Convergence of symmetric Markov chains on $\mathbb{Z}^d$

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Abstract

For each n let Y_t^n be a continuous time symmetric Markov chain with state space $n^{-1}\mathbb{Z}^d$. A condition in terms of the conductances is given for the convergence of the Y_t^n to a symmetric Markov process Y_t on $\mathbb{R}^d$. We have weak convergence of $\{Y_t^n : t \le t_0\}$ for every t_0 and every starting point. The limit process Y has a continuous part and may also have jumps.

1 Introduction

For each n, let Y_t^n be a continuous time symmetric Markov chain with state space $\mathcal{S}_n = n^{-1}\mathbb{Z}^d$ and conductances $C^n(x, y)$. This means that Y^n stays at a state x for an exponential length of time with parameter $\sum_{z\ne x} C^n(x, z)$ and then jumps to the next state y with probability $C^n(x, y)/\sum_{z\ne x} C^n(x, z)$. It is natural to expect that one can give conditions on the conductances such that for each starting point and each t_0, the processes $\{Y_t^n; t \le t_0\}$ converge weakly to a limiting process and that the limiting process be a symmetric Markov process. The purpose of this paper is to give such a theorem.

The earliest convergence theorem of this type is that of [DFGW] in the context of a central limit theorem for random walks in random environment. A more general result is implicit in [SZ]. In [BKu08] the first two authors of the current paper extended the theorem in [SZ] in two ways: chains with unbounded range were allowed and the rather stringent continuity conditions in [SZ] were weakened. A chain with unbounded range is one where there is no bound on the size of the jumps. In all of these papers the limit process is a symmetric diffusion on $\mathbb{R}^d$.

The paper [HK07] considered conductances that were comparable to the distribution of a stable law and the limit process is what is known as a stable-like process. Here the limit process has paths that have no continuous part. A theorem for convergence of pure jump symmetric processes on $\mathbb{R}^d$ can be found in [BKK]; as noted there the methods can be readily modified to give a result on the convergence of symmetric Markov chains whose limiting process has a more general jump structure than stable-like. Finally, we should mention the well-known results of [SV, Chap. 11] on non-symmetric Markov chains.

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The current paper is devoted to proving a fairly general convergence theorem for symmetric Markov chains. We point out three significant differences from earlier work.

- Our Markov chains can have unbounded range and the limit process is associated with a Dirichlet form with both local and non-local components. This means the limit process has a continuous part and may also have a discontinuous part.
- We dispense with any continuity conditions on the conductances. Instead only convergence locally in L^1 is needed.
- The proofs are considerably simpler than previous work.

Let us give a heuristic description of our results, with the main theorem stated precisely in Section 5 as Theorem 5.5. First of all, the limiting symmetric Markov process is associated to the Dirichlet form

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} \nabla f \cdot a \nabla f \, dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(x) - f(y)^2 j(x,y) \, dx \, dy.$$

Here $a_{ij}(x)$ is a symmetric uniformly positive definite and bounded matrix function. The first term on the right hand side represents the continuous part of the limit process; if the second term on the right hand side were not present, one would have a symmetric diffusion, and the Dirichlet form would be the one arising from elliptic operators on $\mathbb{R}^d$ in divergence form. The double integral on the right hand side represents the jump part, and very roughly says that the process jumps from x to y with jump intensity $j(x,y)$.

We write our conductances as $C^n = C^n_C + C^n_J$, where C^n_C and C^n_J are the local (continuous) and non-local (jump) parts, resp. Let us discuss the local part first. If one wants to understand the behavior of the limiting process at a point x, say, to look at $a(x)$, a bit of thought leads to the realization that jumps by the Markov chains that jump over but do not land on x contribute. Thus, in one dimension, one looks at a quantity $a^n(x)$ involving sums of terms involving $C^n_C(y,z)$ with $y \le x \le z$. In higher dimensions one uses a similar idea: one looks at the contribution of $C^n_C(y,z)$ where x lies on the shortest path from y to z; a path here means that at each step the path goes from a point to one of its nearest neighbors. There is no single shortest path in general, so we form $a^n_{ij}(x)$ in terms of an average of expressions involving $C^n_C(y,z)$, the average being over all shortest paths from y to z that pass through x. There are some very mild regularity conditions on C^n, but the main hypothesis is that the $a^n_{ij}(x)$ are uniformly bounded and converge to $a_{ij}(x)$ locally in L^1.

