Two results from Mandelbaum's paper: "The dynamic complementarity problem"

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July 2, 2024

Abstract

A draft of a paper by Mandelbaum, "The dynamic complementarity problem," was circulated in 1987, but has never been published. We give an exposition of two important results from that paper which are not readily accessible in the literature.

The first is an example of a Skorokhod problem in two dimensions in the quadrant for which there is not uniqueness. The second is a proof of uniqueness for the Skorokhod problem in two dimensions in the quadrant in a critical case.

1 Introduction

In 1987 A. Mandelbaum circulated a draft of a paper titled "The dynamic complementarity problem." Although this paper has been cited many times in the ensuing decades, it has never been published. Of particular interest to many is his example of a Skorokhod problem for which a solution exists but for which uniqueness does not hold.

Although this example is of great importance, to the best of our knowledge the only publicly available exposition is in the Ph.D. dissertation of Whitley [Whi03], which is a bit difficult to find (and may require paying to get a copy). See also Stewart [S09], which uses similar methods to handle a related problem. We thought that it would be worthwhile to give an exposition of Mandelbaum's counterexample that is freely available on the Internet.

Less well known is another result in that paper. Consider the matrix R that appears in the Skorokhod problem (details in a moment). Uniqueness has been proved for the Skorokhod problem for a certain class of matrices R by Harrison and Reiman [HR81] and Williams [Wil95] pointed out that the proof works for a much larger class of matrices; see below. In his paper Mandelbaum proves uniqueness for a certain critical case. See Remark 3.3 for a summary of the known results in two dimensions and where things stand in higher dimensions.

We provide proofs for the counterexample and for the critical case. We emphasize that these notes are expository and all the ideas are due to Mandelbaum. We thank him for providing us with a copy of his draft paper. We also would like to thank K. Burdzy and R. Williams for many helpful conversations on the subject of the Skorokhod problem.

Let us turn to describing the Skorokhod problem and the corresponding Skorokhod equation. Except for Remark 3.4, for the remainder of these notes we consider the two-dimensional case only.

For a vector $b = (b_1, b_2) \in \mathbb{R}^2$, we say $b \geq 0$ if $b_1 \geq 0$ and $b_2 \geq 0$. Let $D = \{b \in \mathbb{R}^2 : b \geq 0\}$.

Definition 1.1. *A driving function f is a continuous function from $[0, \infty)$ to $\mathbb{R}^2$ with $f(0) \geq 0$. The Skorokhod problem is to find*
(1) g a continuous function from $[0, \infty) \to D$;
(2) m a continuous function on $[0, \infty)$ with $m(0) = 0$ and each $m_j(t)$ is non-decreasing, $j = 1, 2$;
such that
(3) $g(t) = f(t) + Rm(t)$ for all $t \geq 0$;
and
(4) m_j increases only when $g_j = 0$, $j = 1, 2$.

The equation (3) is known as the Skorokhod equation. It arises as a way to represent reflecting Brownian motion when f is a Brownian path.

Note that (4) is equivalent to

$$\int_0^\infty g_j(t)\, dm_j(t) = 0, \qquad j = 1, 2. \tag{1.1}$$

Finding g and m satisfying (1)–(3) and (1.1) is one type of dynamic complementarity problem.

Mandelbaum is concerned with the following.

Question 1.2. *For which matrices R does there exist a unique solution to the Skorokhod problem for **every** driving function $f(t)$ with $f(0) \geq 0$?*

One can also ask:

Question 1.3. *For which matrices R does there fail to be a unique solution to the Skorokhod problem when the driving function is a typical Brownian path?*

We will not address Question 1.3 in these notes except in Remark 3.5.

Existence of a solution to the Skorokhod problem is thoroughly understood. It is known (see, e.g., [Wil95] or [BeK91]) that there will exist at least one solution for every driving function if and only if R is a completely-$\mathcal{S}$ matrix. In two dimensions this means that the diagonal entries of R are positive and that there exists $x \geq 0$ such that $Rx > 0$.

Suppose $R_{11}, R_{22} > 0$. If we let $\widetilde{m}_i(t) = R_{ii} m_i(t)$ and $\widetilde{R}_{ij} = R_{ij}/R_{jj}$, $i, j = 1, 2$, then the equation $g = f + Rm$ can be rewritten as $g = f + \widetilde{R}\widetilde{m}$. Therefore there is no loss of generality in assuming that the diagonal elements of R are equal to 1.

Suppose from now on that

$$R = \begin{pmatrix} 1 & a_1 \\ a_2 & 1 \end{pmatrix}. \tag{1.2}$$

It is easy to see that R will be completely $\mathcal{S}$ when (1) at least one of a_1, a_2 is positive or when (2) $a_1 a_2 < 1$. If neither (1) nor (2) hold, then R will not be completely-$\mathcal{S}$.

