THE BOUNDARY HARNACK PRINCIPLE FOR NON-DIVERGENCE FORM ELLIPTIC OPERATORS

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Abstract. If L is a uniformly elliptic operator in non–divergence form, the boundary Harnack principle for the ratio of positive L–harmonic functions holds in Hölder domains of order α if $\alpha > 1/2$. A counterexample shows that $1/2$ is sharp. For Hölder domains of order α with $\alpha \in (0, 1]$, the boundary Harnack principle holds provided the domain also satisfies a strong uniform regularity condition.

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1. Introduction.

The boundary Harnack principle (BHP) for the ratio of positive harmonic functions was first proved in Lipschitz domains in 1977 [**1**], [**6**], [**13**]. It is now known that the BHP also holds for the ratio of positive L-harmonic functions in Lipschitz domains when L is a uniformly elliptic operator with bounded coefficients in divergence form [**5**] and in nondivergence form [**7**]. Recently it has been shown [**2**], [**4**] that for harmonic functions and L-harmonic functions where L is in divergence form, the BHP holds even in Hölder domains.

It is the purpose of this paper to show that the BHP also holds in Hölder domains of order α when L is in nondivergence form, provided either that the domain is strongly uniformly regular or that $\alpha > 1/2$. The proofs of [**4**], which depend heavily on the symmetry of the Green function, fail in this case, and a substitute argument is necessary.

We consider uniformly elliptic operators L given by

$$Lf(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial f}{\partial x^i}(x),$$

where the a_{ij} are symmetric, and for some constant $c_L \in (0, \infty)$,

$$c_L^{-1} \sum_{j=1}^{d} (y^j)^2 \le \sum_{i,j=1}^{d} a_{ij}(x) y^i y^j \le c_L \sum_{j=1}^{d} (y^j)^2, \quad \sum_{i=1}^{d} |b_i(x)| \le c_L, \quad x, y \in \mathbb{R}^d. \tag{1.1}$$

We suppose the a_{ij} and b_i are smooth, but all constants depend on L only through the constant c_L and not on the smoothness of the a_{ij} or b_i. A function u is L-harmonic in D if $Lu(x) = 0$ for all $x \in D$.

For the definition of Hölder domain, see Section 2. A domain D is strongly uniformly regular if there exist a constant $c_I \in (0, 1)$ such that

$$|B(x, r) \cap D^c| \ge c_I r^d, \qquad x \in \partial D, r \in (0, 1], \tag{1.2}$$

where $B(x, a)$ is the ball of radius a about x and $|\cdot|$ denotes Lebesgue measure. It is easy to see that Lipschitz domains are strongly uniformly regular, and so our results are an extension of some of those in [**7**].

One of our main theorems is

Theorem 1.1. *Suppose D is a strongly uniformly regular Hölder domain of order α, $\alpha \in (0,1]$. Let K compact be contained in V open. There exists a constant c_H such that if u and v are positive L-harmonic functions in D that vanish continuously on $(\partial D) \cap V$, then*

$$u(y)/v(y) \le c_H u(x)/v(x) \quad \text{for all } x, y \in K \cap D.$$

Our second main theorem is the following.

Theorem 1.2. *Suppose D is a Hölder domain of order α, $\alpha \in (1/2, 1]$. Let K compact be contained in V open. There exists a constant c_H such that if u and v are positive L-harmonic functions in D that vanish continuously at the regular points of $(\partial D) \cap V$ and are bounded in a neighborhood of $(\partial D) \cap V$, then*

$$u(y)/v(y) \le c_H u(x)/v(x) \quad \text{for all } x, y \in K \cap D.$$

The boundedness of u and v in a neighborhood of $(\partial D) \cap V$ is essential, as the Remark following Theorem 3.3 of [**2**] shows.

The last result is sharp because we have the following counterexample.

Theorem 1.3. *For each $\alpha \in (0, 1/2)$ there exist D, K, V, L, u and v which satisfy the assumptions of Theorem 1.2 but $\inf_{x \in K \cap D} u(x)/v(x) = 0$.*

The coefficients of the operator L in Theorem 1.3 are not smooth; see however Remark 5.1 below.

We now state two results that are concerned with properties of operators in non-divergence form which are closely related to the boundary Harnack principle.

We denote the diffusion process corresponding to L (see [**12**]) by (X_t, P^x), and we define

$$T(A) = \inf\{t > 0 : X_t \in A\}.$$

For a positive L-harmonic function h in a domain D, an h-process in D is a diffusion with the transition probabilities

$$P_h^x(X_t \in dy) = P^x(X_t \in dy; T(D^c) > t) h(y)/h(x).$$

Its lifetime is denoted τ, i.e., the h-process is defined on a random time interval $[0, \tau)$ and at time τ it approaches the (Martin) boundary of D.

Theorem 1.4. *Suppose that D is a Hölder domain of order α, $\alpha \in (1/3, 1]$, and L is a uniformly elliptic operator in non-divergence form. Then there exists $c = c(D) < \infty$ such that*

$$E_h^x(\tau) < \infty$$

for all $x \in D$ and all positive L-harmonic functions h in D.

This result has been essentially proved in [**3**] as Theorem 1.1(i)(b)(C). The assumption of the strong uniform regularity made in that theorem is superfluous in view of Remark 3.3(i) of [**3**]. See [**3**] for the history of the "lifetime" problem and references.

Example 1.5. *For each $\alpha \in (0, 1/3)$ there is a Hölder domain D of order α, $x \in D$, a uniformly elliptic operator in non-divergence form, and a positive L-harmonic function h in D such that*

$$P_h^x(\tau = \infty) = 1.$$

The domain D and operator L may be constructed using the ideas from [**3**], Section 4, and Section 5 below (and some new ideas as well). We omit the proof as it is extremely long and complicated.

