

**SELF-INTERSECTION LOCAL TIME:
CRITICAL EXPONENT, LARGE DEVIATIONS,
AND LAWS OF THE ITERATED LOGARITHM**

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Abstract. If β_t is renormalized self-intersection local time for planar Brownian motion, we characterize when $\mathbb{E} e^{\gamma\beta_1}$ is finite or infinite in terms of the best constant of a Gagliardo-Nirenberg inequality. We prove large deviation estimates for β_1 and $-\beta_1$. We establish lim sup and lim inf laws of the iterated logarithm for β_t as $t \rightarrow \infty$.

Subject classifications: Primary 60J55; Secondary 60J55, 60F10

Key words: intersection local time, Gagliardo-Nirenberg inequality, law of the iterated logarithm, critical exponent, self-intersection local time, large deviations

Short title: Self-intersection local time

Research partially supported by NSF grants DMS-9988496 and DMS-0102238.

1. Introduction.

Let β_t be the renormalized self-intersection local time of a planar Brownian motion X_t . Formally,

$$\beta_t = \int_0^t \int_0^s \delta_0(X_s - X_u) du ds - \mathbb{E} \int_0^t \int_0^s \delta_0(X_s - X_u) du ds,$$

where δ_0 is the delta function, and more precisely

$$\beta_t = \lim_{\varepsilon \rightarrow 0} \left[\int_0^t \int_0^s \varphi_\varepsilon(X_s - X_u) du ds - \mathbb{E} \int_0^t \int_0^s \varphi_\varepsilon(X_s - X_u) du ds \right], \quad (1.1)$$

where φ_ε is a suitable approximation to the identity. We have three main results in this paper:

- (1) Le Gall in [L2] showed that there is a critical exponent γ_β such that

$$\mathbb{E} e^{\gamma \beta_1} \quad \begin{cases} < \infty & \text{if } \gamma < \gamma_\beta \\ = \infty & \text{if } \gamma > \gamma_\beta. \end{cases} \quad (1.2)$$

We characterize γ_β in terms of the best constant of one of the Gagliardo-Nirenberg inequalities.

- (2) We prove large deviation estimates for β_1 and $-\beta_1$.
(3) We prove laws of the iterated logarithm for the lim sup and lim inf behavior of β_t .

Self-intersection local time has been an object of much study in recent years. We cite [BK_h2], [Bo], [Dy], [IPV], [L1], [L2], [Ro1], [Ro2], [V], [Y1], [Y2] as an incomplete list of publications in this subject. In addition to probability theory, self-intersection local time has applications to some branches of mathematical physics, e.g., constructive quantum field theories and polymer measures.

The quantity $\int_0^t \int_0^s \varphi_\varepsilon(X_s - X_u) du ds$ will converge almost surely to infinity as $\varepsilon \rightarrow 0$, and in order to get convergence, one must subtract the expectation of this integral. Therefore exponential integrability of β_1 is a subtle issue. In 1994 Le Gall [L2] proved there is a critical value γ_β such that (1.2) holds. This has proved to be of considerable interest to the study of constructive quantum field theories. See also Theorem 2.23 of [Bo] for the discussion in the context of random walks with continuous time but discrete values. Our first main result characterizes γ_β .

Theorem 1.1. $\gamma_\beta = A^{-4}$, where $A > 0$ is the best constant in the inequality

$$\|f\|_4 \leq C \sqrt{\|\nabla f\|_2} \sqrt{\|f\|_2}, \quad f : \mathbb{R}^2 \rightarrow \mathbb{R}. \quad (1.3)$$

Inequality (1.3) is one of a class of inequalities known as Gagliardo-Nirenberg inequalities. The proof of (1.3) is quite simple: begin with the well known Sobolev inequality in \mathbb{R}^2 :

$$\|g\|_2 \leq c_1 \|\nabla g\|_1.$$

Replace g by f^2 , write ∇f^2 as $2f\nabla f$ and apply Cauchy-Schwarz to the right hand side. The best constant in (1.3) appears to be a difficult problem, however, and is currently open. The best constant for Nash's inequality, which is another special case of the Gagliardo-Nirenberg inequalities, was found by [Ca]. Two recent preprints finding the best constants for a class of Gagliardo-Nirenberg inequalities have appeared recently; see [DD] and [CNV]. Numerical values for the best constant in (1.3) were investigated as long ago as 1983 by Weinstein [W] who solved an eigenvalue problem by numerical methods and found that A is approximately $(\pi \cdot 1.86255 \dots)^{-1/4}$. By Theorem 1.1,

$$\gamma_\beta \approx \pi \cdot 1.86255 \dots \approx 5.851373.$$

One could ask the analogous question about the intersection local time of two independent planar Brownian motions. There is a critical exponent γ_α . The critical value in this case was determined in [C] and was found to be the same constant A^{-4} with A as above. As a matter of fact, one will see that the result given in [C] is an important ingredient to the proof of Theorem 1.1 (and Theorem 1.2 below as well).

As part of our proof of Theorem 1.1, we obtain large deviation estimates for β_1 .

Theorem 1.2. *We have*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}(\beta_1 \geq t) = -A^{-4},$$

where A is as in the statement of Theorem 1.1.

One can easily see that Theorem 1.1 is a direct consequence of Theorem 1.2.

Interestingly, the lower tail of β_1 is not exponential but instead double exponential.

Theorem 1.3. *There exists $0 < L < \infty$ such that*

$$\lim_{t \rightarrow \infty} t^{-2\pi} \log \mathbb{P}\{-\beta_1 \geq \log t\} = -L.$$

We also investigate laws of the iterated logarithm for β_t . We have

Theorem 1.4.

$$\limsup_{t \rightarrow \infty} \frac{\beta_t}{t \log \log t} = \frac{1}{\gamma_\beta}, \quad \text{a.s.}$$

For the lim inf behavior we have

Theorem 1.5.

$$\liminf_{t \rightarrow \infty} \frac{\beta_t}{t \log \log \log t} = -\frac{1}{2\pi}, \quad \text{a.s.}$$

Note the triple log in the rate of growth of the lim inf. This is suggested by the double exponential tail of $-\beta_t$. Compare this also with the result of [BKu] on the law of the iterated logarithm for the range of a random walk on \mathbb{Z}^2 ; the rate of growth there also

has a triple log term. For a random walk the number of self-intersections is related to the range of the random walk up to time n , and Theorem 1.5 may provide some further insight into the result of [BKu]. Theorem 1.5 suggests that the right constant in [Bku] should be related to $1/2\pi$; we hope to return to these matters in future research.

Section 2 contains some basic facts about intersection local time. Theorems 1.1, 1.2, and 1.3 are proved in Section 3. Theorem 1.4 is proved in Section 4 and Theorem 1.5 in Section 5.

In all of the proofs a key step is the representation of β as the normalized sum of intersection local times of various pieces of the Brownian path plus sums of self-intersection local times; see Proposition 2.2. What makes the two dimensional case much more difficult than the three dimensional case is that in two dimensions these intersection local times of distinct pieces of the Brownian path are the dominant term.

Acknowledgments. We would like to thank A. Dorogovtsev, J.-F. Le Gall, W. Li, J. Rosen, and Z. Shi for their interest in this work and for helpful discussions. We would also like to thank D. Bakry, E. Carlen, J. Denzler and C. Villani for helpful information about best constants for Gagliardo-Nirenberg inequalities. Finally we thank the referee and associate editor for their exceptionally helpful reports.

2. Preliminaries.

Let us begin with some notation. Let X_t be a planar Brownian motion, let \mathcal{F}_t be the completion of $\sigma\{X_s; s \leq t\}$, and let \mathbb{P}^x denote the law of X when X is started at x . We use \mathbb{P} for \mathbb{P}^0 . The shift operators will be denoted by θ_t as usual. If I is an interval, we write $X(I)$ for the random set $\{X_s; s \in I\}$. The ball of radius r about x is denoted $B(x, r)$, and the letter c with subscripts will be used for positive finite constants whose exact value is unimportant.

