STOCHASTIC DIFFERENTIAL EQUATIONS FOR DIRICHLET PROCESSES

Richard F. Bass [1] and **Zhen-Qing Chen** [2]

Abstract. We consider the stochastic differential equation $dX_t = a(X_t)dW_t + b(X_t)dt$, where W is a one-dimensional Brownian motion. We formulate the notion of solution and prove strong existence and pathwise uniqueness results when a is in $C^{1/2}$ and b is only a generalized function, for example, the distributional derivative of a Hölder function or of a function of bounded variation. When $b = aa'$, that is, when the generator of the SDE is the divergence form operator $\mathcal{L} = \frac{1}{2}\frac{d}{dx}(a^2\frac{d}{dx})$, a result on non-existence of a strong solution and non-pathwise uniqueness is given as well as a result which characterizes when a solution is a semimartingale or not. We also consider extensions of the notion of Stratonovich integral.

Keywords. Stochastic differential equations, SDE, semimartingales, Dirichlet processes, local times, Dirichlet forms, Stratonovich integral, divergence form, energy

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1. Introduction.

Let W_t be a one-dimensional Brownian motion and consider the stochastic differential equation (SDE)

$$dX_t = a(X_t)dW_t + b(X_t)dt. \tag{1.1}$$

Here the stochastic integral is of Itô type. Our goal in this paper is to obtain pathwise existence and uniqueness results for (1.1) for as wide a class of drift terms b as possible when a is a Hölder continuous function of order $\frac{1}{2}$. In fact we allow b to be a generalized function, rather than a function. Of course, it is necessary in this case to formulate what it means to be a solution.

Given a Brownian motion W on a probability space, recall that a strong solution to (1.1) is a continuous process X that is adapted to the filtration generated by W and which solves (1.1). A weak solution of (1.1) is a couple (X, W) on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ such that X_t is adapted to $\mathcal{F}_t$, W_t is an $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion (that is, W_t is $\mathcal{F}_t$-measurable and for $t > s$, $W_t - W_s$ is independent of $\mathcal{F}_s$ and has normal distribution with zero mean and variance $t - s$), and (X, W) satisfies (1.1). We say weak uniqueness holds for (1.1) if whenever (X, W), $(\widetilde{X}, \widetilde{W})$ are two weak solutions of (1.1) and X_0 has the same distribution as $\widetilde{X}_0$, then the process $\{X_t\}_{t\geq 0}$ has the same law as the process $\{\widetilde{X}_t\}_{t\geq 0}$. Pathwise uniqueness is said to hold for (1.1) if whenever (X, W), $(\widetilde{X}, W)$ are two weak solutions of (1.1) with common Brownian motion W (relative to possibly different filtrations) on a common probability space and with common initial value, then $\mathbb{P}(X_t = \widetilde{X}_t \text{ for all } t \geq 0) = 1$. We say that strong uniqueness holds for (1.1) if whenever X and $\widetilde{X}$ are two strong solutions of (1.1) relative to W with common initial condition X_0, then $\mathbb{P}(X_t = \widetilde{X}_t \text{ for all } t \geq 0) = 1$. Clearly pathwise uniqueness implies strong uniqueness. Yamada and Watanabe [20] showed that pathwise uniqueness implies weak uniqueness.

Stroock and Varadhan [18] proved that (1.1) has a unique weak solution if a^2 is bounded away from zero and infinity and b is bounded and measurable. It is known that the existence of a weak solution does not imply the existence of a strong solution. A well-known theorem of Yamada and Watanabe [20] says that if

(i) a is bounded and $|a(x) - a(y)| \leq \rho(|x - y|)$ for an increasing function ρ satisfying $\int_{0+} \rho^{-2}(x)dx = \infty$; and
(ii) b is a bounded Lipschitz function,

then there exists a strong solution to (1.1) and that solution is pathwise unique. (In their paper [20], strong uniqueness is proved. But in fact, their proof also yields pathwise uniqueness, see [12].) Barlow [1] showed that the above condition is nearly optimal for (1.1) when $b = 0$.

Not as well-known is a result of Zvonkin [21] that says if a is bounded below away from 0, is bounded above, and is Hölder continuous of order $\frac{1}{2}$, and b is only bounded and measurable, then strong existence and strong uniqueness holds for (1.1). In [21] the coefficients can also depend on time. Zvonkin's result was extended to the multidimensional case by Veretennikov [19]. Furthermore in [13], LeGall obtained strong existence and strong uniqueness for the SDE (1.1) where $b(x)dx$ is replaced by a finite signed measure $b(dx)$ and a is a right continuous function that is bounded away from zero and is of bounded variation. In earlier work [15] had shown weak existence of Markov solutons under the same hypotheses as LeGall's. For some recent work that is related to the subject of this paper see [4, 5, 6].

The first main result of this paper, in Section 2, concerns the case where we look at

$$dX_t = a(X_t)dW_t + (ba^2)(X_t)dt, \tag{1.2}$$

where a is in $C^{1/2}$ and formally b may be written as the distributional derivative of a function B that is Hölder continuous of order α for some $\alpha > \frac{1}{2}$. Thus b might only be a generalized function rather than a true function. In this case $A_t = \int_0^t (ba^2)(X_s)ds$ does not make sense and a solution X_t might not be a semimartingale. For X_t to be a solution we require X_t to be a Dirichlet process $X_t = X_0 + \int_0^t a(X_s)dW_s + A_t$, where A_t has zero energy (see Section 2 for a definition) and A_t is the limit in a suitable sense of $\int_0^t B_n'(X_s)a^2(X_s)ds$; here the B_n are smooth and converge appropriately to B.

In Section 3 we let $b = a'a$, so the solution to (1.1) corresponds to the diffusion which has infinitesimal generator $\frac{1}{2}(a^2 f')'$, an elliptic operator in divergence form. This is a special case of the situation of Section 2, but not surprisingly more can be said here. Under a condition that is satisfied if a is Hölder continuous of order $\frac{1}{2}$, we construct a symmetric diffusion that is a strong solution to (1.1) and prove pathwise uniqueness in a stronger sense than in Section 2. We also show that for any $\alpha \in (0, 1/2)$, there is an α-Hölder continuous function a that is bounded away from zero and infinity such that (1.1) has no strong solution nor does pathwise uniqueness hold. We also characterize when the solution is a semimartingale or not.

In Section 4 we consider the case where $b = \frac{1}{2}a'a$ so that (1.1) formally becomes

$$dX_t = a(X_t) \circ dW_t, \tag{1.4}$$

where the stochastic integral is of Stratonovich type. We give an interpretation to this SDE and prove strong existence under the assumption that a is positive, bounded, and continuous.

Finally in section 5, we look at the Stratonovich SDE (1.4) from another point of view. We prove strong existence and pathwise uniqueness under this new interpretation, when $a(x)$ is a measurable function on $\mathbb{R}$ that is bounded above and bounded below away from zero.

Let us indicate the idea behind our method by considering (1.1) where b is a bounded continuous function. Let $s(x)$ be the scale function for the operator $\mathcal{L}f(x) = \frac{1}{2}a(x)^2 f''(x) + b(x)f'(x)$, so that $\mathcal{L}s = 0$. In fact, one can take the scale function to be

$$s(x) = c_0 \int_0^x \exp\Big(- \int_0^y \frac{2b(r)}{a^2(r)} dr \Big) dy, \tag{1.5}$$

where $c_0 > 0$ is a constant. If X_t is a solution to (1.1), then by Itô's formula $Y_t = s(X_t)$ is a solution to $dY_t = \widetilde{a}(Y_t)dW_t$, where $\widetilde{a}(y) = (as')(s^{-1}(y))$. If one can show that $\widetilde{a}$ satisfies the Yamada-Watanabe condition, then the paths of Y_t are uniquely determined, and because s is one-to-one, those of X_t are as well.

Throughout W_t will denote a Brownian motion. Stochastic integrals $\int_0^t H_{s-} dW_s$ are of Itô type, while Stratonovich integrals are written $\int_0^t H_{s-} \circ dW_s$. The letter c with subscripts will denote a positive finite constant whose exact value in unimportant. The C^α norm of f is

$$\|f\|_{C^\alpha} = \sup_x |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

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2. Dirichlet processes.

In this section we consider the SDE

$$dX_t = a(X_t)dW_t + dA_t, \qquad X_0 = x_0, \tag{2.1}$$

where A_t is a process of zero energy. Formally we consider A_t as

$$A_t = \int_0^t b(X_s)a^2(X_s)ds,$$

where b is the distributional derivative of a Hölder function B. More precisely, we define a solution to (2.1) as follows.

Define the energy of a right continuous process A_t to be

$$\lim_{\delta \to 0} \sup_{\{\Pi_t:\ mesh(\Pi_t) < \delta\}} \mathbb{E} \sum_{i=0}^{n-1} |A_{t_{i+1}} - A_{t_i}|^2,$$

where $\Pi_t = \{t_0, t_1, \ldots, t_n\}$ denotes a partition of $[0, t]$. A right continuous process X is said to be a Dirichlet process if it has a decomposition

$$X_t = X_0 + M_t + A_t, \quad t \geq 0,$$

where M_t is a local martingale and A_t is a continuous process having zero energy. Clearly such a decomposition is unique for a Dirichlet process.

We define

$$\mathcal{H}_t^{p,\zeta}(A) = \sup_{r\neq s,\ r,s\leq t} \frac{\mathbb{E}\,|A_s - A_r|^p}{|s-r|^\zeta}.$$

$\mathcal{H}_t^{p,\zeta}$ is a type of Hölder semi-norm.