Section 2 contains some probability estimates for strongly uniformly regular domains. Section 3 has a Carleson estimate and the proof of Theorem 1.1, while Theorem 1.2 is proved in Section 4. Our main counterexample, Theorem 1.3, is presented in Section 5.

2. Probability estimates for strongly uniformly regular domains.

We recall the following facts about nondivergence form operators. Throughout the paper all constants c_i are in $(0, \infty)$.

(2.1) [**9**] If Q is the unit cube centered at 0 and $\varepsilon > 0$, then there exists $\delta > 0$ (depending only on ε and c_L) such that if $A \subset Q$ and $|A| \geq \varepsilon$, then

$$\inf_{|x| \leq 1/2} P^x(T(A) < T(\partial Q)) \geq \delta.$$

(2.2) (Harnack inequality; [**10**]). There exists c_1 depending only on c_L such that if $r \le 1$ and h is nonnegative and L-harmonic in $B(y,r)$, then

$$h(x_1) \le c_1 h(x_2), \qquad x_1, x_2 \in B(y, r/2).$$

(2.3) (Support theorem; [**12**], pp. 168–169) If $\psi : [0,t] \to \mathbb{R}^d$ is continuous and $\varepsilon > 0$, there exists $\delta > 0$ (depending only on ψ, t, ε, and c_L) such that

$$P^{\psi(0)}(|X_s - \psi(s)| < \varepsilon \text{ for all } s \in [0,t]) > \delta.$$

We also use the fact that for any $r \in (0,1]$, the operator corresponding to rX_{t/r^2} is another elliptic operator in nondivergence form, with coefficients satisfying (1.1) with the same constant c_L. We refer to this property as "scaling". (If the b_i are all 0, then scaling holds for any $r \in (0,\infty)$.)

An easy consequence of (2.1), (2.3) and scaling is that if $c_2 > 0$, there exists $c_3 > 0$ depending only on c_2 and c_L such that whenever $r \le 1$, $A \subset B(y,r)$ with $|A| \ge c_2 r^d$, then

$$\inf_{|x-y| \le r/2} P^x(T(A) < T(\partial B(y,r))) \ge c_3. \tag{2.4}$$

A domain $D \subset \mathbb{R}^d$ is called a Hölder domain of order α, $\alpha \in (0,1]$, if for every $x \in \partial D$, there exists a neighborhood U of x, an orthogonal coordinate system $CS = CS_x$, and a function $\Gamma = \Gamma_x$, $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$, that is Hölder continuous of order α such that

$$D \cap U = \{y = (y^1, \ldots, y^d) \in U : y^d > \Gamma(y^1, \ldots, y^{d-1}) \text{ in } CS\}.$$

As in [**2**] and [**4**], to prove the BHP it is enough to restrict attention to a domain D lying above the graph of a Hölder function. So we write $x = (x^1, \ldots, x^d) = (\widetilde{x}, x^d)$, we suppose (without loss of generality) that $\alpha \in (0,1)$, that $M \in (1,\infty)$, that $\Gamma : \mathbb{R}^{d-1} \to [-M, M]$ satisfies

$$|\Gamma(\widetilde{x}) - \Gamma(\widetilde{y})| \le M(|\widetilde{x} - \widetilde{y}|^\alpha \wedge 1), \quad \widetilde{x}, \widetilde{y} \in \mathbb{R}^{d-1},$$

and that D is given by

$$D = \{x \in \mathbb{R}^d : x^d > \Gamma(\widetilde{x})\}.$$

We suppose throughout this section and the next that D is a strongly uniformly regular domain. Define $d(z) = \text{dist}(z, \partial D)$. Note that if $z \in D$ with $d(z) < a$, we can choose $z_0 \in \partial D$ so that $|z - z_0| < a$. Then by (1.2),

$$|B(z, 2a) \cap D^c| \geq |B(z_0, a) \cap D^c| \geq c_I a^d.$$

So using (2.4), there exists $\rho \in (0, 1)$ such that

$$P^z(T(\partial B(z, 2a)) < T(\partial D)) \leq \rho. \tag{2.5}$$

Using (2.5) it is not hard to argue that every point of ∂D is regular for the Dirichlet problem for L.

We define

$$Q(x, a, r) = \{y \in D : d(y) < a, |\widetilde{x} - \widetilde{y}| < r\},$$

$$\partial^s Q(x, a, r) = \{y \in \partial Q(x, a, r) : |\widetilde{x} - \widetilde{y}| = r\},$$

and

$$\partial^u Q(x, a, r) = \partial Q(x, a, r) - (\partial D \cup \partial^s Q(x, a, r)).$$

Lemma 2.1. *There exist c_4, c_5 such that if $y \in Q \equiv Q(x, a, r)$ with $|\widetilde{x} - \widetilde{y}| < r/2$ and $r < 1$, then*

$$P^y(T(\partial Q) = T(\partial^s Q)) \leq c_4 \exp(-c_5 r/a).$$

Proof. Let $G_a = \{y \in D : d(y) < a\}$. Let $U_1 = \inf\{t : |X_t - X_0| = 2a\}$, $U_{i+1} = U_i + U_1 \circ \theta_{U_i}$, $i = 1, 2, \ldots$, where θ_t is the usual shift operator. By the strong Markov property, if $z \in G_a$,

$$\begin{aligned} P^z(U_{j+1} < T(\partial G_a)) &\leq P^z(U_1 \circ \theta_{U_j} < T(\partial G_a), U_j < T(\partial G_a)) \\ &= E^z(P^{X_{U_j}}(U_1 < T(\partial G_a)); U_j < T(\partial G_a)) \\ &\leq \sup_{w \in G_a} P^w(U_1 < T(\partial G_a)) P^z(U_j < T(\partial G_a)). \end{aligned}$$

