AN ALMOST SURE INVARIANCE PRINCIPLE FOR RENORMALIZED INTERSECTION LOCAL TIMES

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Abstract. Let $\widetilde{\beta}_k(n)$ be the number of self-intersections of order k, appropriately renormalized, for a mean zero random walk X_n in $\mathbb{Z}^2$ with $2+\delta$ moments. On a suitable probability space we can construct X_n and a planar Brownian motion W_t such that for each $k \geq 2$

$$|\widetilde{\beta}_k(n) - \widetilde{\gamma}_k(n)| = O(n^{-a}), \qquad \text{a.s.}$$

for some $a > 0$ where $\widetilde{\gamma}_k(n)$ is the renormalized self-intersection local time of order k at time 1 for the Brownian motion $W_{nt}/\sqrt{n}$.

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1. Introduction.

If $\{W_t\,;\,t\geq 0\}$ is a planar Brownian motion with density $p_t(x)$, for $x\in\mathbb{R}^2$ set $\gamma_{1,\epsilon}(t,x)=t$ and for $k\geq 2$ and $x=(x_2,\ldots,x_k)\in(\mathbb{R}^2)^{k-1}$ let

$$\gamma_{k,\epsilon}(t,x)=\int_{0\leq t_1\leq\cdots\leq t_k<t}\prod_{i=2}^{k}p_\epsilon(W_{t_i}-W_{t_{i-1}}-x_i)dt_1\cdots dt_k.$$

When $x_i\neq 0$ for all i the limit

$$\gamma_k(t,x)=\lim_{\epsilon\to 0}\gamma_{k,\epsilon}(t,x)$$

exists and for any bounded continuous function $F(x)$ on $\mathbb{R}^{2(k-1)}$ we have

$$\int F(x)\gamma_k(t,x)\,dx=\int_{0\leq t_1\leq\cdots\leq t_k<t}F(W_{t_2}-W_{t_1},\ldots,W_{t_k}-W_{t_{k-1}})dt_1\cdots dt_k. \tag{1.1}$$

(Here we may arbitrarily specify that $\gamma_k(t,x)=\infty$ if any $x_i=0$.) When $x_i\neq 0$ for all i define the renormalized intersection local times as

$$\widetilde{\gamma}_k(t,x)=\sum_{A\subseteq\{2,\ldots,k\}}(-1)^{|A|}\Big(\prod_{i\in A}\frac{1}{\pi}\log(1/|x_i|)\Big)\gamma_{k-|A|}(t,x_{A^c})$$

where $x_{A^c}=(x_{i_1},\ldots,x_{i_{k-|A|}})$ with $i_1<i_2<\cdots<i_{k-|A|}$ and $i_j\in\{2,\ldots,k\}-A$ for each j, that is, the vector $(x_2,\ldots,x_k)$ with all terms that have indices in A deleted. (When $A=\{2,\ldots,k\}$ so that $k-|A|=1$ and $A^c=\emptyset$ we set $\gamma_{k-|A|}(t,x_{A^c})=\gamma_{1,\epsilon}(t)=t$). It is known that the $\widetilde{\gamma}_k(t,x)$ have a continuous extension to all $\mathbb{R}^1_+\times\mathbb{R}^{k-1}$; see [3].

Renormalized self-intersection local time was originally studied by Varadhan [20] for its role in quantum field theory. In Rosen [18] it is shown that $\widetilde{\gamma}_k(t,0)$ can be characterized as the continuous process of zero quadratic variation in the decomposition of a natural Dirichlet process. Renormalized intersection local time turns out to be the right tool for the solution of certain "classical" problems such as the asymptotic expansion of the area of the Wiener sausage in the plane and the range of random walks, [5], [9], [10]. For further work on renormalized self-intersection local times see Dynkin [7], Le Gall [11], Bass and Khoshnevisan [3], Rosen [17] and Marcus and Rosen [14].

Let ξ_i be i.i.d. random variables with values in $\mathbb{Z}^2$ that are mean 0, with covariance matrix equal to the identity, and with $2+\delta$ moments. Let us suppose the ξ_i are symmetric and are strongly aperiodic. Let X_n be the random walk, that is, $X_n=\sum_{i=1}^n\xi_i$. Let $p(n,x,y)$ be the transition probabilities. Let $B_1(n,x)=n$ and for $x\in\mathbb{Z}^2$ set

$$B_2(n,x)=\sum_{0\leq i_1<i_2\leq n}1_{(X_{i_2}=X_{i_1}+x)}.$$

More generally, for $x=(x_2,\dots,x_k)\in(\mathbb{Z}^2)^{k-1}$ let

$$B_k(n,x)=\sum_{0\le i_1<i_2<\dots<i_k\le n}\prod_{j=2}^{k}1_{(X_{i_j}=X_{i_{j-1}}+x_j)}.$$

Note that $B_k(n,x)=0$ for all $n<k-1$.

With $e_1=(1,0)$, let

$$G(x)=\sum_{n=1}^{\infty}[p(n,0,x)-p(n,0,e_1)],$$

and set $G_n(x)=G(x)-G(\sqrt{n}e_1)$. Let

$$\widetilde{B}_k(n,x)=\sum_{A\subset\{2,\dots,k\}}(-1)^{|A|}\Big(\prod_{i\in A}G_n(x_i)\Big)B_{k-|A|}(n,x_{A^c}). \tag{1.2}$$

In particular we have

$$\widetilde{B}_2(n,x)=B_2(n,x)-G_n(x)n. \tag{1.3}$$

Finally we define the renormalized intersection local times for our random walk by

$$\widetilde{\beta}_k(n,x)=\frac{1}{n}\widetilde{B}_k(n,x\sqrt{n}). \tag{1.4}$$

In particular we have

$$\widetilde{\beta}_2(n,x)=\frac{1}{n}B_2(n,x\sqrt{n})-G_n(x\sqrt{n}). \tag{1.5}$$

We note from P12.3 of [19] that for $x\neq 0$

$$\lim_{n\to\infty}G_n(x\sqrt{n})=\lim_{n\to\infty}G(x\sqrt{n})-G(\sqrt{n}e_1)=\frac{1}{\pi}\log(1/|x|). \tag{1.6}$$

We know we can find a version of our random walk and a Brownian motion W_t such that

$$\sup_{s\le 1}|X^n_s-W^n_s|=o(n^{-\zeta}),\qquad a.s. \tag{1.7}$$

for some $\zeta>0$ where $X^n_t=X_{[nt]}/\sqrt{n}$ and $W^n_t=W_{nt}/\sqrt{n}$; see, [6], Theorem 3, for example. Let $\gamma_k(1,x,n)$ and $\widetilde{\gamma}_k(1,x,n)$ be the intersection local times and renormalized intersection local times up to time 1 of order k, resp., for the Brownian motion W^n_t. In this paper we prove the following theorem.

Theorem 1.1. *Let $X_n = \xi_1 + \cdots + \xi_n$ be a random walk in $\mathbb{Z}^2$, where the ξ_i are i.i.d., mean 0, with covariance matrix equal to the identity, with $2+\delta$ moments for some $\delta > 0$, symmetric, and strongly aperiodic. On a suitable probability space we can construct $\{X_n\,;\, n \geq 1\}$ and a planar Brownian motion $\{W_t\,;\, t \geq 0\}$ and we can find $\eta > 0$ such that for each $k \geq 2$*

$$|\widetilde{\beta}_k(n,0) - \widetilde{\gamma}_k(1,0,n)| = o(n^{-\eta}), \qquad a.s.$$

For related work see [4], [5], [16]. In fact, [5] provided much of the motivation for this work; in that paper we proved a strong invariance principle with respect to the L^2 norm.

We give a brief overview of the proof. There is an equation similar to (1.1) when γ_k is replaced by $\widetilde{\gamma}_k$, and also when it is replaced by $\widetilde{\beta}_k$. Since by (1.7) we have X^n_s close to W^n_s for n large, we are able to conclude that $\int F(x)\widetilde{\gamma}_k(1,x)\,dx$ is close to $\int F(x)\widetilde{\beta}_k(n,x)\,dx$ for n large. If F is smooth, with integral 1 and support in a small neighborhood of the origin, then by the continuity of $\widetilde{\gamma}_k(t,x)$ in x, which is proved in [3], we see that $\int F(x)\widetilde{\gamma}_k(1,x)\,dx$ is not far from $\widetilde{\gamma}_k(1,0)$. If we had a similar result for $\widetilde{\beta}_k$, we would then have that $\int F(x)\widetilde{\beta}_k(n,x)\,dx$ is not far from $\widetilde{\beta}(n,0)$, and we would have our proof. So our strategy is to obtain good estimates on $|\widetilde{\beta}_k(n,x) - \widetilde{\beta}_k(n,0)|$. Because of the rate of convergence in (1.7), it turns out we are able to avoid having to find the sharpest estimates on the difference, which simplifies the proof considerably.

Our main tool in obtaining the desired estimates is Proposition 3.2. This proposition may be of independent interest. It has been known for a long time that one way of proving L^p estimates for a continuous increasing process is to prove corresponding estimates for the potential. It is not as well known that one can do the same for continuous processes of bounded variation provided one has some control on the total variation; see, e.g., [1] or [3]. Proposition 3.2 is the discrete time analogue of this result, and is proved in a similar way. Unlike the continuous time version, here it is also necessary to have control on the differences of successive terms.

Section 2 has some estimates on the potential kernel for random walks in the plane, while Section 3 has the proof of the stochastic calculus results we need. Theorem 1.1 in the case when $k = 2$ is proved in Section 4, with the proofs of some lemmas postponed to Section 5. We treat the case $k = 2$ separately for simplicity of exposition. The description of the potentials of intersection local times of random walks in the $k > 2$ case is a bit different than in the $k = 2$ case and this is described in Section 6. Theorem 1.1 in the $k > 2$ case is proved in Section 7, with the proofs of some lemmas given in Sections 8 and 9. Finally in Section 10 we give an extension of our results to L^2 convergence, and more importantly, make a correction to the proof of one of the propositions in [3]. An Appendix contains the detailed proof of that correction. Throughout this paper we use the letter c to denote finite positive constants whose exact value is unimportant and which may vary from line to line.

2. Estimates for random walks.

In this section we prove some estimates for the potential kernel of a random variable. See the forthcoming book by Lawler [13] for further information. Let $\overline{G}$ be the potential kernel for X. Recall that in 2 dimensions, since X is recurrent, the potential kernel is defined somewhat differently than in higher dimensions, and is defined by

$$\overline{G}(x) = \sum_{n=0}^{\infty} [p(n,0,x) - p(n,0,e_1)],$$

where $e_1 = (1,0)$. (Note e_1 can be replaced by any fixed point.) For us it will be more convenient to work with

$$G(x) = \sum_{n=1}^{\infty} [p(n,0,x) - p(n,0,e_1)],$$

which, since $p(0,0,0) = 1$ and $p(0,0,e_1) = 0$, differs from $\overline{G}(x)$ by $1_{\{0\}}(x)$. By Spitzer [19], p. 75, we have

$$p(n,0,x) \leq c/n. \tag{2.1}$$

By [4], Proposition 2.1, if the ξ_i are strongly aperiodic, then

$$|p(n,0,x) - p(n,0,y)| \leq \frac{c|x-y|}{n^{3/2}}. \tag{2.2}$$

Proposition 2.1. *Suppose the ξ_i have $2+\delta$ moments. Then $G(x)$ exists and $|G(x)| \leq c(1+\log^+|x|)$.*

Proof. Using (2.2), we have that

$$|G(0)| \leq \sum_{n=1}^{\infty} \frac{c}{n^{3/2}}$$

is finite. The rest of the assertions follow from

$$\begin{aligned} |G(x)| &\leq \sum_{n=1}^{|x|^2} \frac{c}{n} + \sum_{n=|x|^2+1}^{\infty} \frac{c|x-e_1|}{n^{3/2}} \\ &\leq c + c\log|x| + c\frac{|x|}{(|x|^2+1)^{1/2}}. \end{aligned}$$

□

Proposition 2.2. *For some $c < \infty$*

$$|G(x) - G(y)| \leq c\Big(\frac{|x-y|}{(1+|x|) \wedge (1+|y|)}\Big)^{2/3}, \qquad x, y \in \mathbb{Z}^2.$$

Proof. By [19], P7.10,

$$p(j, 0, x) \leq \frac{c}{|x|^2}. \tag{2.3}$$

Since $p(j, 0, 0) \leq 1$, then we have

$$p(j, 0, x) \leq \frac{c}{1+|x|^2}. \tag{2.4}$$

Suppose $0 < |x| \leq |y|$. Let us set R in a moment. Using (2.4) for $j \leq R$ and (2.2) for $j > R$, we have that

$$|G(x) - G(y)| \leq \sum_{j=1}^{R} \frac{c}{1+|x|^2} + \sum_{j=R+1}^{\infty} \frac{c|x-y|}{j^{3/2}} \leq \frac{cR}{1+|x|^2} + \frac{c|x-y|}{R^{1/2}}.$$

If we select R so that

$$\frac{R}{1+|x|^2} = \frac{|x-y|}{R^{1/2}}, \qquad \text{i.e.,} \qquad R^{3/2} = (1+|x|^2)(|x-y|),$$

the result follows. Since $G(0)$ is finite and $|G(x)| \leq c \log(1+|x|) \leq |x|^{2/3}$, the result holds when either x or y is 0, as well. □

Lemma 2.3. *For some constant κ and any $\rho < \delta/2$,*

$$G(x) = \kappa + \tfrac{1}{\pi} \log(1/|x|) + O(|x|^{-\rho}), \qquad x \in \mathbb{Z}^2.$$