Definition 2.1. *Let $\gamma > 0, p > 1, \zeta > 1$, and $B \in C^\gamma$. We say that X_t is a solution to (2.1) with starting point x_0 if*

(i) $X_t = x_0 + \int_0^t a(X_s)dW_s + A_t$, where A_t is a continuous process having zero energy;

(ii) whenever B_n are C^2 functions converging to B uniformly on $\mathbb{R}$ with $\sup_n \|B_n\|_{C^\gamma} < \infty$, then $A_t^n = \int_0^t B_n'(X_s)a^2(X_s)ds$ converges to A_t uniformly over bounded time intervals in probability;

(iii) whenever B_n are C^2 functions converging to B uniformly on $\mathbb{R}$ with $\sup_n \|B_n\|_{C^\gamma} < \infty$, we have $\sup_n \mathcal{H}_t^{p,\zeta}(A^n) < \infty$ for each t.

When we want to emphasize the values of p and ζ, we will call X_t a (p,ζ)-solution.

Throughout this section we suppose that

$$a \in C^{1/2}, \qquad \gamma \in (\tfrac{1}{2}, 1), \qquad p \in \Big(\frac{2}{1+\gamma}, \frac{2}{2-\gamma}\Big).$$

We show there exists a strong solution to (2.1) and the solution is pathwise unique.

Our first step is to give a candidate for a solution. Motivated by (1.2), define the function s by $s(x) = \int_0^x e^{-2B(y)}dy$. Note $s' > 0$ and s is a Lipschitz function. For typographical convenience we will write σ for s^{-1}.

Let

$$\widetilde{a}(x) = (s'a) \circ \sigma(x).$$

Since $B \in C^\gamma$ with $\gamma > \frac{1}{2}$, then $\widetilde{a} \in C^{1/2}$. Let Y_t solve

$$dY_t = \widetilde{a}(Y_t)dW_t \quad \text{with } Y_0 = s(x_0), \tag{2.2}$$

and

$$X_t = \sigma(Y_t). \tag{2.3}$$

Proposition 2.2. *The process X_t constructed above is a Dirichlet process with $X_0 = x_0$ whose martingale part is $\int_0^t a(X_s)dW_s$. The process X_t is measurable with respect to the σ-fields of W.*

Proof. Since $\widetilde{a} \in C^{1/2}$, we know from [20] that there is a unique pathwise solution to the SDE (2.2) and that Y_t is measurable with respect to the σ-fields of W. Therefore X_t is also measurable with respect to the σ-fields of W with $X_0 = \sigma(s(x_0)) = x_0$.

We next show X is a Dirichlet process having the advertised decomposition. First we examine the martingale term. Let g_n be a sequence of C^2 functions on $\mathbb{R}$ with $g_n(0) = \sigma(0)$ so that g_n' converges uniformly to σ' as $n \to \infty$ with $\sup_n \|g_n'\|_{C^\gamma} < \infty$. By Itô's formula,

$$d[g_n(Y_t)] = g_n'(Y_t)\widetilde{a}(Y_t)dW_t + \tfrac{1}{2} g_n''(Y_t)\widetilde{a}^2(Y_t)dt.$$

Since $X_t = \sigma(Y_t)$, then $Y_t = s(X_t)$, and we can rewrite the above as

$$g_n \circ s(X_t) - g_n \circ s(x_0) = \int_0^t (g_n'\widetilde{a}) \circ s(X_t)dW_t + \tfrac{1}{2}\int_0^t (g_n''\widetilde{a}^2) \circ s(X_t)dt. \tag{2.4}$$

Let $n \to \infty$. Since $g_n' \to \sigma'$ and $g_n(0) = \sigma(0)$, then $g_n \to \sigma$ and so the left hand side of (2.4) converges (uniformly) to $X_t - x_0$. Also

$$(g_n'\widetilde{a}) \circ s(x) = g_n'(s(x))\widetilde{a}(s(x)) = g_n'(s(x))s'(x)a(x) \to a(x),$$

so the stochastic integral term in (2.4) converges to $\int_0^t a(X_s)dW_s$ (uniformly on bounded intervals) in probability.

Since the first three terms in (2.4) converge, then the last term must also converge in probability, say to A_t. It remains to show that A_t has zero energy. We can write

$$g_n(y) - g_n(x) = \int_x^y g_n'(z)dz = g_n'(x)(y-x) + \int_x^y [g_n'(z) - g_n'(x)]dz.$$

Since $c_1 = \sup_n \|g_n'\|_{C^\gamma} < \infty$, the last term is less than $c_1|y-x|^{1+\gamma}$. We then have

$$|g_n(Y_t) - g_n(Y_s) - g_n'(Y_s)(Y_t - Y_s)| \le c_1|Y_t - Y_s|^{1+\gamma}. \tag{2.5}$$

We also have

$$g_n'(Y_s)(Y_t - Y_s) - \int_s^t g_n'(Y_r)dY_r = \int_s^t [g_n'(Y_s) - g_n'(Y_r)]dY_r. \tag{2.6}$$

Write A_t^n for the last term in (2.4). Since Y_t is a martingale with $d\langle Y\rangle_t/dt = \widetilde{a}^2(Y_t)$, which is bounded, and

$$A_t^n - A_s^n = g_n(Y_t) - g_n(Y_s) - \int_s^t g_n'(Y_r)dY_r,$$

then by the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
\mathbb{E}\,(A_t^n - A_s^n)^2 &\le c_2\mathbb{E}\,|Y_t - Y_s|^{2+2\gamma} + c_2\mathbb{E}\int_s^t [g_n'(Y_s) - g_n'(Y_r)]^2\widetilde{a}^2(Y_r)dr \\
&\le c_3|t-s|^{1+\gamma} + c_3(t-s)\mathbb{E}\sup_{r\le s\le t}|Y_s - Y_r|^{2\gamma} \\
&\le c_4|t-s|^{1+\gamma}.
\end{aligned}$$

Using Fatou's lemma,

$$\mathbb{E} \sum (A_{t_{i+1}} - A_{t_i})^2 \le c_4 \sum |t_{i+1} - t_i|^{1+\gamma}.$$

This tends to zero as the mesh of the partition goes to 0. $\square$

Before proceeding to show that the X_t defined in (2.3) is actually a solution to (2.1), we need a lemma giving some estimates about integrals. The proof is modeled on the integrals of L.C. Young.

Lemma 2.3. *(a) Suppose g is continuously differentiable and f is continuous. In the following c_1 does not depend on f or g. If $f \in C^\alpha$, $g \in C^\beta$, and $\alpha + \beta > 1$, then*

$$\Big| \int_0^t f dg \Big| \le c_1 t^\beta (t \vee 1)^\alpha \|f\|_{C^\alpha} \|g\|_{C^\beta}$$

and

$$\Big| \int_s^t f dg \Big| \le c_1 |t-s|^\beta (|t-s| \vee 1)^\alpha \|f\|_{C^\alpha} \|g\|_{C^\beta}.$$

If $\delta \in (0,1)$ is such that $(1-\delta)\alpha + \beta > 1$, then

$$\Big| \int_0^t f dg \Big| \le c_1 t^\beta (t \vee 1)^{(1-\delta)\alpha} \|f\|_\infty^\delta \|f\|_{C^\alpha}^{1-\delta} \|g\|_{C^\beta}.$$

(b) Let H_s, K_s be continuous processes and $p, p', \zeta, \zeta' > 1$ such that

$$\mathcal{H}_t^{p,\zeta}(H) < \infty, \qquad \mathcal{H}_t^{p',\zeta'}(K) < \infty,$$

and $(1/p) + (1/p') > 1$. Let $t > 0$ and

$$J_n = \sum_{k=0}^{2^n - 1} H_{kt/2^n} (K_{(k+1)t/2^n} - K_{kt/2^n}).$$

Then J_n converges in L^1 and the rate of convergence depends only on the quantities $\mathcal{H}_t^{p,\zeta}(H), \mathcal{H}_t^{p',\zeta'}(K), p, p', \zeta, \zeta'$, and t. Moreover, if K_s has paths that are continuously differentiable, then J_n converges to $\int_0^t H_s dK_s$.

Proof. Let $t_k = kt/2^n$ and let I_n be a Riemann sum approximation to $\int f dg$:

$$I_n = \sum_{i=0}^{2^n - 1} f(t_i)(g(t_{i+1}) - g(t_i)).$$

Since f is continuous and g is continuously differentiable, $I_n \to \int_0^t f dg$ as $n \to \infty$. Now

$$I_{n+1} - I_n = \sum_{i\ even} [f(t_{i+1}) - f(t_i)][g(t_{i+2}) - g(t_{i+1})].$$

Using the Cauchy-Schwarz inequality,

$$\begin{aligned}|I_{n+1} - I_n| &\le \Big(\sum_i |f(t_{i+1}) - f(t_i)|^2\Big)^{1/2}\Big(\sum_i |g(t_{i+1}) - g(t_i)|^2\Big)^{1/2} \\ &\le (2^n(t2^{-n})^{2\alpha})^{1/2}\|f\|_{C^\alpha}(2^n(t2^{-n})^{2\beta})^{1/2}\|g\|_{C^\beta} \\ &\le t^{\alpha+\beta}2^{-(\alpha+\beta-1)n}\|f\|_{C^\alpha}\|g\|_{C^\beta}.\end{aligned} \tag{2.7}$$

We also have

$$|I_0| \le \|f\|_{C^\alpha}|g(t_{2^n}) - g(t_0)| \le \|f\|_{C^\alpha}\|g\|_{C^\beta}t^\beta.$$

Since $\alpha + \beta > 1$, summing over n from 0 to N shows

$$|I_N| \le c_2 t^\beta (t \vee 1)^\alpha \|f\|_{C^\alpha}\|g\|_{C^\beta},$$

with c_2 independent of N. Letting N tend to infinity proves the first inequality in (a) and the second is almost identical. For the third inequality in (a), observe that from (2.7),

$$\begin{aligned}|I_{n+1} - I_n| &\le (2\|f\|_\infty)^\delta\Big(\sum_i |f(t_{i+1}) - f(t_i)|^{2-2\delta}\Big)^{1/2}\Big(\sum_i |g(t_{i+1}) - g(t_i)|^2\Big)^{1/2} \\ &\le (2\|f\|_\infty)^\delta(2^n(t2^{-n})^{(2-2\delta)\alpha})^{1/2}\|f\|_{C^\alpha}^{1-\delta}(2^n(t2^{-n})^{2\beta})^{1/2}\|g\|_{C^\beta} \\ &\le (2\|f\|_\infty)^\delta t^{(1-\delta)\alpha+\beta}2^{-((1-\delta)\alpha+\beta-1)n}\|f\|_{C^\alpha}^{1-\delta}\|g\|_{C^\beta}.\end{aligned}$$

Since $(1-\delta)\alpha + \beta > 1$, summing over n from 0 to N shows

$$|I_N| \le c_2 t^\beta (t \vee 1)^{(1-\delta)\alpha}\|f\|_\infty^\delta\|f\|_{C^\alpha}^{1-\delta}\|g\|_{C^\beta},$$

Letting $N \to \infty$ proves the last inequality in (a).