So by (2.5) and induction,

$$P^z(U_j < T(\partial G_a)) \leq \rho^j. \tag{2.6}$$

But if $y \in Q(x,a,r/2)$,

$$P^y(T(\partial Q) = T(\partial^s Q)) \le P^y(T(\partial G_a) > U_{[r/4a]}),$$

which proves the lemma □

Lemma 2.2. *Let $R, N \in (0,1/2)$. There exist c_6, c_7 (depending on R and N) so that if $x \in D$ and $y \in Q(x,2N,R)$, then*

$$P^y(T(\partial Q(x,4N,2R)) = T(\partial^u Q(x,4N,2R))) \ge c_6 \exp(-c_7 d(y)^{\alpha-1}).$$

Proof. Let $y_0 = y$. Define y_i, $i = 1,\dots,n$, by $\widetilde{y}_i = \widetilde{y}$, $y^d_{i+1} = y^d_i + r_i$, where $r_i = (R \wedge d(y_i))/3$, and n is the first i for which $d(y_i) > 4N+1$. By elementary geometric reasoning and the fact that Γ is a Hölder function of order α, we see that the exists c_8 (depending on R and N) such that

$$n \le c_8 d(y)^{\alpha-1}. \tag{2.7}$$

(This is very similar to the proof of Lemma 2.3 of [**2**].)

Let $B_i = B(y_i, 2r_i)$, $B = \bigcup_{i=0}^n B_i$. Let h be the harmonic function that has boundary value 1 on $\partial B \cap \partial B_n$ and value 0 on the remainder of ∂B. By (2.3), there exists c_9 with $h(y_n) \ge c_9$. By (2.2), $h(y_i) \ge c_1^{-1} h(y_{i+1})$. Hence

$$h(y_0) \ge c_9 c_1^{-n}. \tag{2.8}$$

But

$$h(y_0) = P^y(T(\partial B) = T(\partial B_n)) \le P^y(T(\partial Q(x,4N,2R)) = T(\partial^u Q(x,4N,2R)),$$

which, when combined with (2.7) and (2.8) proves the lemma. □

Let $A \in (0,1/8)$, $R \in (2A,1/4)$,

$$H_1 = \{T(\partial Q(0,2A,2R)) = T(\partial^s Q(0,A,2R))\},$$

$$\partial^g Q(0,2A,2R) = \partial Q(0,2A,2R) - (\partial D \cup \partial^s Q(0,A,2R)),$$

and

$$H_2 = \{T(\partial Q(0,2A,2R)) = T(\partial^g Q(0,2A,2R))\}.$$

Theorem 2.3. *There exists c_{10} (depending on A and R) such that if $x \in Q(0,A,R)$,*

$$P^x(H_1) \leq c_{10} P^x(H_2).$$

Proof. Define $b_k = 2^{-k}, r_k = 2R - (R/8)\sum_{i=0}^{k}(1+i)^{-2}$,

$$J_k = \{y \in D : d(y) \in [Ab_{k+1}, Ab_k], |\widetilde{y}| \leq r_k\}, \quad k = 0,1,\ldots.$$

With this definition of J_k, the proof of Theorem 2.4 of [**2**] goes through with minor modifications, provided we replace the Δ's by Q's and we use Lemma 2.1 and 2.2 in place of Lemmas 2.2 and 2.3 of [**2**]. The estimates of Lemma 2.2 allow us to take α in the full range $(0,1]$. □

3. Proof of Theorem 1.1.

We continue to suppose that D is as in Section 2. In place of the path decomposition argument of [**2**], Section 3, we prove the following estimate of Carleson type.

Proposition 3.1. *Let $x \in D$ with $1/3 > R > 2d(x)$. There exists $c_{11} \in (0,\infty)$ (depending on M, α, and c_L) such that if u is L-harmonic in D and vanishes continuously on $\partial D \cap \partial Q(x,3R,3R)$, then $u(y) \leq c_{11} u(x)$ for $y \in Q(x,R,R)$.*

Proof. We normalize u so that $u(x) = 1$. Let $a = d(x)$. By (2.2), u is bounded by a constant c_{12} on $\partial^u Q(x,a,2R)$. Without loss of generality assume $\alpha < 1$.

Suppose $y \in Q(x,a,3R/2)$ and let $r = d(y)$. Let $y_0 = y$ and define $y_i, i = 1,\ldots,n$ and B_i as in Lemma 2.2. Using (2.2) and the chain of balls B_i, we get

$$u(y) \leq c_1^n u(y_n) \leq c_{13} \exp(c_{14} r^{\alpha-1}) \tag{3.2}$$

for constants c_{13}, c_{14} depending only on a and R. Hence if $u(y) > L$, then

$$d(y) \le \left(\frac{\ln(L/c_{13})}{c_{14}}\right)^{1/(\alpha-1)}. \tag{3.3}$$

Since we can bound u on $Q(x,R,R) - Q(x,a,R)$ by using (2.2), to prove our result it suffices to bound u on $Q(x,a,R)$.

Note that since $\alpha \in (0,1)$,

$$S = \sum_{i=1}^{\infty} [\ln(\rho^{-(i-1)}K/c_{13})/c_{14}]^{1/(\alpha-1)} < \infty.$$

Take K large enough so that $S < a \wedge (R/8)$. We will show that u is bounded by K on $Q(x,a,R)$.

Suppose not. Then there exists $z_1 \in Q(x,a,R)$ with $u(z_1) > K$. We construct a sequence $z_1, z_2, \ldots \in Q(x,a,3R/2)$ with $u(z_n) > \rho^{-(n-1)}K$ and $|\widetilde{z}_n - \widetilde{z}_{n-1}| \le 2r_{n-1}$ if $n > 1$, where $r_n = d(z_n)$. Suppose we have the sequence up to z_n. Since $u(z_n) \ge \rho^{-(n-1)}K$, (3.3) tells us that

$$\begin{aligned} |\widetilde{x} - \widetilde{z}_n| &\le |\widetilde{x} - \widetilde{z}_1| + |\widetilde{z}_1 - \widetilde{z}_2| + \ldots + |\widetilde{z}_{n-1} - \widetilde{z}_n| \\ &\le R + 2r_1 + \ldots + 2r_{n-1} \\ &\le R + 2S < 3R/2. \end{aligned} \tag{3.4}$$

Again by (3.3), $r_n < S < a$. Hence $z_n \in Q(x,a,3R/2)$.

