ERRATA: Probabilistic Techniques in Analysis

Updated April 25, 2006

Page 3, line 13. $A_1, \ldots, A_n$ are independent if

$$\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_j}) = \mathbb{P}(A_{_1}) \cdots \mathbb{P}(A_{i_j})$$

for every subset $\{i_1, \ldots, i_j\}$ of $\{1, \ldots, n\}$.

Page 4, line -7. $\lim_{j\to\infty} \mathbb{P}(\cup_{n=j}^\infty A_n)$

Page 9, line 11. $x \in \mathbb{R}^d$, A Borel.

Page 14, line 5. Brownian motion X_t

Page 15, lines 15-16. Let $\mathcal{F}_t^0$ be the σ-field of events that are in the $\mathbb{P}^x$ completion of $\mathcal{F}_t^{00}$ for every x.

Page 15, line -1. $\theta_t(\omega)(s) = \omega(t+s)$

Page 16, 5th display. ... $|\mathcal{F}_2^{00}) = \mathbb{P}^{X_2}($...

Page 18, line 11. In the definition of Y_2 it should be $X_{t_j - s}$.

Page 20, line -7. The last T on this line should be T_n.

Page 33, line 1. $X(\tau_{[a,b]})$

Page 34, line 7. $\sum_{i=1}^\infty$ twice

Page 37, line 16. Add: “with $A_0 \equiv 0$”

Page 43, line 15. $\langle M \rangle_a$

Page 44, line 7. $\mathcal{F}_{a_j}$

Page 44, line 16. $\mathcal{F}_{a_j}$

Page 46, line 14. $R_n \to \infty$

Page 48, line 1. $f''(X_{S_i})$

Page 49, line -14. $\mathbb{P}^x$

Page 52, line 11, $\int_s^{t+s} \mathrm{i}ue^{u^2(r-s-t)/2}e^{\mathrm{i}u(X_r-X_s)}dX_r$

Page 56, line -1. $\frac{1}{2}\sum_{i,j=1}^d$

Page 59, line -9. Since $\sum (a^i t^{i+1}/i!)^{1/2} < \infty$

Page 62, line 15. $\mathbb{E}\,(M_T^*)^p \le c_1 \mathbb{E}\,\langle M\rangle_T^{p/2}$ and $\mathbb{E}\,\langle M\rangle_T^{p/2} \le c_2 \mathbb{E}\,(M_T^*)^p$.

Page 67, line -10. defined for $t \in [k/2^i, (k+1)/2^i)$ by $r_i(t) = +1$

Page 68, line 9. $4\delta N$

Page 68, line 10. $(4\delta N)^{1/2}$

Page 72, line 4. $\mathbb{P}_n$

Page 84, lines -3, -2. If x and y are close together, then $B(x,r) - B(y,r)$ is contained in $B(x, r+|x-y|) - B(x, r-|x-y|)$, and similarly with the roles of x and y reversed, and so

Page 99, line -11. decreasing sequence

Page 109, line -8. V be a nonnegative smooth

Page 129, line 15 to line 18. Remove.

Page 129, line 22. Replace with the following.

So $\|h\|_2^2 = \lim \|Qf\|_2^2 = \beta^2 > 0$, or $h \ne 0$. Note

$$\begin{aligned}\|Qh - \beta h\|_2^2 &= \lim \|Q^2 f_n - \beta Q f_n\|_2^2 \\ &\le \|Q\|_2^2 \lim \|Qf_n - \beta f_n\|_2^2 = 0.\end{aligned}$$

Since both h and Qh are continuous, $Qh = h$. □

Page 132, line -2. $|\varphi(X_{t2^n})|$

Page 205. Theorem 3.6 is incorrect as stated. This was pointed out by T. Lyons. The changes necessary are the following.

Page 205, line 9. $\le c[|x-y|^{2-d} + |y-z|^{2-d}]$.

Page 205, line 10 to Page 206, line 2 should be replaced by the following.

Proof. By the symmetry of (3.10) in x and z, we may suppose $|z-y| \ge |x-y|$.

