Errata for Real Analysis for Graduate Students, Second Edition

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The errata pages for the second edition will no longer be updated.

Page 4, lines 3,4: \( B(x, r/2) \) and \( B(y, r/2) \)

Page 4, line -1: Add:

(3) if \( x \leq y \) and \( y \leq z \), then \( x \leq z \).

Page 11, Exercise 2.5: \( X \) into \( Y \)

Page 11, Exercise 2.6: whenever \( A \in \mathcal{A} \) is non-empty,

Page 15, line -7: such that \( (X, \overline{\mathcal{A}}, \overline{\mu}) \) is complete, where \( \overline{\mu} \) is a measure on \( \overline{\mathcal{A}} \) that is an extension of \( \mu \), that is, \( \overline{\mu}(B) = \mu(B) \) if \( B \in \mathcal{A} \).

Page 16, line -1: Replace by the following:

\( B \in \mathcal{B} \) if and only if there exists \( A \in \mathcal{A} \) and \( N \in \mathcal{N} \) such that \( B = A \cup N \). Define \( \overline{\mu}(B) = \mu(A) \) if \( B = A \cup N \) with \( A \in \mathcal{A} \) and \( N \in \mathcal{N} \). Prove that \( \overline{\mu}(B) \) is uniquely defined for each \( B \in \mathcal{B} \), that \( \overline{\mu} \) is a measure on \( \mathcal{B} \), that \( (X, \mathcal{B}, \overline{\mu}) \) is complete, and that \( (X, \mathcal{B}, \overline{\mu}) \) is the completion of \( (X, \mathcal{A}, \mu) \).

Page 20, line -7: that \( \emptyset \in \mathcal{C} \) and there exist \( D_1, D_2, \ldots \) in \( \mathcal{C} \) such that \( X = \bigcup_{i=1}^{\infty} D_i \). Suppose \( \ell : \mathcal{C} \to [0, \infty] \) with

Page 21, line -4: \( x \geq 0 \)

Page 25, lines 15–19: Replace by the following:

Let \( \delta > 0 \) and let \( A_j, j = 1, 2, \ldots \), be elements of \( \mathcal{C} \) such that \( I_i \subset \bigcup_{j=1}^{\infty} A_j \) and

\[
\sum_{j=1}^{\infty} \ell(A_j) \leq m^*(I_i) + \delta.
\]
Let $C_{ij} = I_i \cap A_j$, which will again be an interval (possibly empty) that is open on the left and closed on the right, and hence in $\mathcal{C}$. Write $J^c = K_1 \cup K_2$, where $K_1 = (-\infty, c]$ and $K_2 = (d, \infty)$.

Note $C_{ij} \cap J$ will be an interval that is open on the left and closed on the right, and the same is true of $C_{ij} \cap K_1$ and $C_{ij} \cap K_2$ (any of these could be empty). Using (4.4) twice,

$$\ell(C_{ij}) = \ell(C_{ij} \cap K_1) + \ell(C_{ij} \cap J) + \ell(C_{ij} \cap K_2).$$

We have that the set $I_i \cap J$ is contained in the union of the countable sub-collection $\{C_{ij} \cap J\}_{j=1}^{\infty}$ of $\mathcal{C}$ and that the set $I_i \cap J^c$ is contained in the union of the countable subcollection $\{C_{ij} \cap K_1, C_{ij} \cap K_2\}_{j=1}^{\infty}$ of $\mathcal{C}$. Therefore

$$m^*(I_i \cap J) + m^*(I_i \cap J^c)$$

$$\leq \sum_{j=1}^{\infty} \ell(C_{ij} \cap J) + \left( \sum_{j=1}^{\infty} \ell(C_{ij} \cap K_1) + \sum_{j=1}^{\infty} \ell(C_{ij} \cap K_2) \right)$$

$$= \sum_{j=1}^{\infty} [\ell(C_{ij} \cap J) + \ell(C_{ij} \cap K_1) + \ell(C_{ij} \cap K_2)]$$

$$= \sum_{j=1}^{\infty} \ell(C_{ij}) \leq \sum_{j=1}^{\infty} \ell(A_j)$$

$$\leq m^*(I_i) + \delta.$$

Since $\delta$ is arbitrary,

$$m^*(I_i \cap J) + m^*(I_i \cap J^c) \leq m^*(I_i).$$

Page 28, line -12: $\inf\{f_0(y) : y \geq x, y \notin C\}$

Page 29, line 22: finite measure

Page 32, line 11: every set in $\mathcal{A}_0$ and every $\mu^*$-null set is $\mu^*$-measurable;

Page 32, line -3: $\mu^*$-measurable. That $\mu^*$-null sets are $\mu^*$-measurable follow by the definition of $\mu^*$-measurable and the fact that $\mu^*$ satisfies Definition 4.1(2).

