A PROBABILISTIC PROOF OF THE BOUNDARY HARNACK PRINCIPLE[1]

by

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1. Introduction. The boundary Harnack principle may be stated as follows (cf. Jerison and Kenig (1982a), Theorem 5.25).

Theorem 1.1. *Let D be a Lipschitz domain and V an open set. For any compact $K \subseteq V$, there exists a constant c_0 such that for all positive harmonic functions u and v in D that vanish continuously on $(\partial D) \cap V$ with $u(x) = v(x)$ for some $x \in K \cap D$,*

$$c_0^{-1} u(y) < v(y) < c_0 u(y) \quad \textit{for all } y \in K \cap D.$$

The boundary Harnack principle was first proved by Dahlberg (1977). Subsequently Wu (1978) and Jerison and Kenig (1982a) gave alternate proofs. The result was extended in many directions, see, e.g., Caffarelli, Fabes, Mortola and Salsa (1981), Fabes, Garofalo and Salsa (1986), Fabes, Garofalo, Marin-Malave, and Salsa (1989) and Jerison and Kenig (1982b).

A related problem is to identify the Martin boundary for Lipschitz domains. Hunt and Wheeden (1970) showed that in a bounded Lipschitz domain the Martin boundary may be identified with the Euclidean one. Jerison and Kenig (1982a) showed how this result follows from the same techniques that they used to prove Theorem 1.1.

The main purpose of this paper is to give a probabilistic proof of Theorem 1.1, one using elementary properties of Brownian motion. We also obtain the fact that the Martin boundary equals the Euclidean boundary as an easy corollary of Theorem 1.1. The boundary Harnack principle may be viewed as a Harnack inequality for conditioned Brownian motion; as an application we prove some new probability bounds for conditioned Brownian motion in Lipschitz domains.

The principal motivation for this work was to give a proof of the boundary Harnack principle and of the Martin boundary result that could be easily extended to domains more general than Lipschitz: ones where locally the boundary is the graph of a continuous function with a modulus of continuity weaker than Lipschitz. See Bass and Burdzy (1989).

In Section 2 the main estimate on Brownian motion in Lipschitz domains is obtained. Theorem 1.1 is proved in Section 3, while the Martin boundary result is given in Section 4. Section 5 contains the estimates on conditioned Brownian motion.

2. The main estimate. Theorem 1.1 is essentially a local result, and for the time being we work with domains lying above the graph of a Lipschitz function.

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So let $\lambda > 0$ and suppose $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ is a bounded Lipschitz function with Lipschitz constant λ. For points $x = (x_1, \dots, x_d)$ in $\mathbb{R}^d$ we write $x = (\widetilde{x}, x_d)$, where $\widetilde{x} = (x_1 \dots, x_{d-1})$. Let

$$D = \{x \in \mathbb{R}^d : x_d > \Gamma(\widetilde{x})\}.$$

We let

$$\Delta(x,a,r) = \{y \in D : \Gamma(\widetilde{y}) < y_d < \Gamma(\widetilde{y}) + a,\ |\widetilde{y} - \widetilde{x}| < r\}, \tag{2.1}$$

$$\partial^u \Delta(x,a,r) = \{y \in \partial\Delta(x,a,r) : y_d = \Gamma(\widetilde{y} + a)\}, \qquad (\text{“}u\text{”} = \text{ upper}),$$

and

$$\partial^s \Delta(x,a,r) = \{y \in \partial\Delta(x,a,r) : |\widetilde{y} - \widetilde{x}| = r\}, \qquad (\text{“}s\text{”} = \text{side}).$$

Let (X_t, P^x) be Brownian motion in $\mathbb{R}^d$. For any Borel set A, let

$$T(A) = \inf\{t : X_t \in A\}.$$

The main estimate that we obtain in this section says that the probability that Brownian motion leaves $\Delta(x,a,r)$ near the boundary of D is bounded by a constant times the probability it leaves far from ∂D. First we have

Lemma 2.1. *There exist a constant $c_1 = c_1(\lambda) \in (0,1)$ such that*
(a) if $a > 0$, $r \geq a$, and $y \in D$ with $\widetilde{y} = \widetilde{x}$ and $y_d \in [\Gamma(\widetilde{x}) + a/2, \Gamma(\widetilde{x}) + a]$, then

$$P^y(T(\partial\Delta(x,a,r)) = T(\partial^u\Delta(x,a,r))) \geq c_1;$$

(b) if $a > 0$ and $y \in \Delta(x,a,a)$ with $\widetilde{y} = \widetilde{x}$, then

$$P^y(T(\partial\Delta(x,a,a)) = T(\partial^s\Delta(x,a,a))) \leq 1 - c_1; \qquad \textit{and}$$