We turn to (b). Again let $t_k = kt/2^n$. As above,

$$J_{n+1} - J_n = \sum_{k\ even} [H_{t_{k+1}} - H_{t_k}][K_{t_{k+2}} - K_{t_{k+1}}].$$

Using Hölder's inequality,

$$\begin{aligned}\mathbb{E}\,|J_{n+1} - J_n| &\le \sum_k (\mathbb{E}\,|H_{t_{k+1}} - H_{t_k}|^p)^{1/p}(\mathbb{E}\,|K_{t_{k+2}} - K_{t_{k+1}}|^{p'})^{1/p'} \\ &\le c_3 \sum (t/2^n)^{\zeta/p}(t/2^n)^{\zeta'/p'} \\ &\le c_4 2^n 2^{-n((1/p)+(1/p'))},\end{aligned}$$

which is summable in n since $(1/p)+(1/p')>1$. The main assertion of (b) is now immediate. Clearly if K_s has paths that are continuously differentiable, then J_n is a Riemann sum approximation of $\int_0^t H_s dK_s$ and so converges to the integral. $\square$

We need the following. Suppose H_n is a sequence of C^2 functions that converges to B uniformly on $\mathbb{R}$ with $\sup_n \|H_n\|_{C^\gamma}<\infty$ and $h_n=H_n'$. Let $G_n\in C^2$ be defined by

$$G_n''(y)=\frac{2h_n}{(s')^2}\circ\sigma(y),\qquad G_n'(0)=\sigma'(0),\qquad G_n(0)=\sigma(0). \tag{2.8}$$

Lemma 2.4. *$G_n'\to\sigma'$ uniformly on bounded intervals.*

Proof. We have

$$G_n'(y)=\int_0^y\frac{2h_n}{(s')^2}\circ\sigma+\sigma'(0).$$

Let $s_n(x)=\int_0^x e^{-2H_n(s)}ds$. So $h_n=-s_n''/2s_n'$. Let $\sigma_n=s_n^{-1}$. Note $s_n\circ\sigma_n(x)=x$, hence $(s_n'\circ\sigma_n)\sigma_n'=1$, or $\sigma_n'=1/(s_n'\circ\sigma_n)$. Differentiating,

$$\sigma_n''=-\frac{1}{(s_n'\circ\sigma_n)^2}(s_n''\circ\sigma_n)\sigma_n'=-\frac{s_n''}{(s_n')^3}\circ\sigma_n.$$

Therefore

$$\int_0^y\frac{2h_n}{(s_n')^2}\circ\sigma_n+\sigma_n'(0)=\sigma_n'(y).$$

What we need to do is to show that the left hand side and G_n' do not differ by much. By a change of variables,

$$G_n'(y)=\int_{\sigma(0)}^{\sigma(y)}\frac{2h_n}{s'}+\sigma'(0),\qquad \sigma_n'(y)=\int_{\sigma_n(0)}^{\sigma_n(y)}\frac{2h_n}{s_n'}+\sigma_n'(0). \tag{2.9}$$

Recall $\sigma(0)=\sigma_n(0)=0$ by the definitions of s and s_n and that $\sigma_n'(0)\to\sigma'(0)$ as $n\to\infty$.

We first get a bound on

$$\Big|\int_0^z\Big[\frac{1}{s'}-\frac{1}{s_n'}\Big]dH_n\Big|.$$

Because H_n converges to B uniformly on $\mathbb{R}$ with $\sup_n\|H_n\|_{C^\gamma}<\infty$, then $1/s_n'$ converges to $1/s'$ uniformly on $\mathbb{R}$ with $\sup_n\|1/s_n'\|_{C^\gamma}<\infty$. The H_n are bounded in C^γ norm and $\gamma>\frac12$, so by Lemma 2.3(a), the expression above is bounded by

$$c_1|z|^\gamma(|z|\vee 1)^{(1-\delta)\gamma}\|(1/s')-(1/s_n')\|_\infty^\delta\|(1/s')-(1/s_n')\|_{C^\gamma}^{1-\delta}\|H_n\|_{C^\gamma}$$

for some $\delta\in(0,1)$. This tends to 0 as $n\to\infty$. Similarly we bound the difference between $\int_0^{\sigma(y)}$ and $\int_0^{\sigma_n(y)}$. Combining proves the lemma. $\square$

Theorem 2.5. *If $p > 2/(1+\gamma)$ and $\zeta = p(1+\gamma)/2$, then the X_t constructed in (2.3) is a (p,ζ)-solution to (2.1).*

Proof. Suppose we have a sequence H_n of C^2 functions converging to B uniformly with $\sup_n \|H_n\|_{C^\gamma} < \infty$. Define $h_n = H_n'$ and define G_n as in (2.8). Since $G_n' \to \sigma'$ by Lemma 2.4, then $G_n \to \sigma$ and $G_n'\widetilde{a} \to a \circ \sigma$. Since $G_n \in C^2$, then by Itô's formula,

$$G_n(Y_t) - G_n(Y_0) = \int_0^t G_n'(Y_s)\widetilde{a}(Y_s)dW_s + \tfrac{1}{2}\int_0^t G_n''(Y_s)\widetilde{a}^2(Y_s)ds.$$

The left hand side converges to $\sigma(Y_t) - \sigma(Y_0) = X_t - X_0$. The stochastic integral term converges to $\int_0^t a \circ \sigma(Y_s)dW_s = \int_0^t a(X_s)dW_s$. Therefore the right-hand term, A_t^n, which is $\int_0^t (h_n a^2) \circ \sigma(Y_s)ds = \int_0^t h_n(X_s)a^2(X_s)ds$, must converge in probability to

$$X_t - X_0 - \int_0^t a(X_s)dW_s = A_t.$$

It remains to bound $\mathcal{H}_t^{p,\zeta}(A^n)$. As in (2.5) and (2.6),

$$|A_t^n - A_s^n| \le c_1|Y_t - Y_s|^{1+\gamma} + \Big|\int_s^t [G_n'(Y_s) - G_n'(Y_r)]dY_r\Big|.$$

By the Burkholder-Davis-Gundy inequalities,

$$\begin{aligned}\mathbb{E}\,|A_t^n - A_s^n|^p \le c_2\mathbb{E}\Big(\int_s^t \widetilde{a}^2(Y_r)dr\Big)^{p(1+\gamma)/2}& \\ + c_2\mathbb{E}\Big(\int_s^t |G_n'(Y_s) - G_n'(Y_r)|^2\widetilde{a}^2(Y_r)dr\Big)^{p/2}&.\end{aligned} \tag{2.10}$$

The first term on the right is bounded by $c_3|t-s|^{p(1+\gamma)/2}$. By (2.9),

$$G_n'(y) = \int_0^{\sigma(y)} \frac{2}{s'}dH_n.$$

Since σ is Lipschitz, by Lemma 2.3(a)

$$|G_n'(y) - G_n'(x)| \le c_4|y-x|^\gamma.$$

Therefore the last term in (2.10) is bounded by

$$c_5\mathbb{E}\Big(\int_s^t |Y_s - Y_r|^{2\gamma}dr\Big)^{p/2} \le c_5\mathbb{E}\,(\sup_{u\in[s,t]} |Y_s - Y_u|^{2\gamma})^{p/2}|t-s|^{p/2}.$$

Since Y_t is a martingale, by the Burkholder-Davis-Gundy inequalities again, this is less than

$$c_6\mathbb{E}\Big(\int_s^t \widetilde{a}^2(Y_r)dr\Big)^{\gamma p/2}|t-s|^{p/2} \le c_7|t-s|^{p(1+\gamma)/2}.$$

Substituting in (2.10),

$$\mathbb{E}\,|A_t^n - A_s^n|^p \le c_8|t-s|^{p(1+\gamma)/2}.$$

□

It remains to prove uniqueness.