We first consider the case when $|x-y| \ge |x-z|/4$. Let $r = |x-z|$. Let x_1 be a point of D such that $\mathrm{dist}\,(x_1, \partial D) \ge c_1 r$ and $|x - x_1| = r/16$, where c_1 depends only on D; the existence of c_1 follows from the fact that D is a bounded Lipschitz domain. By the boundary Harnack principle in $B(x, r/8) \cap D$ with the functions $g_D(\cdot, y)$ and $g_D(\cdot, z)$,

$$g_D(x,y)/g_D(x,z) \le c g_D(x_1,y)/g_D(x_1,z).$$

Similarly, there exists $z_1 \in D$ such that $\operatorname{dist}(z_1, \partial D) \ge c_1 r$, $|z_1 - z| = r/16$, and

$$g_D(y,z)/g_D(x_1,z) \le c g_D(y,z_1)/g_D(x_1,z_1).$$

So it suffices to bound $g_D(x_1,y) g_D(y,z_1)/g_D(x_1,z_1)$.

Let z_2 be a point on $\partial B(z_1, c_1 r/2)$. Both x_1 and z_1 are a distance at least $c_1 r$ from the boundary of D and no more than $9r/8$ apart. Since D is a Lipschitz domain, we can find a curve connecting x_1 and z_2 that always is at least $c_1 r/4$ from $\partial D \cup \{z_1\}$ and whose length is no more than $c_2 r$, where c_2 depends only on the domain D. By scaling and Harnack's inequality (Theorem II.1.20), $g_D(x_1, z_1) \ge c g_D(z_2, z_1)$. Combining with (3.9), $g_D(x_1, z_1) \ge c r^{2-d}$. Since $|y - z_1| \ge |y - z| - |z - z_1| \ge 3r/16$, then $g_D(y, z_1) \le c r^{2-d}$. We also have $|x_1 - y| \ge |x - y| - |x - x_1| \ge |x - y|/2$, so $g_D(x_1, y) \le c|x - y|^{2-d}$. Substituting, we have

$$\begin{aligned} \frac{g_D(x,y) g_D(y,z)}{g_D(x,z)} &\le \frac{c g_D(x_1,y) g_D(y,z_1)}{g_D(x_1,z_1)} \\ &\le c|x-y|^{2-d}. \end{aligned} \tag{3.11}$$

The second case to consider is when $|x - y| < |x - z|/4$. Let $s = |x - y|$. If $w \in \partial B(x, 2s) \cap D$, then by Case 1 above,

$$\frac{g_D(x,y) g_D(y,w)}{g_D(x,w)} \le c|x-y|^{2-d}. \tag{3.12}$$

The expression

$$k(\cdot) = \frac{g_D(x,y) g_D(y,\cdot)}{g_D(x,\cdot)}$$

is the Green function with pole at y for Brownian motion h-path transformed by the function $h(\cdot) = g_D(x, \cdot)$, hence is the Green function for Brownian motion conditioned to go to x before exiting D. Therefore $k(X_t)$ is a martingale under $\mathbb{P}_h^z$ up to time $\tau_D \wedge T_{\{x,y\}}$. Since k is positive in $D - \{x, y\}$ and is 0 on ∂D, by optional stopping

$$\begin{aligned} k(z) = \mathbb{E}_h^z k(X_0) &\le \mathbb{E}_h^z k(X_{T(B(x,2s))}) \\ &\le \sup_{w \in \partial B(x,2s)} k(w). \end{aligned}$$

Comparing with (3.12) completes the proof. □

Page 206, line 5. A sufficient condition

Page 206, lines 6,7. $\int_0^\infty$ twice

Page 206, lines 8-11. Replace by "*Proof.* (a)"

Page 207, line 1. $\le c_1[|x-y|^{2-d}+|y-z|^{2-d}]$.

Page 207, line 6. $\le c_1\Big[\int_D q(y)|x-y|^{2-d}dy+\int_D q(y)|z-y|^{2-d}dy\Big]$

Page 207, line 7. Replace $\int_0^{\tau_D}$ by $\int_0^\infty$ twice.

