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Stochastic calculus and the continuity of local times of Lévy processes

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Stochastic Calculus and the Continuity of Local Times of Lévy Processes

Richard Bass* and Davar Khoshnevisan

1. Introduction. Let Z_t be a one dimensional Lévy process with characteristic function

$$E\exp(iuZ_t) = \exp(-t\psi(u)),$$

where

$$\psi(u) = -iau + \frac{1}{2}\sigma^2 u^2 - \int_{-\infty}^{\infty} (e^{iuz} - 1 - iuz1_{(|z|\le 1)})\nu(dz). \tag{1.1}$$

Here ν satisfies $\int(1 \wedge z^2)\nu(dz) < \infty$.

We are interested in those Lévy processes for which 0 is regular for $\{0\}$ and either $\sigma^2 > 0$ or $\nu(\mathbb{R} - \{0\}) = \infty$. In this case (see [K]) there exists a bounded continuous function g that is a density for the 1-resolvent:

$$\int f(x)g(x-y)dx = E^y \int_0^\infty e^{-t} f(Z_t)dt, \qquad f \ge 0, \quad y \in \mathbb{R}. \tag{1.2}$$

(If $G(x,y)$ is the Green function for Z_t killed at an independent exponential time with parameter 1, the relationship between g and G is given by $g(x) = G(0,x) = G(a, a+x)$ for any $a \in \mathbb{R}$ and $G(x,y) = g(y-x)$.)

For each x,

$$g(x) = \frac{1}{2\pi}\int e^{-iux}\frac{1}{1+\psi(u)}du. \tag{1.3}$$

For each $x, g(x-\cdot)$ is the 1-potential of an additive functional L_t^x that is continuous in t. Moreover, a version of $L_t^x(\omega)$ may be chosen that is jointly measurable in (x,t,ω). See [GK] for details. L_t^x is called the local time of Z_t at x. L_t^x is also a density of occupation time measure: if $f \ge 0$,

$$\int_0^t f(Z_s)ds = \int f(x)L_t^x dx, \text{ a.s..} \tag{1.4}$$

A number of people have studied the question of the continuity of L_t^x in the space variable (see [Bo], [Me], [GK] and [MT]), culminating in the works [B1], [BH], and [B2], where a necessary and sufficient condition for the joint continuity of L_t^x in t and x is given.

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The purpose of this paper is to give a stochastic calculus proof of the following sufficient condition for joint continuity. Let $\varphi : \mathbb{R} \to [0,\infty)$ be defined by

$$\varphi^2(x) = \frac{1}{\pi}\int (1-\cos ux)\mathrm{Re}\frac{1}{1+\psi(u)}du. \tag{1.5}$$

Let $d(a,b) = \varphi(b-a)$ and let $H(u)$ be the logarithm of the smallest number of d-balls of radius less than u that are needed to cover $[-1,1]$. Define

$$F(\delta) = \int_0^\delta (H(u))^{\frac{1}{2}}du. \tag{1.6}$$

Theorem 1.1. *(a) If $F(0+) < \infty$, then L_t^x has a jointly continuous version.*

(b) For each t,

$$\limsup_{\delta\downarrow 0}\ \sup_{\{a,b:\varphi(a-b)<\delta\}}\ \sup_{s\le t}\frac{|L_s^a - L_s^b|}{F(\varphi(a-b))} \le 2(\sup_x L_t^x)^{1/2}, \quad \text{a.s.}$$

Theorem 1.1(a) was first proved in [BH], where it was also remarked that the entropy condition was equivalent to one involving the monotone rearrangement of φ. Part (b) was also proved in [BH], with, however, the constant 2 replaced by a larger constant (namely 416). In [B2] it was shown that part (b) holds with the constant 2 under the additional assumption that φ is regularly varying (but not slowly varying) and that the constant 2 is sharp. (The principle result of [B2] was that the condition $F(0+) < \infty$ is necessary as well as sufficient for joint continuity.) Marcus and Rosen [MR] have recently obtained necessary and sufficient conditions for the joint continuity of local times of certain Markov processes. Theorem 1.1 for symmetric Lévy processes is a special case of their results.

In Section 2 we prove Theorem 1.1 assuming that $\operatorname{ess\,sup}_x L_t^x < \infty$, a.s. We establish this latter fact in Section 3.