Theorem 2.6. *Suppose $p, p' \in (2/(1+\gamma), 2/(2-\gamma))$ and $\zeta, \zeta' > 1$. Suppose X is a (p,ζ)-solution to (2.1) and X' is a (p',ζ')-solution to (2.1). Then $X_t = X'_t$ for all t almost surely.*

Proof. Let $B_m \to B$ uniformly on $\mathbb{R}$ with $\|B_m\|_{C^\gamma} < \infty$, and let s, σ, s_m, σ_m be defined in terms of B and B_m analogously to the above. Let X_t be a (p,ζ)-solution of (2.1). Let $Y_t^m = s_m(X_t)$ and $Y_t = s(X_t)$. Since $s_m \to s$, Y_t^m converges to Y_t. By Itô's formula for Dirichlet processes ([7]),

$$dY_t^m = s'_m a(X_t)dW_t + s'_m(X_t)dA_t + \tfrac{1}{2}(s''_m a^2)(X_t)dt. \tag{2.11}$$

Suppose we show that

$$J(m) = \int_0^t s'_m(X_t)dA_t + \tfrac{1}{2}\int_0^t (s''_m a^2)(X_t)dt \to 0. \tag{2.12}$$

The stochastic integral term in (2.11) is $\int_0^t (s'_m a)(\sigma_m(Y_s^m))dW_s$ and converges to $\int_0^t (s'a)(\sigma(Y_s))dW_s$. So if we show (2.12), then Y_t solves $dY_t = (s'a)(\sigma(Y_t))dW_t$. Since the solution to this equation is unique, $Y_t = s(X_t)$, and s is one-to-one, then the paths of X_t are determined by $X_t = \sigma(Y_t)$. Similarly $X'_t = \sigma(Y_t)$, which would prove uniqueness. So we must show (2.12).

Let $M_t = \int_0^t a(X_t)dW_t$. Using the definition of a (p,ζ)-solution and Fatou's lemma,

$$\mathbb{E}\,|A_t - A_s|^p \le c_1|t-s|^\zeta.$$

Since $p < 2/(2-\gamma)$, we can choose $\tau > 1$ such that $(\gamma/2\tau) + (1/p) > 1$. We estimate

$$\begin{aligned}
|s'_m(X_t) - s'_m(X_s)|^{2\tau/\gamma} &\le \|s'_m\|_{C^\gamma}^{2\tau/\gamma}[|X_t - X_s| \wedge 1]^{2\tau} \\
&\le c_2\|s'_m\|_{C^\gamma}^{2\tau/\gamma}\{[|M_t - M_s| \wedge 1]^{2\tau} + [|A_t - A_s| \wedge 1]^{2\tau}\} \\
&\le c_2\|s'_m\|_{C^\gamma}^{2\tau/\gamma}\{|M_t - M_s|^{2\tau} + |A_t - A_s|^p\} \\
&\le c_3\|s'_m\|_{C^\gamma}^{2\tau/\gamma}(|t-s|^\tau + |t-s|^\zeta),
\end{aligned}$$

using the Burkholder-Davis-Gundy inequalities to bound $\mathbb{E}\,|M_t - M_s|^{2\tau}$. So if let $H_t = s'_m(X_t)$, we have shown that $\mathcal{H}_t^{2\tau/\gamma,\zeta\wedge\tau}(H) < \infty$.

Note that

$$\tfrac{1}{2}(s''_m a^2)(X_t)dt = -(s'_m b_m)(X_t)a^2(X_t)dt = -s'_m(X_t)dA_t^m,$$

where $A_t^m = \int_0^t (b_m a^2)(X_t)dt$. Let $\varepsilon > 0$. Recalling that $(\gamma/2\tau) + (1/p) > 1$, Lemma 2.3(b) tells us that there exists an n_0 independent of m such that if $n \ge n_0$, then

$$\mathbb{P}\Big(\Big|\int_0^t s'_m(X_s)dA_s^m - \sum_{k=0}^{2^n-1} s'_m(X_{kt/2^n})(A^m_{(k+1)t/2^n} - A^m_{kt/2^n})\Big| > \varepsilon\Big) < \varepsilon.$$

with n_0 independent of m. The proof of [7] shows that $\int_0^t s'_m(X_t)dA_t$ is the limit in probability of $\sum_{k=0}^{2^n-1} s'_m(X_{kt/2^n})(A^m_{(k+1)t/2^n} - A^m_{kt/2^n})$ as $n \to \infty$. Using Lemma 2.3(b) again and taking n_0 larger if necessary, if $n \geq n_0$,

$$\mathbb{P}\Big(\Big|\int_0^t s'_m(X_s)dA_s - \sum_{k=0}^{2^n-1} s'_m(X_{kt/2^n})(A_{(k+1)t/2^n} - A_{kt/2^n})\Big| > \varepsilon\Big) < \varepsilon.$$

Therefore, except for a set of probability at most 2ε, we have

$$|J(m)| \leq 2\varepsilon + \sum_{k=0}^{2^n-1} |(A^m_{(k+1)t/2^n} - A^m_{kt/2^n}) - (A_{(k+1)t/2^n} - A_{kt/2^n})|$$

for all m provided we pick $n \geq n_0$. However $(A^m_{(k+1)t/2^n} - A^m_{kt/2^n}) \to (A_{(k+1)t/2^n} - A_{kt/2^n})$ as $m \to \infty$, and since ε is arbitrary, $\limsup_m |J(m)| = 0$ as required. $\square$

Remark 2.7. Suppose instead of (1.2) we consider the SDE

$$dX_t = a(X_t)dW_t + b(X_t)dt, \tag{2.13}$$

where $a \in C^{1/2}$ is bounded above and bounded below away from 0 and b is the distributional derivative of a function $B \in C^\gamma$ for some $\gamma > \frac{1}{2}$. Let

$$D(x) = \int_0^x \frac{1}{a(t)^2} dB_t, .$$

where the integral is defined in the sense of L.C. Young (cf. the proof of Lemma 2.3(a)). By Lemma 2.3(a), D is locally a C^γ function, and (2.13) can be rewritten

$$dX_t = a(X_t)dW_t + (a^2 d)(X_t)dt,$$

where d is the distributional derivative of D. Thus our results provide an interpretation of (2.13) as well as of (1.2).

3. Divergence form operators.

In this section we will give conditions for pathwise existence and uniqueness for Markov processes corresponding to divergence form operators.

Let $\mathcal{L}$ be a divergence form operator on $\mathbb{R}$:

$$\mathcal{L} = \frac{1}{2}\frac{d}{dx}\Big(a^2 \frac{d}{dx}\Big), \tag{3.1}$$

where a is a measurable function on $\mathbb{R}$ and suppose there is a constant $\lambda > 1$ such that

$$0 < 1/\lambda \leq a(x) \leq \lambda < \infty \quad \text{for a.e. } x \in \mathbb{R}. \tag{3.2}$$

The operator $\mathcal{L}$ gives rise to a Dirichlet form $(\mathcal{E}, W^{1,2}(\mathbb{R}))$ on $L^2(\mathbb{R}, dx)$, where

$$\mathcal{E}(f,g) = \tfrac{1}{2} \int_{\mathbb{R}} f'(x) g'(x) a^2(x) dx. \tag{3.3}$$

A Markov process X is said to be associated with $(\mathcal{E}, W^{1,2}(\mathbb{R}))$ on $L^2(\mathbb{R}, dx)$ if its transition semigroup P_t is symmetric in $L^2(\mathbb{R}, dx)$,

$$W^{1,2}(\mathbb{R}) = \{ f \in L^2(dx) : \lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}} f(x)(f(x) - P_t f(x))\, dx < \infty \},$$

and for $f, g \in W^{1,2}(\mathbb{R})$

$$\lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}} g(x)(f(x) - P_t f(x))\, dx = \mathcal{E}(f,g).$$

It is well known that there is a continuous conservative Feller process $(X, \mathbb{P}^x, x \in \mathbb{R})$ associated with $(\mathcal{E}, W^{1,2}(\mathbb{R}))$ on $L^2(\mathbb{R})$ (cf. Example 4.5.2 of [9]). In addition, since $1/\lambda \leq a(x) \leq \lambda$, the capacity induced by X is equivalent to the capacity induced by Brownian motion on $\mathbb{R}$. Therefore each point on $\mathbb{R}$ is non-polar for X. (See Example 4.5.1 of [9].) In what follows we will use X_t^x to denote the process X under $\mathbb{P}^x$ such that $X_0^x = x$. Such a process is unique in distribution in the following sense. If there is another symmetric right continuous strong Markov process Z associated with the Dirichlet form $(\mathcal{E}, W^{1,2}(\mathbb{R}))$, then $\{Z_t^x, t \geq 0\}$ has the same law as $\{X_t^x, t \geq 0\}$ for every $x \in \mathbb{R}$ (cf. Theorem 4.2.7 of [9]). A process Z is said to be a diffusion if it is a continuous strong Markov process.

By applying Fukushima's decomposition to the function $f(x) = x$, which is locally in $W^{1,2}(\mathbb{R})$, the following decomposition

$$X_t = X_0 + \int_0^t a(X_s) dW_s + N_t, \quad t \geq 0, \tag{3.4}$$

holds. Here W is a martingale additive functional of X with $W_0 = 0$ and $\langle W \rangle_t = t$ (so W_t is a Brownian motion under $\mathbb{P}^x$ for every $x \in \mathbb{R}$) and N_t is a continuous additive functional of X that locally has zero energy under the measure $\int_{\mathbb{R}} \mathbb{P}^x(\cdot) dx$ with $N_0 = 0$. Such a decomposition is unique (cf. Theorems 5.5.1 and 5.5.2 in [9]). In fact (3.4) characterizes the symmetric diffusion associated with the Dirichlet form $(\mathcal{E}, W^{1,2}(\mathbb{R}))$, or equivalently, having $\mathcal{L}$ as its infinitesimal generator.

