HÖLDER DOMAINS

AND THE BOUNDARY HARNACK PRINCIPLE

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We will prove the following version of the boundary Harnack principle (see below for the definition of a "Hölder domain," "L-harmonic function," etc.).

Theorem 1. *Suppose that L is a uniformly elliptic operator in divergence form, $D \subset \mathbb{R}^d$, $d \geq 3$, D is a Hölder domain of order α, $\alpha \in (0,1]$, $V \subset \mathbb{R}^d$ is an open set and K is a compact subset of V. Then there exists $c < \infty$ such that whenever u and v are positive and L-harmonic in $V \cap D$, u and v vanish continuously on the regular points of $\partial D \cap V$ and u and v are bounded in a neighborhood of $\partial D \cap V$, then*

$$\frac{u(x)}{v(x)} \leq c\frac{u(y)}{v(y)} \qquad \text{for all } x, y \in K \cap D.$$

The constant c depends on L only through the constant c_L defined in (1) below. Of course, c also depends on D, V and K.

The above result shows that the boundary Harnack principle holds in all Hölder domains. It was first proven for the range $\alpha \in (1/2, 1]$ by Bass and Burdzy (1990a), Theorems 3.5 and 4.5. It was subsequently extended by Bañuelos (unpublished) to the full range of α provided the domain also satisfies a uniform capacity condition on the boundary. In Bañuelos (1990) a class of domains called uniformly Hölder domains of order β was introduced. A boundary Harnack principle may be obtained for such domains (for all $\beta \in (0,1)$) by a variation of the proof of Theorem 1. In order to make this note compact, we refer

Research supported in part by NSF grants DMS 8958449, DMS 8822053 and DMS 8901255 .

the reader to Bass and Burdzy (1990a) for a discussion of the history of the problem and related results. We are going to use several results from Bass and Burdzy (1990a, 1990b) and we will follow the notation of those papers closely for the convenience of the reader.

A domain $D \subset \mathbb{R}^d$ is called a *Hölder domain* of order α, $\alpha \in (0,1]$, if for every $x \in \partial D$ there exist a neighborhood U of x, an orthonormal coordinate system $CS = CS_x$, and a function $\Gamma = \Gamma_x$, $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$, that is Hölder continuous of order α (i.e. $|\Gamma(y) - \Gamma(z)| \le c_x |y-z|^\alpha$ for some $c_x < \infty$ and all y and z) such that

$$D \cap U = \{y = (y^1, \dots, y^d) \in U : y^d > \Gamma(y^1, \dots, y^{d-1}) \text{ in } CS\}.$$

We call L a *uniformly elliptic operator in divergence form* if

$$Lf(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x^i} \left(a_{ij}(x) \frac{\partial f}{\partial x^j} \right)(x)$$

where the a_{ij} are symmetric and for some $c_L < \infty$

$$(1) \qquad c_L^{-1} \sum_{j=1}^{d} (y^j)^2 \le \sum_{i,j=1}^{d} a_{ij}(x) y^i y^j \le c_L \sum_{j=1}^{d} (y^j)^2, \quad x, y \in \mathbb{R}^d.$$

We suppose the a_{ij} are smooth, but all constants below depend on L only through c_L and not on the smoothness of the a_{ij}. A function u is called *L-harmonic* in D if $Lu(x) = 0$ for all $x \in D$.

We will write $x = (x^1, \dots, x^d) = (\widetilde{x}, x^d)$, $\widetilde{x} = (x^1, \dots, x^{d-1})$, for $x \in R^d$. Suppose $\alpha \in (0,1]$, $M \in (1, \infty)$ and $\Gamma : \mathbb{R}^{d-1} \to [-M, M]$ is a function satisfying

$$|\Gamma(\widetilde{x}) - \Gamma(\widetilde{y})| \le M \min(|\widetilde{x} - \widetilde{y}|^\alpha, 1) \quad \text{for } \widetilde{x}, \widetilde{y} \in \mathbb{R}^{d-1}.$$

Let

$$
\begin{aligned}
D &= \{x \in \mathbb{R}^d : x^d > \Gamma(\widetilde{x})\}, \\
\delta(x) &= x^d - \Gamma(\widetilde{x}), \\
B(x,r) &= \{y \in \mathbb{R}^d : |x-y| < r\}, \\
\Delta(x,a,r) &= \{y \in \mathbb{R}^d : |\widetilde{y} - \widetilde{x}| < r, \Gamma(\widetilde{y}) < y^d < \Gamma(\widetilde{y}) + a\}, \\
\partial^s \Delta(x,a,r) &= \{y \in \partial\Delta(x,a,r) : |\widetilde{y} - \widetilde{x}| = r\}, \\
\partial^g \Delta(x,a,r) &= \partial\Delta(x,a,r) \setminus (\partial D \cup \partial^s \Delta(x,a/2,r)).
\end{aligned}
$$

The diffusion process corresponding to L will be denoted (X_t, P^x) and the hitting time of a Borel set Q will be denoted $T(Q)$, i.e.,

$$
T(Q) = \inf\{t > 0 : X_t \in Q\}.
$$

We will tacitly assume that all constants $c_1, c_2, \ldots$ belong to $(0,\infty)$. Their value will remain constant within a lemma and its proof but may change between the proofs.