2. Modulus of continuity. Our proof is modeled after that of [McK]. Let us begin by assuming for this section that $\operatorname{ess\,sup}_x L_t^x < \infty$, a.s. Let R be an exponential variable with parameter 1, independent of Z_t. Since $g(x-\cdot)$ is the 1-potential of L_t^x, we have

$$E^a L_R^b = g(b-a). \tag{2.1}$$

Proposition 2.1. $|g(a) - g(b)| \le \varphi^2(a-b)$.

Proof. Let $T_x = \inf\{t : Z_t = x\}, S = T_a \wedge T_b$. Since L_t^x increases only when Z_t is at x, the strong Markov property at time S yields

$$\begin{aligned}|g(a) - g(b)| &= |E^0 L_R^a - E^0 L_R^b| = |E^0[E^{Z_S} L_R^a - E^{Z_S} L_R^b; S \le R]| \\ &\le E^0|E^{Z_S} L_R^a - E^{Z_S} L_R^b| \\ &= E^0[|E^a L_R^a - E^a L_R^b|; S = T_a] + E^0[|E^b L_R^a - E^b L_R^b|; S = T_b] \\ &= |g(0) - g(b-a)| P^0(S = T_a) + |g(a-b) - g(0)| P^0(S = T_b)\end{aligned}$$

Since $g(x) = E^0 L_R^x \le E^x L_R^x = g(0)$, then

$$|g(a) - g(b)| \le 2g(0) - g(b-a) - g(a-b).$$

By (1.3) and (1.5), the right hand side equals $\varphi^2(a-b)$. □

Using (2.1) and the Markov property,

$$M_t^a = g(a - Z_{t\wedge R}) - g(a - Z_0) - L_{t\wedge R}^a \tag{2.2}$$

is a martingale with $M_0 = 0$. Fix a and b and let $N_t = M_t^a - M_t^b$. Let $L_t^* = \operatorname{ess\,sup}_x L_t^x$.

Proposition 2.2. $\langle N, N\rangle_t \le 2\varphi^2(a-b) L_t^*$

Proof. Let N^c, N^d be the continuous and purely discontinuous parts of N_t, respectively. We first estimate $\langle N^d, N^d\rangle_t$.

Let

$$W(x,z) = [\{g(a-(x+z)) - g(a-x)\} - \{g(b-(x+z)) - g(b-x)\}]. \tag{2.3}$$

Since L_t^a and L_t^b are both continuous in t, the jumps of N_t are the jumps of $g(a-Z_t)$ $-g(b-Z_t)$. Hence

$$[N^d, N^d]_t = \sum_{s\le t} \Delta N_s^2 = \sum_{s \le t\wedge R} (W(Z_{s-}, \Delta Z_s))^2.$$

By the definition of Lévy measure, $E\sum_{s\le t} 1_A(\Delta Z_s) = \nu(A)t$ if A is a subset of $\mathbb{R}$ that is a positive distance from 0. By the Markov property and the translation invariance of the increments of Z_t, $\sum_{s\le t} 1_A(\Delta Z_s) - \nu(A)t$ is a martingale. Taking the stochastic integral of $1_B(Z_{s-})$ with respect to this martingale, we see that $\sum_{s\le t} h(Z_{s-}, \Delta Z_s) -$

$\int_0^t \int h(Z_{s-}, z)\nu(dz)\,ds$ is again a martingale, where $h(x,z) = 1_B(x)1_A(z)$. Taking linear combinations and limits, we deduce that

$$[N^d, N^d]_{t\wedge R} - \int_0^{t\wedge R} \int W(Z_{s-}, z)^2 \nu(dz)\,ds$$

is a local martingale. Hence it follows that

$$\langle N^d, N^d\rangle_t = \int_0^{t\wedge R} \int W(Z_{s-}, z))^2 \nu(dz)ds.$$

Since Z_t has only countably many jumps, we get

$$\begin{aligned}
(2.4) \qquad \langle N^d, N^d\rangle_t &= \int_0^{t\wedge R} \int (W(Z_s, z))^2 \nu(dz)ds \\
&\le \int \int (W(x,z))^2 L_t^x dx\, \nu(dz) \\
&\le L_t^* \int \int (W(x,z))^2 dx\, \nu(dz) \\
&= \frac{L_t^*}{2\pi} \int \int |\widehat{W}(u,z)|^2 du\, \nu(dz) \qquad \text{(Plancherel's theorem)}
\end{aligned}$$

where $\widehat{W}(u,z)$ is the Fourier transform of $W(\cdot, z)$, z fixed.