Theorem 3.1. *Suppose that Z is a diffusion on $\mathbb{R}$ whose transition semigroup is symmetric with respect to Lebesgue measure on $\mathbb{R}$. If Z satisfies (3.4), then Z is a continuous conservative Feller process with infinitesimal generator $\mathcal{L}$ given by (3.1).*

Proof. Since Z is a symmetric diffusion on $\mathbb{R}$, by the Beurling-Deny decomposition (cf. Theorem 3.2.3 of [9]), its associated Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(\mathbb{R}, dx)$ has the expression

$$\widetilde{\mathcal{E}}(f,g) = \int_{\mathbb{R}} f'(x) g'(x) \mu(dx),$$

where μ is a positive Radon measure. By Fukushima's decomposition,

$$Z_t = Z_0 + M_t + \widetilde{N}_t, \quad t \geq 0,$$

where M_t is a continuous local martingale additive functional of Z whose square bracket $\langle M \rangle$ has μ as its Revuz measure. and $\widetilde{N}_t$ is a continuous additive functional of Z locally of zero energy. By the uniqueness of Fukushima's decomposition, we have $M_t = \int_0^t a(Z_s) dW_s$ for some Brownian motion W_t, so $\langle M \rangle_t = \int_0^t a^2(Z_s) ds$. Thus the Revuz measure $\mu(dx)$ is $a^2(x)dx$. This implies $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}) = (\mathcal{E}, W^{1,2}(\mathbb{R}))$ and so $\mathcal{L}$ is the infinitesimal generator of Z. □

The next result says the process associated with the Dirichlet form $(\mathcal{E}, W^{1,2}(\mathbb{R}))$ is continuous with respect to the diffusion coefficient $a(x)$.

Theorem 3.2. *Suppose that $a_n(x)$ and $a(x)$ are measurable functions on $\mathbb{R}$ satisfying (3.2) and $\lim_{n\to\infty} a_n = a$ almost everywhere on $\mathbb{R}$. Denote by X^n and X the symmetric diffusion processes associated with the operators $\mathcal{L}^n = \frac{1}{2}\frac{d}{dx}\left(a_n^2 \frac{d}{dx}\right)$ and $\mathcal{L} = \frac{1}{2}\frac{d}{dx}\left(a^2 \frac{d}{dx}\right)$, respectively. Let $\mathbb{P}_n^x$ and $\mathbb{P}^x$ be the laws of X^n with $X_0^n = x$ and X with $X_0 = x$, respectively. Then for every $x \in \mathbb{R}$, $\mathbb{P}_n^x$ converges weakly to $\mathbb{P}^x$ on the space $C([0,\infty), \mathbb{R})$ equipped with the topology of uniform convergence on compact intervals.*

Proof. It is known (see Lyons and Zhang [14]) that for any smooth function $\phi \geq 0$ with compact support on $\mathbb{R}$, the measure $\int_{\mathbb{R}} \phi(x)\mathbb{P}_n^x(\cdot)dx$ converges weakly to $\int_{\mathbb{R}} \phi(x)\mathbb{P}^x(\cdot)dx$ on $C([0,\infty), \mathbb{R})$. The theorem now follows by the same argument as that in Burdzy and Chen [2], since by Aronson's estimate, the density function $p_t^n(x,y)$ for X_x^n has a Gaussian upper bound independent of n (cf. [17]). □

A natural and open question is: given a Brownian motion W, can one find a symmetric diffusion associated with the Dirichlet form $(\mathcal{E}, W^{1,2}(\mathbb{R}))$ having the decomposition (3.4)? We call such a diffusion, if it exists, a strong solution to the SDE (3.4). In the following we will establish strong existence and pathwise uniqueness for the diffusion X under certain conditions on a, as well as some non-uniqueness results. We will also give necessary and sufficient conditions for the diffusion X to be a semimartingale.

A nonnegative increasing function ρ on $\mathbb{R}_+$ is called moderate if there is a constant $\gamma > 1$ such that $\rho(2x) \leq \gamma\, \rho(x)$ for all $x > 0$. The functions $\rho(x) = x^\alpha$, $0 < \alpha < \infty$ are examples of such functions.

In the next two theorems, in addition to (3.2) we assume that $|a(x) - a(y)| \leq \rho(|x-y|)$ where ρ is an increasing function that satisfies $\int_{0+} \rho^{-2}(x)dx = \infty$ and that $x\rho^2(\sqrt{x})$ is a moderate increasing convex function on $\mathbb{R}_+$. (The functions $\rho(x) = x^\alpha$, $1/2 \leq \alpha < \infty$ have this property.)

Theorem 3.3. *Let $\mathcal{L}$ be the divergence form operator in (3.1) with coefficient a satisfying the above condition. Given a Brownian motion W_t on $\mathbb{R}$, there is a continuous conservative Feller process X associated with $\mathcal{L}$ that is adapted to the filtration of W_t and which has the decomposition*

$$X_t = X_0 + \int_0^t a(X_s)dW_s + N_t, \quad t \geq 0, \tag{3.5}$$

where N_t has zero energy under $\mathbb{P}^x$ for each $x \in \mathbb{R}$.

Proof. Let $a_n(x)$ be smooth functions such that $1/\lambda \leq a_n(x) \leq \lambda$ and $\lim_{n\to\infty} a_n(x) = a(x)$ uniformly in x on compact intervals. Denote by X^n the symmetric diffusion process associated with the operator $\mathcal{L}^n = \frac{1}{2}\frac{d}{dx}(a_n^2 \frac{d}{dx})$ that is driven by the Brownian motion W_t, that is,

$$dX_t^n = a_n(X_t^n)dW_t + (a_n a_n')(X_t)dt. \tag{3.6}$$

Let $s_n(x) = \int_0^x a_n^{-2}(t)dt$. Then $Y_t^n = s_n(X_t^n)$ is the unique strong solution to the SDE

$$dY_t^n = \frac{1}{a_n \circ s_n^{-1}(Y_t^n)} dW_t \tag{3.7}$$

with $Y_0^n = s(X_0^n)$. We will use $X^{n,x}$ and $Y^{n,y}$ to denote the solutions to (3.6) and (3.7) with $X_0^{n,x} = x$ and $Y_0^{n,y} = y$, respectively. Define $s(x) = \int_0^x a^{-2}(t)dt$. For each $y \in \mathbb{R}$, let Y^y be the unique solution to

$$dY_t^y = \frac{1}{a \circ s^{-1}(Y_t)} dW_t \quad \text{with} \quad Y_0^y = y. \tag{3.8}$$

That (3.8) has a strong solution and that the solution is pathwise unique is due to Theorem 1 of Yamada-Watanabe [20], since $(1/a) \circ s^{-1}$ satisfies the Yamada-Watanabe condition. Since $a_n \circ s_n^{-1}$ and $a \circ s_n^{-1}$ are continuous and $a_n \circ s_n^{-1}$ converges to $a \circ s_n^{-1}$ on compact intervals, by Kaneko and Nakao [11], for every compact interval $K \subset \mathbb{R}$ and finite $T > 0$,

$$\lim_{n\to\infty} \sup_{y\in K} \mathbb{E}\left[\max_{0\leq t\leq T} \left| Y_t^{n,y} - Y_t^y \right|^2 \right] = 0.$$

So there is a subsequence k_n such that

$$\lim_{n\to\infty} \sup_{y\in K} \max_{0\leq t\leq T} \left| Y_t^{k_n,y} - Y_t^y \right| = 0 \quad \text{a.s.} \tag{3.9}$$

Define $X^x = s^{-1}(Y^{s(x)})$. Then on every compact interval K and $T > 0$,

$$\lim_{n\to\infty} \sup_{x\in K} \max_{0\leq t\leq T} \left| X_t^{k_n,x} - X_t^x \right| = 0 \quad \text{a.s.} \tag{3.10}$$

Now let $\widetilde{X}$ be the symmetric diffusion associated with the Dirichlet form $(\mathcal{E}, W^{1,2}(\mathbb{R}))$ on $L^2(\mathbb{R})$. Since a_n is uniformly elliptic and $a_n \to a$ as $n \to \infty$, by Theorem 3.2 X and $\widetilde{X}$ have the same distribution whenever X_0 and $\widetilde{X}_0$ have the same distribution. Hence X is a symmetric diffusion associated with the Dirichlet form $(\mathcal{E}, W^{1,2}(\mathbb{R})$ on $L^2(\mathbb{R})$. Clearly X is adapted to the filtration of the prescribed Brownian motion W_t.

Let σ denote the inverse of s. We now show that

$$N_t = X_t - X_0 - \int_0^t a(X_s)dW_s = \sigma(Y_t) - \sigma(Y_0) - \int_0^t \sigma'(Y_s)dY_s$$

has zero energy under $\mathbb{P}^x$ for any $t > 0$ and $x \in \mathbb{R}$. Note that σ is in C^1 and σ' has modulus of continuity function $c_1\rho$. By the mean value theorem,

$$\sigma(y) - \sigma(x) = \sigma'(x)(y-x) + \Big(\sigma'(\theta x + (1-\theta)y) - \sigma'(x)\Big)(y-x)$$

for some $\theta \in [0,\, 1]$, and therefore

$$\Big|\sigma(y) - \sigma(x) - \sigma'(x)(y-x)\Big| \le c_1\rho(|y-x|)|y-x|. \tag{3.11}$$

Thus for $\Pi_t = \{t_0, t_1, \ldots, t_n\}$ a partition of $[0,t]$ with mesh $|\Pi_t| = \max_{1\le k\le n}|t_k - t_{k-1}|$, by (3.11) and the Burkholder-Davis-Gundy equality (cf. Theorem 10.36 of He-Wang-Yan [10]) ,

$$\begin{aligned}
&\mathbb{E}^x\Big[\sum_{k=1}^n (N_{t_k} - N_{t_{k-1}})^2\Big] \\
&\qquad = \sum_{k=1}^n \mathbb{E}^x\Big(\sigma(Y_{t_k}) - \sigma(Y_{t_{k-1}}) - \int_{t_{k-1}}^{t_k} \sigma'(Y_s)dY_s\Big)^2 \\
&\qquad \le 2\sum_{k=1}^n \Big(\mathbb{E}^x\Big(\int_{t_{k-1}}^{t_k} \Big(\sigma'(Y_s) - \sigma'(Y_{t_{k-1}})\Big)dY_s\Big)^2 \\
&\qquad\qquad + c_2\mathbb{E}^x\Big(\rho^2(|Y_{t_k} - Y_{t_{k-1}}|)|Y_{t_k} - Y_{t_{k-1}}|^2\Big)\Big) \\
&\qquad \le c_3\sum_{k=1}^n \Big(\mathbb{E}^x\Big(\int_{t_{k-1}}^{t_k} \Big(\sigma'(Y_s) - \sigma'(Y_{t_{k-1}})\Big)\Big)^2 ds \\
&\qquad\qquad + \rho^2\Big(\sqrt{\mathbb{E}^x(Y_{t_k} - Y_{t_{k-1}})^2}\Big)\mathbb{E}^x(Y_{t_k} - Y_{t_{k-1}})^2\Big) \\
&\qquad \le c_4\,\mathbb{E}^x\Big[\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \Big(\sigma'(Y_s) - \sigma'(Y_{t_{k-1}})\Big)^2 ds\Big] + c_4\,t\,\rho^2\Big(c_5\sqrt{|\Pi_i|}\Big).
\end{aligned}$$

Therefore

$$\lim_{|\Pi_t|\to 0} \mathbb{E}^x\Big[\sum_{k=1}^{n}(N_{t_k} - N_{t_{k-1}})^2\Big] = 0.$$

$\square$

The following is a pathwise uniqueness result for the SDE (3.5).