Proof. Let us begin with the proof of Proposition 3.1 in [2]. We have for $\delta > 0$

$$\Big| e^{i\alpha \cdot x/\sqrt{n}} - 1 - \frac{\alpha \cdot x}{\sqrt{n}} - \frac{|\alpha \cdot x|^2}{2n} \Big| \leq c \Big| \frac{\alpha \cdot x}{\sqrt{n}} \Big|^{2+\delta}.$$

So if ϕ is the characteristic function of a random vector with finite $2+\delta$ moments, mean 0, and the identity as its covariance matrix, then

$$\phi(\alpha/\sqrt{n}) = 1 - \frac{|\alpha|^2}{2n} + E_1(\alpha, n),$$

with

$$|E_1(\alpha, n)| \leq c(|\alpha|/\sqrt{n})^{2+\delta}. \tag{2.5}$$

Applying this also for the characteristic function of a standard normal vector,

$$e^{-|\alpha|^2/2n} = 1 - \frac{|\alpha|^2}{2n} + E_2(\alpha, n),$$

where $E_2(\alpha, n)$ has the same bound as $E_1(\alpha, n)$. If we use this in place of the display in the middle of page 473 of [2], we obtain

$$I_1^{(n)} \le cn^{-\delta/2}(\log n)^{(4+\delta)/2}.$$

So if $E(n,x) = |p(n,0,x) - (2\pi n)^{-1}e^{-|x|^2/2n}|$, following the proof in [2] we obtain

$$\sup_x E(n,x) \le cn^{-1-(\delta/2)}(\log n)^{(4+\delta)/2}.$$

Let us choose $\delta' < \delta$. We then have

$$\sup_x E(n,x) \le cn^{-1-(\delta'/2)}. \tag{2.6}$$

Recall $G(x) = \sum_{k=1}^{\infty}[p(k,0,x) - p(k,0,e_1)]$. It is shown in the proof of Theorem 1.6.2 in [12] that for some constant κ

$$\sum_{k=1}^{\infty}[q(k,0,x) - q(k,0,e_1)] = \kappa + \tfrac{1}{\pi}\log(1/|x|) + O(|x|^{-1}),$$

where $q(k,x,y) = (2\pi k)^{-1}e^{-|x-y|^2/2k}$. Thus, to prove the lemma, it suffices to prove

$$\sum_{k=1}^{\infty}|p(k,x,0) - q(k,x,0)| = O(|x|^{-\rho}) \tag{2.7}$$

for any $\rho < \delta/2$.

To establish (2.7), use [15], p. 60 to observe that

$$p(k,x,0) \le \mathbb{P}(|X_k| > |x|) \le \frac{\mathbb{E}\,|X_k|^{2+\delta}}{|x|^{2+\delta}} \le c\frac{k^{1+\delta/2}}{|x|^{2+\delta}}$$

and a similar estimate is easily seen to hold for $q(k,x,0)$. Therefore, using (2.6) and setting $R = |x|$,

$$\begin{aligned}\sum_{k=1}^{\infty}|p(k,x,0) - q(k,x,0)| &\le \sum_{k=1}^{R}[p(k,x,0) + q(k,x,0)]\\ &\qquad + \sum_{k=R}^{\infty}|p(k,x,0) - q(k,x,0)|\\ &\le \sum_{k=1}^{R} c\frac{k^{1+\delta/2}}{|x|^{2+\delta}} + \sum_{k=R}^{\infty} c\frac{1}{k^{1+\delta'/2}}\\ &\le c\frac{R^{2+\delta/2}}{|x|^{2+\delta}} + c\frac{1}{R^{\delta'/2}}\\ &\le c|x|^{-\rho}.\end{aligned}$$

□

3. Stochastic calculus.

We will use the following propositions; these may be of independent interest. Propositions 3.1 and 3.2 and their proofs are the discrete time analogues of Propositions 6.1 and 6.2 of [3].

Proposition 3.1. *Let A_n be an adapted increasing sequence of random variables with $A_0 = 0$ and $A_\infty = \sup_n A_n$ finite. Suppose that*

$$Y = \sup_n (A_n - A_{n-1})$$

and W is a random variable such that

$$\mathbb{E}\,[A_\infty - A_n \mid \mathcal{F}_n] \le \mathbb{E}\,[W \mid \mathcal{F}_n]$$

for all n. Then for each integer p larger than 1 there exists a constant c such that

$$\mathbb{E}\,A_\infty^p \le c p^p (\|W\|_p + \|Y\|_p)^p.$$

Proof. Since A_n is increasing,

$$pA_n^{p-1} \ge A_n^{p-1} + A_n^{p-2} A_{n-1} + \cdots + A_n A_{n-1}^{p-2} + A_{n-1}^{p-1}.$$

Multiplying by $A_n - A_{n-1}$, we obtain

$$(A_n - A_{n-1}) p A_n^{p-1} \ge A_n^p - A_{n-1}^p.$$

Summing over n we obtain

$$p \sum_{n=1}^{\infty} (A_n - A_{n-1}) A_n^{p-1} \ge A_\infty^p. \tag{3.1}$$

On the other hand, applying the general summation formula

$$A_\infty B_\infty = \sum_{n=1}^{\infty} A_n (B_n - B_{n-1}) + \sum_{n=1}^{\infty} (A_n - A_{n-1}) B_{n-1}$$

with $B_n = A_{n+1}^{p-1}$ we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} (A_n - A_{n-1})\, A_n^{p-1} &= A_\infty^p - \sum_{n=1}^{\infty} A_n (A_{n+1}^{p-1} - A_n^{p-1}) \qquad (3.2) \\ &= \sum_{n=1}^{\infty} (A_\infty - A_n)(A_{n+1}^{p-1} - A_n^{p-1}) + A_\infty A_1^{p-1}. \end{aligned}$$

Here we used the fact that

$$\sum_{n=1}^{\infty} A_\infty (A_{n+1}^{p-1} - A_n^{p-1}) = A_\infty^p - A_\infty A_1^{p-1}.$$

Combining (3.1) and (3.2) we obtain

$$A_\infty^p \le p \sum_{n=1}^{\infty} (A_\infty - A_n)(A_{n+1}^{p-1} - A_n^{p-1}) + p A_\infty A_1^{p-1}. \tag{3.3}$$

Now suppose for the moment that Y is bounded and $A_n = A_{n_0}$ for all $n \ge n_0$ for some n_0. We have

$$\begin{aligned} &\sum_{n=1}^{\infty} (A_\infty - A_n)(A_{n+1}^{p-1} - A_n^{p-1}) \qquad (3.4)\\ &= \sum_{n=1}^{\infty} (A_\infty - A_{n+1})(A_{n+1}^{p-1} - A_n^{p-1}) + \sum_{n=1}^{\infty} (A_{n+1} - A_n)(A_{n+1}^{p-1} - A_n^{p-1}). \end{aligned}$$

and

$$\sum_{n=1}^{\infty} (A_{n+1} - A_n)(A_{n+1}^{p-1} - A_n^{p-1}) \le Y A_\infty^{p-1}.$$

But, $A_1 \le A_\infty$ and also $A_1 \le Y$ so that

$$A_\infty A_1^{p-1} \le Y A_\infty^{p-1}.$$

We write

$$\begin{aligned} \mathbb{E} \sum_{n=1}^{\infty} (A_\infty - A_{n+1})(A_{n+1}^{p-1} - A_n^{p-1}) &= \mathbb{E} \sum_{n=1}^{\infty} \mathbb{E}\,[A_\infty - A_{n+1} \mid \mathcal{F}_{n+1}](A_{n+1}^{p-1} - A_n^{p-1}) \\ &\le \mathbb{E} \sum_{n=1}^{\infty} \mathbb{E}\,[W \mid \mathcal{F}_{n+1}](A_{n+1}^{p-1} - A_n^{p-1}) \\ &= \mathbb{E} \sum_{n=1}^{\infty} W (A_{n+1}^{p-1} - A_n^{p-1}) \\ &\le \mathbb{E}\,[W A_\infty^{p-1}]. \end{aligned}$$

Therefore using Hölder's inequality,

$$\mathbb{E}\, A_\infty^p \le p(\|W\|_p + 2\|Y\|_p)(\mathbb{E}\, A_\infty^p)^{1-\frac{1}{p}}.$$

Our temporary assumptions on A allow us to divide both sides by $(\mathbb{E}\, A_\infty^p)^{1-\frac{1}{p}}$ to obtain our result in this special case.

In general, look at

$$A'_n = \sum_{j=1}^{n} ((A_j - A_{j-1}) \wedge K)$$

and apply the above to $A''_n = A'_{n \wedge n_0}$; note that A'' will satisfy the hypotheses with the same W and Y. Then let $K \uparrow \infty$ and next $n_0 \uparrow \infty$ and use monotone convergence. □

Proposition 3.2. *Suppose Q^1_n and Q^2_n are two adapted nonnegative increasing sequences. Suppose $Q_n = Q^1_n - Q^2_n$, $H_n + Q_n$ is a martingale that is 0 at time 0,*

$$Z = \sup_n |H_n|,$$

$$Y = \sup_n [(Q^1_n - Q^1_{n-1}) + (Q^2_n - Q^2_{n-1})],$$

and

$$W = Q^1_\infty + Q^2_\infty.$$

Then there exists c such that for $p > 1$

$$\mathbb{E} \sup_n |Q_n|^{2p} \le c^p p^{4p} \Big[\mathbb{E}\, Z^{2p} + (\mathbb{E}\, Z^{2p})^{1/2} (\mathbb{E}\, W^{2p})^{1/2} + (\mathbb{E}\, Y^{2p})^{1/2} (\mathbb{E}\, W^{2p})^{1/2} \Big]. \tag{3.5}$$

Proof. There is nothing to prove unless $\mathbb{E}\, W^{2p} < \infty$. Since $\sup_n Q_n \le W$, all the random variables that follow will satisfy the appropriate integrability conditions. Let us temporarily assume that there exists n_0 such that $Q^i_n = Q^i_{n_0}$ if $n \ge n_0, i = 1, 2$.

Let

$$V_m = \mathbb{E}\,[Q_\infty - Q_m \mid \mathcal{F}_m], \qquad M_m = \mathbb{E}\,[Q_\infty \mid \mathcal{F}_m].$$

Note that $V_\infty = 0$, M_m is a martingale, and $Q_m = M_m - V_m$. In fact, in view of our temporary assumption, $V_m = 0$ if $m \ge n_0$.

Our first observation is that since

$$V_m = \mathbb{E}\,[Q_\infty - Q_m \mid \mathcal{F}_m] = \mathbb{E}\,[H_m - H_\infty \mid \mathcal{F}_m],$$

then

$$|V_m| \le 2\mathbb{E}\,[Z \mid \mathcal{F}_m]. \tag{3.6}$$

By Doob's inequality,

$$\mathbb{E} \sup_n V^p_n \le c\mathbb{E}\, Z^p. \tag{3.7}$$

We will use

Lemma 3.3.

$$\mathbb{E}\Big(\sum_{n=0}^{\infty}(M_{n+1}-M_n)^2\Big)^p \le c^p p^{2p}(\mathbb{E}\,[ZW+YW]^p+\mathbb{E}\,Z^{2p}). \tag{3.8}$$

This lemma will be proved shortly. We first show how Proposition 3.2 follows from this lemma. By the Burkholder-Davis-Gundy inequalities, we obtain

$$\mathbb{E}\,\sup_n |M_n|^{2p} \le cp^{4p}(\mathbb{E}\,[ZW+YW]^p+\mathbb{E}\,Z^{2p}). \tag{3.9}$$

Combining with (3.7) and the fact that $Q_m = M_m - V_m$ and then using Cauchy-Schwarz completes the proof of Proposition 3.2 in the special case where the Q^i are constant from some n_0 on. In the general case, let $\overline{Q}^i_n = Q^i_{n\wedge n_0}$ for $i=1,2$, obtain (3.5) for $\overline{Q}_n = \overline{Q}^1_n - \overline{Q}^2_n$, let $n_0\to\infty$, and apply monotone convergence. □

Proof of Lemma 3.3. We now prove (3.8). Simple algebraic manipulations show that

$$V_\infty^2 - V_m^2 = \sum_{n=m}^{\infty}(V_{n+1}-V_n)^2 + 2\sum_{n=m}^{\infty} V_n(V_{n+1}-V_n). \tag{3.10}$$

(Note that the sums are actually finite because $V_m = 0$ if $m \ge n_0$.) Recalling $Q = M - V$ and $V_\infty = 0$, we have

$$\begin{aligned}\sum_{n=m}^{\infty}(M_{n+1}-M_n)^2 &\le 2\sum_{n=m}^{\infty}(V_{n+1}-V_n)^2 + 2\sum_{n=m}^{\infty}(Q_{n+1}-Q_n)^2\\ &= -2V_m^2 - 4\sum_{n=m}^{\infty}V_n(V_{n+1}-V_n) + 2\sum_{n=m}^{\infty}(Q_{n+1}-Q_n)^2\\ &\le -4\sum_{n=m}^{\infty}V_n(V_{n+1}-V_n) + 2\sum_{n=m}^{\infty}(Q_{n+1}-Q_n)^2\\ &=: -4S_1 + 2S_2. \end{aligned} \tag{3.11}$$

We now take the conditional expectation with respect to $\mathcal{F}_m$.