Suppose $A > 0$, $R > 2A$, and define an L-harmonic function h in $\Delta(0,2A,2R)$ by

$$
h(x) = P^x(X_{T(\partial\Delta(0,2A,2R))} \in \partial^g \Delta(0,2A,2R)).
$$

For $a > 0$, $r > 0$ let

$$
\begin{aligned}
W(x,a,r) &= \{y \in \Delta(0,2A,2R) : |\widetilde{y} - \widetilde{x}| < r, h(x) < a\}, \\
\partial^s W(x,a,r) &= \{y \in \partial W(x,a,r) : |\widetilde{y} - \widetilde{x}| = r\}, \\
\partial^u W(x,a,r) &= \partial W(x,a,r) \setminus (\partial D \cup \partial^s W(x,a,r)).
\end{aligned}
$$

Fix some $\alpha_0 \in (0,\alpha)$ and let $\alpha_1 = 1/(\alpha_0 - 1)$.

Lemma 1. *For each $R_1 < 2R$ there are c_1 and $a_0 > 0$ such that*

$$
P^y(T(\partial D) < T(\partial B(y, |\log a|^{\alpha_1}))) > c_1
$$

for all $y \in W(0, a, R_1)$, $a < a_0$.

Proof. (cf. proof of Lemma 2.6 (i) of Bass and Burdzy (1990b)). Let $y \in W(0, a, R_1)$, $b = |\log a|$,

$$\begin{aligned} D_1 &= \{x \in \mathbb{R}^d : |\widetilde{x} - \widetilde{y}| < b^{\alpha_1}/8, \Gamma(\widetilde{y}) < x^d < y^d + 2b^{\alpha_1 \alpha}\}, \\ D_2 &= \{x \in \mathbb{R}^d : |\widetilde{x} - \widetilde{y}| < b^{\alpha_1}/8, |x^d - y^d| < b^{\alpha_1}/2\}, \\ D_3 &= D_1 \cap D_2, \\ N &= \{x \in \mathbb{R}^d : |\widetilde{x} - \widetilde{y}| < b^{\alpha_1}/16, x^d = y^d + (M/16^\alpha + 1) b^{\alpha_1 \alpha}\}. \end{aligned}$$

Note that $D_1 \cup D_2 \subset \Delta(0, 2A, 2R)$ for large b (i.e. for small a).

Suppose that

$$P^y(T(\partial D) < T(\partial B(y, b^{\alpha_1}))) < 1 - c_2. \tag{2}$$

We will show that this assumption leads to a contradiction (for large b) provided $c_2 > 0$ is chosen to be the p_0 of Lemma 2.4 of Bass and Burdzy (1990b). That lemma, scaling, and the following consequence of (2)

$$P^y(T(\partial D_2) = T(\partial D_3)) > c_2$$

imply that

$$P^y(T(N) < T(\partial D_1)) \geq \exp(-c_3 b^{\alpha_1 \alpha} / b^{\alpha_1}). \tag{3}$$

It is elementary to check that $\delta(x) \geq b^{\alpha_1 \alpha}$ for all $x \in N$. An obvious modification of Lemma 2.3 of Bass and Burdzy (1990a) shows that for $x \in N$

$$\begin{aligned} h(x) &= P^x(X_{T(\partial \Delta(0,2A,2R))} \in \partial^g \Delta(0, 2A, 2R)) \\ &\geq c_4 \exp(-c_5 \delta(x)^{1-1/\alpha}) \\ &\geq c_4 \exp(-c_5 b^{\alpha_1 \alpha (1-1/\alpha)}). \end{aligned} \tag{4}$$

The strong Markov property applied at $T(N)$, (3) and (4) yield

$$\begin{aligned} h(y) &= P^y(X_{T(\partial\Delta(0,2A,2R))} \in \partial^g \Delta(0,2A,2R)) \\ &\geq \exp(-c_3 b^{\alpha_1\alpha-\alpha_1} - c_5 b^{\alpha_1\alpha(1-1/\alpha)} + \log c_4) \\ &= \exp(-(c_3+c_5) b^{(\alpha-1)/(\alpha_0-1)} + \log c_4). \end{aligned}$$

Since $(\alpha-1)/(\alpha_0-1) < 1$, for large b we have $h(y) > e^{-b}$ and hence $y \notin W(0,a,R_1)$. This contradiction shows that (2) is false for large b. $\square$

Lemma 2. *There are c_1 and $a_0 > 0$ such that if $r > 0$, $|\widetilde{x}| + r < R_1 < 2R$, $a < a_0$, $y \in W(x,a,r)$, and $\widetilde{y} = \widetilde{x}$ then we have*

$$P^y(X_{T(\partial W(x,a,r))} \in \partial^s W(x,a,r)) \leq \exp(-c_1[r|\log a|^{-\alpha_1}]). \tag{5}$$

Remark. See Lemma 2.2 of Bass and Burdzy (1990a). Here we use [ ] to denote "integer part."