(c) if $k \in \mathbb{Z}^+$, $a > 0$, $r \geq ak$, and $y \in \Delta(x,a,r)$ with $\widetilde{y} = \widetilde{x}$, then

$$P^y(T(\partial\Delta(x,a,r)) = T(\partial^s\Delta(x,a,r))) \leq (1 - c_1)^k.$$

Proof. The proof is elementary. By scaling we may suppose $a = 1$. Choose $c_2 = (\lambda^{-1} \wedge 1)/8$. Let

$$J_1 = \{y : |\widetilde{y} - \widetilde{x}| < c_2,\ \Gamma(\widetilde{x}) + \frac{1}{4} < y_d < \Gamma(\widetilde{x}) + 2\},$$
$$J_2 = \{y : |\widetilde{y} - \widetilde{x}| < c_2,\ \Gamma(\widetilde{x}) - 2 < y_d < \Gamma(\widetilde{x}) + 2\},$$

and

$$\partial^u J = \{y : |\widetilde{y} - \widetilde{x}| < c_2,\ y_d = \Gamma(\widetilde{x}) + 2\}.$$

It is easy to see that there exists c_1 depending only on c_2 such that

$$P^y(T(\partial J_1) = T(\partial^u J)) \geq c_1 \qquad \text{if} \quad \widetilde{y} = \widetilde{x},\ y_d \in (\Gamma(\widetilde{x}) + \frac{1}{2}, \Gamma(\widetilde{x}) + 1)$$

and

$$P^y(T(\partial J_2) = T(\partial^u J)) \geq c_1 \qquad \text{if} \quad \widetilde{y} = \widetilde{x},\ y_d \in (\Gamma(\widetilde{x}), \Gamma(\widetilde{x}) + 1).$$

Note that if $T(\partial J_1) = T(\partial^u J)$, then $T(\partial\Delta(x,1,r)) = T(\partial^u\Delta(x,1,r))$ P^y-a.s. for y such that $\widetilde{y} = \widetilde{x}$, $y_d \in (\Gamma(\widetilde{x}) + \frac{1}{2}, \Gamma(\widetilde{x}) + 1)$; this proves (a). Similarly, if $T(\partial J_2) = T(\partial^u J)$, then $T(\partial\Delta(x,1,1)) \neq T(\partial^s\Delta(x,1,1))$, which proves (b).

Part (c) follows from part (b) by use of the strong Markov property. Using (b),

$$\begin{aligned}
P^y(T(\partial\Delta(x,1,k)) &= T(\partial^s\Delta(x,1,k))) \\
&\le E^y(P^{X(U)}(T(\partial\Delta(X(U),1,1)) = T(\partial^s\Delta(X(U),1,1)); \\
&\qquad\qquad T(\partial\Delta(x,1,k-1)) = T(\partial^s\Delta(x,1,k-1))) \\
&\le (1-c_1)P^y(T(\partial\Delta(x,1,k-1)) = T(\partial^s\Delta(x,1,k-1))),
\end{aligned}$$

where

$$U = T(\partial\Delta(x,1,k-1)).$$

Using induction completes the proof. □

Let

$$F_1 = \{T(\partial\Delta(0,3,3)) = T(\partial^s\Delta(0,1,3))\}.$$

Let

$$\partial^g\Delta(0,3,3) = \partial\Delta(0,3,3) \setminus (\partial D \cup \partial^s\Delta(0,1,3)), \qquad (\text{“}g\text{”} = \text{good}).$$

Let

$$F_2 = \{T(\partial\Delta(0,3,3)) = T(\partial^g\Delta(0,3,3))\}.$$

The main result of this section is

Theorem 2.2. *There exists $c_3 = c_3(\lambda) < \infty$ such that for all $x \in \Delta(0,3,1)$,*

$$P^x(F_1) \le c_3 P^x(F_2).$$

Proof. Choose $M \in \mathbb{Z}^+$ so that $(1-c_1)^M < c_1/2$ and $M \ge 2\sum_{i=1}^\infty i2^{-i}$, where c_1 is the constant of Lemma 2.1. Let

$$J_k = \{y \in D : y_d \in [\Gamma(\widetilde{y}) + M^{-2}2^{-k-1}, \Gamma(\widetilde{y}) + M^{-2}2^{-k}],\ |\widetilde{y}| \le 2 - M^{-1}\sum_{i=1}^k i2^{-i}\}.$$