The conditions on the jump part are even weaker. We form a measure $j^n(x,y)\,dx\,dy$ in terms of the C^n_J. We then require that for each N, the measure $j^n(x,y)\,dx\,dy$ restricted to $B_N = (B(0,N) \times B(0,N)) \setminus (B(0,N^{-1}) \times B(0,N^{-1}))$ converges weakly to the measure $j(x,y)\,dx\,dy$ restricted to B_N, where $B(0,r)$ is the ball of radius r centered at 0.

After giving some definitions and setting up the framework in Section 2, we obtain upper and lower bounds and regularity results for the heat kernels for Y^n in Sections 3 and 4. The formulation of the main theorem is given in Section 5 and the proof is given in Section 6.

2 Framework

For $n \in \mathbb{N}$, let $\mathcal{S}_n = n^{-1}\mathbb{Z}^d$. Let $|\cdot|$ be the Euclidean norm and $B_n(x,r) := \{y \in \mathcal{S}_n : |x-y| < r\}$.

For $n \in \mathbb{N}$, let $C^n(\cdot,\cdot)$ be a symmetric function defined on $(\mathcal{S}_n \times \mathcal{S}_n) \setminus \Delta$ into $\mathbb{R}_+$, where $\Delta = \{(x,x) : x \in \mathcal{S}_n\}$. Here symmetric means $C^n(x,y) = C^n(y,x)$ for all $x \neq y$. We call $C^n(x,y)$ the *conductance* between x and y. Throughout the paper, we assume the following;

(A1) *There exist* $c_1, c_2 > 0$ *independent of* n *such that*

$$c_1 \le \nu_x^n := \sum_{y \in \mathcal{S}_n} C^n(x,y) \le c_2 \quad \textit{for all} \;\; x \in \mathcal{S}_n.$$

(A2) *There exist* $M_0 \ge 1, \delta > 0$ *independent of* n *such that the following holds: for any* $x, y \in \mathcal{S}_n$ *with* $|x-y| = n^{-1}$, *there exist* $N \ge 2$ *and* $x_1, \cdots, x_N \in B_n(x, n^{-1}M_0)$ *such that* $x_1 = x$, $x_N = y$ *and* $C^n(x_i, x_{i+1}) \ge \delta$ *for* $i = 1, \cdots, N-1$.

(A3) *There exists a function* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *so that for any* $n \in \mathbb{N}$,

$$C^n(x,y) \le n^{-(d+2)}\varphi\big(|x-y|\big), \quad x, y \in \mathcal{S}_n \quad \text{and} \quad \int_0^\infty \left(1 \wedge t^2\right) t^{d-1}\varphi(t)dt < \infty.$$

Note that from the assumption (A3), we see for any $x \in \mathcal{S}_n$,

$$\begin{aligned} n^2 \sum_{y \in \mathcal{S}_n} \big(1 \wedge |y|^2\big) C^n(x, x+y) &\le n^2 \sum_{y \in \mathcal{S}_n} \big(1 \wedge |y|^2\big) n^{-(d+2)}\varphi(|y|) = \sum_{y \in \mathcal{S}_n} \big(1 \wedge |y|^2\big)\varphi(|y|)n^{-d} \\ &\le c_d \int_{\mathbb{R}^d} \big(1 \wedge |y|^2\big)\varphi(|y|)dy = c_d' \int_0^\infty (1 \wedge t^2)\, t^{d-1}\varphi(t)dt < \infty. \end{aligned}$$

Thus we have

$$M := \sup_n \sup_{x \in \mathcal{S}_n} \Big(n^2 \sum_{y \in \mathcal{S}_n} \big(1 \wedge |y|^2\big) C^n(x, x+y)\Big) < \infty. \tag{2.1}$$

An example of $C^n(x,y)$ that satisfies (A1), (A2) and (A3) is the following:

$$\begin{aligned} &\frac{c_1 1_{\{1 \ge |x-y| \ge n^{-1}\}}}{n^{d+2}|x-y|^{d+\alpha}} + c_2 1_{\{|x-y| = n^{-1}\}} \le C^n(x,y) \\ &\qquad\qquad \le \frac{c_3 1_{\{1 \ge |x-y| \ge n^{-1}\}}}{n^{d+2}|x-y|^{d+\beta}} + c_4 1_{\{|x-y| = n^{-1}\}} + \frac{c_5 1_{\{|x-y| > 1\}}}{n^{d+2}|x-y|^{d+\alpha}}, \end{aligned}$$

where $0 < \alpha \le \beta < 2$.