Theorem 3.4. *Assume the conditions of Theorem 3.3 hold and let X be a strong solution for the SDE (3.5). Suppose that Z^x is a continuous process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ on which W_t is an $\{\mathcal{F}_t\}_{t\ge 0}$-Brownian motion. If Z^x satisfies equation (3.5) and has the same distribution as that of X^x, then*

$$\mathbb{P}(X_t^x \neq Z_t^x \text{ for some } t \geq 0) = 0.$$

Proof. Let $s(x) = \int_0^x a^{-2}(t)dt$. By a similar argument as that in the proof of Theorem 3.3, we see that

$$s(Z_t^x) - s(x) - \int_0^t s'(Z_s^x)a(Z_s^x)dW_s$$

is a process of zero energy. On the other hand, since $\{s(Z_t^x), t \geq 0\}$ has the same distribution as $\{s(X_t^x), t \geq 0\}$ and the latter is a martingale, $\{s(Z_t^x), t \geq 0\}$ is a martingale as well. Therefore,

$$s(Z_t^x) = s(x) + \int_0^t s'(Z_s^x)a(Z_s^x)dW_s = s(x) + \int_0^t \frac{1}{a(Z_s^x)}dW_s, \quad t \geq 0.$$

Thus both $s(Z_t^x)$ and $s(X_t^x)$ solve the SDE (3.8) with the same initial value $s(x)$. By the pathwise uniqueness for the SDE (3.8) (see Theorem 1 of [20]),

$$\mathbb{P}(s(X_t^x) \neq s(Z_t^x) \text{ for some } t \geq 0) = 0$$

and therefore

$$\mathbb{P}(X_t^x \neq Z_t^x \text{ for some } t \geq 0) = 0.$$

$\square$

Remark 3.5. When $a \in C^\gamma$ with $\gamma > 1/2$ and a is bounded above and bounded below away from 0, the unique solution in Theorems 4.3 and 4.4 coincides with the unique solution in Theorems 2.5 and 2.6 with $B = \frac{1}{2}a^2$, as they are all given by $X_t = s^{-1}(Y_t)$ where Y_t satisfies (2.2).

Theorem 3.6. *Let $(X, \mathbb{P}^x, x \in \mathbb{R})$ be a continuous conservative Feller process with infinitesimal generator $\mathcal{L}$ given by (3.1). Then the following are equivalent.*

(i) X is a semimartingale under $\mathbb{P}^x$ for some $x \in \mathbb{R}$,

(ii) X is a semimartingale under $\mathbb{P}^x$ for all $x \in \mathbb{R}$,

(iii) the distributional derivative of the function $a(x)$ is a signed Radon measure.

If the distributional derivative of the function $a(x)$ is a signed Radon measure $a'(dx)$, then X has the representation

$$X_t = X_0 + \int_0^t a(X_s)dW_s + \int_{\mathbb{R}} a^{-1}(x)L_t^x(X)\, a'(dx), \quad t \geq 0. \tag{3.12}$$

Here W is a Brownian motion and $L_t^x(X)$ is the local time for the semimartingale X at level x, given by (3.2).

Proof. Let

$$A = \{x \in \mathbb{R} : \mathbb{P}^x(s \to N_s \text{ is a process of finite variation}) = 1\}.$$

Note that since for $s, t > 0$, $N_s \circ \theta_t = N_{s+t} - N_t$, where θ_t is the shift operator for the Markov process X, we have $\mathbb{P}^x(X_t \in A) = 1$ for $x \in A$. In other words, $P_t 1_{A^c} = 0$ on A. Since the process X is irreducible, either A or A^c has zero Lebesgue measure. Note that X has continuous transition density functions (in fact, they are Hölder continuous by Nash' well-known result), either $P_t 1_A \equiv 0$ on $\mathbb{R}$ for all $t > 0$ or $P_t 1_A \equiv 1$ on $\mathbb{R}$ for all $t > 0$. Since $x \in A$ if and only if $\lim_{t \downarrow 0} P_t 1_A(x) = 1$, we have either $A = \emptyset$ or $A = \mathbb{R}$. This shows that (i) and (ii) are equivalent.

Since each point of $\mathbb{R}$ is non-polar for X, a smooth measure in the sense of [9] is a Radon measure (see Example 4.5.1 of [9]). By Theorem 3.3, the decomposition (3.5) holds. Note that (3.5) is the Fukushima decomposition for $f(X)$, where $f(x) = x$ is locally in $W^{1,2}(\mathbb{R})$, and that

$$\mathcal{E}(x, v) = \tfrac{1}{2} \int_{\mathbb{R}} a^2(x)v'(x)dx$$

for $v \in C_c^\infty(\mathbb{R})$. Thus by Theorem 44444 in Fukushima-Oshima-Takeda [9], X is a semimartingale under $\mathbb{P}^x$ for every $x \in \mathbb{R}$ if and only if the distributional derivative of the function a^2 is a signed Radon measure. The latter is equivalent to the distributional derivative of the function $a(x)$ being a signed Radon measure $a'(dx)$. In this case N in (3.5) is an additive functional of X having bounded variation whose Revuz measure is $a(x)a'(dx)$.

Let $L_t^x(X)$ be the positive continuous additive functional of X associated with the measure $a^2(x)\delta_{\{x\}}$, where $\delta_{\{x\}}$ is unit mass concentrated at x (see Theorem 5.1.3 of [9]). Given a positive Radon measure ν, it follows from Theorem 5.1.3 of [9] that the positive continuous additive functional of X with Revuz measure $a^2(x)\nu(dx)$ is $\int_{\mathbb{R}} L_t^x(X)\nu(dx)$. On the other hand, it is known that for a Borel measurable function $f \geq 0$, $\int_0^t f(X_s)a^2(X_s)ds$ is a positive continuous additive functional of X having Revuz measure $\nu(dx) = f(x)a^2(x)dx$. Therefore for any Borel measurable function $f \geq 0$,

$$\int_{\mathbb{R}} f(x)L_t^x(X)\,dx = \int_0^t f(X_s)a^2(X_s)ds = \int_0^t f(X_s)d\langle X\rangle_s.$$

This shows that $t \to L_t^x(X)$ is the local time of X at level x (cf. [16, Corollary VI.1.6]).

Now assume that the distributional derivative of the function $a(x)$ is a signed Radon measure $a'(dx)$. As we noted above, N in (3.5) is an additive functional of X having bounded variation whose (signed) Revuz measure is $a(x)a'(dx)$ and so

$$N_t = \int_{\mathbb{R}} a^{-1}(x)L_t^x(X)\,a'(dx), \quad t \geq 0.$$

This completes the proof. □

In the following, we show that the results in Theorems 3.3 and 3.4 above are nearly optimal, by using a result due to Barlow. Let X be a conservative diffusion process associated with the differential operator $\mathcal{L}$ in (3.1).

Theorem 3.7. *Suppose that a is uniformly bounded away from zero and bounded above and let α and β be constants with $\alpha \in (0, \frac{1}{2})$ and $\alpha \leq \beta \leq \alpha + \min\{\frac{\alpha}{2+\alpha}, \frac{\alpha-2\alpha^2}{1+2\alpha}\}$. Suppose on an interval $[x_0, y_0]$ there are positive constants c_1, c_2 and c_3 so that*

$$|a(x) - a(y)| \leq c_1|x-y|^\alpha \quad \text{for } x, y \in [x_0, y_0]$$

and for all $x < y$ in $[x_0, y_0]$, there exist x_1, y_1 with $x < x_1 < y_1 < y$ such that $|x_1 - y_1| \geq c_2|x-y|$ and $|a(x_1) - a(y_1)| > c_3|x_1 - y_1|^\beta$. Then for $x \in [x_0, y_0]$, there is no strong solution nor pathwise uniqueness to the equation

$$Z_t = x + \int_0^t a(Z_s)dW_s + N_t \quad \text{and} \quad Z \text{ has the same distribution as } X^x, \tag{3.13}$$

where N_t is a continuous process having zero energy.

Concrete examples of functions $a(s)$ satisfying the condition of Theorem 3.6 can be found in Barlow [1].