As for uniqueness of the Skorokhod problem, let $Q = I - R$ and let $|Q|$ be the matrix with each coordinate of Q replaced by its absolute value. Thus $|Q| = \begin{pmatrix} 0 & |a_1| \\ |a_2| & 0 \end{pmatrix}$. A calculation shows that the spectral radius of $|Q|$ is $\pm\sqrt{|a_1 a_2|}$. When this spectral radius is strictly less than 1, there is uniqueness for the Skorokhod problem for every driving function by [HR81] as improved by [Wil95].

In Section 2 we take $a_2 = 1$, $a_1 = -2$, and show that there exist two distinct solutions to the Skorokhod problem. The proof is Mandelbaum's, although we are able to streamline it a bit since we are in a special case of his more general results.

In Section 3 we consider the critical case where $|a_1 a_2| = 1$, $a_2 > 0$, and $a_1 < 0$. Mandelbaum's proof of uniqueness is a quite brief sketch, so we flesh out the proof with additional details.

2 A counterexample

We present Mandelbaum's example of a deterministic version of the Skorokhod problem in two dimensions where uniqueness does not hold. We mention that Bernard and el Kharroubi [BeK91] gave an example in three dimensions where the driving function is linear.

Set

$$R = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}.$$

Theorem 2.1. *There exists a driving function $f(t)$ for which there exist two distinct solutions $g(t)$ to the Skorokhod equation*

$$g(t) = f(t) + Rm(t).$$

Proof. We define an auxiliary function $u = (u_1, u_2)$ mapping $[0, 1]$ to $\mathbb{R}^2$ as follows. Let $t_n = 2^{-n}$, $n \geq 0$. Set $u(0) = 0$. For k a non-negative integer, set

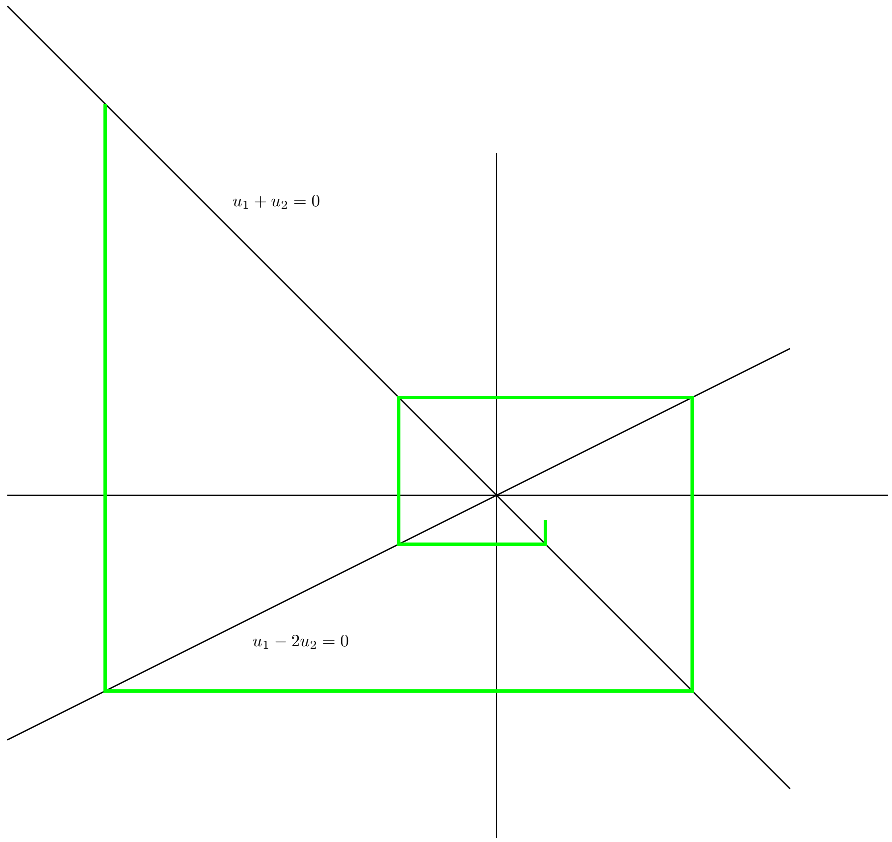
$$u(t_{4k}) = (-2^{-2k}, 2^{-2k}), \qquad u(t_{4k+1}) = (-2^{-2k}, -2^{-2k-1}),$$
$$u(t_{4k+2}) = (2^{-2k-1}, -2^{-2k-1}), \qquad u(t_{4k+3}) = (2^{-2k-1}, 2^{-2k-2}).$$

For t between t_{n+1} and t_n we linearly interpolate between $u(t_{n+1})$ and $u(t_n)$. Thus u is a continuous piecewise linear function of bounded variation that starts at 0 and spirals out away from the origin. Note that each t_n is either on the line $u_1 + u_2 = 0$ or the line $u_1 - 2u_2 = 0$. See Figure 1, which is a drawing of the (u_1, u_2) plane with the graph of $\{u(t) : t_7 \leq t \leq t_0\}$ shown in green.