If X and Y are two independent planar Brownian motions, the intersection local time can be defined formally by

$$\alpha(s, t) = \int_0^s \int_0^t \delta_0(X_r - Y_u) du dr,$$

where δ_0 is the delta function. To make this rigorous, let φ be a smooth nonnegative function in the Schwartz class which integrates to 1, let $\varphi_\varepsilon(x) = \varepsilon^{-2}\varphi(x/\varepsilon)$ (so that φ_ε is an approximation to the identity), and define

$$\alpha(s, t) = \lim_{\varepsilon \rightarrow 0} \int_0^s \int_0^t \varphi_\varepsilon(X_r - Y_u) du dr. \quad (2.1)$$

On the other hand, self-intersection local time can not be defined as simply because the limit

$$\lim_{\varepsilon \rightarrow 0} \int_0^s \int_0^t \varphi_\varepsilon(X_r - X_u)$$

does not exist. A procedure called renormalization is needed. The renormalized self-intersection local time of X is formally defined as

$$\beta_t = \int_0^t \int_0^s \delta_0(X_s - X_u) du ds - \mathbb{E} \int_0^t \int_0^s \delta_0(X_s - X_u) du ds.$$

To give a rigorous definition, let

$$\beta_t = \lim_{\varepsilon \rightarrow 0} \left[\int_0^t \int_0^s \varphi_\varepsilon(X_s - X_u) du ds - \mathbb{E} \int_0^t \int_0^s \varphi_\varepsilon(X_s - X_u) du ds \right]. \quad (2.2)$$

That the limit exists a.s. and is continuous in t is proved in, for instance, [L1], [IPV] and [Y2]. Sometimes slightly different normalizations are used; they differ from ours by at most a constant times t . So there is no difference in the critical exponent or laws of the iterated logarithm, no matter which normalization is used.

If I is an interval, we use $B(I)$ for the renormalized self-intersection for the piece of the path $X(I)$. That is, if $I = [s, t]$, then

$$B(I) = \beta_{t-s} \circ \theta_s. \quad (2.3)$$

If I and J are two intervals whose interiors are disjoint, let $A(I; J)$ denote the intersection local time for the two processes $X(I)$ and $X(J)$. To define this more precisely,

$$A(I; J) = \lim_{\varepsilon \rightarrow 0} \int_I \int_J \varphi_\varepsilon(X_s - X_t) ds dt. \quad (2.4)$$

Remark 2.1. It is immediate by Brownian scaling that $\alpha(t, t)$ is equal in law to $t\alpha(1, 1)$ and β_t is equal in law to $t\beta_1$. Suppose $I = [a, b]$ and $J = [b, c]$. Then $A(I; J)$ measures the intersections of the two independent Brownian motions $X_b - X_{b-s}$ and $X_{b+t} - X_b$, and so $A(I; J)$ is equal in law to $\alpha(b-a, c-b)$ with starting point $(0, 0)$.

Proposition 2.2. *If I is an interval that is the union of subintervals I_j , $1 \leq j \leq n$, such that the interiors of the I_j are pairwise disjoint, then*

$$B(I) = \sum_{j=1}^n B(I_j) + \sum_{i < j} A(I_i; I_j) - \mathbb{E} \sum_{i < j} A(I_i; I_j). \quad (2.5)$$

Proof. We have

$$\begin{aligned} & \int \int_{s, t \in I, s < t} \varphi_\varepsilon(X_t - X_s) ds dt - \mathbb{E} \int \int_{s, t \in I, s < t} \varphi_\varepsilon(X_t - X_s) ds dt \\ &= \sum_{j=1}^n \left[\int \int_{s, t \in I_j, s < t} \varphi_\varepsilon(X_t - X_s) ds dt - \mathbb{E} \int \int_{s, t \in I_j, s < t} \varphi_\varepsilon(X_t - X_s) ds dt \right] \\ & \quad + \sum_{i < j} \int \int_{s \in I_i, t \in I_j} \varphi_\varepsilon(X_t - X_s) ds dt - \sum_{i < j} \mathbb{E} \int \int_{s \in I_i, t \in I_j} \varphi_\varepsilon(X_t - X_s) ds dt. \end{aligned}$$

We now let $\varepsilon \rightarrow 0$. □

Although $\mathbb{E} \alpha(t, t)$ is a constant times t , we need a bit more precision.

Proposition 2.3. Let $\mathbb{P}^{(x_0, y_0)}$ be the joint law of (X_t, Y_t) when X_t is started at x_0 and Y_t is started at y_0 . Then

$$\mathbb{E}^{(x_0, y_0)} \alpha(s, t) \leq \frac{1}{2\pi} [(s+t) \log(s+t) - s \log s - t \log t]. \quad (2.6)$$

If $x_0 = y_0$, then we have equality in (2.6).

Proof. X_r is a 2-dimensional normal random vector with mean x_0 and covariance matrix that is r times the identity and Y_u is a 2-dimensional normal random vector with mean y_0 and covariance matrix that is u times the identity; moreover the two random vectors are independent. Therefore $X_r - Y_u$ is a 2-dimensional normal random vector with mean $x_0 - y_0$ and covariance matrix that is $r + u$ times the identity. Hence

$$\mathbb{E}^{(x_0, y_0)} \int_0^s \int_0^t \varphi_\varepsilon(X_r - Y_u) dr du = \int_0^s \int_0^t \int_{\mathbb{R}^2} \varphi_\varepsilon(z) \frac{1}{2\pi(r+u)} e^{-|z-x_0+y_0|^2/2(r+u)} dz dr du.$$

Letting $\varepsilon \rightarrow 0$ and using (2.1),

$$\mathbb{E} \alpha(s, t) = \int_0^s \int_0^t \frac{1}{2\pi(r+u)} e^{-|x_0-y_0|^2/2(r+u)} dr du.$$

The right hand side is less than or equal to

$$\int_0^s \int_0^t \frac{1}{2\pi(r+u)} dr du$$

with equality when $x_0 = y_0$. Some routine calculus completes the proof. \square

Le Gall [L2] showed that there exists a value γ_β such that

$$\mathbb{E} e^{\gamma\beta_1} \begin{cases} < \infty & \text{if } \gamma < \gamma_\beta, \\ = \infty & \text{if } \gamma > \gamma_\beta. \end{cases} \quad (2.7)$$

In the same article, Le Gall proves that there exists a value γ_α such that

$$\mathbb{E} e^{\gamma\alpha(1,1)} \begin{cases} < \infty & \text{if } \gamma < \gamma_\alpha, \\ = \infty & \text{if } \gamma > \gamma_\alpha. \end{cases} \quad (2.8)$$

He also gives a proof ([L2], p. 178) of a result of Varadhan ([V]) that

$$\mathbb{E} e^{-\gamma\beta_1} < \infty \quad (2.9)$$

for all $\gamma > 0$.

3. Large deviation estimates.

In [C] the large deviations for intersection local time of p independent d -dimensional Brownian motions under the condition $p(d-2) < d$ were studied. Taking $d = p = 2$ in this result,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}\{\alpha(1, 1) \geq t\} = -A^{-4}, \quad (3.1)$$

where $A > 0$ is the best constant in the Gagliardo-Nirenberg inequality

$$\|f\|_4 \leq C \sqrt{\|\nabla f\|_2} \sqrt{\|f\|_2}.$$

Let

$$M = \sup_{f \in \mathbb{F}_2} \left\{ \left(\int_{\mathbb{R}^2} |f(x)|^4 dx \right)^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla f(x)|^2 dx \right\},$$

where \mathbb{F}_2 is the set of absolutely continuous functions on \mathbb{R}^2 satisfying

$$\int_{\mathbb{R}^2} |f(x)|^2 dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^2} |\nabla f(x)|^2 dx < \infty$$

As a special case of Lemma 8.2 in [C],

$$M = \frac{1}{2} A^4. \quad (3.2)$$

In the following result we claim that β_1 satisfies the same large deviation principle as $\alpha(1, 1)$ does.

Theorem 3.1.

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}\{\beta_1 \geq t\} = -A^{-4}. \quad (3.3)$$

In particular,

$$\mathbb{E} e^{\gamma \beta_1} \quad \begin{cases} < \infty & \text{if } \gamma < A^{-4}, \\ = \infty & \text{if } \gamma > A^{-4}. \end{cases}$$

Note that this theorem implies $\gamma_\beta = A^{-4}$ and is a reformulation of Theorems 1.1 and Theorem 1.2.