By (2.3),

$$\begin{aligned}
\widehat{W}(u,z) &= \widehat{g}(-u)(\{e^{iu(a-z)} - e^{iua}\} - \{e^{iu(b-z)} - e^{iub}\}) \\
&= \widehat{g}(-u)e^{iua}(e^{-iuz} - 1)(1 - e^{iu(b-a)}).
\end{aligned}$$

Since $|e^{iu\theta} - 1|^2 = 2(1 - \cos\theta)$,

$$\begin{aligned}
(2.5) \quad \int\int |\widehat{W}(u,z)|^2 \nu(dz)du &= 2\int |\widehat{g}(-u)|^2 |1 - e^{iu(b-a)}|^2 \int (1 - \cos uz)\nu(dz)\,du \\
&= 4\int (1 - \cos(u(b-a)))|\widehat{g}(u)|^2 \mathrm{Re}\,\psi^d(u)\,du,
\end{aligned}$$

where $\psi^d(u) = \psi(u) - \frac{1}{2}\sigma^2 u^2$. Substituting (2.5) in (2.4), we obtain

$$(2.6) \qquad \langle N^d, N^d\rangle_t \le \frac{2L_t^*}{\pi} \int (1 - \cos(u(b-a)))|\widehat{g}(u)|^2 \mathrm{Re}\,\psi^d(u)\,du.$$

Next we estimate $\langle N^c, N^c\rangle_t$. If f is a smooth function and we write K_t for the martingale part of $f(Z_{t\wedge R})$, then by Itô's formula,

$$K_t^c = \int_0^{t\wedge R} f'(Z_{s-})\sigma dB_s,$$

where B_t is a standard Brownian motion. Then

$$
\begin{aligned}
(2.7) \qquad \langle K^c, K^c\rangle_t &= \sigma^2 \int_0^{t\wedge R} (f'(Z_{s-}))^2 ds = \sigma^2 \int_0^{t\wedge R} (f'(Z_s))^2 ds \\
&\le \sigma^2 \int (f'(x))^2 L_t^x dx \\
&\le \sigma^2 L_t^* \int (f'(x))^2 dx \\
&= \sigma^2 L_t^* \frac{1}{2\pi} \int |\widehat{f'}(u)|^2 du \qquad \text{(Plancherel)} \\
&= \sigma^2 L_t^* \frac{1}{2\pi} \int |u|^2 |\hat{f}(u)|^2 du.
\end{aligned}
$$

Approximating $g_{ab}(\cdot) = g(a-\cdot) - g(b-\cdot)$ by smooth functions in a suitable way, taking limits, and noting that $\widehat{g}_{ab}(u) = \widehat{g}(-u)(e^{iua} - e^{iub})$, we get

$$
\begin{aligned}
\langle N^c, N^c\rangle_t &\le \sigma^2 \frac{L_t^*}{2\pi} \int u^2 |\widehat{g}(-u)|^2 |e^{iua} - e^{iub}|^2 du \\
&= \frac{2L_t^*}{\pi} \int \frac{\sigma^2 u^2}{2} |\widehat{g}(u)|^2 (1 - \cos(u(b-a))) du.
\end{aligned}
$$

Adding to (2.6) yields

$$
\begin{aligned}
(2.8) \qquad \langle N, N\rangle_t &= \langle N^c, N^c\rangle_t + \langle N^d, N^d\rangle_t \\
&\le \frac{2L_t^*}{\pi} \int |\widehat{g}(u)|^2 \mathrm{Re}\, \psi(u)(1 - \cos(u(b-a))) du.
\end{aligned}
$$