Since u is 0 on $\partial D \cap Q(x,3R,3R)$,

$$\begin{aligned} \rho^{-(n-1)}K < u(z_n) &= E^{z_n} u(X(T(\partial(D \cap B(z_n,2r_n))))) \\ &\le [\sup_{\partial(D\cap B(z_n,2r_n))} u] P^{z_n}(T(\partial(D \cap B(z_n,2r_n))) < T(\partial D)) \\ &\le \rho[\sup_{\partial(D\cap B(z_n,2r_n))} u]. \end{aligned} \tag{3.5}$$

So there must exist $z_{n+1} \in \partial(D \cap B(z_n,2r_n))$ such that $u(z_{n+1}) > \rho^{-n}K$. Induction gives us our sequence $z_1, \ldots, z_n, z_{n+1}, \ldots$. But $u(z_n) \to \infty$, hence by (3.3), $r_n \to 0$. Yet by (3.4), $z_n \in Q(x,a,3R/2)$. This contradicts the assumption that u vanishes continuously on the boundary of $\partial D \cap Q(x,3R,3R)$. Therefore u is bounded by K on $Q(x,a,R)$. ▫

Proof of Theorem 1.1 Assume $R < 1/4$, $a = d(x) < R/4$, and u and v vanish continuously on $\partial D \cap Q(x, 3R, 3R)$. By multiplying by constants, we may assume $u(x) = v(x) = 1$. We wish to show u/v is bounded in $Q(x, a, R)$. The result of Theorem 1.1 follows from this special case, using (2.2) and standard techniques.

Let $y \in Q \equiv Q(x, a, R)$. Without loss of generality we may assume $x = 0$. Since $u = 0$ on $\partial D \cap Q(x, 3R, 3R)$, Proposition 3.1 and (2.2) yield

$$
\begin{aligned}
(3.6)\quad u(y) &= E^y u(X(T(\partial Q))) \\
&= E^y[u(X(T(\partial Q))); T(\partial Q) = T(\partial^s Q)] + E^y[u(X(T(\partial Q))); T(\partial Q) = T(\partial^u Q)] \\
&\le (\sup_{\partial^s Q} u) P^y(T(\partial Q) = T(\partial^s Q)) + (\sup_{\partial^u Q} u) P^y(T(\partial Q) = T(\partial^u Q)) \\
&\le c_{15} (\sup_{\partial^u Q} u)[P^y(T(\partial Q) = T(\partial^s Q)) + P^y(T(\partial Q) = T(\partial^u Q))].
\end{aligned}
$$

By Theorem 2.3, (2.2), and the fact that $u(x) = v(x) = 1$, the right hand side of (3.6) is

$$
\begin{aligned}
&\le c_{16} (\sup_{\partial^u Q} u) P^y(T(\partial Q) = T(\partial^u Q)) \\
&\le c_{17} (\inf_{\partial^u Q} v) P^y(T(\partial Q) = T(\partial^u Q)) \\
&\le c_{17} E^y[v(X(T(\partial Q))); T(\partial Q) = T(\partial^u Q)] \\
&\le c_{17} E^y v(X(T(\partial Q))) = c_{17} v(y).
\end{aligned}
$$

Hence $u(y)/v(y) \le c_{17}$. □

4. Proof of Theorem 1.2

As before we may assume that D lies above the graph of a Hölder function Γ, but now we assume $\alpha \in (1/2, 1)$ and no longer assume that D is strongly uniformly regular.

Let $\delta(x) = x^d - \Gamma(\widetilde{x})$. We define

$$\Delta(x, a, r) = \{y \in D : \delta(y) < a, |\widetilde{x} - \widetilde{y}| < r\},$$

$$\partial^s \Delta(x, a, r) = \{y \in \partial\Delta(x, a, r) : |\widetilde{x} - \widetilde{y}| = r\},$$

and

$$\partial^u \Delta(x, a, r) = \partial\Delta(x, a, r) - (\partial D \cup \partial^s \Delta(x, a, r)).$$

Using (2.2) and (2.4) we may follow the proofs of Section 2 of [**2**] almost exactly to get the following analogue of Theorem 2.3.

Let $A \in (0, 1/8)$, $R \in (2A, 1/4)$,

$$\widehat{H}_1 = \{T(\partial\Delta(0, 2A, 2R)) = T(\partial^s\Delta(0, A, 2R))\},$$

$$\partial^g\Delta(0, 2A, 2R) = \partial\Delta(0, 2A, 2R) - (\partial D \cup \partial^s\Delta(0, A, 2R)),$$

and

$$\widehat{H}_2 = \{T(\partial\Delta(0, 2A, 2R)) = T(\partial^g\Delta(0, 2A, 2R))\}.$$

Theorem 4.1. *There exists c_{18} (depending on A and R) such that if $x \in \Delta\ (0, A, R)$,*

$$P^x(\widehat{H}_1) \le c_{18} P^x(\widehat{H}_2).$$

We can now obtain a Carleson estimate in this case also, but the reasoning is slightly more delicate.

Proposition 4.1. *Let $x \in D$ with $1/3 > R > 2\delta(x)$. There exists $c_{19} \in (0, \infty)$ such that if u is L-harmonic in D, vanishes continuously at the regular points of $\partial D \cap \partial\Delta(x, 3R, 3R)$, and is bounded in a neighborhood of $\partial D \cap \partial\Delta(x, 3R, 3R)$, then $u(y) \le c_{19} u(x)$ for $y \in \Delta(x, R, R)$.*

Proof. We normalize u so that $u(x) = 1$. Let $a = \delta(x)$. By (2.2), u is bounded by a constant c_{20} on $\partial^u\Delta(x, a, 2R)$. Without loss of generality assume $\alpha < 1$.