$$\begin{aligned}
\mathbb{E}\,[S_1 \mid \mathcal{F}_m] &= \mathbb{E}\Big[\sum_{n=m}^{\infty} V_n(V_{n+1}-V_n) \mid \mathcal{F}_m\Big] \\
&= \mathbb{E}\Big[\sum_{n=m}^{\infty} \mathbb{E}\,[V_{n+1}-V_n \mid \mathcal{F}_n] V_n \mid \mathcal{F}_m\Big] \\
&= -\mathbb{E}\Big[\sum_{n=m}^{\infty} \mathbb{E}\,[Q_{n+1}-Q_n \mid \mathcal{F}_n] V_n \mid \mathcal{F}_m\Big] \\
&= -\mathbb{E}\Big[\sum_{n=m}^{\infty} V_n(Q_{n+1}-Q_n) \mid \mathcal{F}_m\Big] \\
&= -\mathbb{E}\Big[\sum_{n=m}^{\infty} V_{n+1}(Q_{n+1}-Q_n) \mid \mathcal{F}_m\Big] \\
&\qquad + \mathbb{E}\Big[\sum_{n=m}^{\infty} (V_{n+1}-V_n)(Q_{n+1}-Q_n) \mid \mathcal{F}_m\Big] \\
&=: I_1 + I_2.
\end{aligned}$$

Since $V_{n+1} \le 2\mathbb{E}\,[Z \mid \mathcal{F}_{n+1}]$ by (3.6), and recalling that $Q_n = Q_n^1 - Q_n^2$, we have

$$|I_1| \le 2\mathbb{E}\Big[\sum_{n=m}^{\infty} Z|Q_{n+1}-Q_n| \mid \mathcal{F}_m\Big] \le 2\mathbb{E}\,[ZW \mid \mathcal{F}_m].$$

Recalling that $V_n = M_n - Q_n$ and setting $S_3 = \sum_{n=m}^{\infty}(M_{n+1}-M_n)(Q_{n+1}-Q_n)$ we see that

$$I_2 = -\mathbb{E}\,[S_2 \mid \mathcal{F}_m] + \mathbb{E}\,[S_3 \mid \mathcal{F}_m].$$

Since

$$\sum_{n=m}^{\infty} (Q_{n+1}-Q_n)^2 \le YW \tag{3.12}$$

we have

$$\mathbb{E}\,[S_2 \mid \mathcal{F}_m] \le \mathbb{E}\,[YW \mid \mathcal{F}_m].$$

Let $J = \mathbb{E}\,[\sum_{n=m}^{\infty}(M_{n+1}-M_n)^2 \mid \mathcal{F}_m]$. By Cauchy-Schwarz and (3.12),

$$|\mathbb{E}\,[S_3 \mid \mathcal{F}_m]| \le J^{1/2}\Big(\mathbb{E}\,[YW \mid \mathcal{F}_m]\Big)^{1/2}.$$

We therefore conclude

$$J \le c\mathbb{E}\,[ZW + YW \mid \mathcal{F}_m] + c_1 J^{1/2}\Big(\mathbb{E}\,[YW \mid \mathcal{F}_m]\Big)^{1/2}. \tag{3.13}$$

Using the inequality $A^{1/2}x^{1/2} \le (A+x)/2$ with $x = J$ and $A = c_1^2 \mathbb{E}[YW \mid \mathcal{F}_m]$, we see that

$$J \le c\mathbb{E}[ZW + YW \mid \mathcal{F}_m] + c_1^2\Big(\mathbb{E}[YW \mid \mathcal{F}_m]\Big)/2 + J/2 \tag{3.14}$$

and therefore

$$J = \mathbb{E}\Big[\sum_{n=m}^{\infty}(M_{n+1}-M_n)^2 \mid \mathcal{F}_m\Big] \le c\mathbb{E}[ZW + YW \mid \mathcal{F}_m]. \tag{3.15}$$

We have $|Q_{n+1} - Q_n| \le Y$ for all n and so

$$\mathbb{E}\sup_n |Q_{n+1} - Q_n|^p \le \mathbb{E}\, Y^p. \tag{3.16}$$

Using (3.7), (3.16) and the fact that $Q_m = M_m - V_m$ and $Y \le (YW)^{1/2}$ we then have that

$$\begin{aligned}\mathbb{E}\sup_n |M_{n+1} - M_n|^{2p} &\le c^p \mathbb{E}\sup_n |Q_{n+1} - Q_n|^{2p} + c^p\mathbb{E}\sup_n |V_{n+1} - V_n|^{2p}\\ &\le c^p(\mathbb{E}\, Z^{2p} + \mathbb{E}\,(YW)^p).\end{aligned} \tag{3.17}$$

(3.8) then follows using (3.15) and Proposition 3.1 with $A_n = \sum_{j=1}^n (M_j - M_{j-1})^2$. □

4. The $k = 2$ case.

Proposition 4.1. *If*

$$\widetilde{U}_2(n,x) = \sum_{i=0}^{n-1} G_n(X_n - X_i - x),$$

then

$$M_n = \widetilde{U}_2(n,x) + \widetilde{B}_2(n,x)$$

is a martingale with $M_0 = 0$.

Proof. If

$$U_2(n,x) = \sum_{i=0}^{n-1} G(X_n - X_i - x),$$

we have $\widetilde{U}_2(n,x) = U_2(n,x) - nG(e_1\sqrt{n})$ so that

$$M_n = U_2(n,x) + B_2(n,x) - nG(x).$$

Abbreviating $\bar{B}_n = B_2(n,x) - nG(x)$ we have

$$\bar{B}_n - \bar{B}_{n-1} = \sum_{i=0}^{n-1} 1_{(X_n = X_i + x)} - G(x).$$

So

$$\begin{aligned}\mathbb{E}\left[\bar{B}_n - \bar{B}_{n-1} + G(x) \mid \mathcal{F}_{n-1}\right] &= \sum_{i=0}^{n-1} \mathbb{P}(X_n - X_{n-1} + X_{n-1} - X_i = x \mid \mathcal{F}_{n-1}) \\ &= \sum_{i=0}^{n-1} p(1, 0, X_{n-1} - X_i - x)\end{aligned} \tag{4.1}$$

Abbreviating $U_n = U_2(n, x)$ we have

$$U_n - U_{n-1} - G(x) = \sum_{i=0}^{n-1} [G(X_n - X_i - x) - G(X_{n-1} - X_i - x)].$$

Now for any $i \leq n-1$

$$\begin{aligned}\mathbb{E}\left[G(X_n - X_i - x) \mid \mathcal{F}_{n-1}\right] &= \mathbb{E}\left[G(X_n - X_{n-1} + X_{n-1} - X_i - x) \mid \mathcal{F}_{n-1}\right] \\ &= \sum_y G(y + X_{n-1} - X_i - x)\mathbb{P}(X_n - X_{n-1} = y \mid \mathcal{F}_{n-1}) \\ &= \sum_y G(y + X_{n-1} - X_i - x)\mathbb{P}(X_n - X_{n-1} = y) \\ &= \sum_y G(y + X_{n-1} - X_i - x)p(1, 0, y) \\ &= P_1 G(X_{n-1} - X_i - x)\end{aligned} \tag{4.2}$$

where P_j is the transition operator associated to $p(j, x, y)$. Hence

$$\mathbb{E}\left[U_n - U_{n-1} - G(x) \mid \mathcal{F}_{n-1}\right] = \sum_{i=0}^{n-1} [P_1 G(X_{n-1} - X_i - x) - G(X_{n-1} - X_i - x)].$$

Comparing with (4.1) and using

$$P_1 G(z) - G(z) = -p(1, 0, z), \qquad z \in \mathbb{Z}^2, \tag{4.3}$$

we see that

$$\mathbb{E}\left[M_n - M_{n-1} \mid \mathcal{F}_{n-1}\right] = \mathbb{E}\left[U_n - U_{n-1} + \bar{B}_n - \bar{B}_{n-1} \mid \mathcal{F}_{n-1}\right] = 0$$

as required. □

The key to proving Theorem 1.1 in the $k = 2$ case is the following proposition.

Proposition 4.2. *We have*

$$\mathbb{E}\,|\widetilde{\beta}_2(n,x)-\widetilde{\beta}_2(n,x')|^p \le c(p)(\log n)^p n|x-x'|^{p/3} \tag{4.4}$$

for each integer $p>1$ and $x,x'\in\mathbb{Z}^2/\sqrt{n}$ with $|x|,|x'|\le 1$.

Let

$$W_2(n)=|B_2(n,x)|+|B_2(n,x')|,$$

$$Y_2(n)=\max_{i\le n}\{|B_2(i,x)-B_2(i-1,x)|+|B_2(i,x')-B_2(i-1,x')|\},$$

and

$$\widetilde{Z}_2(n)=\sup_{j\le n}|\widetilde{U}_2(j,x)-\widetilde{U}_2(j,x')|.$$

In the proof of Proposition 4.2 we will need the following three lemmas, whose proofs are deferred until the next section.

Lemma 4.3. *For any x,x' with $|x|,|x|'\le\sqrt{n}$*

$$\mathbb{E}\,W_2(n)^p\le c(\log n)^p n^p. \tag{4.5}$$

Lemma 4.4. *For any x,x' with $|x|,|x|'\le\sqrt{n}$*

$$\mathbb{E}\,Y_2(n)^p\le cn(\log n)^p. \tag{4.6}$$

Lemma 4.5. *For any x,x' with $|x|,|x|'\le\sqrt{n}$*

$$\mathbb{E}\,\widetilde{Z}_2(n)^p\le cn^p\Big|\frac{x-x'}{\sqrt{n}}\Big|^{2p/3}, \tag{4.7}$$

Proof of Proposition 4.2. Converting from $\widetilde{\beta}$'s to $\widetilde{B}$'s, estimate (4.4) for $k=2$ is equivalent to

$$\mathbb{E}\,|\widetilde{B}_2(n,x)-\widetilde{B}_2(n,x')|^p\le c(p)(\log n)^p n^{p+1}\Big(\frac{|x-x'|}{\sqrt{n}}\Big)^{p/3} \tag{4.8}$$

for $x,x'\in\mathbb{Z}^2$ with $|x|,|x'|\le\sqrt{n}$. We want to apply Proposition 3.2. We fix an n. We use the notation $f^+(x)=\max(f(x),0)$, $f^-(x)=\max(-f(x),0)$ so that $f(x)=f^+(x)-f^-(x)$. Take for $i\le n$

$$Q_i^1=B_2(i,x)+(G^+(x')+G^-(x))i,\qquad Q_i^2=B_2(i,x')+(G^+(x)+G^-(x'))i, \tag{4.9}$$

so that Q^1 and Q^2 are increasing and $Q_i=Q_i^1-Q_i^2=\widetilde{B}_2(i,x)-\widetilde{B}_2(i,x')$. For $i\ge n$ and $j=1,2$, set $Q_i^j=Q_n^j$. We set $H_i=\widetilde{U}_2(i,x)-\widetilde{U}_2(i,x')$. By Proposition 4.1, Q_i+H_i is a martingale.

From Proposition 2.1, Lemmas 4.3 and 4.4, and the fact that $|x|,|x'|\le\sqrt{n}$, we see that

$$\mathbb{E}\,[(Q_n^1+Q_n^2)^p]\le c(\log n)^p n^p \tag{4.10}$$

and

$$\mathbb{E}\,[(\max_{i\le n}\{[Q_i^1-Q_{i-1}^1]+[Q_i^2-Q_{i-1}^2]\})^p]\le cn(\log n)^p. \tag{4.11}$$

Combining (4.10), (4.11), Lemma 4.5, and the fact that $\frac{1}{\sqrt{n}}\le\frac{|x-x'|}{\sqrt{n}}\le 2$ unless $x=x'$ with Proposition 3.2, we obtain

$$\mathbb{E}\sup_{j\le n}|\widetilde{B}_2(j,x)-\widetilde{B}_2(j,x')|^p\le c(p)(\log n)^p n^{p+1}\Big(\frac{|x-x'|}{\sqrt{n}}\Big)^{p/3} \tag{4.12}$$

for $x,x'\in\mathbb{Z}^2$ with $|x|,|x'|\le\sqrt{n}$, which implies (4.8). This is the bound we need. $\square$

Proof of Theorem 1.1, the $k=2$ case. Let $f:\mathbb{R}^2\to[0,\infty)$ be a nonnegative C^∞ function with support in $\{y:\frac{1}{2}\le|y|\le 1\}$, and with integral 1. Let $f_\tau(x)=\tau^{-2}f(x/\tau)$. The gradient of f_τ is bounded by a constant times τ^{-3}. Set $\tau_n=n^{-\zeta/4}$. Then recalling (1.7),

$$\int_0^1\int_0^t|f_{\tau_n}(X_t^n-X_s^n)-f_{\tau_n}(W_t^n-W_s^n)|ds\,dt\le c\tau_n^{-3}n^{-\zeta}\le cn^{-\zeta/4}. \tag{4.13}$$

We also have by Lemma 2.3 that for some $\bar{\delta}>0$

$$\Bigg|\sum_{x\in\mathbb{Z}^2/\sqrt{n}}f_{\tau_n}(x)G_n(x\sqrt{n})\frac{1}{n}-\sum_{x\in\mathbb{Z}^2/\sqrt{n}}f_{\tau_n}(x)\frac{1}{\pi}\log(1/|x|)\frac{1}{n}\Bigg|\le cn^{-\bar{\delta}} \tag{4.14}$$

and it is easy to see from the support properties of $f_{\tau_n}(x)$ that

$$\Bigg|\int f_{\tau_n}(x)\tfrac{1}{\pi}\log(1/|x|)\,dx-\sum_{x\in\mathbb{Z}^2/\sqrt{n}}f_{\tau_n}(x)\frac{1}{\pi}\log(1/|x|)\frac{1}{n}\Bigg|\le cn^{-\bar{\delta}}. \tag{4.15}$$