Page 207, line 14. $\int_0^\infty$

Page 231, line 11. Replace y by r.

Page 233, line 17. proposition

Page 257, line -1. $\partial_j\partial_k Uf$ is in C^α.

Page 260, line 5. Using (4.5) and the fact that $|B_y|=cy^{d+1}$,

Page 260, line 10. $c\int_{C(0)} s^{1-d}|\nabla u(x,s)|^2 dx\, ds$

Page 262, line 4. $|\mathcal{F}_r]\, dr\Big]$

Page 317, proof of Corollary 1.12. The proof should be replaced by the following.

Proof. Let $F(z)=f(z)/g(z)$. The hypothesis implies that $|F(z)-1|<1$ on γ. Let $\mathit{\Gamma}$ be the image of γ under F. Then $\mathit{\Gamma}$ is contained in the open disk of radius 1 centered at 1. $1/w$ is analytic in that disk, so by Cauchy's integral theorem and a change of variables,

$$\int_\gamma \frac{f'(z)}{f(z)}dz-\int_\gamma \frac{g'(z)}{g(z)}dz=\int_\gamma \frac{F'(z)}{F(z)}dz=\int_{\mathit{\Gamma}}\frac{dw}{w}=0.$$

By the argument principle, we see that the number of zeroes of f and g in D must be the same. □

Pages 321-322. The proof of Proposition V.2.2 is incorrect. The support theorem which is applied on page 322, line 18, can be used only if Z_{T_i} is at least some fixed distance from b, which will not always be the case. This was pointed out to us by K. Burdzy. (Incidentally, the proof in Durrett [1] also contains an error, although it is a much more subtle one.)

To repair the proof, delete the text from line 5 of page 321 to the end of page 322 and replace with the following.

Proof of Theorem 2.1. Suppose the range of f omits more than two points of $\mathbb{C}$, say a and b. By looking at $2[(f(z)-a)/(b-a)-\frac{1}{2}]$, we may assume that the two points omitted are -1 and 1.

Since f is entire, $|f'|>0$ except for a countable set, and by the recurrence of Brownian motion, $\int_0^t |f'(Z_s)|^2 ds \to \infty$ as $t\to\infty$ (cf. Exercise 1). Hence $f(Z_t)$ is the time change of a Brownian motion (not killed or stopped). In particular, the range of f must be dense in $\mathbb{C}$.

Let ε be small and z_0 chosen so that if $|z-z_0|<\varepsilon$, $f(z)$ can be connected to 0 by a curve lying in $f(\mathbb{C})\cap B(0,1/8)$. Without loss of generality, we may assume $z_0=0$. Infinitely often $Z_t\in B(0,\varepsilon)$. Let L_t be the straight line segment connecting Z_t to 0. The curve consisting of adding the line segment L_t to the end of Z_t is a closed curve homotopic to the single point 0. So the curve consisting of adding $f(L_t)$ to the end of $f(Z_t)$ must also be homotopic to a single point. By Proposition 2.2, for t sufficiently large, this curve is not homotopic to a single point, a contradiction. □

(2.2) Proposition. *Let L_t be as in the proof above. With probability one, there exists t_0 (depending on ω) such that if $t>t_0$, the curve formed by adding $f(L_t)$ to the end of $f(Z_t)$ is not homotopic to a single point.*

Proof. **Step 1.** First we define the Brownian *word*. Let

$$\begin{aligned}
H_0 &= [-1/2,1/2),\\
H_1 &= (-\infty,-1/2),\\
H_2 &= \{-\tfrac{1}{2}+y\mathrm{i} : y>0\},\\
H_3 &= \{\tfrac{1}{2}+y\mathrm{i} : y>0\},\\
H_4 &= (\tfrac{1}{2},\infty),\\
H_5 &= \{\tfrac{1}{2}+y\mathrm{i} : y<0\},\\
H_6 &= \{-\tfrac{1}{2}+y\mathrm{i} : y<0\}.
\end{aligned}$$

Let $T_1=0$. Let $b_i\in\{0,1,2,3,4,5,6\}$ be such that $f(Z_{T_i})\in H_{b_i}$, and

$$T_{i+1}=\inf\{t>T_i : f(Z_t)\in \cup_{j=0}^{6} H_j - H_{b_i}\}.$$

Thus the T_is are the times to hit one of the sets H_j different from the last one hit.