Suppose $y \in \Delta(x, a, 3R/2)$ and let $s = \delta(y)$. Let $y_0 = y$ and define $y_i, i = 1, \dots, n$ and B_i as in Lemma 2.3 of [**2**]. Using (2.2) and the chain of balls B_i, we get

$$u(y) \le c_{21} \exp(c_{22} s^{1-1/\alpha}) \tag{4.1}$$

for constants c_{21}, c_{22} depending only on a and R. Hence if $u(y) > L$, then

$$\delta(y) \le \left(\frac{\ln(L/c_{21})}{c_{22}}\right)^{\alpha/(\alpha-1)}. \tag{4.2}$$

By Lemma 2.2 of [**2**] and (2.4), there exists $c_{23} \in (1, \infty)$ such that for each $A \le 1$,

$$P^z(T(\partial\Delta(z, A, c_{23}A)) = T(\partial^s\Delta(z, A, c_{23}A))) \le 1/4.$$

Choose N large so that $\beta \equiv N^{1-1/\alpha} < 1$. This is possible since $1 - 1/\alpha < 0$. Then take K large so that if

$$S = \sum_{i=1}^{\infty} [\ln(2^{i-1}K/c_{21})/c_{22}]^{\alpha/(\alpha-1)},$$

then $S < a \wedge (R/8Nc_{23})$. Take K larger if necessary so that $K^{1-\beta} \ge 2c_{21}^{1-\beta}$.

As in Proposition 3.1, it suffices to bound u on $\Delta(x, a, R)$. If u is not bounded by K on $\Delta(x, a, R)$, we construct a sequence $z_1, z_2, \ldots$ with $u(z_n) > 2^{n-1}K$ and $|\widetilde{z}_n - \widetilde{z}_{n-1}| < Nc_{23}s_{n-1}$ if $n > 1$, where

$$s_n = [\ln(u(z_n)/c_{21})/c_{22}]^{\alpha/(\alpha-1)}.$$

Suppose we have the sequence up to z_n. Just as in (3.4), $z_n \in \Delta(x, a, 3R/2)$. Now consider $\Delta \equiv \Delta(z_n, Ns_n, c_{23}Ns_n)$. If $y \in \partial^u\Delta$, then $\delta(y) = Ns_n$. So by (4.1),

$$\begin{aligned} u(y) &\le c_{21}\exp(c_{22}(Ns_n)^{1-1/\alpha}) = c_{21}\exp(\beta\ln(u(z_n)/c_{21})) \\ &= c_{21}^{1-\beta}u(z_n)^{\beta} \le u(z_n)/2, \end{aligned}$$

recalling $u(z_n) > K$.

Let L' be the operator in divergence form defined by

$$L'f(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x^i}\left(a_{ij}\frac{\partial f}{\partial x^j}\right)(x).$$

Since we are assuming the a_{ij} and b_i are smooth, by the Girsanov transformation (see [**12**]), the law of the diffusion corresponding to L' started at x is equivalent to the law of the diffusion corresponding to L started at x, for each $x \in \mathbb{R}^d$. Hence a point is regular for a set with respect to L if and only if it is regular for the set with respect to L', and a set is polar with respect to L if and only if it is polar with respect to L'. By the argument of [**11**], p. 44, trivially modified to apply to divergence form operators, the irregular points of D^c (with respect to L') are polar (with respect to L'). Hence the same is true with L' replaced by L. In particular,

$$u(X(T(\partial\Delta))) = 0 \text{ on } \{T(\partial\Delta) = T(\partial D)\}. \tag{4.3}$$

We then write

$$
\begin{aligned}
u(z_n) &= E^{z_n} u(X(T(\partial\Delta))) \\
&= E^{z_n}(u(X(T(\partial\Delta))); T(\partial\Delta) = T(\partial^s\Delta)) + E^{z_n}(u(X(T(\partial\Delta))); T(\partial\Delta) = T(\partial^u\Delta)) \\
&\le (\sup_\Delta u) P^{z_n}(T(\partial\Delta) = T(\partial^s\Delta)) + (\sup_{\partial^u\Delta} u) \\
&\le (1/4)(\sup_\Delta u) + u(z_n)/2.
\end{aligned}
$$

Therefore

$$\sup_\Delta u \ge 2u(z_n) > 2^n K.$$

Hence we can use induction and construct our sequence $z_1, \dots, z_n,\ z_{n+1}, \dots$ with $u(z_n) \to \infty$ and $z_n \in \Delta(x, a, 3R/2)$ for all n. By (4.2), $\delta(z_n) \to 0$. But this contradicts the assumption that u is bounded in a neighborhood of $\partial D \cap \partial\Delta(x, 3R, 3R)$. Therefore u must be bounded by K on $\Delta(x, a, R)$. □

Proof of Theorem 1.2. The proof is almost exactly the same as the proof of Theorem 1.1 in Section 3. We need to replace the Q's by Δ's, to use Theorem 4.1 in place of Theorem 2.3, to use Proposition 4.2 in place of Proposition 3.1, and to recall (4.3) in the analogue of (3.6). □

5. Counterexample.

Proof of Theorem 1.3. Fix some $\alpha \in (0, 1/2)$. We will construct a Hölder domain D_* of order α and a uniformly elliptic operator L_* in non-divergence form such that the boundary Harnack principle (in the sense of Theorem 1.2) fails for D_* and L_*. In order to simplify the notation we will discuss only the 3-dimensional case. The same idea works in higher dimensions.