Proof. To establish the upper bound, we consider the decomposition

$$\beta_1 = \beta_{1/2} + B([1/2, 1]) + A([0, 1/2]; [1/2, 1]) - \mathbb{E} A([0, 1/2]; [1/2, 1]).$$

Recall that $B([1/2, 1])$ and $\beta_{1/2}$ are equal in law to $\frac{1}{2}\beta_1$ and $A([0, 1/2]; [1/2, 1])$ is equal in law to $\frac{1}{2}\alpha(1, 1)$. Moreover $B([1/2, 1])$ is independent of $\beta_{1/2}$. Given $\varepsilon > 0$,

$$\mathbb{P}\{\beta_1 \geq t\} \leq \mathbb{P}\{\alpha(1, 1) - \mathbb{E} \alpha(1, 1) \geq (1 - \varepsilon)t\} + \mathbb{P}\{\beta_1 + \beta'_1 \geq (1 + \varepsilon)t\},$$

where β'_1 is an independent copy of β_1 . In view of (3.1),

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}\{\beta_1 \geq t\} \leq \max \left\{ -(1 - \varepsilon)A^{-4}, \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}\{\beta_1 + \beta'_1 \geq (1 + \varepsilon)t\} \right\}.$$

We now need the simple fact that (1.2) is equivalent to

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}\{\beta_1 \geq t\} = -\gamma_\beta.$$

Also notice that

$$\mathbb{E} \exp\{\gamma(\beta_1 + \beta'_1)\} = \left(\mathbb{E} \exp\{\gamma\beta_1\} \right)^2 \begin{cases} < \infty & \gamma < \gamma_\beta \\ = \infty & \gamma > \gamma_\beta. \end{cases}$$

So

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}\{\beta_1 + \beta'_1 \geq (1 + \varepsilon)t\} = -(1 + \varepsilon)\gamma_\beta$$

and therefore

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}\{\beta_1 \geq t\} \leq -(1 - \varepsilon)A^{-4}.$$

Letting $\varepsilon \rightarrow 0^+$ we obtain

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}\{\beta_1 \geq t\} \leq -A^{-4}. \quad (3.4)$$

By scaling we have the upper bound of (3.3).

By scaling, Theorem 3.1 is equivalent to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}\{\beta_n \geq \theta n^2\} = -\theta A^{-4}, \quad \theta > 0. \quad (3.5)$$

Let

$$C_n = \sum_{k=1}^{n-1} A([0, k]; [k, k+1]), \quad n = 1, 2, \dots$$

Then by Proposition 2.2

$$\beta_n = C_n - \mathbb{E} C_n + \sum_{k=1}^n \beta([k-1, k]).$$

Notice that $\{\beta([k-1, k])\}$ is an i.i.d. sequence with the same distribution as β_1 . Since the moment generating function of β_1 exists in a neighborhood of the origin, Cramér's theorem implies that for any $\delta > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}\left\{ \sum_{k=1}^n \beta([k-1, k]) \geq \delta n^2 \right\} = -\infty. \quad (3.6)$$

Also, using Proposition 2.3 a calculation implies

$$\mathbb{E} C_n = \frac{1}{2\pi} n \log n. \quad (3.7)$$

By Theorem 4.2.13 in [DZ], (3.5) is then equivalent to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P} \{ C_n \geq \theta n^2 \} = -\theta A^{-4}, \quad \theta > 0. \quad (3.8)$$

We now claim that Theorem 3.1 will hold provided

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \exp \{ \lambda C_n^{1/2} \} \geq \frac{\lambda^2 A^2}{4}, \quad \lambda > 0. \quad (3.9)$$

Indeed, from the upper bound (3.4) we can improve (3.9) into equality. In the case $\lambda < 0$, we use Jensen's inequality:

$$\begin{aligned} & \mathbb{E} \exp \{ \lambda C_n^{1/2} \} \\ & \geq \exp \{ \lambda \mathbb{E} C_n^{1/2} \} \\ & \geq \exp \{ \lambda (\mathbb{E} C_n)^{1/2} \} \\ & = \exp \{ -O(\sqrt{n \log n}) \}, \end{aligned}$$

where the last step follows from (3.7). Therefore, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \exp \{ \lambda C_n^{1/2} \} = \psi(\lambda)$$

for any real number λ , where

$$\psi(\lambda) = \begin{cases} \frac{\lambda^2 A^4}{4} & \lambda \geq 0, \\ 0 & \lambda < 0. \end{cases}$$

By the Gärtner-Ellis theorem (Theorem 2.3.6 in [DZ]),

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P} \{ C_n^{1/2} \geq \theta n \} \\ & = -\sup_{\lambda \in \mathbb{R}} \{ \lambda \theta - \psi(\lambda) \} = -\sup_{\lambda > 0} \left\{ \lambda \theta - \frac{\lambda^2 A^4}{4} \right\} = -\theta^2 A^{-4}, \quad \theta > 0, \end{aligned}$$

which is equivalent to (3.8).

We now prove (3.9). Some of the ideas come from [CL]. We start with the fact (see, e.g., [Re]) that for any measurable, bounded function f on \mathbb{R}^2 ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \exp \left\{ \int_0^n f(X_t) dt \right\} = \sup_{g \in \mathbb{F}_2} \left\{ \int_{\mathbb{R}^2} f(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla g(x)|^2 dx \right\}.$$

For any $\varepsilon > 0$, let $p_\varepsilon(x)$ be the density of X_ε and write

$$L(t, x, \varepsilon) = \int_0^t p_\varepsilon(X_s - x) ds, \quad x \in \mathbb{R}^2, \quad t \geq 0.$$

It is easy to see from the semigroup property that

$$\begin{aligned} \left(\iint_{0 \leq s \leq t \leq n} p_{2\varepsilon}(X_s - X_t) ds dt \right)^{1/2} &= \frac{1}{\sqrt{2}} \left(\int_{\mathbb{R}^2} L^2(n, x, \varepsilon) dx \right)^{1/2} \\ &\geq \frac{1}{\sqrt{2}} \int_{\mathbb{R}^2} f(x) L(n, x, \varepsilon) dx = \frac{1}{\sqrt{2}} \int_0^n f_\varepsilon(X_t) dt \end{aligned}$$

for any measurable f on \mathbb{R}^2 with

$$\int_{\mathbb{R}^2} f^2(x) dx = 1,$$

where

$$f_\varepsilon(x) = \int_{\mathbb{R}^2} f(x - y) p_\varepsilon(y) dy.$$

Therefore,

$$\begin{aligned} &\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \exp \left\{ \lambda \left(\iint_{0 \leq s \leq t \leq n} p_{2\varepsilon}(X_s - X_t) ds dt \right)^{1/2} \right\} \\ &\geq \sup_{g \in \mathbb{F}_2} \left\{ \frac{\lambda}{\sqrt{2}} \int_{\mathbb{R}^2} f_\varepsilon(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla g(x)|^2 dx \right\} \\ &= \sup_{g \in \mathbb{F}_2} \left\{ \frac{\lambda}{\sqrt{2}} \int_{\mathbb{R}^2} f(x) \left(\int_{\mathbb{R}^2} g^2(x - y) p_\varepsilon(y) dy \right) dx - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla g(x)|^2 dx \right\}. \end{aligned}$$

Taking the supremum over f with $\|f\|_2 = 1$ and using the fact that the dual of L^2 is L^2 gives

$$\begin{aligned} &\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \exp \left\{ \lambda \left(\iint_{0 \leq s \leq t \leq n} p_{2\varepsilon}(X_s - X_t) ds dt \right)^{1/2} \right\} \\ &\geq \sup_{g \in \mathbb{F}_2} \left\{ \frac{\lambda}{\sqrt{2}} \left[\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} g^2(x - y) p_\varepsilon(y) dy \right)^2 dx \right]^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla g(x)|^2 dx \right\} \end{aligned} \tag{3.10}$$

for any $\lambda > 0$.

On the other hand, write

$$\xi_k(\varepsilon) = \iint_{\{k-1 \leq s \leq t \leq k\}} p_{2\varepsilon}(X_s - X_t) ds dt, \quad k = 1, 2, \dots$$

and

$$D_n = \bigcup_{k=1}^{n-1} [0, k] \times (k, k+1] \quad n = 1, 2, \dots$$

Then $\{\xi_k(\varepsilon)\}_{k \geq 1}$ is an i.i.d. sequence and

$$\iint_{\{0 \leq s \leq t \leq n\}} p_{2\varepsilon}(X_s - X_t) ds dt = \iint_{D_n} p_{2\varepsilon}(X_s - X_t) ds dt + \sum_{k=1}^n \xi_k(\varepsilon).$$

Let $p, q > 1$ be such that $p^{-1} + q^{-1} = 1$. By the triangle inequality and Hölder's inequality,

$$\begin{aligned} & \mathbb{E} \exp \left\{ p^{-1} \lambda \left(\iint_{\{0 \leq s \leq t \leq n\}} p_{2\varepsilon}(X_s - X_t) ds dt \right)^{1/2} \right\} \\ & \leq \left[\mathbb{E} \exp \left\{ \lambda \left(\iint_{D_n} p_{2\varepsilon}(X_s - X_t) ds dt \right)^{1/2} \right\} \right]^{1/p} \left[\mathbb{E} \exp \left\{ qp^{-1} \lambda \left(\sum_{k=1}^n \xi_k(\varepsilon) \right)^{1/2} \right\} \right]^{1/q}. \end{aligned}$$

It is easy to see from standard large deviation theory that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \exp \left\{ qp^{-1} \lambda \left(\sum_{k=1}^n \xi_k(\varepsilon) \right)^{1/2} \right\} = 0.$$