On the other hand, recalling the notation $X_t^n=X_{[nt]}/\sqrt{n}$

$$\begin{aligned}\sum_{x\in\mathbb{Z}^2/\sqrt{n}}f_{\tau_n}(x)B_2(n,\sqrt{n}x)\frac{1}{n^2}&=\frac{1}{n^2}\sum_{x\in\mathbb{Z}^2/\sqrt{n}}f_{\tau_n}(x)\sum_{0\le i_1<i_2\le n}1_{(X_{i_2}=X_{i_1}+\sqrt{n}x)}\\&=\frac{1}{n^2}\sum_{0\le i_1<i_2\le n}f_{\tau_n}\Big(\frac{(X_{i_2}-X_{i_1})}{\sqrt{n}}\Big)\\&=\frac{1}{n^2}\int_0^n\int_0^t f_{\tau_n}\Big(\frac{(X_{[t]}-X_{[s]})}{\sqrt{n}}\Big)ds\,dt\\&=\int_0^1\int_0^t f_{\tau_n}(X_t^n-X_s^n)ds\,dt\end{aligned} \tag{4.16}$$

so that

$$\sum_{x\in\mathbb{Z}^2/\sqrt{n}} f_{\tau_n}(x)\widetilde{\beta}_2(n,x)\frac{1}{n} = \int_0^1\int_0^t f_{\tau_n}(X_t^n - X_s^n)ds\,dt - \sum_{x\in\mathbb{Z}^2/\sqrt{n}} f_{\tau_n}(x)G_n(x\sqrt{n})\frac{1}{n}.$$

By [3],

$$\int f_{\tau_n}(x)\widetilde{\gamma}_2(1,x,n)\,dx = \int_0^1\int_0^t f_{\tau_n}(W_t^n - W_s^n)\,ds\,dt - \int f_{\tau_n}(x)\tfrac{1}{\pi}\log(1/|x|)\,dx. \tag{4.17}$$

(This conforms with the definition given in Section 1 above; the definition in [3] is very slightly different and would yield $\int_0^1\int_0^1 f_{\tau_n}(W_t^n - W_s^n)\,ds\,dt$ instead.) Combining the above,

$$\left|\sum_{x\in\mathbb{Z}^2/\sqrt{n}} f_{\tau_n}(x)\widetilde{\beta}_2(n,x)\,\frac{1}{n} - \int f_{\tau_n}(x)\widetilde{\gamma}_2(1,x,n)\,dx\right| = O(n^{-\zeta/4}) + O(n^{-\overline{\delta}}). \tag{4.18}$$

Recall $\int f_{\tau_n}(x)dx = 1$. Without loss of generality we may assume $\zeta < 1/2$ is small enough so that $\psi_n =: \sum_{x\in\mathbb{Z}^2/\sqrt{n}} f_{\tau_n}(x)\frac{1}{n} = 1 + O(n^{-\overline{\delta}})$ for some $\overline{\delta} > 1/4$. (If ζ were too large, then τ_n would tend to 0 too quickly, and then the above estimate for ψ_n might not be valid. In general one has $\psi_n = 1 + O(n^{-1/2}\tau_n^{-3})$.) Jensen's inequality and estimates (4.4), (4.5) imply that

$$\begin{aligned}
&\mathbb{E}\left|\sum_{x\in\mathbb{Z}^2/\sqrt{n}} f_{\tau_n}(x)\widetilde{\beta}_2(n,x)\frac{1}{n} - \psi_n\widetilde{\beta}_2(n,0)\right|^p \qquad (4.19)\\
&\quad\le c(p)\mathbb{E}\left|\sum_{x\in\mathbb{Z}^2/\sqrt{n}} \frac{1}{n}f_{\tau_n}(x)[\widetilde{\beta}_2(n,x) - \widetilde{\beta}_2(n,0)]\right|^p + c(p)(\psi_n - 1)^p\mathbb{E}\,\widetilde{\beta}_2(n,0)^p\\
&\quad\le c(p)\sum_{x\in\mathbb{Z}^2/\sqrt{n}} f_{\tau_n}(x)\mathbb{E}\,|\widetilde{\beta}_2(n,x) - \widetilde{\beta}_2(n,0)|^p\frac{1}{n} + c(p)n^{-p\overline{\delta}}(\log n)^p\\
&\quad\le c(p)(\log n)^p n(\tau_n)^{p/3}.
\end{aligned}$$

If we take p big enough, then using Chebyshev's inequality

$$\mathbb{P}\Big(\Big|\sum_{x\in\mathbb{Z}^2/\sqrt{n}} f_{\tau_n}(x)\widetilde{\beta}_2(n,x)\frac{1}{n} - \psi_n\widetilde{\beta}_2(n,0)\Big| \ge n^{-\zeta/24}\Big) \le c\frac{(\log n)^p n(\tau_n)^{p/3}}{n^{-\zeta p/24}} \le \frac{c}{n^2}.$$

By Borel-Cantelli, we conclude that

$$\sum_{x\in\mathbb{Z}^2/\sqrt{n}} f_{\tau_n}(x)\widetilde{\beta}_2(n,x)\frac{1}{n} - \psi_n\widetilde{\beta}_2(n,0) = O(n^{-\zeta/24}), \qquad a.s. \tag{4.20}$$

Using (4.10),

$$\mathbb{E}\left[|\widetilde{B}_2(n,0)^p|\right] \le c(\log n)^p n^p,$$

so

$$\mathbb{P}(|\widetilde{\beta}_2(n,0)| > n^{\overline{\delta}/2}) \le \frac{c(\log n)^p}{n^{\overline{\delta}p/2}},$$

and if we take p large enough, Borel-Cantelli tells us that

$$\widetilde{\beta}_2(n,0) = O(n^{\overline{\delta}/2}), \qquad a.s.$$

So then

$$\widetilde{\beta}_2(n,0) - \psi_n \widetilde{\beta}_2(n,0) = O(n^{-\overline{\delta}/2}), \qquad a.s. \tag{4.21}$$

A very similar argument to the above also shows that we have

$$\int f_{\tau_n}(x)\widetilde{\gamma}_2(1,x,n)\,dx - \widetilde{\gamma}_2(1,0,n) = O(n^{-\zeta/24}), \qquad a.s. \tag{4.22}$$

the analogue to estimate (4.4) is in [3]. Combining (4.18), (4.20), (4.21) and (4.22) we conclude that

$$\widetilde{\beta}_2(n,0) - \widetilde{\gamma}_2(1,0,n) = O(n^{-\zeta/24}) + O(n^{-\overline{\delta}/2}), \qquad a.s.$$

□

Remark 4.6. To see the importance of renormalization, note that if we also had the estimate (4.8) for $B_2(n,x) - B_2(n,x')$, this would imply that uniformly in n

$$|G(x) - G(x')| \le c(p)(\log n)\, n^{1/p} \Big(\frac{|x-x'|}{\sqrt{n}}\Big)^{1/3}$$

which is impossible if $p > 6$ and n is sufficiently large.

5. Proofs of Lemmas 4.3-4.5.

Proof of Lemma 4.3. We have

$$\begin{aligned} &\mathbb{E}\left\{(B_2(n,x))^p\right\} \\ &\quad \le \mathbb{E}\left\{\Big(\sum_{i,j=0}^{n} 1_{(X_j - X_i = x)}\Big)^p\right\} \\ &\quad = \sum_{s\in\mathcal{S}} \sum_{0\le i_1\le \ldots i_{2p}\le n} \sum_{z_1,\ldots,z_p\in\mathbb{Z}^2} \prod_{j=1}^{2p} p(i_j - i_{j-1}, z_{s(j-1)} + x_{c(j-1)}, z_{s(j)} + x_{c(j)}) \end{aligned} \tag{5.1}$$

where s runs over the set of maps $\mathcal{S}$ from $\{1,\dots,2p\}$ to $\{1,\dots,p\}$ such that $s^{-1}(j)=2$ for each $1\le j\le p$, $c(j)=\sum_{i=1}^{j}1_{\{s(i)=s(j)\}}$ and $x_1=0, x_2=x$. Here we use the conventions $i_0=0, z_0=0, c(0)=0$. Setting

$$g_n(x)=\sum_{i=0}^{n}p(i,0,x)\le c\log n \tag{5.2}$$

for $x\le\sqrt{n}$ by Proposition 2.1, and using the obvious fact that

$$\sum_{x\in\mathbb{Z}^2}g_n(x)=\sum_{x\in\mathbb{Z}^2}\sum_{i=0}^{n}p(i,0,x)=n+1$$

we can bound (5.1) by

$$\begin{aligned}&\sum_{s\in\mathcal{S}}\sum_{z_1,\dots,z_p\in\mathbb{Z}^2}\prod_{j=1}^{2p}g_n(z_{s(j)}+x_{c(j)}-z_{s(j-1)}-x_{c(j-1)})\\ &\quad\le c(\log n)^p\sum_{s\in\mathcal{S}}\sum_{z_1,\dots,z_p\in\mathbb{Z}^2}\prod_{j\,:\,c(j)=1}g_n(z_{s(j)}+x_{c(j)}-z_{s(j-1)}-x_{c(j-1)})\\ &\quad\le c(n\log n)^p.\end{aligned} \tag{5.3}$$

□

Proof of Lemma 4.4. Let

$$C(m,x)=[B_2(m,x)-B_2(m-1,x)]=\sum_{i=0}^{m-1}1_{(X_m=X_i+x)}.$$

We show that for any $1\le m\le n$

$$\mathbb{E}\,C(m,x)^p\le c(\log n)^p; \tag{5.4}$$

using

$$Y_2(n)^p\le\sum_{m=1}^{n}c(p)C(m,x)^p,$$

we are then done.

But

$$
\begin{aligned}
&\mathbb{E}\,\{(\sum_{i=0}^{m-1} 1_{(X_m=X_i+x)})^p\} \\
&\le p! \sum_{0\le i_1\le \ldots i_p\le m-1} \sum_{z_1,\ldots,z_p,y\in\mathbb{Z}^2} \prod_{j=1}^p p(i_j-i_{j-1},z_{j-1},z_j)1_{(y=z_j+x)}p(m-i_p,z_p,y) \\
&= p! \sum_{0\le i_1\le \ldots i_p\le m-1} \sum_{y\in\mathbb{Z}^2} p(i_1,0,y-x) \prod_{j=2}^p p(i_j-i_{j-1},0,0)p(m-i_p,0,x) \\
&= p! \sum_{0\le i_1\le \ldots i_p\le m-1} \prod_{j=2}^p p(i_j-i_{j-1},0,0)p(m-i_p,0,x) \\
&\le c(\log m)^p
\end{aligned}
$$

which gives (5.4). □

Proof of Lemma 4.5. We begin by estimating

$$\mathbb{E}\,(1+|X_i|^2)^{-b/2}$$

with $0<b<2$.

First

$$\mathbb{E}\,[(1+|X_i|^2)^{-b/2};|X_i|=0] = 1\cdot\mathbb{P}(X_i=0)\le c/i\le ci^{-b/2}.$$

Next, using (2.1)

$$
\begin{aligned}
\mathbb{E}\,[(1+|X_i|^2)^{-b/2};0<|X_i|<\sqrt{i}] &\le \sum_{\{x\in\mathbb{Z}^2,0<|x|\le\sqrt{i}\}} |x|^{-b}p(i,0,x) \\
&\le \frac{c}{i} \sum_{\{x\in\mathbb{Z}^2,0<|x|\le\sqrt{i}\}} |x|^{-b} \\
&= \frac{c}{i} i^{1-b/2} = ci^{-b/2}.
\end{aligned}
$$

Finally,

$$\mathbb{E}\,[(1+|X_i|^2)^{-b/2};\sqrt{i}\le|X_i|\,]\le(1+i)^{-b/2}\mathbb{P}(\sqrt{i}\le|X_i|)\le ci^{-b/2}. \tag{5.5}$$

We conclude that for any $0<b<2$

$$\mathbb{E}\,[(1+|X_i|^2)^{-b/2}]\le ci^{-b/2}. \tag{5.6}$$

Using the estimate

$$|G(X_i+x)-G(X_i+y)| \le \frac{c|x-y|^{2/3}}{(1+|X_i+x|^2)^{1/3}} + \frac{c|x-y|^{2/3}}{(1+|X_i+y|^2)^{1/3}}$$

of Proposition 2.2, the fact that symmetry tells us that $\mathbb{E}\,[(1+|X_i+x|^2)^{-1/3}]$ is largest when $x=0$, and the estimate (5.6) above, we obtain

$$\mathbb{E}\sum_{i=1}^n |G(X_i+x)-G(X_i+y)| \le c|x-y|^{2/3}\sum_{i=1}^n i^{-1/3} \le cn(|x-y|/\sqrt{n})^{2/3}.$$

So by independence, using $\bar{X}_i, \bar{\mathbb{E}}$ to denote an independent copy of X_i and its expectation operator,

$$\begin{aligned} &\mathbb{E}\Big[\sum_{i=m+1}^n |G(X_i+x)-G(X_i+y)| \ \mid \mathcal{F}_m\Big] \qquad (5.7)\\ &\le \bar{\mathbb{E}}\sum_{i=1}^n |G(\bar{X}_i+X_m+x)-G(\bar{X}_i+X_m+y)| \le cn\Big(\frac{|x-y|}{\sqrt{n}}\Big)^{2/3}. \end{aligned}$$

If $|x|\le\sqrt{n}$, by Proposition 2.1 and Doob's inequality

$$\begin{aligned} \mathbb{E}\sup_{i\le n}&|G(X_i+x)|^p \le c\mathbb{E}\sup_{i\le n}(1+\log^+|X_i+x|)^p \qquad (5.8)\\ &\le c(\log n)^p\mathbb{P}(\sup_{i\le n}|X_i+x|\le n) + c\sum_{m=[\log n]}^\infty m^p\mathbb{P}(e^m\le\sup_{i\le n}|X_i+x|\le e^{m+1})\\ &\le c(\log n)^p + c\sum_{m=[\log n]}^\infty m^p\frac{\mathbb{E}\,|X_n+x|^2}{e^{2m}}\\ &\le c(\log n)^p. \end{aligned}$$

If $x\ne y$, then $|x-y|\ge 1$ and (5.8) then implies that

$$\mathbb{E}\sup_{i\le n}|G(X_i+x)-G(X_i+y)|^p \le c(\log n)^p \le c\Big(n\Big(\frac{|x-y|}{\sqrt{n}}\Big)^{2/3}\Big)^p.$$

Using Proposition 3.1 with $A_j=\sum_{i=1}^{j\wedge n}|G(X_i+x)-G(X_i+y)|$, if $|x|,|y|\le\sqrt{n}$, then

$$\Big\|\sum_{i=1}^n |G(X_i+x)-G(X_i+y)|\Big\|_p \le cn(|x-y|/\sqrt{n})^{2/3}. \qquad (5.9)$$

Replacing x and y by $-x$ and $-x'$, resp., and using the fact that $\sum_{i=1}^n G(X_i-x)$ is equal in law to $\sum_{i=0}^{n-1} G(X_n-X_i-x)$ yields the L^p estimate that we want. □

6. The martingale connection: $k > 2$.

Let $\widetilde{B}_{1,m}(j,x) = j$ and for $k \geq 2$ define

$$\widetilde{B}_{k,m}(j,x) = \sum_{A \subset \{2,\ldots,k\}} (-1)^{|A|} \Big(\prod_{i \in A} G_m(x_i) \Big) B_{k-|A|}(j, x_{A^c}). \tag{6.1}$$

Note that $\widetilde{B}_{k,n}(n,x) = \widetilde{B}_k(n,x)$.