Page 33, line 4: $E \in \sigma(\mathcal{A}_0)$

Page 35, line 17: Add:

This is known as the Steinhaus theorem.
Exercise 4.16 (1) Give an example of a set $X$ and a finite outer measure $\mu^*$ on $X$, subsets $A_n \uparrow A$ of $X$, and subsets $B_n \downarrow B$ of $X$ such that $\mu^*(A_n)$ does not converge to $\mu^*(A)$ and $\mu^*(B_n)$ does not converge to $\mu^*(B)$.

(2) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and define $\mu^*$ as in Exercise 4.3. Show that if $A_n \uparrow A$ for subsets $A_n, A$ of $X$, then $\mu^*(A_n) \uparrow \mu^*(A)$.

Page 39, line 13: provided they are finite.

Page 44, Exercise 5.6: $f : \mathbb{R} \to \mathbb{R}$

Page 49, line -1: = instead of $\to$

Page 50, Exercise 6.7: Let $(X, \mathcal{A}, \mu)$ be a finite measure space

Page 58, line 12: Add:

For this problem you may use the fact that if $f$ is continuous on $[a,b]$ and $F$ is differentiable on $[a,b]$ with derivative $f$, then $\int_a^b f(x) \, dx = F(b) - F(a)$. This follows by the results of the next chapter and the fundamental theorem of calculus.

Page 76, between lines 15 and 16: Add:

If $A = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} A_j$, then $x \in A$ if and only if $x$ is in infinitely many of the $A_j$. Sometimes one writes $A = \{A_j \text{ i.o.}\}$.

Page 83, line -1: $\mu(t_y(E_n))$

Page 85, line 10: Insert before “If either”:

Suppose $\mu$ and $\nu$ are $\sigma$-finite measures on $X$ and $Y$, resp.

Page 88, lines -1,-2: Change exponent from $3/2$ to $3/4$ in both lines

Page 93, line -10: Insert before “whenever”:

with absolute convergence of the series when $\mu(\cup_{i=1}^{\infty} A_i)$ is finite

Page 96, lines 23, 24: $E = (\frac{1}{2}, 1]$ and $F = [0, \frac{1}{2}]$

Page 98, Exercise 12.4: Move to just before Exercise 13.9 in the next chapter.

Page 98, Exercise 12.6: $|\mu + \nu|(A) \leq |\mu|(A) + |\nu|(A)$
Page 105, Exercise 13.8: Suppose \( \mu, \nu, \) and \( \rho \) are finite measures, \( \nu \ll \mu, \) and \( \rho \ll \nu. \)

Page 114, line 6: for almost every \( x \) (with respect to the measure \( m \))

Page 114, line -6: Change \( \lambda((x, r)) \) to \( \lambda(B(x, r)) \)

Page 136, line 10: dense in \( L^p(\mathbb{R}) \) for \( 1 \leq p < \infty. \)

Page 143, Exercise 15.15: Let \( p \in [1, \infty) \) and suppose \( \mu \) is a finite measure.

Page 156, Exercise 16.6, line -2: completion of \( L \times \cdots \times L \)

Page 174, lines -3,-2: \( x_0 \) not in \( M \) such that \( \inf_{x \in M} |x - x_0| > 0, \) we can define \( f(x + \lambda x_0) = \lambda \) for \( x \in M, \)

Page 175, line -3: and \( x_n \) both

Page 179, Exercise 18.3, line 5: times continuously differentiable

Page 180, Exercise 18.13: Let \( X \) be the space of continuously differentiable functions with the supremum norm and

Page 186, line -8: \( +2\|y\| \)

Page 187, line -11: Insert the following after line -11 and before line -10:

**Lemma 19.8.1** Let \( M \) be a closed subspace of \( H \) with \( M \neq H. \) Then \( M^\perp \) contains a non-zero element.

**Proof.** Choose \( x \in H \) with \( x \notin M. \) Let \( E = \{w - x : w \in M\}. \) It is routine to check that \( E \) is a closed and convex subset of \( H. \) By Lemma 19.8, there exists an element \( y \in E \) of smallest norm.

Note \( y + x \in M \) and we conclude \( y \neq 0 \) because \( x \notin M. \)

We show \( y \in M^\perp \) by showing that if \( w \in M, \) then \( \langle w, y \rangle = 0. \) This is obvious if \( w = 0, \) so assume \( w \neq 0. \) We know \( y + x \in M, \) so for any real number \( t \) we have \( tw + (y + x) \in M, \) and therefore \( tw + y \in E. \) Since \( y \) is the element of \( E \) of smallest norm,

\[
\langle y, y \rangle = \|y\|^2 \leq \|tw + y\|^2 = \langle tw + y, tw + y \rangle = t^2 \langle w, w \rangle + 2t \text{Re} \langle w, y \rangle + \langle y, y \rangle,
\]

which implies

\[t^2 \langle w, w \rangle + 2t \text{Re} \langle w, y \rangle \geq 0\]
for each real number $t$. Choosing $t = -\text{Re} \langle w, y \rangle / \langle w, w \rangle$, we obtain

\[-\frac{\lvert \text{Re} \langle w, y \rangle \rvert^2}{\langle w, w \rangle} \geq 0,
\]

from which we conclude $\text{Re} \langle w, y \rangle = 0$.