Finally, from (1.1), $\mathrm{Re}\, \psi(u) \ge 0$. So

$$
\begin{aligned}
(2.9) \qquad \varphi^2(x) &= \frac{1}{\pi} \int (1 - \cos ux) \mathrm{Re} \frac{1}{1+\psi(u)} du \\
&= \frac{1}{\pi} \int (1 - \cos ux) \frac{\mathrm{Re}(1 + \overline{\psi(u)})}{|1+\psi(u)|^2} du \\
&= \frac{1}{\pi} \int (1 - \cos ux) |\widehat{g}(u)|^2 (1 + \mathrm{Re}\, \psi(u)) du,
\end{aligned}
$$

since $\widehat{g}(u) = (1+\psi(u))^{-1}$. Comparing (2.9) to (2.8) proves the proposition. □

Proposition 2.3. *Let $\epsilon > 0$. There exists $J_0 > 0$ depending on ϵ such that if X_t is any square integrable martingale with jumps bounded in absolute value by J_0 and with $\langle X, X\rangle_t$ continuous, then $\exp(X_t - (1+\epsilon)\langle X, X\rangle_t/2)$ is a positive supermartingale.*

Proof. Take J_0 small enough so that $|e^x - 1 - x| \le (1+\epsilon)x^2/2$ if $|x| \le J_0$. Let

$$Y_t = X_t - (1+\epsilon)\langle X, X\rangle_t/2.$$

By Itô's formula,

$$e^{Y_t} = 1 + \int_0^t e^{Y_{s-}} dY_s + \frac{1}{2}\int_0^t e^{Y_{s-}} d\langle Y^c, Y^c\rangle_s + \sum_{s\le t}(e^{Y_s} - e^{Y_{s-}} - e^{Y_{s-}}\Delta Y_s)$$

$$= 1 + \int_0^t e^{Y_{s-}} dX_s - \frac{(1+\epsilon)}{2}\int_0^t e^{Y_{s-}} d\langle X, X\rangle_s + \frac{1}{2}\int_0^t e^{Y_{s-}} d\langle X^c, X^c\rangle_s$$
$$+ \sum_{s\le t} e^{Y_{s-}}(e^{\Delta Y_s} - 1 - \Delta Y_s)$$

$$= 1 + \text{ local martingale } - \frac{\epsilon}{2}\int_0^t e^{Y_{s-}} d\langle X^c, X^c\rangle_s - \frac{(1+\epsilon)}{2}\int_0^t e^{Y_{s-}} d\langle X^d, X^d\rangle_s$$
$$+ \sum_{s\le t} e^{Y_{s-}}(e^{\Delta X_s} - 1 - \Delta X_s)$$

Since $\langle X^d, X^d\rangle_t - \sum_{s\le t}(\Delta X_s)^2$ is a local martingale,

$$\text{(2.10)}\quad e^{Y_t} = 1 + \text{ local martingale } - \frac{\epsilon}{2}\int_0^t e^{Y_{s-}} d\langle X^c, X^c\rangle_s + \text{ local martingale}$$
$$- \frac{1+\epsilon}{2}\sum_{s\le t} e^{Y_{s-}}(\Delta X_s)^2 + \sum_{s\le t} e^{Y_{s-}}(e^{\Delta X_s} - 1 - \Delta X_s).$$

But $e^{\Delta X_s} - 1 - \Delta X_s - (1+\epsilon)(\Delta X_s)^2/2 \le 0$ by our selection of J_0. Hence (2.10) exhibits $\exp(Y_t)$ as a local martingale minus an increasing process. □

Write P for P^0.

Corollary 2.4. $P(\sup_{s\le t}|X_s| > \lambda + (1+\epsilon)\langle X, X\rangle_t/2) \le 2e^{-\lambda}$.

Proof. Reducing the continuous part of X_t by stopping times, we may assume X_t bounded, as long as our probability bound does not depend on the L^∞ norm of X_t. We can then write $e^{Y_t} = K_t - V_t$, where K_t is a martingale with $K_0 \equiv 1$ and V_t an increasing process with $V_0 \equiv 0$. Then by Doob's inequality,

$$P(\sup_{s\le t} e^{Y_s} > e^\lambda) \le P(\sup_{s\le t} K_s > e^\lambda)$$
$$\le e^{-\lambda}EK_t = e^{-\lambda}EK_0 = e^{-\lambda}.$$

This proves $P(\sup_{s\le t} X_s > \lambda + (1+\epsilon)\langle X, X\rangle_t/2) \le \exp(-\lambda)$. Applying the same argument to $-X$ proves the corollary. □

Under the assumption $L_t^* < \infty$, a.s., we can now prove Theorem 1.1.