If $x = (x_2, \ldots, x_{k-1}, x_k)$, let $x_{k^c} = (x_2, \ldots, x_{k-1})$.

Proposition 6.1. *Let $k > 2$. If*

$$\widetilde{U}_{k,m}(n,x) = \sum_{i=1}^{n} G_m(X_n - X_i - x_k)[\widetilde{B}_{k-1,m}(i, x_{k^c}) - \widetilde{B}_{k-1,m}(i-1, x_{k^c})],$$

then for each m

$$M_{n,m} = \widetilde{U}_{k,m}(n,x) + \widetilde{B}_{k,m}(n,x), \qquad n = 0, 1, 2, \ldots$$

is a martingale with $M_{0,m} = 0$.

Proof. We will show that for each k

$$N_{k,m}(n) = U_{k,m}(n,x) + B_{k,m}(n,x) - G_m(x_k) B_{k-1,m}(n, x_{k^c}), \qquad n = 0, 1, 2, \ldots$$

is a martingale where

$$\begin{aligned} U_{k,m}(n,x) &= \sum_{i=1}^{n} [G(X_n - X_i - x_k) - G(e_1\sqrt{m})][B_{k-1}(i, x_{k^c}) - B_{k-1}(i-1, x_{k^c})] \\ &= \sum_{i=1}^{n} G(X_n - X_i - x_k)[B_{k-1}(i, x_{k^c}) - B_{k-1}(i-1, x_{k^c})] \\ &\qquad - G(e_1\sqrt{m}) B_{k-1}(n, x_{k^c}). \end{aligned}$$

This will prove the proposition since, with the notation $D_k = D \cup \{k\}$,

$$\begin{aligned} &\widetilde{U}_{k,m}(n,x) + \widetilde{B}_{k,m}(n,x) \\ &= \sum_{i=1}^{n} G_m(X_n - X_i - x_k) \\ &\quad \sum_{D \subset \{2,\ldots,k-1\}} (-1)^{|D|} \Big(\prod_{l \in D} G_m(x_l) \Big) [B_{k-|D_k|,m}(i, x_{D_k^c}) - B_{k-|D_k|,m}(i-1, x_{D_k^c})] \\ &\quad + \sum_{A \subset \{2,\ldots,k\}} (-1)^{|A|} \Big(\prod_{i \in A} G_m(x_i) \Big) B_{k-|A|}(n, x_{A^c}) \\ &= \sum_{D \subset \{2,\ldots,k-1\}} (-1)^{|D|} \Big(\prod_{l \in D} G_m(x_l) \Big) \\ &\qquad [U_{k-|D^c|,m}(n, x_{D^c}) + B_{k-|D^c|,m}(n, x_{D^c}) - G_m(x_k) B_{k-|D^c|-1,m}(n, x_{D_k^c})]. \end{aligned}$$

If we set

$$\bar{U}_k(n,x) = \sum_{i=1}^{n} G(X_n - X_i - x_k)[B_{k-1}(i, x_{k^c}) - B_{k-1}(i-1, x_{k^c})]$$

we have that

$$N_{k,m}(n) = \bar{U}_k(n,x) + B_k(n,x) - G(x_k)B_{k-1}(n, x_{k^c}).$$

Abbreviating $\bar{U}_n = \bar{U}_k(n,x)$ and $\bar{B}_n = B_k(n,x) - G(x_k)B_{k-1}(n, x_{k^c})$, we have that

$$N_{k,m}(n) = \bar{U}_n + \bar{B}_n.$$

Setting

$$H_i = B_{k-1}(i, x_{k^c}) - B_{k-1}(i-1, x_{k^c}) = \sum_{0 \le i_1 < i_2 < \ldots < i_{k-1} = i} \prod_{j=2}^{k-1} 1_{(X_{i_j} = X_{i_{j-1}} + x_j)}$$

we have

$$\bar{B}_n - \bar{B}_{n-1} = \sum_{i=1}^{n-1} 1_{(X_n = X_i + x_k)} H_i - G(x_k)H_n.$$

So using (4.1)

$$\begin{aligned} &\mathbb{E}\left[\bar{B}_n - \bar{B}_{n-1} + G(x_k)H_n \mid \mathcal{F}_{n-1}\right] \\ &\qquad = \sum_{i=1}^{n-1} p(1, 0, X_{n-1} - X_i - x_k)H_i. \end{aligned} \tag{6.2}$$

From the definition of $\bar{U}_n$ we have

$$\begin{aligned} \bar{U}_n - \bar{U}_{n-1} = G(x_k)H_n& \\ &+ \sum_{i=1}^{n-1} [G(X_n - X_i - x_k) - G(X_{n-1} - X_i - x_k)]H_i. \end{aligned}$$

Recalling (4.2)

$$\begin{aligned} &\mathbb{E}\left[\bar{U}_n - \bar{U}_{n-1} - G_n(x_k)H_n \mid \mathcal{F}_{n-1}\right] \\ &\qquad = \sum_{i=1}^{n-1} [P_1 G(X_{n-1} - X_i - x_k) - G(X_{n-1} - X_i - x_k)]H_i. \end{aligned}$$

Comparing with (6.2) and using (4.3)

$$P_1 G(x) - G(x) = -p(1, 0, x),$$

we see that

$$\mathbb{E}\,[\bar{U}_n - \bar{U}_{n-1} + \bar{B}_n - \bar{B}_{n-1} \mid \mathcal{F}_{n-1}] = 0$$

as required. □

Remark. The statement of Proposition 6.1 is not an exact analogue of that of Proposition 4.1. Consider the summands in the definition of $\widetilde{U}_{k,m}(n,x)$:

$$G_m(X_n - X_i - x_k)[\widetilde{B}_{k-1,m}(i, x_{k^c}) - \widetilde{B}_{k-1,m}(i-1, x_{k^c})]. \tag{6.3}$$

When $k = 2$ and $i = n$, this is nonrandom, whereas this is not the case when $k > 2$ and $i = n$. On the other hand, recalling that $B_{k-1}(i,x) = 0$ if $i > k-1$, it is natural to define $B_{k-1}(-1,x)$ to be 0. It is also natural to define $\widetilde{B}_{1,m}(i,x) = i$ for $i \geq 0$. Then (6.3) will be 0 if $i = 0$ for all $k \geq 2$, but the $i = 0$ term in the statement of Proposition 4.1 is not zero.

7. The case of general k.

The key to proving Theorem 1.1 for the case of general k is the following proposition.

Proposition 7.1. *For any $k \geq 2$ we have*

$$\mathbb{E}\,|\widetilde{\beta}_k(n,x) - \widetilde{\beta}_k(n,x')|^p \leq c(p)(\log n)^{p(k-1)} n^k |x - x'|^{8^{-k}p} \tag{7.1}$$

for each integer $p > 1$ and $x, x' \in (\mathbb{Z}^2)^{k-1}/\sqrt{n}$ with $|x|, |x'| \leq 1$.

Let

$$W_k(n) = |B_k(n,x)| + |B_k(n,x')|,$$

$$\widetilde{W}_k(n) = |\widetilde{B}_k(n,x)| + |\widetilde{B}_k(n,x')|,$$

$$Y_k(n) = \max_{i \leq n}\{|B_k(i,x) - B_k(i-1,x)| + |B_k(i,x') - B_k(i-1,x')|\},$$

$$\widetilde{Y}_k(n) = \max_{i \leq n}\{|\widetilde{B}_{k,n}(i,x) - \widetilde{B}_{k,n}(i-1,x)| + |\widetilde{B}_{k,n}(i,x') - \widetilde{B}_{k,n}(i-1,x')|\},$$

and

$$\widetilde{Z}_k(n) = \sup_{j \leq n} |\widetilde{U}_{k,n}(j,x) - \widetilde{U}_{k,n}(j,x')|.$$

In the proof of Proposition 7.1 we will need the following three lemmas, whose proofs are deferred until the next two sections.

Lemma 7.2. *For any x, x' with $|x|, |x|' \leq \sqrt{n}$*

$$\mathbb{E}\, W_k(n)^p \leq c(\log n)^{p(k-1)} n^p \tag{7.2}$$

and

$$\mathbb{E}\, \widetilde{W}_k(n)^p \leq c(\log n)^{p(k-1)} n^p. \tag{7.3}$$

Lemma 7.3. *For any x, x' with $|x|, |x|' \leq \sqrt{n}$*

$$\mathbb{E}\, Y_k(n)^p \leq cn(\log n)^{p(k-1)} \tag{7.4}$$

and

$$\mathbb{E}\, \widetilde{Y}_k(n)^p \leq cn(\log n)^{p(k-1)}. \tag{7.5}$$

Lemma 7.4. *For any x, x' with $|x|, |x|' \leq \sqrt{n}$*

$$\mathbb{E}\, \widetilde{Z}_k(n)^p \leq c(\log n)^{p(k-1)} n^{p+k} \Big(\frac{|x-x'|}{\sqrt{n}}\Big)^{8^{-k}2p}, \tag{7.6}$$

Proof of Proposition 7.1. Converting from $\widetilde{\beta}$'s to $\widetilde{B}$'s, estimate (7.1) is equivalent to

$$\mathbb{E}\, |\widetilde{B}_{k,n}(n,x) - \widetilde{B}_{k,n}(n,x')|^p \leq c(p)(\log n)^{p(k-1)} n^{p+k} \Big(\frac{|x-x'|}{\sqrt{n}}\Big)^{8^{-k}p} \tag{7.7}$$

for $x, x' \in (\mathbb{Z}^2)^{k-1}$ with $|x|, |x'| \leq \sqrt{n}$. We want to apply Proposition 3.2. We fix an n. Let

$$\mathcal{A}_+(x) = \Big\{ A \subset \{2, \ldots, k\} \,\big|\, (-1)^{|A|} \prod_{i \in A} G_n(x_i) > 0 \Big\}$$

$$\widetilde{B}_{k,n,+}(j,x) = \sum_{A \in \mathcal{A}_+(x)} (-1)^{|A|} \Big(\prod_{i \in A} G_n(x_i) \Big) B_{k-|A|}(j, x_{A^c}). \tag{7.8}$$

$$\widetilde{B}_{k,n,-}(j,x) = \sum_{A \in \mathcal{A}_+^c(x)} (-1)^{|A|} \Big(\prod_{i \in A} G_n(x_i) \Big) B_{k-|A|}(j, x_{A^c}). \tag{7.9}$$

For $i \leq n$ set

$$Q_i^1 = \widetilde{B}_{k,n,+}(i,x) - \widetilde{B}_{k,n,-}(i,x'), \qquad Q_i^2 = \widetilde{B}_{k,n,+}(i,x') - \widetilde{B}_{k,n,-}(i,x),$$

so that Q^1 and Q^2 are increasing and $Q_i = Q_i^1 - Q_i^2 = \widetilde{B}_{k,n}(i,x) - \widetilde{B}_{k,n}(i,x')$. For $i \geq n$ set $Q_i^j = Q_n^j$, $j = 1, 2$. We set $H_i = \widetilde{U}_{k,n}(i,x) - \widetilde{U}_{k,n}(i,x')$. By Proposition 6.1, $Q_i + H_i$ is a martingale. Using Lemmas 7.2-7.4 and Proposition 2.1 to bound the right hand side of (3.5) in Proposition 3.2 and using the fact that $\frac{1}{\sqrt{n}} \leq \frac{|x-x'|}{\sqrt{n}} \leq 2$ unless $x = x'$ we obtain

$$\mathbb{E} \sup_{j \leq n} |\widetilde{B}_{k,n}(j,x) - \widetilde{B}_{k,n}(j,x')|^p \leq c(p)(\log n)^{p(k-1)} n^{p+k} \Big(\frac{|x-x'|}{\sqrt{n}}\Big)^{8^{-k}p} \tag{7.10}$$

for $x, x' \in (\mathbb{Z}^2)^{k-1}$ with $|x|, |x'| \le \sqrt{n}$, which implies (7.7). This is the bound we need. $\square$