Since $w \in M$, then $iw \in M$, and if we repeat the argument with $w$ replaced by $iw$, then we get $\text{Re} \langle iw, y \rangle = 0$, and so

\[\text{Im} \langle w, y \rangle = -\text{Re} (i \langle w, y \rangle) = -\text{Re} \langle iw, y \rangle = 0.\]

Therefore $\langle w, y \rangle = 0$ as desired. \qed

If in the proof above we set $Px = y + x$ and $Qx = -y$, then $Px \in M$ and $Qx \in M^\perp$, and we can write $x = Px + Qx$. We call $Px$ and $Qx$ the orthogonal projections of $x$ onto $M$ and $M^\perp$, resp. It is an exercise to show that each element of $H$ can be written as the sum of an element of $M$ and an element of $M^\perp$ in exactly one way.

Page 191, lines 4,5: basis, then, is a subset of $H$

Page 194, Exercise 19.3: $M$ is a closed subspace of

Page 194, Exercise 19.5: Remove exercise. (This is now Lemma 19.8.1.)

Page 201, line -3: that is dense in $X$.

Page 207, line 11: and by the

Page 216, line 6: $\leq 4\varepsilon$.

Page 224, line 3: $\{G_1, G_2, \ldots\}$

Page 231, line 6: $2\|g\|_{\infty} \int_{|g| > \delta} \varphi_\delta(y) \, dy$.

Page 232, line -2 through Page 233, line 16: Replace with the following:

**Lemma 20.42.** Suppose $\mathcal{A}$ is an algebra of functions in $C(X)$ such that $\mathcal{A}$ separates points and vanishes at no point. If $x$ and $y$ are two distinct points in $X$ and $a, b$ are two real numbers, there exists a function $f \in \mathcal{A}$ (depending on $x, y, a, b$) such that $f(x) = a$ and $f(y) = b$.

**Proof.** Let $g$ be a function in $\mathcal{A}$ such that $g(x) \neq g(y)$. Let $h_x$ and $h_y$ be functions in $\mathcal{A}$ such that $h_x(x) \neq 0$ and $h_y(y) \neq 0$. Define $u$ and $v \in \mathcal{A}$ by

\[u(z) = g(z)h_x(z) - g(y)h_x(z)\]

and

\[v(z) = g(z)h_y(z) - g(x)h_y(z)\]
and
\[ v(z) = g(z)h_y(z) - g(x)h_y(z). \]
Note that \( u(x) \neq 0, u(y) = 0, v(x) = 0, \) and \( v(y) \neq 0. \) Now set
\[ f(z) = \frac{a}{u(x)}u(z) + \frac{b}{v(y)}v(z). \]
This \( f \) is the desired function.

**Theorem 20.43.** Let \( X \) be a compact Hausdorff space and let \( A \) be a lattice of real-valued continuous functions with the property that whenever \( x \neq y \) and \( a, b \in \mathbb{R}, \) then there exists \( f \in A \) (depending on \( x, y, a, \) and \( b \)) such that \( f(x) = a \) and \( f(y) = b. \) Then \( A \) is dense in \( C(X). \)
Since \( r > 1 \), then \( x \to |x|^r \) is continuously differentiable, and so \( w \in C^1_k \). We observe

\[
\lim_{x \to 0, N \to \infty} \int_{x < |x| < N} \frac{y_j}{|y|^{n+1}} f(x - y) \, dy
\]

We observe to those of

\[
\leq \frac{c_1}{\lambda_2^2} \|g\|^2_2
\]

Suppose we have found eigenvectors \( z_1, \ldots, z_n \) with corresponding eigenvectors \( \lambda_1, \ldots, \lambda_n \).

\[
Y = X_n^\perp
\]

If \( x \in Y \) and \( k \leq n \), then

\[
\leq 2|a| \|x\| \|Ax\| 
\]

\[
\| (\lambda - A)(x_1 - x_2) \|^2
\]

Add at end:

Now use the right hand equality to define \( L_{x,y}f \) for all \( f \) that are bounded and Borel measurable.

\[
= \langle E(\sigma(A))x, y \rangle.
\]

\( n^{th} \) largest non-negative eigenvalue

\( n^{th} \) largest non-negative eigenvalue

\[
\text{supp} (f)
\]

finitely many sets

\[
\|f\|_{C^m(K)} \leq 1/m
\]

there exist a non-negative integer \( L \) and

\[
|D^j f(x) - D^j f(-x_0)|
\]

only possible limit is equal to \( g \) a.e. Therefore we may assume that
\[ = (-1)^k D^{2k}G_g(f) \]

Page 391, line 14: \( D^k \mathcal{F}f \) is a continuous function, and hence \( \mathcal{F}f \in \)

Page 395, item [2]: Bourdon, and W. Ramey.

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