We form the sequence $0b_1b_2\dots b_n$, and then form the reduced sequence as follows:

(∗) Starting from the beginning of the sequence, the first time the pattern $b_{i_1}b_{i_2}b_{i_1}$ appears, delete the $b_{i_2}b_{i_1}$. Thus $\dots3454\dots$ becomes $\dots34\dots$ and $\dots2165654\dots$ becomes $\dots21654\dots$.

We apply the rule (∗) repeatedly until the sequence cannot be reduced any further, and that is our reduced sequence.

Next, whenever in the reduced sequence we have the pair $23, 32, 56$, or 65, replace it by $203, 302, 506,$, or 605, respectively. Then reduce this new sequence to get our final sequence.

Between successive 0s in the final sequence, we can only have $216, 612$, 345, or 543. Replace these by a, a^{-1}, b, b^{-1}, respectively, remove the 0s, and this will be our Brownian word. Let $w(s,t)$ be the Brownian word derived from the path of Z between times s and t.

The *content* of a Brownian word will be the number of times the patterns $ab^{-1}, b^{-1}a, ba^{-1}$, or $a^{-1}b$ appear. For example, the word $ab^{-1}aaab^{-1}$ will have a content of 3.

Step 2. We show that given $\eta > 0$, there exists $\delta < 1/2$ such that if $z \in B(0,\delta)$ and D is $\mathcal{F}_\infty$-measurable, then

$$\begin{aligned}(1-\eta)\mathbb{P}^0(D \circ \theta_{\tau(B(0,1/2))}) &\le \mathbb{P}^z(D \circ \theta_{\tau(B(0,1/2))}) \\ &\le (1+\eta)\mathbb{P}^0(D \circ \theta_{\tau(B(0,1/2))}).\end{aligned} \tag{2.2.1}$$

To see this, by the strong Markov property

$$\mathbb{P}^0(D \circ \theta_{\tau(B(0,1/2))}) = \mathbb{E}^{\,z}\varphi(Z_{\tau(B(0,1/2))}),$$

where $\varphi(w) = \mathbb{P}^w(D)$. We obtain (2.2.1) from Theorem I.1.19, the Harnack inequality, by taking $R = \frac{1}{2}$ and δ small enough.

Step 3. Let $U_0 = 0$, $T_{i+1} = \inf\{t > U_i : |Z_t| > 1/2\}$, $U_i = \inf\{t > T_i : Z_t \in (-\delta,\delta)\}$. We show there exists c_1 independent of δ such that the content of the Brownian word $w(0,U_1)$ will be at least one with probability at least c_1. To show this, by the support theorem, there is probability at least $c_2 > 0$ that started at 0, the Brownian motion goes once clockwise around -1, once counterclockwise around 1, and then hits $(-\frac{1}{4}, \frac{1}{4})$, say at time S, without any further windings about -1 or 1. So the Brownian word $w(0,S)$ is ab^{-1}. There is some chance that the first letters of the Brownian word $w(S,U_1)$ will be b. By symmetry there is equal probability that the first letter will be b^{-1}. Thus there is probability at most $\frac{1}{2}$ that the first letter will be b, and hence by the strong Markov property, the probability that $w(0,U_1)$ will have content at least one is $c_1 = c_2/2$.

Step 4. Fix $\eta < c_1/8$ and choose δ so that (2.2.1) holds. Let C_n be the content of $w(0,U_n)$. Let $X_n = C_{n+1} - C_n$. We will find a lower bound for $\mathbb{E}\,[X_n \mid \mathcal{F}_{U_n}]$.