Arguing just as in the proof of Lemma 2.1(a), there exists a constant $c_4 = c_4(\lambda) \in (0,1)$ such that

$$P^z(F_2) \ge c_4, \qquad z \in \Delta(0,3,2) \setminus \Delta(0,M^{-2}/4,2). \tag{2.2}$$

Our first goal is to prove that

$$P^z(F_2) \ge c_4 c_1^{m-1} \quad \text{for } z \in J_m. \tag{2.3}$$

We use induction. By (2.2), we have that (2.3) holds for $m = 1$. Suppose (2.3) holds for m, and suppose $z \in J_{m+1}$. For the remainder of the proof, write

$$\Delta_m = \Delta(z, M^{-2}2^{-m}, mM^{-1}2^{-m}), \quad \text{and} \quad U_m = T(\partial\Delta_m). \tag{2.4}$$

By the strong Markov property,

$$P^z(F_2) \geq E^z(P^{X(U_m)}(F_2); X(U_m) \in \partial^u \Delta_m).$$

Since $\partial^u \Delta_m \in J_m$ when $z \in J_{m+1}$ and since $m \geq 1 \geq M^{-1}$, then by Lemma 2.1 (a) and the induction hypothesis,

$$P^z(F_2) \geq c_4 c_1^{m-1} P^z(X(U_m) \in \partial^u \Delta_m) \geq c_4 c_1^m.$$

So (2.3) is proved.

Now let

$$d_m = \sup_{z \in J_m} P^z(F_1)/P^z(F_2).$$

By (2.2),

$$P^z(F_1) \leq 1 \leq c_4^{-1} P^z(F_2), \qquad z \in \Delta(0,3,2) \setminus \Delta(0, M^{-2}/4, 2).$$

Hence $d_1 < \infty$, and so to prove the theorem, it suffices to prove that $\sup_m d_m < \infty$, since $\Delta(0, M^{-2}/2, 1) \subset \bigcup_{m=1}^\infty J_m$.

Consider $z \in J_{m+1}$. Using the strong Markov property, we have

$$P^z(F_1) \leq E^z(P^{X(U_m)}(F_1);\ X(U_m) \in \partial^u \Delta_m) + P^z(X(U_m) \in \partial^s \Delta_m) \tag{2.5}$$

and

$$P^z(F_2) \geq E^z(P^{X(U_m)}(F_2);\ X(U_m) \in \partial^u \Delta_m). \tag{2.6}$$

Since $\partial^u \Delta_m \subseteq J_m$, the definition of d_m says that the first term on the right of (2.5) is bounded by

$$d_m E^z(P^{X(U_m)}(F_2);\ X(U_m) \in \partial^u \Delta_m) \leq d_m P^z(F_2).$$

By Lemma 2.1 (c), the second term on the right of (2.5) is bounded by

$$(1 - c_1)^{mM} \leq (c_1/2)^m \leq 2^{-m} c_4^{-1} P^z(F_2),$$

using (2.3).

Hence, substituting in (2.5),

$$P^z(F_1) \leq (d_m + 2^{-m} c_4^{-1}) P^z(F_2).$$

Thus $d_{m+1} \leq d_m + c_4^{-1} 2^{-m}$, or $\sup_m d_m \leq d_1 + c_4^{-1} \sum_{m=1}^\infty 2^{-m} < \infty$ as required. $\square$

3. Boundary Harnack principle. We first borrow an elementary lemma from Jerison and Kenig (1982a), Lemma 5.4. The notation is as in Section 2.

Lemma 3.1. *There exists a constant* $c_5 = c_5(\lambda) > 0$ *such that if* u *is positive and harmonic in* $\Delta(x,5,5)$ *and vanishes continuously on* $\Delta(x,5,5) \cap \partial D$*, where* $x_d = \Gamma(\widetilde{x}) + 1$*, then* u *is bounded in* $\Delta(x,3,3)$ *by* $c_5 u(x)$*.*

Proof. Fix x. Without loss of generality, assume $u(x) = 1$. Let

$$J_k = \Delta(x, 2^{-k}, 4) \setminus \Delta(x, 2^{-k-1}, 4), \qquad k = 1, 2, \dots .$$

By the usual Harnack inequality, u is bounded in $\Delta(x,4,4) \setminus \bigcup_{k=2}^{\infty} J_k$ by a constant $c_6 = c_6(\lambda)$.