For $j = 1, 2$, each u_j is of bounded variation, and we write

$$u_j(t) = m_j(t) - \overline{m}_j(t),$$

Figure 1: Graph of $u(t)$



where $m_j, \overline{m}_j$ both are non-decreasing and start at 0. Define

$$f(t) = -(Rm(t) \wedge R\overline{m}(t)), \tag{2.1}$$

where this equation means that $f_j(t) = -((Rm(t))_j \wedge (R\overline{m}(t))_j)$ for $j = 1, 2$. Set

$$\begin{aligned} g(t) &= (Rm(t) - R\overline{m}(t))^+, \\ \overline{g}(t) &= (Rm(t) - R\overline{m}(t))^-, \end{aligned} \tag{2.2}$$

where again these equations are interpreted component-wise.

We claim (g, f, m) and $(\overline{g}, f, \overline{m})$ are two distinct solutions to the Skorokhod equation. Clearly $g, \overline{g} \geq 0$, $g \neq \overline{g}$, $m, \overline{m}$ both start at 0 and are non-decreasing, and all the functions are continuous. That

$$g(t) = f(t) + Rm(t), \qquad \overline{g}(t) = f(t) + R\overline{m}(t)$$

follow from the identities $a - (a \wedge b) = (a - b)^+$ and $b - (a \wedge b) = (a - b)^-$.

It remains to show that m_j only increases when $g_j = 0$, $j = 1, 2$, and the same with $\overline{m}_j, \overline{g}_j$. We do the case where m_2 increases, the other three cases being exactly similar. m_2 increases only when t is in an interval $[t_{4k+1}, t_{4k}]$ for some non-negative integer k. In that interval u moves vertically from the line $(Ru)_1 = u_1 - 2u_2 = 0$ to the line $(Ru)_2 = u_1 + u_2 = 0$. Since $Rm(t) - R\overline{m}(t) = Ru(t)$, for such t

$$(Rm(t))_2 - (R\overline{m}(t))_2 = (Ru(t))_2 \leq 0.$$

Therefore for such t, using (2.2) we have $g_2(t) = 0$ as required. $\square$

Remark 2.2. The same argument works if in the matrix R we replace -2 by any real that is strictly less than -1.

Remark 2.3. In the above proof, for each interval $[t_{n+1}, t_n]$ only one of u_1, u_2 changes, and for whichever u_j that changes, either u_j increases or decreases over the entire time interval. It is thus straightforward to see how the decomposition of u into the difference of non-decreasing functions m and $\overline{m}$ occurs.

3 Uniqueness - the critical case

We examine the critical case when the spectral radius is exactly 1.

Lemma 3.1. *Suppose $C > 0$. There is a unique solution for every continuous driving function for the deterministic Skorokhod problem with matrix $R = \begin{pmatrix} 1 & a_1 \\ a_2 & 1 \end{pmatrix}$ if and only if there is a unique solution for every continuous driving function for the Skorokhod problem with matrix $S = \begin{pmatrix} 1 & Ca_1 \\ a_2/C & 1 \end{pmatrix}$.*

Proof. If we write out $g = f + Rm$ in coordinates and multiply the second equation by $1/C$, we get

$$\begin{aligned} g_1 &= f_1 + m_1 + a_1 m_2, \\ \frac{1}{C} g_2 &= \frac{1}{C} f_2 + \frac{1}{C} a_2 m_1 + \frac{1}{C} m_2. \end{aligned}$$

Let $\widetilde{m}_1 = m_1$, $\widetilde{g}_1 = g_1$, $\widetilde{f}_1 = f_1$, and

$$\widetilde{m}_2 = \frac{1}{C} m_2, \quad \widetilde{g}_2 = \frac{1}{C} g_2, \quad \widetilde{f}_2 = \frac{1}{C} f_2.$$

We then have

$$\widetilde{g} = \widetilde{f} + S\widetilde{m}.$$

It follows that there will be two distinct solutions to the Skorokhod problem for a driving function f with respect to the matrix R if and only if there are two distinct solutions to the Skorokod problem with driving function $\widetilde{f}$ with respect to the matrix S. □

Theorem 3.2. *If $|a_1 a_2| = 1$, $a_2 > 0$, and $a_1 < 0$, there is a unique solution for every driving function f to the Skorokhod problem when R is of the form* (1.2).