Therefore, by (3.10) we have

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \exp \left\{ \lambda \left(\iint_{D_n} p_{2\varepsilon}(X_s - X_t) ds dt \right)^{1/2} \right\} \\ & \geq p \sup_{g \in \mathbb{F}_2} \left\{ \frac{p^{-1} \lambda}{\sqrt{2}} \left[\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} g^2(x-y) p_\varepsilon(y) dy \right)^2 dx \right]^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla g(x)|^2 dx \right\}. \end{aligned}$$

Letting $p \rightarrow 1^+$ gives

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \exp \left\{ \lambda \left(\iint_{D_n} p_{2\varepsilon}(X_s - X_t) ds dt \right)^{1/2} \right\} \\ & \geq \sup_{g \in \mathbb{F}_2} \left\{ \frac{\lambda}{\sqrt{2}} \left[\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} g^2(x-y) p_\varepsilon(y) dy \right)^2 dx \right]^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla g(x)|^2 dx \right\}. \end{aligned} \tag{3.11}$$

For any $m \geq 0$, let $k \geq 0$ be the integer such that $2k \leq m \leq 2(k+1)$. By Lemma 3.4 below,

$$\begin{aligned} \mathbb{E} (C_n^{\frac{m+2}{2}}) & \geq [\mathbb{E} C_n^{k+1}]^{\frac{m+2}{2(k+1)}} \\ & \geq \left[\mathbb{E} \left(\iint_{D_n} p_{2\varepsilon}(X_s - X_t) ds dt \right)^{k+1} \right]^{\frac{m+2}{2(k+1)}} \geq \left[\mathbb{E} \left(\iint_{D_n} p_{2\varepsilon}(X_s - X_t) ds dt \right)^{m/2} \right]^{\frac{m+2}{m}}. \end{aligned}$$

As $n \rightarrow \infty$, it is clear that $C_n \rightarrow \infty$. Using Lemma 3.4 one can also see that there is a $N > 0$ and $\varepsilon_0 > 0$ such that

$$\mathbb{E} \left(\iint_{D_n} p_{2\varepsilon}(X_s - X_t) ds dt \right)^{m/2} \geq 1, \quad m = 0, 1, \dots$$

if $n \geq N$ and $\varepsilon \leq \varepsilon_0$. Hence

$$\mathbb{E} (C_n^{\frac{m+2}{2}}) \geq \mathbb{E} \left(\iint_{D_n} p_{2\varepsilon}(X_s - X_t) ds dt \right)^{m/2}.$$

Using the Taylor series expansion for $e^{\lambda x}$, for each $0 < \delta < \lambda$

$$\mathbb{E} (C_n e^{(\lambda-\delta)C_n^{1/2}}) \geq \mathbb{E} \exp \left\{ (\lambda - \delta) \left(\iint_{D_n} p_{2\varepsilon}(X_s - X_t) ds dt \right)^{1/2} \right\}.$$

By the fact that $e^{\lambda\sqrt{x}} \geq xe^{(\lambda-\delta)\sqrt{x}}$ for sufficiently large $x > 0$ and in view of (3.11) (with λ being replaced by $\lambda - \delta$), the above estimate implies

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \exp \{ \lambda C_n^{1/2} \} \\ & \geq \sup_{g \in \mathbb{F}_2} \left\{ \frac{\lambda - \delta}{\sqrt{2}} \left[\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} g^2(x-y) p_\varepsilon(y) dy \right)^2 dx \right]^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla g(x)|^2 dx \right\}. \end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$ on the right hand side gives

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \exp \{ \lambda C_n^{1/2} \} \\ & \geq \sup_{g \in \mathbb{F}_2} \left\{ \frac{\lambda - \delta}{\sqrt{2}} \left(\int_{\mathbb{R}^2} |g(x)|^4 dx \right)^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla g(x)|^2 dx \right\} \\ & = \frac{(\lambda - \delta)^2}{2} \sup_{f \in \mathbb{F}_2} \left\{ \left(\int_{\mathbb{R}^2} |f(x)|^4 dx \right)^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla f(x)|^2 dx \right\} = \frac{(\lambda - \delta)^2 A^4}{4} \end{aligned} \tag{3.12}$$

for any $0 < \delta < \lambda$, where the second step follows from the substitution

$$g(x) = \frac{\lambda - \delta}{\sqrt{2}} f\left(\frac{\lambda - \delta}{\sqrt{2}} x\right)$$

and the last from (3.2). Finally, letting $\delta \rightarrow 0^+$ gives (3.9). □

Theorem 3.2. *There is a $0 < L \leq \infty$ such that*

$$\lim_{t \rightarrow \infty} t^{-2\pi} \log \mathbb{P}\{-\beta_1 \geq \log t\} = -L.$$

Theorem 3.2 will prove part of Theorem 1.3.

Proof. For any positive integers m and n , by Proposition 2.2

$$\begin{aligned}\beta_{m+n} &= \beta_n + B([n, n+m]) + A([0, n]; [n, n+m]) - \mathbb{E} A([0, n]; [n, n+m]) \\ &\geq \beta_n + B([n, n+m]) - \mathbb{E} A([0, n]; [n, n+m]),\end{aligned}$$

and β_n and $B([n, n+m])$ are independent. Hence

$$\begin{aligned}\mathbb{E} \exp \{ -2\pi(m+n)\beta_1 \} &= \mathbb{E} \exp \{ -2\pi\beta_{m+n} \} \\ &\leq \exp \{ 2\pi\mathbb{E} A([0, n]; [n, n+m]) \} \mathbb{E} \exp \{ -2\pi B([n, n+m]) \} \mathbb{E} \exp \{ -2\pi\beta_n \} \\ &= \exp \{ 2\pi\mathbb{E} A([0, n]; [n, n+m]) \} \mathbb{E} \exp \{ -2\pi\beta_m \} \mathbb{E} \exp \{ -2\pi\beta_n \} \\ &= \exp \{ 2\pi\mathbb{E} A([0, n]; [n, n+m]) \} \mathbb{E} \exp \{ -2\pi m\beta_1 \} \mathbb{E} \exp \{ -2\pi n\beta_1 \}.\end{aligned}$$

Using this and Proposition 2.3,

$$\begin{aligned}(m+n)^{-(m+n)} \mathbb{E} \exp \{ -2\pi(m+n)\beta_1 \} \\ \leq \left(m^{-m} \mathbb{E} \exp \{ -2\pi m\beta_1 \} \right) \left(n^{-n} \mathbb{E} \exp \{ -2\pi n\beta_1 \} \right).\end{aligned}$$

If we write

$$a(n) = \log \left(n^{-n} \mathbb{E} \exp \{ -2\pi n\beta_1 \} \right), \quad n = 1, 2, \dots,$$

then we have proved that for any positive integers m and n ,

$$a(n+m) \leq a(m) + a(n).$$

Consequently,

$$\lim_{n \rightarrow \infty} \frac{1}{n} a(n) = \inf_{m \geq 1} \left\{ \frac{1}{m} a(m) \right\}.$$

By Stirling's formula, this is equivalent to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left((n!)^{-1} \mathbb{E} \exp \{ -2\pi n\beta_1 \} \right) = 1 + \inf_{m \geq 1} \left\{ \frac{1}{m} a(m) \right\}.$$

By Lemma 2.3 of [KM], (2002),

$$\limsup_{t \rightarrow \infty} t^{-1} \log \mathbb{P} \left\{ \exp \{ -2\pi\beta_1 \} \geq t \right\} = -\exp \left\{ -1 - \inf_{m \geq 1} \left\{ \frac{1}{m} a(m) \right\} \right\} = -L,$$

where

$$L = \exp \left\{ -1 - \inf_{m \geq 1} \left\{ \frac{1}{m} a(m) \right\} \right\}.$$

□

Remark 3.3. In fact $L < \infty$. This is established in Corollary 5.7 below.