Proof of Theorem 1.1. Let $N_t = M_t^a - M_t^b$ as above, $F(\delta)$ defined by (1.5). Since the potentials of $L_{t\wedge R}^a$ and $L_{t\wedge R}^b$ are bounded. N_t is square integrable ([DM], p.193).

Clearly $F(\delta) \to 0$ as $\delta \to 0$. Also, $\varphi(\delta) \to 0$ as $\delta \to 0$ by the continuity of g, hence $H(u) \to \infty$ as $u \to 0$, hence $\delta/F(\delta) \to 0$ as $\delta \to 0$.

Let α, β be > 0 such that $\alpha\beta > 1$, let $\epsilon > 0$, set $\delta = |b-a|$, and set $\eta = \varphi(\delta)$. Let

$$X_t = \beta F(\eta)\eta^{-2} N_t.$$

Since the jumps of N_t are bounded by $2\sup_x |g(x-a) - g(x-b)| \le 2\varphi^2(b-a)$, the jumps of X_t are bounded by $2\beta F(\eta)$, which will be less than the J_0 of Proposition 2.3 if δ is small.

Now apply Corollary 2.4: if δ is sufficiently small,

$$\begin{aligned}
(2.11)\quad & P(\sup_{s\le t} |M_s^a - M_s^b| > \alpha F(\eta) + (1+\epsilon)\beta F(\eta) L_t^*) \\
& \le P(\sup_{s\le t} |N_s| > \alpha F(\eta) + \frac{(1+\epsilon)}{2}\beta \frac{F(\eta)}{\eta^2}\langle N, N\rangle_t) \qquad \text{(Proposition 2.2)} \\
& = P(\sup_{s\le t} |X_s| > \alpha\beta F^2(\eta)/\eta^2 + \frac{1+\epsilon}{2}\langle X, X\rangle_t) \\
& \le \exp(-\alpha\beta F^2(\eta)/\eta^2).
\end{aligned}$$

A standard metric entropy argument (see, e.g., [D]) and (2.11) shows that we can find a version of M_t^x that is jointly continuous in $t \in [0, R)$ and $x \in \mathbb{Q}$ and such that for each $K > 0$,

$$(2.12)\qquad P(\limsup_{\eta\downarrow 0} \sup_{\{a,b\in\mathbb{Q}\cap[-K,K]:\varphi(a-b)<\eta\}} \sup_{s\le t} \frac{|M_s^a - M_s^b|}{F(\varphi(a-b))} > c(\alpha + (1+\epsilon)\beta L_t^*) = 0$$

for each α, $\beta > 0$ such that $\alpha\beta > 1$. Here $\mathbb{Q}$ denotes the rationals. By being a bit more careful with the constants in the metric entropy argument, one can show that one can in fact take $c = 1$.

Fix an ω not in the null set for any α, β, ϵ rational, K a positive integer, take $K \ge \sup_{s\le t}(|Z_s| + 1)$, $\alpha \in [(L_t^*(\omega))^{1/2}, (1+\epsilon)(L_t^*(\omega))^{1/2}]$, and $\beta = (1+\epsilon)/\alpha$, and then let $\epsilon \to 0$. We thus get

$$(2.13)\qquad \limsup_{\eta\downarrow 0} \sup_{\{a,b\in\mathbb{Q}:\varphi(a-b)<\eta\}} \sup_{s\le t} \frac{|M_s^a - M_s^b|}{F(\varphi(a-b))} \le 2(L_t^*)^{1/2}, \quad \text{a.s.}$$

By Proposition 2.1, $|g(x-a) - g(x-b)| \le \varphi^2(\delta)$. Since $\eta = o(F(\eta))$ as $\eta \to 0$, (2.2) yields (2.13) with $M_s^a - M_s^b$ replaced by $L_{s\wedge R}^a - L_{s\wedge R}^b$. Arguing as in [GK], one can find a version of L_t^x that is jointly continuous in $t \in [0, R)$ and $x \in \mathbb{R}$, that is still an occupation time density, and that satisfies

$$\limsup_{\eta \downarrow 0} \sup_{\{a,b:\varphi(a-b)<\eta\}} \sup_{s \le t \wedge R} \frac{|L_s^a - L_s^b|}{F(\varphi(a-b))} \le 2(L_t^*)^{1/2}, \quad \text{a.s.}$$