Proof. Suppose we are given a Brownian motion W_t. Suppose that (3.14) has a strong solution Z. Define $s(x) = \int_0^x a^{-2}(t)dt$. Since $a(x)$ is bounded away from zero and infinity,

$x \to s(x)$ is a one-to-one bi-Lipschitz map from $\mathbb{R}$ into $\mathbb{R}$. Let $Y_t = s(Z_t)$. Then Y_t is a strong solution to the following one dimensional SDE:

$$Y_t = s(x) + \int_0^t \widetilde{a}(Y_s) dW_s, \tag{3.14}$$

where $\widetilde{a}(y) = 1/a(s^{-1}(y))$. Note that $\widetilde{a}$ satisfies the same hypotheses as does a, but with x_0, y_0 being replaced by $s(x_0)$ and $s(y_0)$, respectively. Thus by Theorem 1.3 of Barlow [1], the SDE (3.15) does not have pathwise uniqueness. Since weak uniqueness holds for (3.15), by a result of Gikhman and Skorokhod (see Theorem 1.2 in [1]), (3.15) can not have a strong solution. This is a contradiction. Therefore equation (3.14) does not have a strong solution.

The same argument shows that solutions to (3.14) do not have the pathwise uniqueness property either. □

4. Stratonovich SDEs.

In this section we consider the case where $b = \frac{1}{2}a'a$, so that formally (1.1) becomes the Stratonovich SDE (1.4). It is well known that when a is C^2, the SDE (1.4) has a unique strong solution. We will show in this section that if a is bounded and continuous, then (1.4) has a strong solution. Of course in this case, we need to give an interpretation to this Stratonovich SDE as a is not differentiable.

The following result is a special case of the generalized Ito's formula in [8], where the same formula is proved for continuous f with $f' \in L^2$. Here we give a simple proof for C^1 functions f.

Theorem 4.1. *For a C^1 function f,*

$$f(W_t) = f(W_0) + \lim_{n\to\infty} \sum_{k=1}^n \frac{\nabla f(W_{(k-1)t/n}) + \nabla f(W_{kt/n})}{2} (W_{kt/n} - W_{(k-1)t/n})$$

The limit is in the sense of convergence in probability with respect to $\mathbb{P}^x$ for every $x \in \mathbb{R}^n$.

Proof. Write $S_n(f) = \sum_{k=1}^n (\nabla f(W_{(k-1)t/n}) + \nabla f(W_{kt/n}))(W_{kt/n} - W_{(k-1)t/n})$. As $f \in C^1$, given any $\varepsilon > 0$, there is a $g \in C^2$ with $|\nabla(f-g)| < \varepsilon$ everywhere. It is straightforward that the theorem holds for C^2 functions, so if we can show $\mathbb{E}^x(S_n(f-g))^2 < C\varepsilon^2$, where C is a constant independent of f and g, the theorem will follow.

Fix n, and write $\phi = \nabla(f-g)$, so that $|\phi| < \varepsilon$ everywhere. Then $S_n(f-g) = X + Y$, where $X = \sum_{k=1}^n \phi(W_{(k-1)t/n})(W_{kt/n} - W_{(k-1)/n})$ and $Y = \sum_{k=1}^n \phi(W_k)(W_{kt/n} - W_{(k-1)t/n})$. We write

$$\mathbb{E}^x(X^2) = \frac{t}{n} \sum_{k=1}^n \mathbb{E}^x(\phi(W_{(k-1)t/n}))^2 \le \varepsilon^2 t.$$

Also

$$Y^2 = \sum_{k=1}^n \phi(W_{kt/n})^2 (W_{kt/n} - W_{(k-1)t/n})^2$$
$$+ 2 \sum_{1 \le j < k \le n} \phi(W_{jt/n})\phi(W_{kt/n})(W_{jt/n} - W_{(j-1)t/n})(W_{kt/n} - W_{(k-1)/n}).$$

We need the following observation. If Z_s is a Brownian bridge tied down to be a at time 0 and b at time u, then there exists a Brownian motion B_s such that

$$Z_s = a + B_s - \frac{s}{u}(B_u - (b-a)).$$

So

$$\mathbb{E}\left[Z_u - Z_s\right] = \frac{u-s}{u}(b-a).$$

If $1 \le j < k \le n$ and we condition on $W_0, W_{jt/n}$, and $W_{kt/n}$, we have two Brownian bridges, one from time 0 to time jt/n and the other from time jt/n to time kt/n. We then have

$$\begin{aligned}
&\mathbb{E}^x((W_{jt/n} - W_{(j-1)t/n})(W_{kt/n} - W_{(k-1)t/n})\big| W_0, W_{jt/n}, W_{kt/n}) \\
=&\frac{1}{j(k-j)}(W_{jt/n} - W_0)(W_{kt/n} - W_{jt/n}),
\end{aligned}$$

and so

$$\begin{aligned}
&|\mathbb{E}^x(\phi(W_{jt/n})\phi(W_{kt/n})(W_{jt/n} - W_{(j-1)t/n})(W_{kt/n} - W_{(k-1)t/n}))| \\
=&\frac{1}{j(k-j)}|\mathbb{E}^x(\phi(W_{jt/n})\phi(W_{kt/n})(W_{jt/n} - W_0)(W_{kt/n} - W_{jt/n}))| \\
\le&\frac{\varepsilon^2}{j(k-j)}\mathbb{E}^x(|W_{jt/n} - W_0|\,|W_{kt/n} - W_{jt/n}|).
\end{aligned}$$

By Cauchy-Schwarz, this is less than

$$\frac{2\varepsilon^2 t}{n\sqrt{j(k-j)}}.$$

Hence

$$\mathbb{E}^x(Y^2) \le \varepsilon^2 t + \frac{4\varepsilon^2 t}{n} \sum_{1 \le j < k \le n} \frac{1}{\sqrt{j(k-j)}} \le C\varepsilon^2 t$$

as required. □

Let us define the Stratonovich integral $\int_0^t H_{s-} \circ dW_s$ as the limit of the Riemann sums

$$\sum_{k=1}^n \frac{H_{(k-1)t/n} + H_{kt/n}}{2}(W_{kt/n} - W_{(k-1)t/n})$$

as $n \to \infty$, as long as the Riemann sums converge in probability. Then the above theorem can be rephrased as saying

$$f(W_t) = f(W_0) + \int_0^t \nabla f(W_s) \circ dW_s, \quad t \geq 0.$$

Clearly when f is smooth this definition of Stratonovich integral is consistent with that in the literature. For the one dimensional case, we have a strong existence result.

Theorem 4.2. *Suppose a is a positive, bounded continuous function on $\mathbb{R}$. Then the Stratonovich SDE*

$$dX_t = a(X_t) \circ dW_t$$

has a strong solution. In fact, given a Brownian motion W_t with $W_0 = 0$ and $x_0 \in \mathbb{R}$, there is a continuous process $X_t = s^{-1}(s(x_0) + W_t)$ that solves

$$X_t = x_0 + \int_0^t a(X_s) \circ dW_s, \tag{4.1}$$

where $s(x) = \int_0^x \frac{1}{a(t)} dt$. This particular solution $X = s^{-1}(s(x_0) + W_t)$ is a semimartingale if and only if the distributional derivative of a is a signed measure $a'(dx)$. In this case, X has the representation

$$X_t = x_0 + \int_0^t a(X_s) dW_s + \frac{1}{2} \int_{\mathbb{R}} a^{-1}(x) L_t^x(X)\, a'(dx), \quad t \geq 0, \tag{4.2}$$

where $L_t^x(X)$ is the local time of the semimartingale X at level x.

Proof. Define

$$s(x) = \int_0^x \frac{1}{a(t)} dt.$$

$s(x)$ is a C^1 function that maps $\mathbb{R}$ onto $\mathbb{R}$ and so is its inverse $\sigma = s^{-1}$. Let $X_t = \sigma(s(x_0) + W_t)$. It follows from Theorem 4.1 that

$$X_t = x_0 + \int_0^t \sigma'(s(x) + W_s) \circ dW_s = x_0 + \int_0^t a(X_s) \circ dW_s$$

By the same argument in proving the equivalence of (i) and (ii) in Theorem 3.6, it can be shown that $X_t = \sigma(s(x) + W_t)$ is a semimartingale for some $x \in \mathbb{R}$ if and only if $X_t = \sigma(s(x) + W_t)$ is a semimartingale for all $x \in \mathbb{R}$. Thus by Example 5.5.1 in Fukushima-Oshima-Takeda [9], $X_t = \sigma(s(x_0) + W_t)$ is a semimartingale if and only if the second order distributional derivative of s^{-1} is a signed Radon measure, which happens if and only if the distributional derivative of a is a signed Radon measure. Assume the distributional derivative of a is a signed Radon measure $a'(dx)$. Note that $\sigma'(y) = a(\sigma(y))$ and $d\sigma'(y) = da(\sigma(y))$. By Example 5.5.1 of [9], X has the following representation:

$$X_t = x + \int_0^t a(X_s)dW_s + \tfrac{1}{2}\int_{\mathbb{R}} L_t^{y-s(x)}(W)\, d\sigma'(dy), \quad t \geq 0,$$

where $L_t^y(W)$ is the local time for Brownian motion W at level y. Note that

$$L_t^{y-s(x)}(W) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t 1_{[y-s(x),y-s(x)+\varepsilon)}(W_s)ds.$$

W_s is between $y - s(x)$ and $y - s(x) + \varepsilon$ if and only if $X_s = \sigma(s(x) + W_s$ is between $\sigma(y)$ and $\sigma(y + \varepsilon)$. Therefore

$$\begin{aligned} L_t^{y-s(x)}(W) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t 1_{[\sigma(y),\sigma(y)+\varepsilon)}(X_s)ds \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t 1_{[\sigma(y),\sigma(y)+\varepsilon)}(X_s)a^{-2}(X_s)d\langle X\rangle_s \\ &= \sigma'(y)a^{-2}(\sigma(y))L_t^{a(y)}(X), \end{aligned}$$

where $L_t^z(X) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t 1_{[z,z+\varepsilon)}(X_s)d\langle X\rangle_s$ is the local time for $X_t = a(s(x) + W_t)$ at level z. Since $\sigma'(y) = a(\sigma(y))$,

$$L_t^{y-s(x)}(W) = \frac{1}{a(\sigma(y))}L_t^{a(y)}(X),$$

and therefore

$$\int_{\mathbb{R}} L_t^{y-s(x)}(W)\, d\sigma'(dy) = \int_{\mathbb{R}} \frac{1}{a(\sigma(y))}L_t^{\sigma(y)}(X)\, da(\sigma(y)) = \int_{\mathbb{R}} a^{-1}(z)L_t^z(X)\, a'(dz).$$

This completes the proof of the theorem. □

As $a'(dx)$ has no atoms, it can be shown that the local time $L_t^x(X)$ in (4.2) is the same as the symmetric local time of X. Note also that when $a \in C^\gamma$ with $\gamma > 1/2$ and a is bounded above and bounded below away from 0, the unique solution as defined in Section 2 with $B = a^2/4$ solves the Stratonovich SDE (4.1).