Lemma 3.4. For any positive numbers ε and ε' with $\varepsilon > \varepsilon'$, any $D \subset \{(s, t); s \leq t\}$, and integer $m \geq 1$,

$$\mathbb{E} \left[\iint_D p_{\varepsilon'}(X_t - X_s) ds dt \right]^m \geq \mathbb{E} \left[\iint_D p_{\varepsilon}(X_t - X_s) ds dt \right]^m$$

Further, if D is a finite union of disjoint rectangles contained in $\{(s, t); s \leq t\}$:

$$D = \bigcup_{k=1}^n (I_k \times J_k)$$

then

$$\mathbb{E} \left[\iint_D p_{\varepsilon}(X_t - X_s) ds dt \right]^m \leq \mathbb{E} \left[\sum_{k=1}^n A(I_k; J_k) \right]^m$$

Proof. By the Fourier transform,

$$p_{\varepsilon}(X_t - X_s) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} d\xi \exp \{ -i\xi \cdot (X_t - X_s) \} \exp \left\{ -\frac{\varepsilon}{2} |\xi|^2 \right\}.$$

Hence

$$\begin{aligned} & \mathbb{E} \left[\iint_D p_{\varepsilon}(X_t - X_s) ds dt \right]^m \\ &= \frac{1}{(2\pi)^{2m}} \int_{D^m} ds_1 dt_1 \cdots ds_m dt_m \int_{(\mathbb{R}^2)^m} d\xi_1 \cdots d\xi_m \\ & \quad \times \mathbb{E} \left[\prod_{k=1}^m \exp \{ -i\xi_k \cdot (X_t - X_s) \} \exp \left\{ -\frac{\varepsilon}{2} |\xi_k|^2 \right\} \right] \\ &= \frac{1}{(2\pi)^{2m}} \int_{D^m} ds_1 dt_1 \cdots ds_m dt_m \int_{(\mathbb{R}^2)^m} d\xi_1 \cdots d\xi_m \\ & \quad \times \exp \left\{ -\frac{1}{2} \text{Var} \left[\sum_{k=1}^m \xi_k \cdot (X_t - X_s) \right] \right\} \exp \left\{ -\frac{\varepsilon}{2} \sum_{k=1}^m |\xi_k|^2 \right\} \end{aligned}$$

which leads to the first half of the lemma.

As for the second half of the lemma, by Theorem 4, p. 191, in [L1],

$$\mathbb{E} \left[\sum_{k=1}^n A(I_k; J_k) \right]^m < \infty \quad m = 0, 1, \dots$$

and

$$\iint_D p_{\varepsilon}(X_t - X_s) ds dt \longrightarrow \sum_{k=1}^n A(I_k; J_k) \quad (\varepsilon \rightarrow 0^+)$$

in L^m -norm for all integers $m \geq 1$. (In fact, Le Gall proves the above convergence with p_ε replaced by the uniform density on the disk of radius ε . It can be seen from his argument that it remains true in our case.) Therefore, letting $\varepsilon' \rightarrow 0^+$ leads to the second half of the lemma. □

4. The lim sup result.

In this section we establish Theorem 1.4.

Lemma 4.1. *There exist constants c_1, c_2 such that for all $\lambda > 0$ and all $a \in (0, 1)$*

$$\mathbb{P}(\alpha(1, a) > \lambda) \leq c_1 e^{-c_2 \lambda / \sqrt{a}}.$$

Proof. Let $m \geq 1$ be an integer. We first prove there exist constants c_3, c_4 such that

$$\mathbb{E}[\alpha(1, a)^m] \leq c_3 c_4^m a^{m/2} m! \tag{4.1}$$

To establish this, write

$$\alpha(1, a) = \lim_{\varepsilon \rightarrow 0} \int_0^1 \int_0^a p(\varepsilon, 0, X_v - Y_u) dv du,$$

where $p(\varepsilon, x, y)$ is the transition density of planar Brownian motion. As mentioned in the last paragraph of the proof of Lemma 3.4, the convergence takes place in L^p for every p . By the semigroup property,

$$p(\varepsilon, 0, X_v - Y_u) = p(\varepsilon, X_v, Y_u) = \int_{\mathbb{R}^2} p(\varepsilon/2, x, X_v) p(\varepsilon/2, x, Y_u) dx,$$

and so

$$\begin{aligned} & \left[\int_0^1 \int_0^a p(\varepsilon, 0, X_v - Y_u) dv du \right]^m \\ &= \int_{(\mathbb{R}^2)^m} dx_1 \cdots dx_m \left(\prod_{k=1}^m \int_0^1 p(\varepsilon/2, x_k, X_v) dv \right) \left(\prod_{k=1}^m \int_0^a p(\varepsilon/2, x_k, Y_u) du \right). \end{aligned}$$

Using the independence of X and Y , the expectation is equal to

$$\int_{(\mathbb{R}^2)^m} dx_1 \cdots dx_m \mathbb{E} \left[\prod_{k=1}^m \int_0^1 p(\varepsilon/2, x_k, X_v) dv \right] \mathbb{E} \left[\prod_{k=1}^m \int_0^a p(\varepsilon/2, x_k, Y_u) du \right].$$

By Cauchy-Schwarz this less than $J_1(\varepsilon)^{1/2} J_2(\varepsilon)^{1/2}$, where

$$J_1(\varepsilon) = \int_{(\mathbb{R}^2)^m} dx_1 \cdots dx_m \left(\mathbb{E} \left[\prod_{k=1}^m \int_0^1 p(\varepsilon/2, x_k, X_v) dv \right] \right)^2$$

and

$$J_2(\varepsilon) = \int_{(\mathbb{R}^2)^m} dx_1 \cdots dx_m \left(\mathbb{E} \left[\prod_{k=1}^m \int_0^a p(\varepsilon/2, x_k, Y_u) du \right] \right)^2$$

By Brownian scaling,

$$\lim_{\varepsilon \rightarrow 0} J_2(\varepsilon) = a^m \lim_{\varepsilon \rightarrow 0} J_1(\varepsilon). \quad (4.2)$$

To estimate $J_1(\varepsilon)$, we rewrite it as

$$\int_{(\mathbb{R}^2)^m} dx_1 \cdots dx_m \mathbb{E} \left[\prod_{k=1}^m \int_0^1 p(\varepsilon/2, x_k, X_v) dv \right] \mathbb{E} \left[\prod_{k=1}^m \int_0^1 p(\varepsilon/2, x_k, Y_u) du \right],$$

and so by the argument above in reverse order,

$$J_1(\varepsilon) = \mathbb{E} \left[\int_0^1 \int_0^1 p(\varepsilon, X_v, Y_u) dv du \right]^m.$$

Therefore $\lim_{\varepsilon \rightarrow 0} J_1(\varepsilon) = \mathbb{E} [\alpha(1, 1)^m]$. Lemma 2 of [L2] together with (4.2) and an application of Fatou's lemma completes the proof of (4.1).

We then obtain

$$\mathbb{E} e^{\alpha(1, a)/(2c_4\sqrt{a})} = \sum_{m=0}^{\infty} \left(\frac{1}{2c_4\sqrt{a}} \right)^m \frac{\mathbb{E} [\alpha(1, a)^m]}{m!} \leq c_7,$$

where c_7 does not depend on a . Finally,

$$\mathbb{P}(\alpha(1, a) > \lambda) \leq e^{-\lambda/(2c_4\sqrt{a})} \mathbb{E} e^{\alpha(1, a)/(2c_4\sqrt{a})} \leq c_7 e^{-\lambda/(2c_4\sqrt{a})},$$

which is what we wanted. □

The key to the upper bound is to obtain an estimate of the following form.

Proposition 4.2. *If $\gamma < \gamma_\beta$, there exists c_1 such that*

$$\mathbb{P}(\sup_{t \leq 1} \beta_t > \lambda) \leq c_1 e^{-\gamma\lambda}, \quad \lambda > 0. \quad (4.3)$$

Proof. By Proposition 2.2,

$$\beta_t - \beta_s = B([s, t]) + A([0, s]; [s, t]) - \mathbb{E} A([0, s]; [s, t]).$$

Let γ' be the midpoint of (γ, γ_β) and let $\varepsilon > 0$ be chosen so that $\gamma'(1 - \varepsilon)$ is the midpoint of (γ, γ') . Note

$$\mathbb{P}(\beta_t - \beta_s > \lambda) \leq \mathbb{P}(B([s, t]) > \lambda/2) + \mathbb{P}(A([0, s]; [s, t]) > \lambda/2). \quad (4.4)$$

Since $B([s, t])$ equals β_{t-s} in law, which equals $(t-s)\beta_1$ in law, the first probability on the right is bounded by

$$c_2 \exp \left\{ -\frac{\gamma'\lambda}{2(t-s)} \right\}. \quad (4.5)$$

$A([0, s]; [s, t])$ is equal in law to $\alpha(s, t-s)$, which is smaller than $\alpha(1, t-s)$. So by Lemma 4.1 there exists c_3 not depending on s or t such that

$$\mathbb{P}(A([0, s]; [s, t]) > \lambda/2) \leq \exp \left\{ -\frac{c_3\lambda}{(t-s)^{1/2}} \right\}. \quad (4.6)$$

Fix $n = 2^N$. Since

$$\sup_{k \leq n} \mathbb{E} e^{\gamma' \beta_{k/n}} = \sup_{k \leq n} \mathbb{E} e^{(\gamma' k/n) \beta_1} \leq c_4,$$

where c_4 does not depend on n , then

$$\mathbb{P}(\sup_{k \leq n} \beta_{k/n} > (1 - \varepsilon)\lambda) \leq n \sup_{k \leq n} \mathbb{P}(\beta_{k/n} > (1 - \varepsilon)\lambda) \leq n e^{-\gamma'(1-\varepsilon)\lambda} \leq n e^{-\gamma\lambda}. \quad (4.7)$$