If u is harmonic and positive in $\Delta(y,1,1)$ and $y_d = \Gamma(\widetilde{y}) + \frac{1}{2}$, then by the usual Harnack inequality there exists $c_7 = c_7(\lambda) > 0$ so that u is bounded on $\Delta(y,1,1) \setminus \Delta(y, \frac{1}{4}, 1)$ by $c_7 u(y)$.

Using this observation together with scaling, we see that

$$\sup_{J_{k+1}} u \le c_7 \sup_{J_k} u,$$

and hence

$$\sup_{J_k} u \le c_6 c_7^k. \tag{3.1}$$

This implies that there exist constants $c_8 = c_8(\lambda), \beta = \beta(\lambda) > 0$ such that if

$$r(y) = y_d - \Gamma(\widetilde{y}),$$

then

$$r(y) \le c_8 u(y)^{-\beta}. \tag{3.2}$$

Suppose $y \in \Delta(x,3,3)$. Arguing as in Lemma 2.1 (a), there is a constant $c_9 = c_9(\lambda) > 0$ such that

$$P^y(T(\partial\Delta(y, 2r(y), 2r(y)) = T(\partial D)) \ge c_9.$$

Now let $M = (1-c_9)^{-1}$ and let N be a large real to be chosen later. Suppose there exists $x^{(1)} \in \Delta(x,3,3)$ with $u(x^{(1)}) \ge NM$. We now show that this implies there exist $x^{(2)}, \dots, x^{(n)}, \dots \in \Delta(x,4,4)$ with $u(x^{(n)}) \ge NM^n$, $x^{(k+1)} \in \Delta(x^{(k)}, 3r_k, 3r_k)$, where $r_k = r(x^{(k)})$. We use induction. Suppose we have $x^{(1)}, x^{(2)}, \dots, x^{(n)}$.

Write Δ_n for $\Delta(x^{(n)}, 2r_n, 2r_n)$. Note

$$\begin{aligned} u(x^{(n)}) = E^{x^{(n)}} u(X_{T(\partial\Delta_n)}) &\le (\sup_{\partial\Delta_n} u) P^{x^{(n)}}(T(\partial\Delta_n) \ne T(\partial D)) \\ &\le (\sup_{\partial\Delta_n} u)(1 - c_9). \end{aligned}$$

Hence there exists $x^{(n+1)} \in \partial\Delta_n \subseteq \Delta(x^{(n)}, 3r_n, 3r_n)$ with

$$u(x^{(n+1)}) \ge (1-c_9)^{-1} u(x^{(n)}) \ge NM^{n+1}.$$

By (3.2),

$$r_{n+1} \le c_8(NM^{n+1})^{-\beta},$$

and so provided we take N sufficiently large so that $\sum_{i=1}^{\infty} c_8(NM^i)^{-\beta} < \frac{1}{4}$, then $x^{(n+1)} \in \Delta(x,4,4)$.

We thus have our sequence $x^{(n)}$ in $\Delta(x,4,4)$ with $u(x^{(n)}) \to \infty$. Moreover, by (3.2), $r_n \to 0$. But this contradicts the assumption that u vanishes continuously on $(\partial D) \cap \Delta(x,5,5)$. So we must have u bounded on $\Delta(x,3,3)$ by NM. □

We now prove the following special case of the boundary Harnack principle.

Theorem 3.2. *There exists a constant* $c_{10} = c_{10}(\lambda) > 0$ *such that if* $x \in D$ *with* $x_d = \Gamma(\widetilde{x})+1$*,* u *and* v *are positive and harmonic on* $\Delta(x,5,5)$*, vanish continuously on* $\partial D \cap \Delta(x,5,5)$*, and* $u(x) = v(x) = 1$*, then*

$$c_{10}^{-1} u(y) < v(y) < c_{10} u(y) \quad \text{for all } y \in \Delta(x,3,1).$$

Proof. Recall the definitions of F_1 and F_2 of Theorem 2.2. By Lemma 3.1, u is bounded on $\Delta(x,3,3)$ by c_5. Then if $y \in \Delta(x,3,1)$,

$$\begin{aligned} u(y) = E^y u(X_{T(\partial\Delta(x,3,3))}) &\le c_5 P^y(T(\partial\Delta(x,3,3)) \ne T(\partial D)) \\ &\le c_5(P^y(F_1) + P^y(F_2)) \\ &\le c_5(1+c_3)P^y(F_2) \end{aligned} \tag{3.3}$$

by Theorem 2.2.