Let D_n be the content of $w(U_n, U_{n+1})$. Let A_n^{-1} be the indicator of the event that the first letter of $w(U_n,U_{n+1})$ is a^{-1} and similarly B_n^{-1}, B_n, A_n. On the event that the last letter of $w(0,U_n)$ is a, we have

$$\begin{aligned}\mathbb{E}^{\,0}[X_n \mid \mathcal{F}_{U_n}] &\ge \mathbb{E}\,[D_n(A_n+B_n) + (D_n+1)B_n^{-1} - (D_n+1)A_n^{-1} \mid \mathcal{F}_{U_n}] \\ &= \mathbb{E}^{\,Z(U_n)}[[D_0(A_0+B_0) + (D_0+1)B_0^{-1} - (D_0+1)A_0^{-1}] \\ &\ge (1-\eta)\mathbb{E}^{\,0}[D_0(A_0+B_0+B_0^{-1}) + B_0^{-1}] \\ &\qquad - (1+\eta)\mathbb{E}^{\,0}[D_0A_0^{-1} + A_0^{-1}] \\ &= (1-\eta)3\mathbb{E}^{\,0}D_0/4 + (1-\eta)\mathbb{E}^{\,0}B_0^{-1} \\ &\qquad - (1+\eta)\mathbb{E}^{\,0}D_0/4 - (1+\eta)\mathbb{E}^{\,0}A_{-1}^{0} \\ &= (\tfrac{1}{2}-\eta)\mathbb{E}^{\,0}D_0 - 2\eta\mathbb{E}^{\,0}B_0^{-1},\end{aligned}$$

using the fact that we have symmetry about both the real and imaginary axes when we start from 0. Since $\mathbb{E}^0 D_0 \geq c_1$ by Step 3, we have that the conditional expectation is greater than $\frac{1}{4}c_1 - 2\eta = c_3$, which is positive by our choice of η.

We have the same estimate when the last letter of $w(0, U_n)$ is a^{-1}, b, or b^{-1}. If $w(0, U_n)$ has zero length, then we have

$$\begin{aligned}\mathbb{E}^0[X_n \mid \mathcal{F}_{U_n}] &= \mathbb{E}^0[D_n \mid \mathcal{F}_{U_n}] = \mathbb{E}^{Z(U_n)} D_0 \\ &\geq (1-\eta)\mathbb{E}^0 D_0 \geq c_1/2.\end{aligned}$$

So in any case $\mathbb{E}[X_n \mid \mathcal{F}_{U_n}] \geq c_3 > 0$.

Step 5. We want to show that $\mathbb{E} X_n^2 \leq c_4 < \infty$. Whenever in the Brownian word there appears the pair ab^{-1}, the Brownian motion must have intersected H_0. Similarly this happen if the pairs $b^{-1}a$, ba^{-1}, or $a^{-1}b$ appear. By the support theorem, Brownian motion started in H_0 will hit $(-\delta, \delta)$ with probability $c_5 > 0$ before hitting both the lines $\{x = -1\}$ and $\{x = 1\}$. So the probability that D_n will be larger than $2m$ is less than the probability that for m times the Brownian motion will hit H_0 and then hit both $\{x = -1\}$ and $\{x = 1\}$ before hitting $(-\delta, \delta)$. By the strong Markov property the probability of this is less than $(1 - c_5)^m$. Hence X_n has moments of all orders, and in particular, has a finite second moment c_4 that does not depend on n.

We can make a slightly stronger statement. Let $\widetilde{D}_n$ be the largest the content of $w(U_n, t)$ ever gets for $U_n \leq t \leq U_{n+1}$. Then $\widetilde{D}_n$ also has a second moment bounded by c_4 be the same argument.

Step 6. We prove $\liminf(C_n/n) > 0$ a.s. Let $Y_n = X_n - \mathbb{E}[X_n \mid \mathcal{F}_{U_n}]$. Then $\mathbb{E}[Y_n \mid \mathcal{F}_{U_n}] = 0$ and so $\sum_{i=1}^n (Y_i/i)$ is a martingale with second moment bounded by $c_4 \sum_{i=1}^\infty i^{-2} < \infty$. By the martingale convergence theorem, this martingale converges a.s. By Kronecker's lemma (see Chung [1], p. 123), $n^{-1}(\sum_{i=1}^n Y_i) \to 0$ a.s., or $\liminf C_n/n > 0$ a.s.