Proof. Let $b = |\log a|$,

$$\begin{aligned} T_1 &= \inf\{t : |X_t - X_0| = b^{\alpha_1}\}, \\ T_j &= \inf\{t > T_{j-1} : |X_t - X_{T_{j-1}}| = b^{\alpha_1}\}, \quad j \geq 2. \end{aligned}$$

By Lemma 1,

$$P^y(T(\partial W(x,a,r)) > T_1) \leq c_2 < 1.$$

The strong Markov property applied at T_1 implies

$$P^y(T(\partial W(x,a,r)) > T_2) \leq c_2^2,$$

and by induction we obtain

$$P^y(T(\partial W(x,a,r)) > T_j) \leq c_2^j = \exp(-c_3 j). \tag{6}$$

Since the event in (5) is contained in $\{T(\partial W(x,a,r)) > T_{[rb^{-\alpha_1}]}\}$, the lemma follows from (6). $\square$

Let

$$H_1 = \{X_{T(\partial\Delta(0,2A,2R))} \in \partial^s\Delta(0,A,2R)\},$$
$$H_2 = \{X_{T(\partial\Delta(0,2A,2R))} \in \partial^g\Delta(0,2A,2R)\}.$$

Lemma 3. *There is c_1 such that $P^x(H_1) \le c_1 P^x(H_2)$ for all $x \in \Delta(0,A,R)$.*

Proof. (cf. proof of Theorem 2.4 of Bass and Burdzy (1990a)) Let

$$r_k = 2R - (R/8)\sum_{i=0}^{k}(1+i)^{-2}, \quad k \ge 0.$$

Note that $r_0 < 15R/8$ and $\inf_k r_k > R$. Let

$$J_0 = \{y \in D : |\widetilde{y}| \le r_0, h(y) \ge e^{-2}\},$$
$$J_k = \{y \in D : |\widetilde{y}| \le r_k, h(y) \in [\exp(-2^{k+1}), \exp(-2^k)]\}, \quad k \ge 1.$$

By definition of the function h,

$$P^z(H_2) \ge e^{-2} \ge e^{-2}P^z(H_1)$$

for $z \in J_0$.

Let $d_m = \sup_{z\in J_m} P^z(H_1)/P^z(H_2)$. We have just proved that $d_0 \le e^{-2}$. Since $\Delta(0,A,R) \subset \bigcup_{k=0}^{\infty} J_k$, it will suffice to show that $\sup_m d_m < \infty$.

Fix some m and $z \in J_{m+1}$. Let

$$W_m = W(z, \exp(-2^{m+1}), r_m - r_{m+1}),$$
$$U_m = T(\partial W_m).$$

By the strong Markov property applied at U_m,

$$P^z(H_1) \le E^z(P^{X(U_m)}(H_1), X_{U_m} \in \partial^u W_m) + P^z(X_{U_m} \in \partial^s W_m). \tag{7}$$

Observe that $\partial^u W_m \subset J_m$ since h is a continuous function. It follows that

$$\begin{aligned} E^z(P^{X(U_m)}(H_1), X_{U_m} \in \partial^u W_m) &\le d_m E^z(P^{X(U_m)}(H_2), X_{U_m} \in \partial^u W_m) \\ &\le d_m P^z(H_2). \end{aligned} \tag{8}$$

For all m, we obviously have

$$P^z(X_{U_m} \in \partial^s W_m) \le 1 \le P^z(H_2)\exp(2^{m+2}). \tag{9}$$

Since $\alpha_1 < -1$, Lemma 2 implies that for large m

$$\begin{aligned} P^z(X_{U_m} \in \partial^s W_m) &\le \exp(-c_2[(r_m - r_{m-1})2^{-(m+1)\alpha_1}]) \\ &\le \exp(-c_3 m^{-2} 2^{-(m+1)\alpha_1}) \\ &\le m^{-2}\exp(-2^{m+2}) \le m^{-2} P^z(H_2). \end{aligned} \tag{10}$$

Combine (7), (8) and (10) to see that

$$P^z(H_1) \le d_m P^z(H_2) + m^{-2} P^z(H_2)$$

for large m. Hence

$$d_{m+1} \le d_m + m^{-2} \tag{11}$$

for large m. For any m, an analogous estimate using (9) in place of (10) shows that

$$d_{m+1} \le d_m + \exp(2^{m+2}).$$

Thus $d_m < \infty$ for every m, and in view of (11), $\sup_m d_m < \infty$. $\square$

Proof of Theorem 1. The theorem may be proved exactly like Theorems 3.5 and 4.5 of Bass and Burdzy (1990a). All we have to do is to use our present Lemma 3 in place of Theorem 2.4 of that paper. $\square$

References

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