On the other hand, by the usual Harnack inequality, there exists $c_{11} = c_{11}(\lambda) > 0$ such that v is bounded below by c_{11} on $\partial\Delta(0,3,3) \setminus (\partial D \cup \partial^s \Delta(0,1,3))$. Then

$$v(y) = E^y v(X_{T(\partial\Delta(x,3,3))}) \ge c_{11} P^y(F_2). \tag{3.4}$$

Comparing (3.3) and (3.4) gives the left hand inequality, and reversing the roles of u and v gives the right hand inequality. □

However, Theorem 3.2 is actually equivalent to Theorem 1.1. We first recall the definition of a Lipschitz domain.

A bounded domain D is a Lipschitz domain if for each $x \in \partial D$ there is a Lipschitz function $\Gamma_x : \mathbb{R}^{d-1} \to \mathbb{R}$, a coordinate system CS_x, and $r_x > 0$ such that if $y = (y_1, \dots, y_d)$ in CS_x coordinates, then

$$D \cap B(x, r_x) = \{y : y_d > \Gamma_x(y_1, \dots, y_{d-1})\} \cap B(x, r_x).$$

Proof of Theorem 1.1. Theorem 1.1 follows from using scaling, Theorem 3.2 and the usual Harnack principle repetitively. □

4. Martin boundary. In this section we prove that the Martin boundary of a Lipschitz domain may be taken to be the Euclidean boundary. For details about Martin boundary, see Doob (1984).

Suppose D is a bounded Lipschitz domain. We denote the Green function for D by $G(x,y)$.

Let us fix $x_0 \in D$ and suppose $\varepsilon < \operatorname{dist}(x_0, \partial D)/4$.

Lemma 4.1. *Suppose* $x \in D$ *with* $|x - x_0| > 4\varepsilon$*. There exists a constant* $c_{12}(\varepsilon, D, x_0, x)$ *such that*

$$G(x,y)/G(x_0,y) \le c_{12} \quad \text{for } y \text{ in } D \setminus (\overline{B(x_0,\varepsilon)} \cup \overline{B(x,\varepsilon)}).$$

Moreover $c_{12}(\varepsilon, D, x_0, x) \to 0$ *uniformly as* $\operatorname{dist}(x, \partial D) \to 0$.

Proof. Pick $y_0 \in \partial B(x_0, 2\varepsilon)$. If G^0 is the Green function for Brownian motion killed on exiting $\partial B(x_0, 3\varepsilon)$, then

$$G(x_0, y_0) \ge G^0(x_0, y_0) \ge \delta(\varepsilon) > 0.$$

(See Section 1.11 of Durrett (1984) for an explicit expression for G^0.)

On the other hand, $G(x, y_0)$ is bounded above by the Newtonian potential evaluated at x, y_0; hence $G(x, y_0)$ is bounded above by a constant depending on ε (use the Green function for a large ball containing D instead of the Newtonian potential in the case $d = 2$). Moreover $G(x, y_0) \to 0$ uniformly as $\operatorname{dist}(x, \partial D) \to 0$.

Thus the ratio $G(x, y_0)/G(x_0, y_0)$ is bounded above. But Theorem 1.1 says that $G(x, y)/G(x_0, y)$ is comparable to $G(x, y_0)/G(x_0, y_0)$ for all points y in $D \setminus (B(x, \varepsilon/2) \cup B(x_0, \varepsilon/2))$; here $u = G(x, \cdot), v = G(x_0, \cdot)G(x, y_0)/G(x_0, y_0)$. The lemma follows. $\square$

We now prove that for fixed x_0, x, the ratio $G(x, y)/G(x_0, y)$ is Hölder continuous in y.