Theorem 4.3. *Suppose that a is a positive, bounded continuous function and has a distributional derivative that is a Radon measure $a'(dx)$. Then for any $x_0 \in \mathbb{R}$, there is a strong solution X_t to the SDE (4.2) with $X_0 = x_0$ and the solution is pathwise unique.*

Proof. The existence of a strong solution is already proved in Theorem 4.2 so we only need to show the pathwise uniqueness. Now suppose (X, W) is a solution to (4.2) with

$X_0 = x_0$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ such that X_t is $\mathcal{F}_t$-measurable and W is an $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion. Define $s(x) = \int_0^x 1/a(t)dt$. Then $s'(x) = 1/a(x)$ and the distributional derivative of $s'(x)$ is a signed measure $s''(dx) = -a^{-2}(x)a'(dx)$. By the generalized Itô formula and the fact that local time process $t \to L_t^x(X)$ increases only when $X_t = x$,

$$
\begin{aligned}
s(X_t) &= s(x_0) + \int_0^t s'(X_s)dX_s + \tfrac{1}{2}\int_0^t L_t^x(X)s''(dx) \\
&= s(x_0) + W_t + \tfrac{1}{2}\int_{\mathbb{R}} a^{-1}(x)a^{-1}(x)L_t^x(X)a'(dx) - \tfrac{1}{2}\int_R a^{-2}(x)L_t^x(X)a'(dx) \\
&= s(x_0) + W_t.
\end{aligned}
$$

Thus the paths of $s(X_t)$ are uniquely determined, and hence so are those of X. □

An interesting and natural question is whether pathwise uniqueness holds for the SDE (4.1) when a is only assumed to be positive and continuous. We will present an answer to this question in next question.

5. Another view of Stratonovich SDEs.

Assume in this section that $a(x)$ is a measurable function on $\mathbb{R}$ that is bounded above and bounded below away from zero. Formally, the Stratonovich SDE

$$
dX_t = a(X_t) \circ dW_t, \quad X_0 = x_0, \tag{5.1}
$$

has generator

$$
\mathcal{L} = \frac{a(x)^2}{2}\frac{d^2}{dx^2} + \tfrac{1}{2}a(x)a'(x)\frac{d}{dx} = \frac{a(x)}{2}\frac{d}{dx}\left(a(x)\frac{d}{dx}\right).
$$

So $\mathcal{L}$ is the infinitesimal generator corresponding to the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}, a^{-1}dx)$, where $\mathcal{F} = W^{1,2}(\mathbb{R})$ and

$$
\mathcal{E}(f, g) = \frac{1}{2}\int_{\mathbb{R}} a(x)f'(x)g'(x)dx \quad \text{for } f, g \in \mathcal{F}.
$$

It is well known (cf. [9]) that there is a diffusion process X associated with $(\mathcal{E}, \mathcal{F})$ with symmetrizing measure $a(x)^{-1}dx$.

Theorem 5.1. *Given a Brownian motion W_t, there is a continuous conservative Feller process X associated with $\mathcal{L}$ that is adapted to the filtration of W_t. Furthermore if a is continuous, then for each $x \in \mathbb{R}$ $\mathbb{P}^x$-a.s.,*

$$
X_t - x = \lim_{n\to\infty}\sum_{k=1}^n \frac{a(X_{(k-1)t/n}) + a(X_{kt/n})}{2}(W_{kt/n} - W_{(k-1)t/n}). \tag{5.2}
$$

Proof. Let
$$s(x) = \int_0^x \frac{1}{a(t)} dt. \tag{5.3}$$
Then s is $\mathcal{L}$-harmonic; in fact, $\mathcal{E}(s, f) = 0$ for $f \in C_c^1(\mathbb{R})$. Suppose Y is a diffusion process associated with $(\mathcal{E}, \mathcal{F})$; by Fukushima's decomposition, $s(Y_t) = s(Y_0) + M_t^s$ with $\langle M^s \rangle_t = t$. So $s(Y_t)$ is a Brownian motion starting from $s(Y_0)$.

Now suppose a Brownian motion W_t is given. Let σ denote the inverse function of $s(x)$. Define
$$X_t = \sigma(s(X_0) + W_t). \tag{5.4}$$
Then X_t is a continuous conservative Feller process X associated with $\mathcal{L}$. Since $W_t = s(X_t) - s(X_0)$, by Lyons and Zheng's forward and backward martingale decomposition (see [9]) or by the generalized Ito's formula in [8], we have (5.2). □

Note that the above X is a Dirichlet process. Its associated Dirichlet form is the $(\mathcal{E}, W^{1,2}(\mathbb{R}))$ given just below (5.1) and satisfies Fukushima's decomposition
$$X_t = X_0 + \int_0^t a(X_s) dW_s + N_t, \tag{5.5}$$
where N_t has zero energy under $\mathbb{P}^x$ for each $x \in \mathbb{R}$.

It is natural to formulate the following definition of solution to the Stratonovich SDE (5.1).

Definition 5.2. *Given a Brownian motion W_t on $\mathbb{R}$, we say X_t is a strong solution to (5.1) with starting point x_0 if*

(i) X_t is adapted to the filtration generated by W_t;

(ii) Whenever a_n is a sequence of C^2 functions that converges to a a.e. on $\mathbb{R}$ with
$$\lambda^{-1} \le a_n(x) \le \lambda \qquad \text{a.e. on } \mathbb{R} \tag{5.6}$$
for some $\lambda > 0$ and all $n \ge 1$, then with probability one $\sup_{0\le s\le t} |X_s^n - X_s|$ converges to zero for each $t > 0$. Here X_t^n is the unique solution to $dX_t^n = a(X_t^n) \circ dW_t$ with $X_0^n = x_0$.

Remark 5.3. Definition 5.2(ii) is equivalent to

(ii′) *There is a sequence of C^2 functions $\{a_n\}$ that converges to a a.e. on $\mathbb{R}$ and satisfies condition (5.6), and with probability one $\sup_{0\le s\le t} |X_s^n - X_s|$ converges to zero for each $t > 0$. Here X_t^n is the unique solution to $dX_t^n = a(X_t^n) \circ dW_t$ with $X_0^n = x_0$.*

Similarly to the proof of Theorem 3.4, onc can show that

Theorem 5.4. *Let X be defined by (5.3) and (5.4) with $X_0 = x$. Suppose that Z^x is a continuous process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ on which W_t is an $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion. If Z^x satisfies equation (5.5) and has the same distribution as that of X^x, then*

$$\mathbb{P}(X_t^x \neq Z_t^x \text{ for some } t \geq 0) = 0.$$

Theorem 5.5. *Suppose that $a_n(x)$ is a sequence of C^2 functions converging to $a(x)$ a.e. on $\mathbb{R}$ and satisfying condition (5.6). Denote by X^n the unique strong solution to*

$$X_t^n = x_0 + \int_0^t a_n(X_s^n) \circ dW_s$$

and let X be defined by (5.3) and (5.4) with $X_0 = x_0$. Then almost surely

$$\lim_{n\to\infty} \sup_{0\leq s\leq t} |X_s^n - X_s| = 0 \tag{5.7}$$

for every $t > 0$.

Proof. Define $s_n(x) = \int_0^x \frac{1}{a_n(t)} dt$. We use σ_n and σ to denote the inverse functions of s_n and s, respectively. Clearly $s_n \to s$ uniformly on bounded intervals and so $\sigma_n \to \sigma$. We see from the proof of Theorem 5.1 that $X_t^n = \sigma_n(s_n(x_0) + W_t)$ and as $X_t = \sigma(s(x_0) + W_t)$ (5.7) follows. □

Combining Theorems 5.4 and 5.5 we have

Theorem 5.6. *For every $x_0 \in \mathbb{R}$, there is a strong solution to the Stratonovich SDE (5.1) and the solution is pathwise unique.*

By a proof similar to that of Theorem 4.2, we have

Theorem 5.7. *The solution to the Stratonovich SDE (5.1) is a semimartingale if and only if the distributional derivative of a is a signed Radon measure $a'(dx)$. In this case, X has the representation*

$$X_t = x_0 + \int_0^t a(X_s) dW_s + \tfrac{1}{2} \int_{\mathbb{R}} a^{-1}(x) L_t^x(X)\, a'(dx), \quad t \geq 0, \tag{5.8}$$

where $L_t^x(X)$ is the local time of the semimartingale X at level x.

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Richard Bass
Department of Mathematics
University of Connecticut
Storrs, CT 06269, USA
Email: bass@math.uconn.edu
http://www.math.uconn.edu/~bass

Zhen-Qing Chen
Department of Mathematics
University of Washington
Box 354350
Seattle, WA 98195-4350, USA
Email: zchen@math.washington.edu
http://www.math.washington.edu/~zchen