Proof of Theorem 1.1, the general case. The proof is quite similar to the $k = 2$ case. Let $f : \mathbb{R}^2 \to [0, \infty)$ be a nonnegative C^∞ function with support in $\{y : \frac{1}{2} \le |y| \le 1\}$, and with integral 1. Let $f_\tau(x) = \tau^{-2} f(x/\tau)$. Set $\tau_n = n^{-\zeta/4k}$. Set $g_n(f) = \sum_{x \in \mathbb{Z}^2/\sqrt{n}} f(x) G_n(x\sqrt{n}) \frac{1}{n}$ and $l(f) = \frac{1}{\pi} \int f(x) \log(1/|x|) dx$. As in (4.14), (4.15)

$$|g_n(f_{\tau_n}) - \frac{1}{\pi} l(f_{\tau_n})| \le c(\tau_n \sqrt{n})^{-\rho}. \tag{7.11}$$

Using (1.1) and setting $F_{\tau_n}(x_2, \ldots, x_k) = \prod_{i=2}^k f_{\tau_n}(x_i)$ we have

$$\begin{aligned} \int F_{\tau_n}(x) \widetilde{\gamma}_k(1, x)\, dx = & \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \Big(l(f_{\tau_n}) \Big)^j \\ & \times \int_{0 \le t_1 \le \cdots \le t_{k-j} < 1} \prod_{i=2}^{k-j} f_{\tau_n}(W_{t_i} - W_{t_{i-1}}) dt_1 \cdots dt_{k-j}. \end{aligned} \tag{7.12}$$

On the other hand, as in (4.16), it is easily checked that we have

$$\begin{aligned} \sum_{x \in \mathbb{Z}^{2(k-1)}/\sqrt{n}} F_{\tau_n}(x) \widetilde{\beta}_k(n, x) \frac{1}{n^{k-1}} = & \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \Big(g_n(f_{\tau_n}) \Big)^j \\ & \times \int_{0 \le t_1 \le \cdots \le t_{k-j} < 1} \prod_{i=2}^{k-j} f_{\tau_n}(X^n_{t_i} - X^n_{t_{i-1}}) dt_1 \cdots dt_{k-j}. \end{aligned} \tag{7.13}$$

Since the gradient of f_τ is bounded by a constant times τ^{-3}

$$\begin{aligned} \int_{0 \le t_1 \le \cdots \le t_{k-j} < 1} \Big| \prod_{i=2}^{k-j} f_{\tau_n}(W^n_{t_i} - W^n_{t_{i-1}}) - \prod_{i=2}^{k-j} f_{\tau_n}(X^n_{t_i} - X^n_{t_{i-1}}) \Big| dt_1 \cdots dt_{k-j} \\ \le c\tau_n^{-3-2(k-j-2)} n^{-\zeta} \le cn^{-\zeta/4}. \end{aligned} \tag{7.14}$$

Combining (7.11), (7.14), and the fact that both $|g_n(f_{\tau_n})|$ and $|l(f_{\tau_n})|$ are bounded by $c \log n$ we see that

$$\begin{aligned} \Big| \int F_{\tau_n}(x) \widetilde{\gamma}_k(t, x, n)\, dx - \sum_{x \in \mathbb{Z}^{2(k-1)}/\sqrt{n}} F_{\tau_n}(x) \widetilde{\beta}_k(t, x) \frac{1}{n^{k-1}} \Big| \\ \le c\tau_n^{-2(k-2)} (\tau_n \sqrt{n})^{-\rho} + c(\log n)^{k-1} n^{-\zeta/4} \le cn^{-\zeta/8} \end{aligned} \tag{7.15}$$

if we take $\zeta > 0$ sufficiently small.

Since $\int F_{\tau_n}(x)dx = 1$, we have $\psi_{k,n} =: \sum_{x\in\mathbb{Z}^{2(k-1)}/\sqrt{n}} F_{\tau_n}(x)\frac{1}{n^{k-1}} = 1 + O(n^{-\overline{\delta}})$, provided we assume, as we may without loss of generality, that ζ is sufficiently small. Jensen's inequality and estimates (7.1), (7.2) imply that

$$\begin{aligned}
&\mathbb{E}\Big|\sum_{x\in\mathbb{Z}^{2(k-1)}/\sqrt{n}} F_{\tau_n}(x)\widetilde{\beta}_k(n,x)\frac{1}{n^{k-1}} - \psi_{k,n}\widetilde{\beta}_k(n,0)\Big|^p \\
&\le c(p)\sum_{x\in\mathbb{Z}^{2(k-1)}/\sqrt{n}} F_{\tau_n}(x)\mathbb{E}\,|\widetilde{\beta}_k(n,x) - \widetilde{\beta}_k(n,0)|^p\frac{1}{n^{k-1}} + c(p)(\psi_{k,n}-1)^p \\
&\le c(p)(\log n)^{p(k-1)}n^2(\tau_n)^{8^{-k}p}. \qquad (7.16)
\end{aligned}$$

If we take p big enough, then

$$\begin{aligned}
&\mathbb{P}\Big(\Big|\sum_{x\in\mathbb{Z}^{2(k-1)}/\sqrt{n}} F_{\tau_n}(x)\widetilde{\beta}_k(n,x)\frac{1}{n^{k-1}} - \psi_{k,n}\widetilde{\beta}_k(n,0)\Big| \ge n^{-8^{-k}\zeta/8k}\Big) \\
&\le c\frac{(\log n)^{p(k-1)}n^2(\tau_n)^{8^{-k}p}}{n^{-8^{-k}p\zeta/8k}} \le \frac{c}{n^2}.
\end{aligned}$$

By Borel-Cantelli, we conclude that

$$\sum_{x\in\mathbb{Z}^{2(k-1)}/\sqrt{n}} F_{\tau_n}(x)\widetilde{\beta}_k(n,x)\frac{1}{n^{k-1}} - \psi_{k,n}\widetilde{\beta}_k(n,0) = O(n^{-8^{-k}\zeta/8k}), \qquad a.s. \qquad (7.17)$$

We use (7.3) and the same argument as in the $k = 2$ case to show

$$\widetilde{\beta}_k(n,0) - \psi_{k,n}\widetilde{\beta}_k(n,0) = O(n^{-\overline{\delta}/2}), \qquad a.s.$$

for some $\overline{\delta} > 0$. This with (7.17) yields

$$\sum_{x\in\mathbb{Z}^{2(k-1)}/\sqrt{n}} F_{\tau_n}(x)\widetilde{\beta}_k(n,x)\frac{1}{n^{k-1}} - \widetilde{\beta}_k(n,0) = O(n^{-8^{-k}\zeta/8k}) + O(n^{-\overline{\delta}/2}), \qquad a.s. \quad (7.18)$$

A similar argument shows that we have (7.18) holding with the $\widetilde{\beta}_k(n,x)$ replaced by $\widetilde{\gamma}_k(1,x,n)$; the analogue to estimate (7.1) is in [3]. Combining, we conclude that

$$\widetilde{\beta}_k(n,0) - \widetilde{\gamma}_k(1,0,n) = O(n^{-8^{-k}\zeta/8k}) + O(n^{-\overline{\delta}/2}), \qquad a.s.$$

□

8. Proofs of Lemmas 7.2-7.3.

These are again similar to the $k = 2$ case.

Proof of Lemma 7.2. Using Proposition 2.1 it suffices to prove (7.2) for all k.

We have

$$
\begin{aligned}
&\mathbb{E}\left\{(B_k(n,x))^m\right\} &(8.1)\\
&\quad\le \mathbb{E}\left\{\left(\sum_{i_1,\ldots,i_k=0}^{n}\prod_{j=2}^{k}1_{(X_{i_j}=X_{i_{j-1}}+x_j)}\right)^m\right\}\\
&\quad= \sum_{s\in S(k,m)}\sum_{0\le i_1\le\ldots i_{km}\le n}\sum_{z_1,\ldots,z_m\in\mathbb{Z}^2}\prod_{j=1}^{km}p(i_j-i_{j-1},z_{s(j-1)}+\bar{x}_{c(j-1)},z_{s(j)}+\bar{x}_{c(j)})
\end{aligned}
$$

where s runs over the set of maps $S(k,m)$ from $\{1,\ldots,km\}$ to $\{1,\ldots,m\}$ such that $s^{-1}(j)=k$ for each $1\le j\le m$, $c(j)=\sum_{i=1}^{j}1_{\{s(i)=s(j)\}}$, $\bar{x}_j=\sum_{l=2}^{j}x_l$ and $\bar{x}_1=0$. Here we use the conventions $i_0=0, z_0=0, c(0)=0$. Setting

$$
g_n(x)=\sum_{i=0}^{n}p(i,0,x)\le c\log n \tag{8.2}
$$

and using the obvious fact that $\sum_{x\in\mathbb{Z}^2}g_n(x)=\sum_{x\in\mathbb{Z}^2}\sum_{i=0}^{n}p(i,0,x)=n+1$ we can bound (8.1) by

$$
\begin{aligned}
&\sum_{s\in S(k,m)}\sum_{z_1,\ldots,z_m\in\mathbb{Z}^2}\prod_{j=1}^{km}g_n(z_{s(j)}+\bar{x}_{c(j)}-z_{s(j-1)}-\bar{x}_{c(j-1)})\\
&\qquad\le c(\log n)^{(k-1)m}\sum_{s\in\mathcal{S}}\sum_{z_1,\ldots,z_k\in\mathbb{Z}^2}\prod_{j\,:\,c(j)=1}g_n(z_{s(j)}+\bar{x}_{c(j)}-z_{s(j-1)}-\bar{x}_{c(j-1)})\\
&\qquad\le cn^m(\log n)^{(k-1)m}.
\end{aligned} \tag{8.3}
$$

□

Proof of Lemma 7.3. Using Proposition 2.1 it suffices to prove (7.4) for all k.

Let

$$
C_k(n,x)=[B_k(n,x)-B_k(n-1,x)]=\sum_{0\le i_1<i_2<\ldots<i_k=n}\prod_{j=2}^{k}1_{(X_{i_j}=X_{i_{j-1}}+x_j)}.
$$

If we show

$$
\mathbb{E}\,C_k(i,x)^p\le c(\log n)^{(k-1)p},\qquad i\le n, \tag{8.4}
$$

then using

$$
Y_k(n)^p\le\sum_{i=1}^{n}c(p)C_k(i,x)^p,
$$

we are done.

But

$$\mathbb{E}\Big\{\Big(\sum_{0\le i_1<i_2<\ldots<i_k=n}\prod_{j=2}^{k}1_{(X_{i_j}=X_{i_{j-1}}+x_j)}\Big)^m\Big\}$$
$$= \sum_{s\in S(k-1,m)}\sum_{(i_1,\ldots,i_{(k-1)m})\in D(s)}\sum_{z_1,\ldots,z_m,y\in\mathbb{Z}^2} \tag{8.5}$$
$$\prod_{j=1}^{(k-1)m} p(i_j-i_{j-1}, z_{s(j-1)}+\bar{x}_{c(j-1)}, z_{s(j)}+\bar{x}_{c(j)})$$
$$\Big(\prod_{j\,:\,c(j)=k-1}1_{(y=z_{s(j)}+\bar{x}_k)}\Big)p(n-i_{(k-1)m}+\bar{x}_{k-1}, z_{s((k-1)m)}, y)$$

where $S(k-1,m), c(j), \bar{x}_j$ are defined in the last section and for each $s\in S(k-1,m)$

$$D(s)=\{(i_1,\ldots,i_{(k-1)m})\,:\,0\le i_1\le\ldots\le i_{(k-1)m}<n \text{ and }$$
$$i_{j-1}<i_j \text{ whenever } s(j-1)=s(j)\}.$$

We can then see that (8.5) equals

$$\sum_{s\in S(k-1,m)}\sum_{(i_1,\ldots,i_{(k-1)m})\in D(s)}\sum_{y\in\mathbb{Z}^2}$$
$$p(i_1,0,y-\bar{x}_k)\prod_{j=2}^{(k-1)m}p(i_j-i_{j-1},\bar{x}_{c(j-1)},\bar{x}_{c(j)})\,p(n-i_{(k-1)m},0,x_k)$$
$$\le c(\log n)^{(k-1)m}$$

which is (7.4). □

9. Proof of Lemma 7.4.

This proof is substantially different from the proof of Lemma 4.5.

Proof of Lemma 7.4. We use induction on k. We already know (7.6) for $k=2$. Thus assume (7.6) has been proved with k replaced by i for all $i\le k-1$. Then as explained above in the proof of Proposition 7.1 we will have that (7.10) holds with k replaced by i for all $i\le k-1$.

We will show that

$$\mathbb{E}\,[|\widetilde{U}_{k,n}(m,x)-\widetilde{U}_{k,n}(m,x)|^p]\le c(\log n)^{(k-1)p}n^{p+k-1}\Big(\frac{|x-x'|}{\sqrt{n}}\Big)^{8^{-k}2p} \tag{9.1}$$

for $m \leq n$. This and the inequality

$$\mathbb{E}\,[\max_{m\leq n}|C_m|^p] \leq \mathbb{E}\,\sum_{m=1}^{n}|C_m|^p = \sum_{m=1}^{n}\mathbb{E}\,[|C_m|^p] \leq n\max_{m\leq n}\mathbb{E}\,[|C_m|^p]$$

yields (7.6).