Lemma 4.2. *Let x, x_0, ε be as above. Then $G(x, y)/G(x_0, y)$ is a Hölder continuous function of y for $y \in D \setminus (\overline{B(x_0, \varepsilon)} \cup \overline{B(x, \varepsilon)})$, the constants depending only on x, x_0, ε, and D.*

Proof. For a set A, define

$$\operatorname*{Osc}_A f = \sup_A f - \inf_A f.$$

Let $f(y) = G(x, y)/G(x_0, y)$. Let $y_0 \in D_\varepsilon = D \setminus (\overline{B(x_0, \varepsilon)} \cup \overline{B(x, \varepsilon)})$. Since f is bounded by c_{12} on the region D_ε by Lemma 4.1, then $\operatorname*{Osc}_{D_\varepsilon} f \le c_{12}$. So to prove the lemma, it will suffice to show that there exists $\rho = \rho(D, \varepsilon, x, x_0) < 1$ such that

$$\operatorname*{Osc}_{D \cap B(y_0, r)} f \le \rho \operatorname*{Osc}_{D \cap B(y_0, 2r)} f, \qquad r < \varepsilon/4. \tag{4.1}$$

Suppose $r < \varepsilon/4$, and let g be the ratio of any two positive harmonic functions on $D_{\varepsilon/4}$ vanishing continuously on ∂D. By considering $ag + b$ for suitable a and b, we may assume

$$\sup_{D \cap B(y_0, 2r)} g = 1, \qquad \inf_{D \cap B(y_0, 2r)} g = 0.$$

If $\sup_{D \cap B(y_0, r)} g \le \frac{1}{2}$, then since $\inf_{D \cap B(y_0, r)} g \ge 0$,

$$\operatorname*{Osc}_{D \cap B(y_0, r)} g \le \frac{1}{2}.$$

If $\sup_{D \cap B(y_0, r)} g \ge \frac{1}{2}$, there exists a point y_1 in $D \cap B(y_0, r)$ with $g(y_1) \ge \frac{1}{2}$. But then by Theorem 1.1, there exists a constant $c_{13} = c_{13}(\varepsilon, D, x, x_0) \in (0, 1)$ such that

$$\inf_{D \cap B(y_0, r)} g \ge c_{13} g(y_1).$$

Since $\sup_{D \cap B(y_0, r)} g \le 1$, in this case we have

$$\operatorname*{Osc}_{D \cap B(y_0, r)} g \le 1 - c_{13}/2.$$

Since $\operatorname*{Osc}_{D \cap B(y_0, 2r)} g = 1$, we have (4.1) with $\rho = \max(\frac{1}{2}, 1 - c_{13}/2)$. $\square$

To construct the Martin boundary of a domain, one first compactifies D by adding all limit points of the ratios $G(x, y)/G(x_0, y)$ as $y \to z$, $z \in \partial D$ (see Doob(1984)). But Lemmas 4.1 and 4.2 show that $G(x, y)/G(x_0, y)$ converges to a single value as $y \to z \in \partial D$. Thus we have proved

Theorem 4.3. *The Martin boundary of a Lipschitz domain may be identified with a subset of the Euclidean boundary.*

To complete the identification of the Martin boundary, one needs to show that a proper subset of the Euclidean boundary will not suffice. We will write

$$K(x,z) = \lim_{y\in D, y\to z} G(x,y)/G(x_0,y) \qquad \text{for } x \in D,\ z \in \partial D.$$

We will also show that $K(x,z)$ is a minimal harmonic function for each $z \in \partial D$; that is, if u is harmonic in D satisfying $u(x) \le K(x,z)$ for all $x \in D$, then $u = cK(\,\cdot\,,z)$ for some constant c.

Theorem 4.4. *The Martin boundary of a Lipschitz domain may be identified with the Euclidean boundary.*

Proof. We first show that if $w \in \partial D$, then $K(x,w) \to 0$ uniformly as $\operatorname{dist}(x, \partial D \setminus B(x,2\varepsilon)) \to 0$. To see this, pick $y_0 \in D \cap B(w,\varepsilon)$. Let $\delta > 0$. By Lemma 4.1, we can make $G(x,y_0)/G(x_0,y_0) < \delta$ by taking $\operatorname{dist}(x, \partial D \setminus B(w,2\varepsilon))$ small enough. By Theorem 1.1,

$$G(x,y)/G(x_0,y) \le c_0\delta, \qquad y \in D \cap B(w,\varepsilon).$$

Now let $y \to w$ to get $K(x,w) \le c_0\delta$.