Step 1. First we discuss some properties of a certain class of operators and their associated diffusions. The drift coefficients $b_i(x)$ will be identically 0 for all operators considered in this section and will not be mentioned further. Let $\beta \in (0, 1) \cup (1, 2) \cup (2, \infty)$ and set

$\nu = (\beta - 2)/(\beta - 1)$. Suppose that the coefficients of an operator $\widetilde{L}$ are given by

$$
\begin{aligned}
&\widetilde{a}_{ij}(x) = 0 && \text{if } i = 3 \text{ or } j = 3, \quad i \neq j, \\
&\widetilde{a}_{33}(x) = 1/2, && \\
&\widetilde{a}_{ii}(x) = \frac{1 - \nu(x^i/|\widetilde{x}|)^2}{2(1-\nu)}, && i = 1, 2, \\
&\widetilde{a}_{ij}(x) = \frac{-\nu x^i x^j/|\widetilde{x}|^2}{2(1-\nu)}, && i, j = 1 \text{ or } 2, \quad i \neq j,
\end{aligned}
\tag{5.1}
$$

for $x \in \mathbb{R}^3$, $\widetilde{x} \neq 0$. The value of $a_{ij}(x)$ for $\widetilde{x} = 0$ is irrelevant to us but for the sake of definiteness let $a_{ij}(x) = \delta_{ij}$ for $\widetilde{x} = 0$. Note $\beta = (2 - \nu)/(1 - \nu)$.

The process $(X_t, \widetilde{P}^x)$, $X_t = (X_t^1, X_t^2, X_t^3) = (\widetilde{X}_t, X_t^3)$, corresponding to $\widetilde{L}$ has 2 independent components, $\widetilde{X}_t$ and X_t^3. The latter is a standard Brownian motion. The first one has a skew-product decomposition (see [**8**], 7.15). Its radial part is a β-dimensional Bessel process. The angular part of $\widetilde{X}_t$ is a Brownian motion on a circle run with a clock determined by the radial part but otherwise independent of the radial part.

We will consider only positive $\beta \neq 2$. It is standard to check that $|\widetilde{X}_t|^{2-\beta}$ is a local martingale and, therefore, $f(r) = r^{2-\beta}$ is a harmonic function for the β-dimensional Bessel process. Let

$$F(q) = \{x \in \mathbb{R}^3 : |\widetilde{x}| = q\}.$$

Then

$$\widetilde{P}^x(T(F(q_0)) < T(F(q_2))) = \frac{q_1^{2-\beta} - q_2^{2-\beta}}{q_0^{2-\beta} - q_2^{2-\beta}} \tag{5.2}$$

for $x \in F(q_1)$, $0 < q_0 < q_1 < q_2$. In particular, X hits $F(0)$ with probability 1 if $\beta < 2$. Bessel diffusions have the scaling property, i.e., $|r\widetilde{X}_{t/r^2}|$ is also a β-dimensional Bessel process. Hence, if $\beta < 2$ then

$$\widetilde{P}^x(T(F(0)) < q^2) > c_1 > 0 \tag{5.3}$$

for $x \in F(q)$, where c_1 does not depend on q.

Step 2. The domain D_* will be assembled from an infinite number of building blocks. In this step we define such a block.

Recall that $\alpha < 1/2$ and fix some $\delta \in (\alpha, 1/2)$. We will define a domain $D = D(r)$ for all $r > 0$; the estmates in Steps 2-4 are valid, however, only if $r < 1$ and $r^{1/\alpha} < r^{1/\delta}/4$. Let

$$\mathbb{Y} = \mathbb{Y}(r, \delta) \stackrel{\text{df}}{=} \{(mr^{1/\delta}, nr^{1/\delta}) : m, n \text{ integers}\}.$$

Let

$$f_1(x) = r - |x|^\alpha \qquad \text{for } x \in \mathbb{R}^2,$$

$$A = \{x \in \mathbb{R}^2 : |x^1| \geq 2 \text{ or } |x^2| \geq r\},$$

$$f_2(x) = \sup_{y \in \mathbb{Y} \cup A} (f_1(x - y) \vee 0) \qquad \text{for } x \in \mathbb{R}^2,$$

$$D = D(r) = \{x \in \mathbb{R}^3 : x^3 > f_2(x^1, x^2)\}.$$

Next we define an operator $L = L_r$. Its coefficients $a_{ij}(x)$ are defined by (5.1) with $\beta = 3/2$ for $x \in \mathbb{R}^3$ such that $|x^1| < r^{1/\delta}/2$, $|x^2| < r^{1/\delta}/2$ and $x^3 > r/2$. For x such that $|x^1| < r^{1/\delta}/2$, $|x^2| < r^{1/\delta}/2$ and $x^3 \leq r/2$ we let $a_{ij}(x)$ be defined by (5.1) with $\beta = \gamma$, where γ is a large number which will be specified later. We extend a_{ij}'s periodically by letting $a_{ij}(x) = a_{ij}(x - y)$ for all $x \in \mathbb{R}^3$ and all y such that $y^3 = 0$, $\widetilde{y} \in \mathbb{Y}$. On the set of x for which $a_{ij}(x)$'s are still undefined (the set consists of the union of planes) we let $a_{ij}(x) = \delta_{ij}$.