Abbreviating $\Delta_i\widetilde{B}_{k-1,n}(i,x_{k^c}) := \widetilde{B}_{k-1,n}(i,x_{k^c}) - \widetilde{B}_{k-1,n}(i-1,x_{k^c})$, where x_{k^c} is the same as $(x_2,\ldots,x_{k-1})$, we have

$$\begin{aligned}
&\|\widetilde{U}_{k,n}(m,x) - \widetilde{U}_{k,n}(m,x')\|_p \qquad (9.2)\\
&\leq \|\sum_{i=1}^{m}\Big(G(X_m - X_i - x_k) - G(X_m - X_i - x'_k)\Big)\Big(\Delta_i\widetilde{B}_{k-1,n}(i,x_{k^c})\Big)\|_p\\
&\quad + \|\sum_{i=1}^{m}\Big(G(X_m - X_i - x'_k) - G(e_1\sqrt{n})\Big)\Big(\Delta_i\widetilde{B}_{k-1,n}(i,x_{k^c}) - \Delta_i\widetilde{B}_{k-1,n}(i,x'_{k^c})\Big)\|_p.
\end{aligned}$$

Then with $m \leq n$

$$\begin{aligned}
&\|\sum_{i=1}^{m}\Big(G(X_m - X_i - x_k) - G(X_m - X_i - x'_k)\Big)\Big(\Delta_i\widetilde{B}_{k-1,n}(i,x_{k^c})\Big)\|_p\\
&\qquad \leq \|\sum_{i=1}^{m}|G(X_m - X_i - x_k) - G(X_m - X_i - x'_k)|\,\|_{2p}\,\|\widetilde{Y}_{k-1}(m)\|_{2p} \qquad (9.3)\\
&\qquad \leq cn\Big|\frac{x-x'}{\sqrt{n}}\Big|^{2/3} n^{1/2p}(\log n)^{(k-2)}.
\end{aligned}$$

by (5.9) and (7.5).

After interchanging x' and x for convenience it remains to bound

$$\begin{aligned}
&\Big\|\sum_{i=1}^{m}\Big(G(X_m - X_i - x_k) - G(e_1\sqrt{n})\Big)\Big(\Delta_i\widetilde{B}_{k-1,n}(i,x_{k^c}) - \Delta_i\widetilde{B}_{k-1,n}(i,x'_{k^c})\Big)\Big\|_p\\
&\leq \Big\|\sum_{i=1}^{m}G(X_m - X_i - x_k)\Big(\Delta_i\widetilde{B}_{k-1,n}(i,x_{k^c}) - \Delta_i\widetilde{B}_{k-1,n}(i,x'_{k^c})\Big)\Big\|_p \qquad (9.4)\\
&+ \Big\|G(\sqrt{n}e_1)[\widetilde{B}_{k-1,n}(m,x_{k^c}) - \widetilde{B}_{k-1,n}(m,x'_{k^c})]\Big\|_p
\end{aligned}$$

Using Proposition 2.1 and our inductive hypothesis concerning (7.10) we see that

$$\begin{aligned}
&\Big\|G(\sqrt{n}e_1)[\widetilde{B}_{k-1,n}(m,x_{k^c}) - \widetilde{B}_{k-1,n}(m,x'_{k^c})]\Big\|_p \qquad (9.5)\\
&\leq c(\log n)^{(k-1)}n^{1+(k-1)/p}\Big(\frac{|x-x'|}{\sqrt{n}}\Big)^{8^{-k+1}}.
\end{aligned}$$

To complete the proof of (9.1) it therefore suffices to show that

$$\begin{aligned}\| \sum_{i=1}^{m} G(X_m - X_i - x))[\Delta_i \widetilde{B}_{k-1,n}(i, x_{k^c}) - \Delta_i \widetilde{B}_{k-1,n}(i, x'_{k^c})]\|_p \\ \le c(\log n)^{(k-1)} n^{1+(k-1)/p} \Big(\frac{|x-x'|}{\sqrt{n}} \Big)^{8^{-k}2}.\end{aligned} \tag{9.6}$$

By Propositions 2.1 and 2.3, $G(x)$ is bounded above (but $G(x) \to -\infty$ as $|x| \to \infty$). Let

$$J(x) = G(x) \vee (-9 \log m), \qquad H(x) = G(x) - J(x).$$

Let $K_i = J(X_m - X_i - x)$ for $i = 0, \dots, m$ and let $K_i = J(X_m - x)$ for $i < 0$. Let B be a small positive real to be chosen later and let

$$L_i = \frac{K_i + \cdots + K_{i-Bm}}{Bm}.$$

We see that

$$\begin{aligned}\Big| \sum_{i=1}^{m} (K_i - L_i)[\Delta_i \widetilde{B}_{k-1,n}(i, x_{k^c})] \Big| \\ \le \widetilde{Y}_{k-1}(n) \sum_{i=1}^{n} |K_i - L_i|.\end{aligned} \tag{9.7}$$

Since J is bounded in absolute value by $c \log m$, the same is true for K_i and L_i for any i, i.e.

$$\sup_i |K_i - L_i| \le c \log m. \tag{9.8}$$

Note that L_i and K_i are independent of $\mathcal{F}_h$ for $i \ge h + Bm$, and thus

$$\mathbb{E} \Big[\sum_{i=h}^{m} |K_i - L_i| \mid \mathcal{F}_h \Big] \le \mathbb{E} \sum_{i=h+2Bm}^{m} |K_i - L_i| + cBm \log m. \tag{9.9}$$

Now by Proposition 2.2

$$\begin{aligned}&\mathbb{E}\, |J(X_m - X_i - x) - J(X_m - X_j - x)| \\ &\quad \le \mathbb{E}\, |G(X_m - X_i - x) - G(X_m - X_j - x)| \\ &\quad \le \mathbb{E} \Big[\frac{c|X_i - X_j|^{2/3}}{(1 + |X_m - X_i - x|^2)^{1/3}} + \frac{c|X_i - X_j|^{2/3}}{(1 + |X_m - X_j - x|^2)^{1/3}} \Big].\end{aligned}$$

By (5.6) and symmetry

$$\mathbb{E} \Big((1 + |X_m - X_i - x|^2)^{-1/2} \Big) \le \mathbb{E} \Big((1 + |X_m - X_i|^2)^{-1/2} \Big) \le 1 \wedge c(m-i)^{-1/2}.$$

Then using Holder's inequality in the form $|\mathbb{E}\,(fg)| \leq \|f\|_3\,\|g\|_{3/2}$ we obtain from the last two displays that

$$\begin{aligned}\mathbb{E}\,|J(X_m - X_i - x) - J(X_m - X_j - x)| \qquad\qquad\qquad\qquad\qquad \\ \leq \frac{c|i-j|^{1/3}}{1 \vee |m-i|^{1/3}} + \frac{c|i-j|^{1/3}}{1 \vee |m-j|^{1/3}}.\end{aligned} \tag{9.10}$$

Thus for $i \geq 2Bm$, summing over j from $i - Bm$ to i and dividing by Bm shows

$$\mathbb{E}\,|K_i - L_i| \leq c(Bm)^{1/3}(1 \vee |m-i|)^{-1/3}.$$

Therefore,

$$\mathbb{E} \sum_{i=h+2Bm}^{m} |K_i - L_i| \leq \sum_{i=h+2Bm}^{m} \frac{c(Bm)^{1/3}}{(1 \vee |m-i|)^{1/3}} \leq cmB^{1/3}.$$

Recalling (9.8)–(9.9) and then using Proposition 3.1 we have that

$$\mathbb{E}\Big|\sum_{i=1}^{m} |K_i - L_i|\Big|^p \leq c(\log m)^p + cm^p B^{p/3} \leq c(\log n)^p n^p B^{p/3} \tag{9.11}$$

for n large. Combining with this with (9.7), (7.5) and Cauchy-Schwarz, the left hand side of (9.7) is bounded in L^p norm by

$$c(\log n)^{k-1} n^{1+(1/2p)} B^{1/3}. \tag{9.12}$$

We use summation by parts on

$$\sum_{i=1}^{m} L_i[\Delta_i \widetilde{B}_{k-1,n}(i, x_{k^c}) - \Delta_i \widetilde{B}_{k-1,n}(i, x'_{k^c})] \tag{9.13}$$

and we see that it is equal to

$$\begin{aligned}&L_m[\widetilde{B}_{k-1,n}(m, x_{k^c}) - \widetilde{B}_{k-1,n}(m, x'_{k^c})] \\ &- \sum_{i=1}^{m} [\widetilde{B}_{k-1,n}(i-1, x_{k^c}) - \widetilde{B}_{k-1,n}(i-1, x'_{k^c})]\,[L_i - L_{i-1}].\end{aligned} \tag{9.14}$$

Write $w = |x_{k^c} - x'_{k^c}|/\sqrt{n} \leq 1$. Using the fact that L_m is bounded by $c \log m$ and our inductive hypothesis concerning (7.10), we can bound the L^p norm of the first term of (9.14) by

$$c(\log n)^{k-1} n^{1+(k-1)/p} w^{8^{-k+1}}.$$

Since K_i is bounded by $c \log n$, then $L_i - L_{i-1}$ is bounded by $c \log n/(Bn)$. Hence using once again our inductive hypothesis concerning (7.10)

$$\begin{aligned}\|[\widetilde{B}_{k-1,n}(i-1, x_{k^c}) - \widetilde{B}_{k-1,n}(i-1, x'_{k^c})]\,[L_i - L_{i-1}]\|_p & \\ &\le \frac{c \log n}{Bn} \|\widetilde{B}_{k-1,n}(i-1, x_{k^c}) - \widetilde{B}_{k-1,n}(i-1, x'_{k^c})\|_p \\ &\le \frac{c \log n}{Bn} (\log n)^{k-2} n^{1+(k-1)/p} w^{8^{-k+1}}.\end{aligned}$$

Since there are n summands in the sum in (9.14), we bound the L^p norm of the left hand side of (9.13) by

$$\frac{c}{B} (\log n)^{k-1} n^{1+(k-1)/p} w^{8^{-k+1}}. \tag{9.15}$$

Notice that

$$\Big|\sum_{i=1}^m H(X_m - X_i - x)[\Delta_i \widetilde{B}_{k-1,n}(i, x_{k^c})]\Big| \le m\widetilde{Y}_{k-1}(n) \sup_{1\le i\le m} |H(X_m - X_i - x)|. \tag{9.16}$$

By Proposition 2.3, $H(z)$ is 0 unless $|z| \ge e^{8 \log m}$. By hypothesis we have $|x| \le \sqrt{n}$. Therefore using $|H(z)|^{2p} \le c|\log z|^{2p} \le c|z|$ for $|z| \ge e^{8 \log m}$

$$\begin{aligned}\mathbb{E}\,|H(X_m - X_i - x)|^{2p} &\le c(p)\mathbb{E}\,[|X_m - X_i - x|; |X_m - X_i - x| \ge e^{8 \log n}] \\ &\le c(p)e^{-8 \log m}\mathbb{E}\,|X_m - X_i - x|^2 \\ &\le c(p)m^2 e^{-8 \log m} = c(p)m^{-6}.\end{aligned}$$

Hence $\mathbb{E}\,\sup_{i\le m} |H(X_m - X_i - x)|^{2p} \le c(p)m^{-5}$. Since $w \ge 1/\sqrt{n}$, then this estimate, (9.16), (7.5) and Cauchy-Schwarz imply that the left hand side of (9.16) is bounded in L^p norm by $c(\log n)^{k-2} n^{1/2p} \le c(\log n)^{k-2} n^{1+1/2p} w^2 \le c(\log n)^{k-1} n^{1+1/2p} w^{8^{-k+1}}$.

Combining our estimates (9.12), (9.15), and our last estimate for (9.16), we have

$$\begin{aligned}&\|\sum_{i=1}^m G(X_m - X_i - x))[\Delta_i \widetilde{B}_{k-1,n}(i, x_{k^c}) - \Delta_i \widetilde{B}_{k-1,n}(i, x'_{k^c})]\|_p \\ &\le c(\log n)^{(k-1)} n^{1+(k-1)/p}[w^{8^{-k+1}} + B^{1/3} + w^{8^{-k+1}} B^{-1}].\end{aligned} \tag{9.17}$$

If we take $B = w^{6(8^{-k})}$, we obtain (9.6). Together with (9.2)–(9.5) we obtain (9.1). □

10. Other results.

A. L^2 norms. By Section 3 of [5] we see that we can choose W_t and X_n such that

$$\|\sup_{s\le 1} |X^n_s - W^n_s|\,\|_2 = o(n^{-\zeta})$$

for some $\zeta > 0$. If we then use this (in place of (1.7)), our proof shows that we obtain

$$\|\widetilde{\beta}_k(n,0) - \widetilde{\gamma}_k(1,0,n)\|_2 = o(n^{-\eta}) \tag{10.1}$$

for some $\eta > 0$.

B. A correction. We take this opportunity to correct an error in [3]. In the statement of (8.3) in Theorem 8.1 of that paper, $G^\vee := \max_{1\le j\le k-1}|G(x_j)|$ should be replaced by $N^\vee := \max_{1\le j\le k-1}|x_j|^{-1}$. The term $G^\vee$ also needs to be replaced by $N^\vee$ throughout the proof of (8.3).

Proposition 9.2 of that paper is correct as stated. Where the proof of this proposition says to follow the lines of the proof of (8.3), it is to be understood that here one uses $G^\vee$ throughout.

For the convenience of the interested reader we give a complete proof of that proposition in the following Appendix.

Appendix. Proof of Proposition 9.2 of [3].

The proof of Proposition 9.2 in [3] is perhaps a bit confusing due to an error in the statement of (8.3) in Theorem 8.1 of that paper. This Appendix provides a complete proof of Proposition 9.2 of [3].