Suppose that $K(\,\cdot\,,w) = K(\,\cdot\,,z)$ for some $w, z \in \partial D$, $w \ne z$, and let $\varepsilon = |w - z|/8$. By the above argument, we have $K(x,w) \to 0$ uniformly when $\operatorname{dist}(x, \partial D \setminus B(w,2\varepsilon)) \to 0$ and when $\operatorname{dist}(x, \partial D \setminus B(z,2\varepsilon)) \to 0$. Thus, $K(x,w) \to 0$ uniformly as $\operatorname{dist}(x,\partial D) \to 0$. By the maximum principle, the positive harmonic function $K(\,\cdot\,,w)$ vanishes on D, contrary to the fact that $K(x_0,w) = 1$. This contradiction shows that the Martin kernels corresponding to w and z are distinct. □

Theorem 4.5. *If $z \in \partial D$, $K(\,\cdot\,,z)$ is a minimal harmonic function.*

Proof. Fix $z \in \partial D$ and suppose $u \le K(\,\cdot\,,z)$, where u is positive and harmonic. By Theorem 4.3, it follows that

$$u(\,\cdot\,) = \int K(\,\cdot\,,w)\mu(dw)$$

for some measure μ on ∂D. If μ is not a multiple of point mass at z, then there exists a finite measure $\widehat{\mu} \le \mu$ such that $\operatorname{dist}(z, \operatorname{supp}(\widehat{\mu})) > 0$. Let

$$\widehat{u}(\,\cdot\,) = \int K(\,\cdot\,,w)\widehat{\mu}(dw).$$

Then $\widehat{u}$ is positive, harmonic, and bounded by $K(\,\cdot\,,z)$.

Recall from the proof of Theorem 4.4 that $K(x,z) \to 0$ uniformly as $\operatorname{dist}(x, \partial D \setminus B(z,\varepsilon)) \to 0$. So the same is true of $\widehat{u}$. But for each $w \in \operatorname{supp}(\widehat{u})$, we see that $K(x,w) \to 0$ uniformly as $\operatorname{dist}(x, \partial D \cap B(z,2\varepsilon)) \to 0$ provided $2\varepsilon < \operatorname{dist}(z, \operatorname{supp}(\widehat{\mu}))$. So it follows by dominated convergence that $\widehat{u}(x) \to 0$ as $\operatorname{dist}(x, \partial D \cap B(z,2\varepsilon)) \to 0$. But then $\widehat{u}$ is a positive harmonic function vanishing continuously on ∂D, or $\widehat{u}$ is identically 0. This implies that $\widehat{\mu}$ is 0, or that μ must be point mass at z. □

5. Conditioned Brownian motion. Let h be a positive harmonic function on D and let (X_t, P_h^x) be conditioned Brownian motion, that is, the h path transform of Brownian motion. See Doob (1984) for more information about conditioned Brownian motion. In this section we prove the analog of Theorem 2.2 for (X_t, P_h^x) and we obtain an exponential bound on $P_h^x(\sup_{s\le t}|X_s - x| \ge r)$ similar to the bound for Brownian motion. We suppose we are in the set-up of Section 2 where D is the region above the graph of a Lipschitz function. The definition of F_1 and F_2 are as in Theorem 2.2.

Theorem 5.1. *Suppose h is positive, harmonic in $\Delta(0,5,5)$, and h vanishes continuously on $\partial D \cap \Delta(0,5,5)$. Then there exists $c_{14} = c_{14}(\lambda) > 0$ such that for all $x \in \Delta(0,3,1)$*

$$P_h^x(F_1) \le c_{14} P_h^x(F_2).$$

Proof. Let z be such that $\widetilde{z} = 0$ and $z_d = \Gamma(\widetilde{z})+1$. Since multiplying h by a constant does not change P_h^x, let us assume $h(z) = 1$. Let $u(x) = P_h^x(F_1), v(x) = P_h^x(F_2)$. By the usual Harnack inequality, there exists $c_{15} = c_{15}(\lambda)$ such that $h \ge c_{15}$ on $\partial^g\Delta(0,3,3) = \partial\Delta(0,3,3) \setminus (\partial D \cup \partial^s\Delta(0,1,3))$. As in Lemma 2.1 (a), there exists $c_{16} = c_{16}(\lambda)$ such that $P^z(F_2) \ge c_{16}$. Then using basic properties of h path transforms,

$$v(z) = E^z(h(X_{T(\partial^g\Delta(0,3,3))}); F_2)/h(z) \ge c_{15}c_{16}.$$

Since $u(z) \le 1$, then $u(z) \le (c_{15}c_{16})^{-1}v(z)$.