Proof. Letting $C = 1/a_2$ and applying Lemma 3.1 we see that it suffices to look at R given by

$$R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{3.1}$$

Suppose there are two solutions (g, m) and $(\overline{g}, \overline{m})$. Let $u = m - \overline{m}$. Then

$$g - \overline{g} = Ru.$$

Using (1.1), for $j = 1, 2$

$$\begin{aligned} (Ru)_j \, du_j &= (g_j - \overline{g}_j) \, d(m_j - \overline{m}_j) \\ &= g_j \, dm_j - g_j \, d\overline{m}_j - \overline{g}_j \, dm_j + \overline{g}_j \, d\overline{m}_j \\ &= -g_j \, d\overline{m}_j - \overline{g}_j \, dm_j \\ &\le 0, \end{aligned}$$

where we write $d\mu \le 0$ for a signed measure μ if $\mu(B) \le 0$ for all Borel sets B, or equivalently, if μ has no positive part.

With our choice of R, this equation is the same as

$$\begin{aligned}(u_1+u_2)\,du_2 &\le 0 \\ (u_1-u_2)\,du_1 &\le 0.\end{aligned} \tag{3.2}$$

Divide the (u_1, u_2) plane into 4 pieces by using the lines $u_2 = u_1$ and $u_2 = -u_1$. We label clockwise the pieces N, E, S, W (for north, east, south, and west). To assign the boundaries, we define

$$\begin{aligned} N &= \{(u_1,u_2) : u_2 > 0, -u_2 < u_1 \le u_2\}, \\ E &= \{(u_1,u_2) : u_1 > 0, -u_1 < u_2 \le u_1\}, \\ S &= \{(u_1,u_2) : u_2 < 0, u_2 < u_1 \le -u_2\}, \\ W &= \{(u_1,u_2) : u_1 < 0, u_1 < u_2 \le -u_1\}. \end{aligned}$$

Note that we include one side of the boundary of N in N but not the other. This is true for each of the four sectors.

Let $v = \max(|u_1|, |u_2|)$. For $(u_1, u_2) \in N$ we see that $v = u_2$ and $u_1 + u_2 > 0$. Using the first line of (3.2) this shows $dv = du_2 \le 0$.

For $(u_1, u_2) \in W$ we have $-(u_1 - u_2) = -u_1 + u_2 > 0$, and using the second line of (3.2) we conclude $du_1 \ge 0$, so $dv = -du_1 \le 0$. Note that the ray $u_2 = -u_1, u_2 > 0$ is included in W but not N.

We argue similarly for E and S. Hence $v(0) = \max(|u_1(0)|, |u_2(0)|) = 0$, $v(t) \ge 0$ for all t, and $dv \le 0$ on the set $\{t : v(t) \ne 0\}$. This means that v is non-increasing on $\{t : v(t) \ne 0\}$, so v must be identically 0. Therefore u is identically 0, so $m = \overline{m}$, and then $g = \overline{g}$. $\square$

Remark 3.3. The results concerning Question 1.2 in two dimensions are the following. Suppose R is given by (1.2). There are five cases to consider:
(1) $|a_1a_2| < 1$;
(2) $|a_1a_2| = 1$, a_1, a_2 are of opposite signs;
(3) $|a_1a_2| = 1$, a_1, a_2 are both positive;
(4) $|a_1a_2| > 1$, a_1, a_2 are of opposite signs;
(5) $|a_1a_2| > 1$, a_1, a_2 are both positive.
(R will not be completely-$\mathcal{S}$ if $|a_1a_2| \ge 1$ and a_1, a_2 are both negative.)

Uniqueness holds in Case (1) by [HR81, Wil95]. The results of Mandelbaum discussed above together with Lemma 3.1 take care of Cases (2) and (4). K. Burdzy and I [BB24b] have recently resolved Cases (3) and (5) (uniqueness for g but not m in Case (3); non-uniqueness for Case (5)).

Remark 3.4. For dimensions larger than 2, the result of [HR81] and [Wil95] still holds: if the spectral radius of $|Q|$ is strictly less than 1, uniqueness holds. As far as we know, the cases where the spectral radius is greater than or equal to one are largely open.

It is not known if the proof given in this section can be extended to higher dimensions for the case where the spectral dimension is exactly one.

Remark 3.5. When there is uniqueness for every driving function, there will be uniqueness when f is replaced by the path of a Brownian motion. It is conceivable, however, that a matrix R could be such that there is not uniqueness for every driving function, but that there is uniqueness almost surely when f is a Brownian path. There is not much known here. See [BB24a], where it is shown that for a large class of matrices R pathwise uniqueness does not hold for almost every Brownian path.

There is a notion of weak uniqueness in probability theory which, not surprisingly, is weaker than the notion of pathwise uniqueness. Weak uniqueness holds for every matrix R which is completely–$\mathcal{S}$; see Taylor and Williams [TW93] for definitions and proofs.

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