Step 3. We will estimate the probability that the L-diffusion X will hit $Z \stackrel{\text{df}}{=} \{x \in \mathbb{R}^3 : x^3 = r\}$ before hitting D^c. Let

$$Z_j = \{x \in \mathbb{R}^3 : X^3 = r - jr^{1/\delta}\}.$$

Suppose $1 \leq j \leq (r/2)/r^{1/\delta} - 1$ and $x \in Z_j$. Find $y_x \in \mathbb{Y}$ such that $|(y_x, x^3) - x| < 2r^{1/\delta}$ and let

$$J_0(x) = \{z \in \mathbb{R}^3 : \widetilde{z} = y_x\},$$

$$J_1(x) = \{z \in \mathbb{R}^3 : |z^3 - x^3| < r^{1/\delta}/2, |\widetilde{z} - y_x| < r^{1/\delta}/4\}.$$

It follows easily from the support theorem (2.3) that there is $c_2 > 0$ such that

$$P^x(T(J_1(x)) < T(Z_{j-1})) > c_2 \tag{5.4}$$

for $x \in Z_j$. For all $x \in Z_j$

$$P^x(\sup_{0 \le s \le (r^{1/\delta}/4)^2} |X_0^3 - X_s^3| < r^{1/\delta}/2) = c_3 > 0 \tag{5.5}$$

where c_3 does not depend on r, by Brownian scaling. By (5.3), for $z \in J_1(x)$,

$$P^z(T(J_0(x)) < (r^{1/\delta}/4)^2) > c_1.$$

This, (5.5) and the independence of $\widetilde{X}$ and X^3 imply

$$P^z(T(J_0(x)) < T(Z_{j-1})) > c_1 c_3 \quad \text{for } z \in J_1(x),$$

which combined with (5.4) and the strong Markov property applied at $T(J_1(x))$ yields

$$P^x(T(D^c) < T(Z_{j-1})) \ge P^x(T(J_0(x)) < T(Z_{j-1})) > c_1 c_3 c_2 \stackrel{\text{df}}{=} c_4 > 0$$

for $x \in Z_j$. A repeated application of the last inequality and the strong Markov property at the stopping times $T(Z_j), 1 \le j \le (r/2)/r^{1/\delta} - 1 \stackrel{\text{df}}{=} r_*$, shows that

$$P^x(T(Z) < T(D^c)) \le (1 - c_4)^{r_*} = \exp(-c_5 r^{1-1/\delta})$$

for some $c_5 > 0$.

Step 4. Consider an operator $\hat{L}$ with coefficients $\hat{a}_{ij}(x) \stackrel{\text{df}}{=} a_{ij}((\widetilde{x}, r/4))$. The probabilities corresponding to $\hat{L}$ will be denoted $\hat{P}^x$. By (5.2), for $x \in F(r^{1/\delta}/4)$,

$$\hat{P}^x(T(F(r^{1/\alpha})) < T(F(r^{1/\delta}/2))) = \frac{(r^{1/\delta}/4)^{2-\gamma} - (r^{1/\delta}/2)^{2-\gamma}}{(r^{1/\alpha})^{2-\gamma} - (r^{1/\delta}/2)^{2-\gamma}} \le c_6 r^{(2-\gamma)(1/\delta - 1/\alpha)}. \tag{5.6}$$

We have assumed that $r^{1/\alpha} < r^{1/\delta}/4$, so c_6 may be chosen independently of r.

It follows from the support theorem (2.3) and scaling that

$$\hat{P}^x(T(F(2q)) < T(F(q/2))) > c_7 > 0$$

for $x \in F(q)$, $q > 0$. This inequality, when applied repeatedly at the stopping times $T(F(2^j \cdot r^{1/\delta}/2))$ for $j < k_0 \stackrel{\text{df}}{=} 1 + \log_2(8r/(r^{1/\delta}/2))$, yields, for r sufficiently small,

$$\hat{P}^x(T(F(8r)) < T(F(r^{1/\delta}/4))) > c_7^{k_0} \ge r^{-c_8(1-1/\delta)} \tag{5.7}$$

for $x \in F(r^{1/\delta}/2)$ and some $c_8 > 0$.

Let $S_1 = 0$,

$$T_k = \inf\{t > S_k : X_t \in F(r^{1/\delta}/2)\}, \quad k \geq 1,$$

$$S_k = \inf\{t > T_{k-1} : X_t \in F(r^{1/\delta}/4)\}, \quad k \geq 2.$$

The event $\{T(F(r^{1/\alpha})) < T(F(8r))\}$ is the union of events

$$C_k \stackrel{\text{df}}{=} \{S_k < T(F(r^{1/\alpha})) \circ \theta_{S_k} < T_k < T(F(8r))\}, \quad k \geq 1,$$

$\hat{P}^x$-a.s. for $x \in F(r^{1/\delta}/4)$, where θ is the usual shift operator. By the strong Markov property, (5.6), and (5.7),

$$\hat{P}^x(C_k) \leq (1 - r^{-c_8(1-1/\delta)})^{k-1} c_6 r^{(2-\gamma)(1/\delta - 1/\alpha)}$$

and so

$$\begin{aligned} (5.8) \qquad \hat{P}^x(T(F(r^{1/\alpha})) < T(F(8r))) &\leq c_6 r^{(2-\gamma)(1/\delta-1/\alpha)} \sum_{k=1}^{\infty} (1 - r^{-c_8(1-1/\delta)})^{k-1} \\ &= c_6 r^{(2-\gamma)(1/\delta-1/\alpha)} r^{c_8(1-1/\delta)} \end{aligned}$$

for $x \in F(r^{1/\delta}/4)$. By the strong Markov property applied at $T(F(r^{1/\delta}/4))$, formula (5.8) remains true for $|\widetilde{x}| \geq r^{1/\delta}/4$.

Let

$$M = \bigcup_{y \in \mathbb{Y}} \{x \in \mathbb{R}^3 : |\widetilde{x} - y| < r^{1/\alpha}\},$$

$$D_1 = \{x \in \mathbb{R}^3 : |x^1| < r + r^{1/\delta}/2, |x^2| < r/2\} - M.$$

There are no more than $(4r^{1-1/\delta})^2$ points $y \in \mathbb{Y}$ such that $|y^1| < r + r^{1/\delta}/2$ and $|y^2| < r/2$. The estimate (5.8) may be applied to each set $\{x \in \mathbb{R}^3 : |\widetilde{x} - y| < r^{1/\alpha}\}$, $y \in \mathbb{Y}$, by the periodicity of the coefficients of $\hat{L}$, so that

$$\hat{P}^x(T(M) \leq T(D_1^c)) \leq c_6 r^{(2-\gamma)(1/\delta-1/\alpha)} r^{c_8(1-1/\delta)} (4r^{1-1/\delta})^2$$

for x such that $\text{dist}(x, M) \geq r^{1/\delta}/4$. Now we choose large γ so that

$$\eta \stackrel{\text{df}}{=} (2-\gamma)(1/\delta - 1/\alpha) + c_8(1 - 1/\delta) + 2(1 - 1/\delta) > 0.$$