The functions uh and vh are positive and harmonic (with respect to P^x) on $\Delta(0,3,3)$ and vanish continuously on $\partial D \cap \Delta(0,3,3)$ since u and v are bounded being probabilities. By the boundary Harnack principle, there exists c_{14} so that

$$(uh)(x) \le c_{14}(vh)(x) \quad \text{for } x \in \Delta(0,3,1).$$

Dividing both sides by $h(x)$ proves the theorem. $\square$

We now obtain the following exponential bound

Theorem 5.2. *Suppose h is as in Theorem 5.1. Then there exist $r_0 = r_0(\lambda) > 0$, $c_{17} = c_{17}(\lambda) > 0$ and $c_{18} = c_{18}(\lambda) > 0$ such that*

$$P_h^x(\sup_{s\le t}|X_s - x| > r) \le c_{17}\exp(-r^2/c_{18}t), \qquad r < r_0.$$

Proof. Since $P_h^x(F_1) + P_h^x(F_2) = 1$, then by Theorem 5.1

$$P_h^x(F_2) \ge (1+c_{14})^{-1} \qquad \text{for } x \in \Delta(0,3,1). \tag{5.1}$$

Define

$$\tau_r = \inf\{t : |X_t - X_0| \ge r\}.$$

We have

$$P^y(\tau_\beta > 1) \ge c_{19}$$

for a constant $c_{19} = c_{19}(\lambda,\beta) > 0$. We have assumed in the proof of Theorem 5.1 that $h(z) = 1$. It follows that h is bounded above and below by constants depending only on λ on the set $\Delta(0,4,4) \setminus \Delta(0,\frac{1}{2},4)$ and

$$P_h^y(\tau_\beta > 1) \le E^y(h(X_1);\ \tau_\beta > 1)/h(y) \ge c_{20}, \qquad y \in \partial^g\Delta(0,3,3), \tag{5.2}$$

for a constant $c_{20} = c_{20}(\lambda, \beta) > 0$ provided β is taken small enough so that

$$\text{dist}\,(\partial\Delta(0,1/2,4), \partial^g\Delta(0,3,3)) \geq 2\beta.$$

So by the strong Markov property, (5.1), and (5.2),

$$P_h^x(T(\partial\Delta(0,4,4)) > 1) \geq (1+c_{14})^{-1}c_{20}, \qquad x \in \Delta(0,3,1).$$

By scaling and the fact that $y_d - \Gamma(\widetilde{y})$ is comparable to dist $(y, \partial D)$ for $y \in D$, we then get the existence of constants $c_{21} = c_{21}(\lambda) > 0$ and $p = p(\lambda) \in (0,1)$ such that

$$P_h^x(\tau_1 \leq c_{21}) \leq p. \tag{5.3}$$

Without loss of generality we may assume $c_{21} \leq 1$.

Let n be a positive integer to be chosen later. Let $U_1 = \tau_{1/n}, U_{i+1} = U_i + \tau_{1/n} \circ \theta_{U_i}$, where θ is the usual shift operator. Clearly $U_n \leq \tau_1$.

By (5.3) and scaling, $P_h^x(\tau_{1/n} \leq c_{21}n^{-2}) \leq p$. Hence

$$\begin{aligned} P_h^x(U_{n+1} - U_n \leq z \mid U_1, \dots, U_n) &\leq \begin{cases} p & \text{if} \quad z \leq c_{21}n^{-2} \\ 1 & \text{if} \quad z > c_{21}n^{-2} \end{cases} \\ &\leq p + (1-p)zn^2/c_{21} \quad \text{if} \quad z > 0. \end{aligned}$$

By Barlow and Bass (1989), Lemma 1.1, then

$$P_h^x(\tau_1 < z) = \exp(an^{3/2}z^{1/2} - bn),$$

where

$$a = 2((1-p)/pc_{21})^{1/2} \qquad \text{and} \qquad b = \log(\frac{1}{p}).$$

Taking n to be the integer part of $4b^2/9a^2z$, for z sufficiently small we get

$$P_h^x(\tau_1 < z) \leq \exp(-c_{22}/z), \tag{5.4}$$

$c_{22} = c_{22}(\lambda) > 0$.

Using (5.4) and scaling gives Theorem 5.2, provided we take c_{17} sufficiently large.
□

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