Finally, using the strong Markov property at R and performing a renewal argument yields Theorem 1.1. □

3. Essential boundedness. It remains to show that $L_t^* < \infty$, a.s., under the hypotheses of Theorem 1.1. Let

$$J_t^x(r) = \int_0^t \frac{1}{2r} 1_{[x-r,x+r]}(Z_s)ds. \tag{3.1}$$

Clearly $J_t^x(r)$ is bounded by $t/2r$. Let $\rho(x) = 1_{[-1,1]}(x)/2$, $\rho_r(x) = r^{-1}\rho(x/r)$. Note $J_t^x(r) = L_t^{\cdot} * \rho_r(x)$, where $*$ denotes convolution.

Proposition 3.1. *If $\epsilon > 0$, there exists $K > 0$ such that*

$$\sup_{r \le 1} P(\sup_x J_t^x(r) > K) < \epsilon.$$

Proof. Since $J_t^x(r) = L_t^{\cdot} * \rho_r(x)$, the 1–potential of $J_t^x(r)$ is $g * \rho_r(x - \cdot)$. By Proposition 2.1,

$$|g * \rho_r(a) - g * \rho_r(b)| \le \varphi^2(a-b). \tag{3.2}$$

Let $W_r(x,z) = \int W(y,z)\rho_r(x-y)dy$. If we let $N_t(r)$ be the martingale part of $g * \rho_r(a - Z_{t\wedge R}) - g * \rho_r(b - Z_{t\wedge R})$, then as in the proof of Proposition 2.2,

$$\langle N^d(r), N^d(r)\rangle_t \le \int\int (W_r(x,z))^2 L_t^x dx\, \nu(dz).$$

But by Jensen's inequality,

$$(W_r(x,z))^2 \le \int W(y,z)^2 \rho_r(x-y)dy,$$

hence

$$\begin{aligned}\langle N^d(r), N^d(r)\rangle_t &\le \int\int\int W(y,z)^2 \rho_r(x-y) L_t^x dx\, \nu(dz)\, dy \\ &= \int\int W(y,z)^2 J_t^y(r) \nu(dz) dy \\ &\le (\sup_x J_t^x(r)) \int\int W(y,z)^2 dy\, \nu(dz)\end{aligned}$$

With a similar change to estimate $\langle N^c(r), N^c(r)\rangle_t$, we get

$$\langle N(r), N(r)\rangle_t \leq 2\varphi^2(a-b)\sup_x J_t^x(r). \tag{3.3}$$

Proceeding as in Section 2, we get the joint continuity of $J_t^x(r)$ in x, with probability estimates independent of r. Take K_0 large so that $P(\sup_{s\leq t}|Z_s| > K_0 - 1) < \epsilon/2$. Using the probability estimates for the continuity of $J_t^x(r)$ in x, take $K > K_0$ large enough so that

$$P(\sup_{|x|\leq K_0} J_t^x(r) > K) < \epsilon/2.$$

This proves the proposition. □

Theorem 3.1. *Under the assumptions of Theorem 1.1,* $\operatorname{ess\,sup}_x L_t^x < \infty$, a.s.

Proof. Let $r_n = 2^{-n}$. Let $\epsilon > 0$ and choose K as in Proposition 3.1. Let

$$A_n = \{\sup_x J_t^x(r_n) > K\}$$

If $J_t^x(r_n) > K$ for some x, then since

$$J_t^x(r_n) = \frac{1}{2}[J_t^{x+r_{n+1}}(r_{n+1}) + J_t^{x-r_{n+1}}(r_{n+1})],$$

we get $\sup_y J_t^y(r_{n+1}) > K$. Therefore $A_n \subseteq A_{n+1}$.

It follows that

$$P(\sup_{x,n} J_t^x(r_n) = \infty) \leq P(\bigcup_{n=1}^{\infty} A_n) = \lim_n P(A_n) \leq \epsilon.$$

Since ϵ is arbitrary, $\sup_{x,n} J_t^x(r_n) < \infty$, a.s. Our result then follows by Lebesgue's differentiation theorem. □

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