Then for small $r > 0$, some $c_9 > 0$ and x such that $\mathrm{dist}(x, M) \geq r^{1/\delta}/4$ we have

$$\hat{P}^x(T(M) \leq T(D_1^c)) \leq c_9 r^\eta. \tag{5.9}$$

Let

$$D_2 = \{x \in D_1 : x^3 \in (0, r/2)\},$$
$$N_1 = \{x \in \mathbb{R}^3 : x^1 = r^{1/\delta}/2, |x^2| < r/4, x^3 \in [r/8, 3r/8]\},$$
$$N_k = N_1 + (k-1)(r, 0, 0), \quad k \geq 2.$$

By the support theorem (2.3),

$$\hat{P}^x(T(N_2) \leq T(D_2^c)) \geq c_{10} > 0$$

for $x \in N_1$. For small $r > 0$, in view of (5.9),

$$\hat{P}^x(T(N_2) = T(D_2^c) < T(M)) \geq c_{11} \stackrel{\mathrm{df}}{=} c_{10}/2 > 0$$

for $x \in N_1$. Since $D_2 - M \subset D$ and $\hat{L}$ and L have identical coefficients on $D_2 - M$, we see that

$$P^x(T(N_2) < T(D^c)) \geq c_{11}$$

for $x \in N_1$ and for similar reasons

$$P^x(T(N_{k+1}) < T(D^c)) \geq c_{11}$$

for $x \in N_k$, $k \geq 1$, assuming $N_{k+1} \subset D$. There are fewer than $3/2r$ such k's. If $Q = \{x \in \partial D : x^1 = 1\}$ then the last estimate and the repeated application of the strong Markov property at the stopping times $T(N_k)$ show that for some $c_{12} > 0$,

$$P^x(T(Q) \leq T(D^c)) \geq c_{11}^{3/2r} \geq \exp(-c_{12} r^{-1})$$

for $x \in N_1$.

Step 5. For each $r > 0$ we have defined a domain $D = D(r)$ in Step 2. Let

$$D_* = \{x \in \mathbb{R}^3 : |x| < 100, |x^1| < 1\} \cap \bigcup_{k=2}^{\infty} [D(2^{-k}) + (0, 2 - 2^{-k+2}, -2^{-k})].$$

Let L_* be an operator whose coefficients are defined by

$$a^*_{ij}(x) = a^k_{ij}(x - (0, 2 - 2^{-k+2}, -2^{-k}))$$

for $x \in D(2^{-k}) + (0, 2 - 2^{-k+2}, -2^{-k})$, where a^k_{ij} are the coefficients of $L = L_{2^{-k}}$ (see Step 2). For the remaining x, we let $a^*_{ij}(x) = \delta_{ij}$. The probabilities corresponding to L_* will be denoted P^x_*. Let

$$\begin{aligned}
Z_* &= \{x \in \partial D_* : x^3 \geq 1\}, \\
Q_* &= \{x \in \partial D_* : x^1 = 1\}, \\
z_k &= ((2^{-k})^{1/\delta}/2, 2 - 2^{-k+2}, -2^{-k} + 2^{-k}/4), \\
u(x) &= P^x_*(T(Z_*) \leq T(D^c_*)), \\
v(x) &= P^x_*(T(Q_*) \leq T(D^c_*)).
\end{aligned}$$

By the results of Steps 3 and 4,

$$u(z_k) \leq \exp(-c_5 2^{-k(1-1/\delta)})$$

and

$$v(z_k) \geq \exp(-c_{12} 2^k)$$

for large k. Since $\delta < 1/2$,

$$\lim_{k \to \infty} u(z_k)/v(z_k) = 0.$$

Let

$$\begin{aligned}
V &= \{x \in \mathbb{R}^3 : |x| < 99, |x^1| < 1/2\}, \\
K &= \{x \in \mathbb{R}^3 : |x| \leq 98, |x^1| \leq 1/4\}.
\end{aligned}$$

It is easy to check that D_*, L_*, V, K, u and v satisfy the assumptions of Theorem 1.2 (except that $\alpha < 1/2$). But the ratio u/v takes arbitrarily small values in $D_* \cap K$. So Theorem 1.3 is proved. ◻

Remark 5.1. (i) In Theorem 1.2 we assume that the coefficients a_{ij} and b_i of the operator L are smooth, but the operator L_* in the above counterexample has non-smooth coefficients. One may slightly modify the example so that it gives some information about operators with smooth coefficients.

For a fixed n, it is possible to modify the coefficients of L_* in D_* to obtain a new operator L_*^n, which has smooth coefficients which satisfy (1.1) with the same constant c_L and such that the estimates of Steps 3 and 4 for the hitting probabilities in

$$\bigcup_{k=2}^{n} [D(2^{-k}) + (0, 2 - 2^{-k+2}, -2^{-k})]$$

remain essentially unchanged. In this way we obtain a sequence of operators L_*^n such that $\inf_{x \in D_* \cap K} u(x)v(0,0,1)/v(x)u(0,0,1)$ goes to 0 as n tends to ∞. This shows that Theorem 1.2 cannot be true for $\alpha < 1/2$ and operators with smooth coefficients if we insist that c_H depend on L only through c_L.

(ii) Our proof of Theorem 1.3 relies heavily on operators with large ellipticity constant c_L. This suggests the following

Problem 5.1. What is the critical value of $\alpha = \alpha(\lambda)$ in Theorems 1.2-1.3 if we limit ourselves to operators L with ellipticity constant c_L bounded by λ?

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