Now we use metric entropy. If $t \in (0, 1)$, let t_j be the largest multiple of 2^{-j} that is less than or equal to t . Write

$$\beta_t = \beta_{t_N} + (\beta_{t_{N+1}} - \beta_{t_N}) + (\beta_{t_{N+2}} - \beta_{t_{N+1}}) + \dots$$

If $\beta_t > \lambda$ for some $t \leq 1$, either

- (a) for some $k \leq n$ we have $\beta_{k/n} > (1 - \varepsilon)\lambda$; or
- (b) for some $j \geq N$ and some $s < t$ with $t - s = 2^{-j}$ and both s, t integer multiples of 2^{-j} we have

$$\beta_t - \beta_s > \varepsilon\lambda/(100j^2). \quad (4.8)$$

The probability of possibility (a) is bounded by (4.7). Using (4.5) and (4.6), the probability of possibility (b) is bounded by

$$c_5 \sum_{j=N}^{\infty} 2^j \left[e^{-\varepsilon\gamma'\lambda 2^j/(200j^2)} + e^{-c_3\varepsilon\lambda 2^{j/2}/(200j^2)} \right]. \quad (4.9)$$

The 2^j in front of the brackets comes about because there are 2^j pairs (s, t) to consider. It is not hard to see that the sum in (4.9) is bounded by

$$c_6 \left[e^{-\varepsilon \gamma' \lambda 2^N / (400N^2)} + e^{-c_3 \varepsilon \lambda 2^{N/2} / (400N^2)} \right].$$

If we choose N large enough so that $2^N \varepsilon / (400N^2) > 1$ and $c_3 2^{N/2} \varepsilon / (400N^2) > \gamma'$, we then have that the probability of possibility (b) is bounded by

$$2c_7 e^{-\gamma' \lambda} \leq 2c_7 e^{-\gamma \lambda}.$$

If we combine this with (4.7), we have (4.3). \square

Using Borel-Cantelli it is now straightforward to get

Theorem 4.3.

$$\limsup_{t \rightarrow \infty} \frac{\beta_t}{t \log \log t} \leq \frac{1}{\gamma_\beta}, \quad \text{a.s.}$$

Proof. Let $M > 1/\gamma_\beta$. Choose $\varepsilon > 0$ small and $q > 1$ close to 1 so that $M(\gamma_\beta - 2\varepsilon)/q > 1$. Let $t_n = q^n$ and let $C_n = \{\sup_{s \leq t_n} \beta_s > M t_{n-1} \log \log t_{n-1}\}$. By Proposition 4.2 and scaling the probability of C_n is bounded by

$$c_1 e^{-(\gamma_\beta - \varepsilon) M t_{n-1} \log \log t_{n-1} / t_n}.$$

By our choices of ε and q this is summable, so by Borel-Cantelli the probability that C_n happens infinitely often is zero. To complete the proof we point out that if $\beta_t > M t \log \log t$ for some $t \in [t_{n-1}, t_n]$, then the event C_n occurs. \square

To finish the proof of Theorem 4.3 we prove

Theorem 4.4.

$$\limsup_{t \rightarrow \infty} \frac{\beta_t}{t \log \log t} \geq \frac{1}{\gamma_\beta}, \quad \text{a.s.}$$

Jointly, Theorem 4.3 and Theorem 4.4 are a reformulation of Theorem 1.4.

Proof. Let $\gamma > \gamma_\beta$ and let γ' be the midpoint of (γ_β, γ) . Then by Theorem 3.1

$$\mathbb{P}(\beta_1 \geq a \log \log n) \geq c_2 e^{-\gamma' a \log \log n}, \quad a > 0. \quad (4.10)$$

Let $\delta > 0$ be small enough so that $(1 + \delta)\gamma'/\gamma < 1$ and set $t_n = e^{n^{1+\delta}}$. By (2.5),

$$\begin{aligned} \beta_{t_n} &= B([0, t_n]) \\ &= B([t_{n-1}, t_n]) + B([0, t_{n-1}]) \\ &\quad + A([0, t_{n-1}]; [t_{n-1}, t_n]) - \mathbb{E} A([0, t_{n-1}]; [t_{n-1}, t_n]) \\ &\geq B([t_{n-1}, t_n]) + B([0, t_{n-1}]) - \mathbb{E} A([0, t_{n-1}]; [t_{n-1}, t_n]). \end{aligned}$$

By scaling

$$\mathbb{E} A([0, t_{n-1}]; [t_{n-1}, t_n]) \leq \mathbb{E} \alpha(t_n, t_n) = t_n \mathbb{E} \alpha(1, 1) = o(t_n \log \log t_n), \quad n \rightarrow \infty.$$

Since $A \geq 0$, we need only to prove

$$\limsup_{n \rightarrow \infty} \frac{B([t_{n-1}, t_n])}{t_n \log \log t_n} \geq \frac{1}{\gamma_\beta}, \quad \text{a.s.} \quad (4.11)$$

and

$$\lim_{n \rightarrow \infty} \frac{|B([0, t_{n-1}])|}{t_n \log \log t_n} = 0, \quad \text{a.s.} \quad (4.12)$$

Using (4.10) and scaling, it is straightforward to obtain

$$\sum_{n=1}^{\infty} \mathbb{P}(B([t_{n-1}, t_n]) > \frac{1}{\gamma} t_n \log \log t_n) = \infty.$$

Using the fact that different pieces of a Brownian path are independent and Borel-Cantelli,

$$\limsup_{n \rightarrow \infty} \frac{B([t_{n-1}, t_n])}{t_n \log \log t_n} > \frac{1}{\gamma} \quad \text{a.s.}$$

Letting $\gamma \rightarrow \gamma_\beta^+$ gives (4.11).

Let $\epsilon > 0$. By (2.9) there exists $c_4 > 0$ such that

$$\mathbb{P}\{-\beta_1 \geq \epsilon \log \log n\} \leq c_4 e^{-2 \log \log n}. \quad (4.13)$$

So (4.12) follows from Theorem 3.1, (4.13), scaling, and Borel-Cantelli. □

Remark 4.5. Theorems 4.3 and 4.4 together imply Theorem 1.4.

5. The lim inf result.

Let us write D_t for $-\beta_t$. We know $\mathbb{E} e^{\gamma D_1} < \infty$ for every $\gamma > 0$. But in fact we have the following.

Proposition 5.1. $\mathbb{E} e^{\gamma \sup_{s \leq 1} D_s} < \infty$ for every $\gamma > 0$.

Proof. Fix $\gamma > 0$. Choose N a fixed integer so that $2^N / (1600N^3) > 2$. If $s < t \leq 1$, we know $\mathbb{E} A([0, s]; [s, t]) \leq c_1(t-s)L(t-s) \leq c_2$, where $L(x) = 1 + |\log(1/x)|$. Suppose

$\lambda > c_3$, where c_3 is chosen so that $c_3/400j^2 > 2c_12^{-j}L(2^{-j})$ for each $j \geq 0$. Let $s_j = \inf\{k/2^j : s \leq k/2^j\}$. If $s \in [0, 1]$ we can write

$$D_s = D_{s_N} + (D_{s_{N+1}} - D_{s_N}) + (D_{s_{N+2}} - D_{s_{N+1}}) + \dots$$

So if $D_s > \lambda$ for some $s \in [0, 1]$, then either

- (a) for some $k \leq 2^N$ we have $D_{k/2^N} > \lambda/2$; or
- (b) for some $j > N$ and some $s < t$ both multiples of 2^{-j} with $t - s = 2^{-j}$,

$$D_t - D_s > \frac{\lambda}{200j^2}.$$

We have $\mathbb{P}(D_{k/2^N} > \lambda/2) = \mathbb{P}(D_1 > 2^N \lambda/2k) \leq \mathbb{P}(D_1 > \lambda/2) \leq c_4 e^{-2\gamma\lambda}$ since $\mathbb{E} e^{4\gamma D_1} < \infty$. So the probability of possibility (a) is bounded by

$$c_4 2^N e^{-2\gamma\lambda}. \tag{5.1}$$

By Proposition 2.2,

$$D_t - D_s = -B([s, t]) + \mathbb{E} A([0, s]; [s, t]) - A([0, s]; [s, t]) \leq -B([s, t]) + c_1(t - s)L(t - s).$$

Since $\lambda/400j^2 > 2c_1(t - s)L(t - s)$, then for $D_t - D_s$ to be larger than $\lambda/200j^2$, we must have $-B([t - s]) > \lambda/400j^2$. Since $-B([s, t])/(t - s)$ is equal in law to D_1 , then

$$\mathbb{P}(-B([t - s]) > \lambda/(400j^2)) \leq c_5 e^{-\gamma\lambda 2^j L(2^j)/(800j^2)}.$$

Since for each j there are 2^j pairs (s, t) to consider, the probability of (b) is bounded by

$$\sum_{j=N}^{\infty} c_5 2^j e^{-\gamma\lambda 2^j L(2^j)/(800j^2)}.$$

This is summable and can be bounded by

$$c_6 e^{-\gamma\lambda 2^N/(1600N^3)}$$

for some c_6 . By our choice of N , this is less than

$$c_6 e^{-2\gamma\lambda}. \tag{5.2}$$

Combining (5.1) and (5.2), we have

$$\mathbb{P}(\sup_{s \leq 1} D_s > \lambda) \leq c_7 e^{-2\gamma\lambda}$$

if $\lambda > c_3$. Our result follows immediately from this. \square

Theorem 5.2. *With probability 1*

$$\limsup_{t \rightarrow \infty} \frac{D_t}{t \log \log \log t} \leq \frac{1}{2\pi}.$$

Theorem 5.2 and Theorem 5.5 (given below) together are just a reformulation of Theorem 1.5.