Step 7. We fill in the gaps between the times U_n. The above shows that for n large, the content of C_n will be larger than $c_6 n$. Let $M > 0$. If the content of $w(0, t)$ is to be less than M infinitely often for t arbitrarily large, then infinitely often $\widetilde{D}_n$ must be larger than $c_6 n/2$. But for $z \in (-\delta, \delta)$

$$\mathbb{P}^z(\widetilde{D}_n > c_6 n/2) \leq (1 + \eta)\mathbb{P}^0(\widetilde{D}_n > c_6 n/2) \leq c_7/n^2,$$

which is summable. By the Borel-Cantelli lemma, this only happens finitely often. Therefore for sufficiently large t, the content of $w(0, t)$ is greater than M; since M is arbitrary, then the content of $w(0, t)$ tends to infinity almost surely. The conclusion of the proposition follows. □

Page 345, line 17. $\widetilde{A}$ is a

Page 345, line -5. $\sum_{j=k_m}^{\infty}[\text{dist}\,(z_{\theta_j}, \partial D]^{1+\varepsilon}$

Page 346, line 6. continuous on $\overline{\mathbb{D}}$

Page 335. The following changes are needed to Proposition 3.11.

Line 9. Should read $\frac{1}{2\pi}\int_0^{2\pi} e^{\lambda|f(e^{i\theta})-f(0)|}\,d\theta \le 2e^{2\lambda^2\|g^*(f)\|_\infty^2}$.

Line 10. ...= 1. Write $f = u + iv$. Because $|\nabla u| = |\nabla v| = |f'|$, $g^*(u) = g^*(v) = g^*(f)$, where $g^*(u)$ and $g^*(v)$ are defined analogously to $g^*(f)$ with f' replaced by ∇u or ∇v. Since ...

From line 12 to line -3, replace every appearance of f by u.

Line -3. Replace $\square$ by the following:

Replacing u by $-u$ and adding,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\lambda|u(e^{i\theta})|}\,d\theta \le 2e^{\lambda^2/2}.$$

Repeating the argument with u replaced by v and using $|f| \le |u| + |v|$ together with the Cauchy-Schwarz inequality,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} e^{\lambda|f(e^{i\theta})|}\,d\theta &\le \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda|u(e^{i\theta})|}e^{\lambda|v(e^{i\theta})|}\,d\theta \\ &\le \Big(\frac{1}{2\pi}\int_0^{2\pi} e^{2\lambda|u(e^{i\theta})|}\,d\theta\Big)^{1/2}\Big(\frac{1}{2\pi}\int_0^{2\pi} e^{2\lambda|v(e^{i\theta})|}\,d\theta\Big)^{1/2} \\ &\le 2e^{(2\lambda)^2/2}.\end{aligned}$$

$\square$

Page 336, line 1. $\sup_{0\le s\le r}|f_s(e^{i\theta})|$

Page 337, line 5. $2\exp(2\lambda^2c^2\|g^*(f_r)\|_\infty^2) = 2\exp(2\lambda^2c^2\log(1/(1-r)))$

Page 337, line 7. $\le 2e^{-\lambda\alpha}$ ...

Page 338, line -14. $(E^- - \rho(E^+))$

Page 361, line -12. sun

Page 371, line 7. Show $\int_0^t |f'(Z_s)|^2\,ds \to \infty$ a.s.

Page 376, line -8. *Canad. Bull. Math.* **39** (1996) 138–150.

Page 377, line 6. **101** (1995) 479–493.

Page 381, line -4. *Stoch. Proc. and Applic.* **57** (1995) 319–337.

Page 385, line 13. **123** (1995) 1075–1082.

Page 386, line 5. In: *Essays on Fourier Analysis in Honor of Elias M. Stein*, 321–384. Princeton Univ. Press, Princeton, 1995.