Write $g(y) = \frac{1}{\pi}\log(1/|y|)$, $\widetilde{\gamma}_1(t,x) = t$, and for $x = (x_2,\dots,x_k) = (x_{k^c}, x_k) \in (\mathbb{R}^2)^{k-1}$, set

$$g^\vee(x) = \max_{2\le i\le k}|g(x_i)|$$

and

$$\overline{U}_k(t,x) = \int_0^t g(W_t - W_r - x_k)\widetilde{\gamma}_{k-1}(dr, x_{k^c}).$$

Proposition A.1. *Let $M \ge 1$, let $x, x' \in (\mathbb{R}^2)^{k-1}$ with $|x_i|, |x_i'| \le M$ for $i = 2,\dots,k$, and let $g^+ = g^\vee(x) + g^\vee(x') + 1$. There exist a_k and ν_k such that for $k \ge 2$*

(a)
$$\mathbb{E}\,|\overline{U}_k(t,x) - \overline{U}_k(t,x')|^p \le c(g^+)^{\nu_k p}|x - x'|^{a_k p}.$$

(b)
$$\mathbb{E}\,|\overline{U}_k(t,x) - \overline{U}_k(s,x)|^p \le c(g^+)^{\nu_k p}|t - s|^{a_k p}.$$

Except for the restriction on the size of x, x', this is Proposition 9.2 of [3] translated to the notation of this paper. Using the argument of [3] this is sufficient to prove the joint continuity of $\widetilde{\gamma}_k(t,x)$ over $t \in [0,1], x \in (B(0,M))^{k-1}$ for each k and M. For almost every path of Brownian motion, $\{W_s : s \in [0,1]\}$ is contained in $B(0,M)$ for some M (depending on the path), and hence for $|x| > M$ we have $\widetilde{\gamma}_k(t,x) = 0$. The joint continuity of $\widetilde{\gamma}_k(t,x)$

over $t \in [0,1], x \in (\mathbb{R}^2)^{k-1}$ follows. For the purposes of this paper we only need the case $M = 1$.

Note that renormalization allows us to use $\overline{U}_k(t,x)$ in place of

$$U_k^*(t,x) = \int_0^t [g(W_t - W_r - x_k) - g(-x_k)]\widetilde{\gamma}_{k-1}(dr, x_{k^c}).$$

If one were to try to use $U_k^*(t,x)$ in Proposition A.1, the right hand sides of (a) and (b) would have to have g^+ replaced by $N^\vee$, which is not a good enough bound for the joint continuity argument.

Proof. Since g^+ is infinite if any component of x or x' is zero, we may assume that no component of either is 0.

Let $A \in (0, \frac{1}{2}]$ be chosen later and let

$$g_A(x) = \big(g(x) \wedge \tfrac{1}{\pi}\log(1/A)\big) \vee \big(-\tfrac{1}{\pi}\log(1/A)\big),$$

$$h_A(x) = [g(x) - g_A(x)]1_{(|x|<A)}, \qquad j_A(x) = [g_A(x) - g(x)]1_{(|x|>A^{-1})},$$

so $g = g_A + h_A - j_A$. With $C_k = C \cup \{k\}$ set

$$L_k(t,x) = \sum_{C \subset \{2,\dots,k-1\}} \Big(\prod_{i\in C} |g(x_i)|\Big)\gamma_{k-1-|C|}(t, x_{C_k^c}).$$

The proof is by induction. We start with $k = 2$. In preparation for general k we retain the general notation, but note that when $k = 2$, we have $L_2(t,x) = t$, x_{k^c} is superfluous, and we have $\widetilde{\gamma}_{k-1}(dr, x_{k^c}) = dr$.

$$\begin{aligned}
&\overline{U}_k(t,x) - \overline{U}_k(t,x') \qquad\qquad (A.1)\\
&= \int_0^t [g_A(W_t - W_r - x_k) - g_A(W_t - W_r - x_k')]\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&\quad + \int_0^t h_A(W_t - W_r - x_k)\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&\quad - \int_0^t h_A(W_t - W_r - x_k')\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&\quad - \int_0^t j_A(W_t - W_r - x_k)\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&\quad + \int_0^t j_A(W_t - W_r - x_k')\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&=: I_1 + I_2 - I_3 - I_4 + I_5.
\end{aligned}$$

If we connect x, x' by a curve Γ of length $c|x-x'|$ that never gets closer to 0 than $|x| \wedge |x'|$, use the fact that $|\nabla g_A| \le A^{-1}$, and use inequality (8.1) of [3] (this is only needed for $k > 2$)

$$\mathbb{E}\,|I_1|^p \le cA^{-p}|x-x'|^p \mathbb{E}\,L_k(t,x)^p \le cA^{-p}(g^+)^{\nu_1 p}|x-x'|^p.$$

By Proposition 5.2 of [3], for some constants b_1 and ν_1'

$$\mathbb{E}\,|I_2|^p \le cA^{b_1 p}(g^+)^{\nu_1' p} \tag{A.2}$$

and similarly for I_3.

We next turn to I_4. Standard estimates on Brownian motion tells us that

$$\mathbb{P}(\sup_{0\le r\le t\le 1} |W_t - W_r| > \lambda) \le ce^{-c'\lambda^2}. \tag{A.3}$$

Since $|x_k| \le M$, it follows that

$$\mathbb{E}\,[\sup_{0\le r\le t\le 1} |\log(1/|W_t - W_r - x_k|)|^p] \le c(p, M) \tag{A.4}$$

for each $p \ge 1$. If $|A| \le (2M)^{-1}$, then $|W_t - W_r - x_k| \ge A^{-1}$ only if $|W_t - W_r| \ge (2A)^{-1}$. So by Cauchy-Schwarz, (A.3), and (A.4),

$$\begin{aligned}
&\mathbb{E}\,[\sup_{0\le r\le t\le 1} |j_A(W_t - W_r - x_k)|^p] \\
&\quad \le c\mathbb{E}\,[\sup_{0\le r\le t\le 1} |\log(1/|W_t - W_r - x_k|)|^p 1_{(\sup_{0\le r\le t\le 1}|W_t - W_r - x_k| \ge A^{-1})}] \\
&\quad \le c(\mathbb{E}\,[\sup_{0\le r\le t\le 1} |\log(1/|W_t - W_r - x_k|)|^{2p}])^{1/2}(\mathbb{P}(\sup_{0\le r\le t\le 1} |W_t - W_r| \ge (2A)^{-1}))^{1/2} \\
&\quad \le cA^p
\end{aligned}$$

for each $p \ge 1$. Using the fact that

$$|I_4| \le \big(\sup_{0\le r\le t\le 1} |j_A(W_t - W_r - x_k)|\big) L_k(t,x),$$

another application of Cauchy-Schwarz shows that

$$\mathbb{E}\,|I_4|^p \le (g^+)^{\overline{\nu}_1 p} A^p.$$

I_5 is handled the same way.

Combining shows the left hand side of (A.1) is bounded by

$$c(g^+)^{\nu_2 p}[A^{-p}|x-x'|^p + A^{b_1 p} + A^p]$$

for some constant ν_2, and we obtain (a) for $k=2$ by setting $A = |x-x'|^{1/2} \wedge (2M)^{-1}$.

Next we look at (b) for the $k=2$ case. We write

$$\begin{aligned}
&\overline{U}_k(t,x) - \overline{U}_k(s,x) \qquad\qquad (A.5)\\
&\quad = \int_0^s [g_A(W_t - W_r - x_k) - g_A(W_s - W_r - x_k)]\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&\qquad + \int_s^t g_A(W_t - W_r - x_k)\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&\qquad + \int_0^t h_A(W_t - W_r - x_k)\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&\qquad - \int_0^s h_A(W_s - W_r - x_k)\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&\qquad - \int_0^s j_A(W_s - W_r - x_k)\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&\qquad + \int_0^s j_A(W_s - W_r - x_k')\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&\quad =: I_6 + I_7 + I_8 - I_9 - I_{10} + I_{11}.
\end{aligned}$$

Using Cauchy-Schwarz, for $s,t \le 1$ we have for some constant ν_1''

$$\mathbb{E}\,|I_6|^p \le cA^{-p}s^p\Big(\mathbb{E}\,|W_t - W_s|^{2p}\Big)^{1/2}\Big(\mathbb{E}\,L(s,x_{k^c})^{2p}\Big)^{1/2} \le cA^{-p}(g^+)^{\nu_1''p}|t-s|^{p/2}.$$

We bound I_7 by

$$\mathbb{E}\,|I_7|^p \le c(\log(1/A))^p|t-s|^p(g^+)^{\nu'''p}.$$

We bound I_8 and I_9 just as we did I_2 and bound I_{10} and I_{11} as we did I_4. Combining, the left hand side of (A.5) is bounded by

$$c(g^+)^{\nu_2 p}[A^{-p}|t-s|^{p/2} + A^{b_1 p} + A^p],$$

and (b) follows by setting $A = |t-s|^{1/4} \wedge (2M)^{-1}$.

We now turn to the case when $k>2$. We suppose (a) and (b) hold for $k-1$ and prove them for k. We prove (a) in two cases, when $x_{k^c} = x'_{k^c}$ and when $x_k = x'_k$; the general case follows by the triangle inequality. Suppose first that $x_{k^c} = x'_{k^c}$. Using the induction hypothesis, the proof is almost exactly the same as the proof of (a) in the case $k=2$.

Suppose next that $x_k = x'_k$. Let

$$V_A = \{|W_{s+u} - W_u| \ge u^{1/4}/A \text{ for some } s,u \in [0,1]\}.$$

Standard estimates on Brownian motion show that

$$\mathbb{P}(V_A) \le c_1 e^{-c_2/A^2}.$$

We write

$$\begin{aligned}
&\overline{U}_k(t,x) - \overline{U}_k(t,x') \qquad\qquad (A.6)\\
&\quad = 1_{V_A}\int_0^t g_A(W_t - W_r - x_k)\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&\qquad - 1_{V_A}\int_0^t g_A(W_t - W_r - x_k)\widetilde{\gamma}_{k-1}(dr, x'_{k^c})\\
&\qquad + \int_0^t h_A(W_t - W_r - x_k)\widetilde{\gamma}_{k-1}(dr, x_{k^c})\\
&\qquad - \int_0^t h_A(W_t - W_r - x_k)\widetilde{\gamma}_{k-1}(dr, x'_{k^c})\\
&\qquad - \int_0^t j_A(W_t - W_r - x_k)\widetilde{\gamma}_{k-1}(dr, x'_{k^c})\\
&\qquad + \int_0^t j_A(W_t - W_r - x'_k)\widetilde{\gamma}_{k-1}(dr, x'_{k^c})\\
&\qquad + 1_{V_A^c}\int_0^t g_A(W_t - W_r - x_k)[\widetilde{\gamma}_{k-1}(dr, x_{k^c}) - \widetilde{\gamma}_{k-1}(dr, x'_{k^c})]\\
&\quad =: I_{12} - I_{13} + I_{14} - I_{15} - I_{16} + I_{17} + I_{18}.
\end{aligned}$$

Since $|g_A| \le \log(1/A)$, for some constant ν_{k-1}

$$\begin{aligned}
\mathbb{E}\,|I_{12}|^p &\le c(\log(1/A))^p\mathbb{E}\,[L_k(t,x)^p; V_A]\\
&\le c(\log(1/A))^p\Big(\mathbb{E}\,L_k(t,x)^{2p}\Big)^{1/2}\Big(\mathbb{P}(V_A)\Big)^{1/2} \le c(g^+)^{\nu_{k-1}p}A^p,
\end{aligned}$$

and similarly for I_{13}. We bound I_{14} and I_{15} just as we did I_2 and bound I_{16} and I_{17} as we did I_4.

We turn to I_{18}. Let $f(r) = g_A(W_t - W_r - x_k)$ and

$$f_A(t) = \frac{1}{A^{12}}\int_t^{t+A^{12}} f(u)\,du.$$

On V_A^c we have

$$\begin{aligned}
|f(r) - f(s)| &= |g_A(W_t - W_r - x_k) - g_A(W_t - W_s - x_k)|\\
&\le A^{-1}|W_r - W_s| \le A^{-2}|r-s|^{1/4},
\end{aligned}$$

and therefore

$$|f(t) - f_A(t)| \le A^{-2}(A^{12})^{1/4} = A.$$

Using integration by parts, we write

$$\begin{aligned}
I_{18} = 1_{V_A^c} \int_0^t & [f(r) - f_A(r)]\widetilde{\gamma}_{k-1}(dr, x_{k^c}) \\
& - 1_{V_A^c} \int_0^t [f(r) - f_A(r)]\widetilde{\gamma}_{k-1}(dr, x'_{k^c}) \\
& + 1_{V_A^c} f_A(t)[\widetilde{\gamma}_{k-1}(t, x_{k^c}) - \widetilde{\gamma}_{k-1}(t, x'_{k^c})] \\
& - 1_{V_A^c} \int_0^t [\widetilde{\gamma}_{k-1}(r, x_{k^c}) - \widetilde{\gamma}_{k-1}(r, x'_{k^c})] f_A(dr) \\
=: & \; I_{19} - I_{20} + I_{21} - I_{22}.
\end{aligned}$$

We bound

$$\mathbb{E}\,|I_{19}|^p \le cA^p (g^+)^{\nu'_{k-1} p}$$

for some constant ν'_{k-1} and similarly for I_{20}. By the induction hypothesis and the fact that $|f_A|$ is bounded by $\log(1/A) \le cA^{-p}$,

$$\mathbb{E}\,|I_{21}|^p \le cA^{-p}(g^+)^{\nu''_{k-1} p} |x_{k^c} - x'_{k^c}|^{a_{k-1} p}.$$

Finally, since $|f'_A| \le \|f_A\|_\infty A^{-12} \le cA^{-13}$,

$$\mathbb{E}\,|I_{22}|^p \le c(g^+)^{\nu'''_{k-1} p} A^{-13p} |x_{k^c} - x'_{k^c}|^{a'_{k-1} p}.$$

If we combine all the terms, we see that the left hand side of (A.6) is bounded by

$$c(g^+)^{\nu_k p}[A^{b_k p} + A^{-b'_k p}|x - x'|^{a''_{k-1} p} + A^p].$$

Setting $A = |x - x'|^{a''_{k-1}/(2b'_k)} \wedge (2M)^{-1}$ completes the proof of (b) for $k > 2$.

The proof of (b) for $k > 2$ is almost identical to the $k = 2$ case. □

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