Proof. Let $K = \lceil \log \log t \rceil$, $R = t/K$, and $I_j = [(j-1)R, jR]$. Let

$$E_j = \sup_{(j-1)R \leq t \leq jR} (-B([(j-1)R, t])).$$

By Proposition 2.2, if $s < t$ and $(\ell-1)R \leq s < \ell R$, then

$$\begin{aligned} D_s &\leq \sum_{j < \ell} (-B(I_j)) + (-B([(\ell-1)R, s])) \\ &\quad + \sum_{i < j < \ell} \mathbb{E} A(I_i; I_j) + \sum_{j < \ell} \mathbb{E} A([(\ell-1)R, s]; I_j) \\ &\leq \sum_{j=1}^K E_j + \sum_{i < j \leq K} \mathbb{E} A(I_i; I_j) \\ &= \sum_{j=1}^K E_j + \sum_{j=1}^K \mathbb{E} A([0, (j-1)R]; I_j). \end{aligned}$$

By Proposition 2.3 and Remark 2.1 the last term on the last line is bounded by

$$\sum_{j=1}^K \frac{1}{2\pi} [jR \log(jR) - (j-1)R \log((j-1)R) - R \log R],$$

which is easily seen to be equal to

$$\frac{1}{2\pi} t \log \log \log t.$$

Then for $\varepsilon > 0$ and t large enough, we have

$$\begin{aligned}
& \mathbb{P}(\sup_{s \leq t} D_s > (1 + 2\varepsilon) \frac{1}{2\pi} t \log \log \log t) \\
& \leq \mathbb{P}(\sum_{j=1}^K E_j > \varepsilon t \log \log \log t) \\
& = \mathbb{P}(\sum_{j=1}^K \frac{E_j}{R} > \frac{\varepsilon t \log K}{R}) \\
& = \mathbb{P}(\sum_{j=1}^K \frac{E_j}{R} > \varepsilon K \log K) \\
& \leq c_6 e^{-\varepsilon K \log K} \mathbb{E} e^{\sum_{j=1}^K E_j/R} \\
& = c_6 e^{-\varepsilon K \log K} (\mathbb{E} e^{E_1/R})^K,
\end{aligned}$$

using the independence of the E_j . Since E_1/R is equal in law to $\sup_{s \leq 1} D_s$, then by Proposition 5.1 the above is bounded by

$$c_6 e^{-\varepsilon K \log K} (c_7)^K.$$

If we take t large enough, we have the bound

$$c_8 e^{-2K}.$$

We apply this with $t_n = q^n$ with $q > 1$ close to 1 so that $(1 + 3\varepsilon)/q > 1 + 2\varepsilon$. Since $e^{-2 \log \log t_n} = O(n^{-2})$, we have

$$\mathbb{P}(\sup_{s \leq t_n} D_s > (1 + 3\varepsilon) \frac{1}{2\pi} t_n \log \log \log t_n) \leq \frac{c_9}{n^2}$$

for n large. If $D_s > (1 + 4\varepsilon) \frac{1}{2\pi} s \log \log \log s$ for some $s \in [t_{n-1}, t_n]$, then it follows that $\sup_{s \leq t_n} D_s > \frac{1+3\varepsilon}{q} \frac{1}{2\pi} t_n \log \log \log t_n$. By Borel-Cantelli, it follows that

$$\limsup_{t \rightarrow \infty} \frac{D_t}{t \log \log \log t} \leq \frac{1}{2\pi} (1 + 4\varepsilon), \quad \text{a.s.}$$

Since ε is arbitrary, our result follows. □

We now turn to the lower bound.

Lemma 5.3. *The quantity*

$$\mathbb{P}\left(\sup_{x \in B(0,3), r < 2} \frac{1}{r} \int_0^1 1_{B(x,r)}(X_s) ds > \lambda\right)$$

tends to 0 as $\lambda \rightarrow \infty$.

Proof. Let

$$U_t(x, r) = \int_0^t 1_{B(x,r)}(X_s) ds.$$

By symmetry the expectation of $\mathbb{E}^y U_1(x, r)$ will be largest when $y = x$. We have

$$\mathbb{E}^x \int_0^1 1_{B(x,r)}(X_s) ds = \int_0^1 \int_{B(x,r)} \frac{1}{2\pi s} e^{-|z-x|^2/2s} dz ds,$$

and a straightforward calculation shows this is bounded by $c_1 r^2 (1 + \log^+(1/r))$, where c_1 can be chosen to be independent of x and r . Then by the Markov property

$$\mathbb{E}[U_1(x, r) - U_t(x, r) \mid \mathcal{F}_t] = \mathbb{E}^{X_t} U_{1-t}(x, r) \leq c_1 r^2 (1 + \log^+(1/r)).$$

By [B], Theorem I.6.11, since $U_t(x, r)$ has continuous paths and is nondecreasing, there exists c_2 such that

$$\mathbb{E} e^{c_2 U_1(x,r)/r^2(1+\log^+(1/r))} \leq 2. \quad (5.3)$$

Set $r_k = 2^{-k}$ and let \mathcal{A}_k be the set of points in $B(0, 4)$ such that each coordinate is an integer multiple of 2^{-k} . The cardinality of \mathcal{A}_k is less than $c_3 2^{2k}$. By Chebyshev's inequality,

$$\mathbb{P}\left(\sup_{x_k \in \mathcal{A}_k} \frac{1}{r_k} U_1(x, r_k) > \lambda/4\right) \leq c_4 2^{2k} e^{-c_5 \lambda / (r_k (1 + \log^+(1/r_k)))}.$$

This is summable in k , so

$$\mathbb{P}\left(\sup_{k \geq -1} \sup_{x_k \in \mathcal{A}_k} \frac{1}{r_k} U_1(x, r_k) > \lambda/4\right)$$

tends to 0 as $\lambda \rightarrow 0$. If $x \in B(0, 3)$ and $r < 2$, then $B(x, r) \subset B(x_k, r_k)$ for some $x_k \in \mathcal{A}_k$ and some k such that $r_k/4 \leq r \leq r_k$. Our result now follows. \square

Lemma 5.4. *Suppose μ is a measure supported in $B(0, 2)$ such that for all $r \leq 2$ and all $x \in \mathbb{R}^2$ we have $\mu(B(x, r)) \leq c_1 r$. There exists c_2 such that for all $x \in \mathbb{R}^2$*

$$\int_0^1 \int p(s, x, y) \mu(dy) ds \leq c_1 c_2,$$

where $p(s, x, y) = \frac{1}{2\pi s} e^{-|x-y|^2/2s}$ is the transition density of two dimensional Brownian motion.

Proof. (cf. [BKhl]) Substituting $t = |x - y|^2/2s$ shows that

$$\int_0^1 p(s, x, y) ds \leq c_3(1 + \log^+(1/|x - y|)).$$

We then use Fubini's theorem to write

$$\begin{aligned} \int_0^1 \int p(s, x, y) \mu(dy) ds &\leq c_4 \sum_{k=-1}^{\infty} \int_{B(x, 2^{-k}) \setminus B(x, 2^{-k-1})} (1 + \log^+(1/|x - y|)) \mu(dy) \\ &\leq c_5 \sum_{k=-1}^{\infty} (2 + k) \mu(B(x, 2^{-k})) \\ &\leq c_6 c_1 \sum_{k=-1}^{\infty} (2 + k) 2^{-k} \leq c_7 c_1, \end{aligned}$$

as required. □

Theorem 5.5. *We have*

$$\limsup_{t \rightarrow \infty} \frac{-\beta_t}{t \log \log \log t} \geq \frac{1}{2\pi}, \quad \text{a.s.}$$

Proof. Let $K = [b \log \log t]$ and $R = t/K$, where b will be chosen later. Let $I_j = [(j-1)R, jR]$. Let $\mathcal{G}_j = \sigma(X_s : s \leq jR)$.

By (2.5) we have

$$\begin{aligned} -\beta_t &= \sum_{j=1}^K -B(I_j) - \sum_{j=1}^K A(I_j; [0, (j-1)R]) + \sum_{j=1}^K \mathbb{E} A(I_j; [0, (j-1)R]) \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Recall that $A(I_j; [0, (j-1)R])$ is equal in law to $\alpha(R, (j-1)R)$. By Proposition 2.3 and Remark 2.1,

$$J_3 = \sum_{j=1}^K \frac{1}{2\pi} [jR \log(jR) - R \log R - (j-1)R \log((j-1)R)],$$

and it is straightforward to see that this is equal to

$$\frac{1}{2\pi} t \log K.$$

Define the following sets.

$$D_{j1} = \{X_{jR} \in B(j\sqrt{R}, \sqrt{R}/16)\},$$

$$D_{j2} = \{X(I_j) \subset [(j-1)\sqrt{R} - (\sqrt{R}/8), j\sqrt{R} + (\sqrt{R}/8)] \times [-\sqrt{R}/8, \sqrt{R}/8]\},$$

$$D_{j3} = \{B(I_j) \leq \kappa_1 R\},$$

$$D_{j4} = \left\{ \int_{(j-1)R}^{jR} 1_{B(x, r\sqrt{R})}(X_s) ds \leq \kappa_2 rR \text{ for all } x \in B(j\sqrt{R}, 3\sqrt{R}), 0 < r < 2\sqrt{R} \right\},$$

$$D_{j5} = \{A(I_{j-1}; I_j) \leq \kappa_3 R\}.$$

$\kappa_1, \kappa_2, \kappa_3$ are constants which will be chosen later and will not depend on j, b, t, R . Let

$$C_j = D_{j1} \cap D_{j2} \cap D_{j3} \cap D_{j4} \cap D_{j5}$$

and

$$E = \bigcap_{j=1}^K C_j.$$

We want to show

$$\mathbb{P}(C_j \mid \mathcal{G}_{j-1}) \geq c_1 \tag{5.4}$$

on the set $C_1 \cap \dots \cap C_{j-1}$, where $c_1 > 0$ does not depend on j, b, t, R . Once we have (5.4), then

$$\mathbb{P}\left(\bigcap_{i=1}^m C_i\right) = \mathbb{E}\left(\mathbb{P}(C_m \mid \mathcal{G}_{m-1}); \bigcap_{i=1}^{m-1} C_i\right) \geq c_1 \mathbb{P}\left(\bigcap_{i=1}^{m-1} C_i\right),$$

and by induction,

$$\mathbb{P}\left(\bigcap_{i=1}^K C_i\right) \geq c_1^K = c_1^{b \log \log t} = e^{b \log \log t \log c_1}. \tag{5.5}$$

On the set E we have

$$-J_1 = \sum_{j=1}^K B(I_j) \leq \kappa_1 KR = \kappa_1 t.$$

Since for each j we are on the set D_{j2} , then on the event E we have $X(I_i)$ disjoint from $X(I_j)$ if $|i - j| > 1$. Therefore

$$-J_2 = \sum_{j=1}^K A(I_{j-1}; I_j) \leq \kappa_3 KR = \kappa_3 t.$$

So on E we have altogether that

$$J_1 + J_2 + J_3 \geq \frac{1}{2\pi} t \log \log \log t - (\kappa_1 + \kappa_3)t.$$

We now proceed to show (5.4). By the support theorem for planar Brownian motion and scaling (see [B], Theorem I.6.6),

$$\mathbb{P}(D_{j1} \cap D_{j2} \mid \mathcal{G}_{j-1}) > c_2.$$

By scaling,

$$\mathbb{E} B(I_j) = R \mathbb{E} B([0, 1]) \leq c_3 R.$$

So if κ_1 is chosen large enough,

$$\mathbb{P}(B(I_j) > \kappa_1 R) < c_2/6.$$

Now let us look at D_{j4} . By scaling and Lemma 5.3 it follows that $\mathbb{P}(D_{j4}) \leq c_2/6$ if we choose κ_2 large enough.

Next we look at D_{j5} . We have an estimate on $\mathbb{E} A(I_{j-1}; I_j)$, but what we actually need is an estimate on $\mathbb{E}[A(I_{j-1}; I_j) \mid \mathcal{G}_{j-1}]$. To show

$$\mathbb{P}(A(I_{j-1}; I_j) > \kappa_3 R \mid \mathcal{G}_{j-1}) \leq c_2/6$$

if κ_3 is large enough, it is enough to show

$$\mathbb{E}[A(I_{j-1}; I_j) \mid \mathcal{G}_{j-1}] \leq c_4 R \tag{5.6}$$

on the set $\cap_{i=1}^{j-1} C_i$. By [BK2], we can let μ be the measure on \mathbb{R}^2 defined by

$$\mu(F) = \int_0^1 1_F(X_s) ds$$

and consider $A([0, 1]; [1, 2])$ as an additive functional of Brownian motion corresponding to the measure μ . So to show (5.6), by scaling and translation invariance it is enough to show

$$\mathbb{E}[A([0, 1]; [1, 2]) \mid \mathcal{F}_1] \leq c_5 \tag{5.7}$$

on the set where

$$\mu(B(x, r)) \leq \kappa_1 r$$

for all $x \in B(0, 3)$ and all $r \in (0, 2)$. The conditional expectation (5.7) is bounded by

$$\sup_y \int_0^1 \int p(s, y, z) \mu(dz) ds,$$

where $p(s, y, z)$ is the transition density for planar Brownian motion. Using Lemma 5.4 we have (5.7).

Setting $c_6 = c_2/2$ gives the desired lower bound (5.4).

Let $t_n = e^{n^\gamma}$ for some $\gamma > 1$ and let $\varepsilon > 0$. Provided we take b (in the definition of K) small enough, Borel-Cantelli tells us that

$$-B([t_{n-1}, t_n]) \geq (\frac{1}{2\pi} - \varepsilon)(t_n - t_{n-1}) \log \log \log(t_n - t_{n-1}) \quad (5.8)$$

infinitely often with probability 1. By (2.5) we have

$$\begin{aligned} -B([0, t_n]) &= -B([0, t_{n-1}]) - B([t_{n-1}, t_n]) - A([0, t_{n-1}]; [t_{n-1}, t_n]) \\ &\quad + \mathbb{E} A([0, t_{n-1}]; [t_{n-1}, t_n]). \end{aligned} \quad (5.9)$$

By the upper bound for $-\beta_t$ from Theorem 4.3, we know

$$B([0, t_{n-1}]) = O(t_{n-1} \log \log \log t_{n-1}) = o(t_n \log \log \log t_n) \quad a.s. \quad (5.10)$$

By scaling

$$\tilde{A}_n = A([0, t_{n-1}]; [t_{n-1}, t_n]) \quad (5.11)$$

is equal in law to $\Delta_n \alpha(1, t_{n-1}/\Delta_n)$, where $\Delta_n = t_n - t_{n-1}$. By Lemma 4.1,

$$\mathbb{P}(\tilde{A}_n > t_n) \leq \exp\left(-c_8 \frac{t_n}{\Delta_n} \sqrt{\Delta_n/t_{n-1}}\right),$$

which is summable. Using Borel-Cantelli, we have

$$\tilde{A}_n = o(t_n \log \log \log t_n) \quad a.s. \quad (5.12)$$

Substituting this, (5.8), (5.10), and (5.12) in (5.9) proves the theorem. \square

Remark 5.6. Theorem 1.5 follows immediately from Theorems 5.4 and 5.5.

The following Corollary 5.7 completes the proof of Theorem 1.3.

Corollary 5.7. *Let L be as in the statement of Theorem 3.2. Then $L < \infty$.*

Proof. In the proof of Theorem 5.5 we showed that the event E had probability at least $e^{b \log \log t \log c_1}$ and that on the event E we had $-\beta_t \geq \frac{1}{2\pi} t \log \log \log t - c_2 t$ provided t was large enough. Choose t so that $\frac{1}{2\pi} \log \log \log t - c_2 = \log s$. Using scaling, we then have

$$\mathbb{P}(-\beta_1 > \log s) \geq e^{-c_3 \log \log t} = e^{-c_4 s^{2\pi}}.$$

Now take logarithms of both sides and divide by $s^{2\pi}$. \square

Remark 5.8. Theorem 1.3 follows immediately from Theorem 3.2 and Corollary 5.7.

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