Let $\mu_x^n \equiv n^{-d}$ for all $x \in \mathcal{S}_n$ and for each $A \subset \mathcal{S}_n$, define $\mu^n(A) = \sum_{y \in A} \mu_y^n$ and $\nu^n(A) = \sum_{y \in A} \nu_y^n$. Note that $L^2(\mathcal{S}_n, \mu^n) = L^2(\mathcal{S}_n, \nu^n)$ by (A1). Now, for each $f \in L^2(\mathcal{S}_n, \mu^n)$, define

$$\mathcal{E}^n(f,f) = \frac{n^{2-d}}{2} \sum_{x,y \in \mathcal{S}_n} (f(y) - f(x))^2 C^n(x,y), \tag{2.2}$$

$$\mathcal{F}^n = \{f \in L^2(\mathcal{S}_n, \mu^n) : \mathcal{E}^n(f,f) < \infty\}. \tag{2.3}$$

For $p \ge 1$, define $\|f\|_{p,n}^p = \sum_{y \in \mathcal{S}_n} \big|f(x)\big|^p \mu_x^n$. The following lemma is standard.

Lemma 2.1 *For each n, $\mathcal{F}^n = L^2(\mathcal{S}_n, \mu^n)$, and for $f \in L^2(\mathcal{S}_n, \mu^n)$, we have*

$$\mathcal{E}^n(f,f) \le 2n^2 M ||f||_{2,n}^2,$$

where M is the constant appearing in (2.1).

Proof. Let $f \in L^2(\mathcal{S}_n, \mu^n)$. Since $|x-y| \ge 1/n$ for any $x,y \in \mathcal{S}_n$ with $x \ne y$, we have

$$\begin{aligned}
\frac{n^{2-d}}{2} \sum_{\substack{x,y\in\mathcal{S}_n \\ x\ne y}} \big(f(x)-f(y)\big)^2 C^n(x,y) &\le n^{2-d} \sum_{\substack{x,y\in\mathcal{S}_n \\ |x-y|\ge 1/n}} \big(f(x)^2+f(y)^2\big) C^n(x,y) \\
&\le 2n^{2-d} \sum_{x\in\mathcal{S}_n} f(x)^2 \sum_{\substack{y\in\mathcal{S}_n \\ |x-y|\ge 1/n}} C^n(x,y) \\
&\le 2n^{4-d} \sum_{x\in\mathcal{S}_n} f(x)^2 \sum_{\substack{y\in\mathcal{S}_n \\ |x-y|\ge 1/n}} \big(1\wedge|x-y|^2\big) C^n(x,y) \\
&\le 2n^{2-d} M \sum_{x\in\mathcal{S}_n} f(x)^2 = 2n^2 M ||f||_{2,n}^2.
\end{aligned}$$

□

Using Lemma 2.1, it is easy to check that $(\mathcal{E}^n, \mathcal{F}^n)$ is a regular Dirichlet form on $L^2(\mathcal{S}_n, \mu^n)$. Further, $\mathcal{F}^n = L^2(\mathcal{S}_n, \mu^n)$ is equal to the closure of the space of compactly supported functions on $\mathcal{S}_n$ with respect to $(\mathcal{E}^n(\cdot,\cdot) + ||\cdot||_{2,n}^2)^{1/2}$. Let $Y_t^{(n)}$ be the corresponding continuous time Markov chains on $\mathcal{S}_n$ and let $p^n(t,x,y)$ be the transition density for $Y_t^{(n)}$ with respect to μ^n. The infinitesimal generator of $Y_t^{(n)}$ can be written as

$$\mathcal{A}^n f(x) = \sum_{y\in\mathcal{S}_n} (f(y)-f(x)) C^n(x,y) n^2 = \sum_{y\in\mathcal{S}_n} (f(y)-f(x)) \frac{C^n(x,y) n^{2-d}}{\mu_x^n},$$

for each $f \in L^2(\mathcal{S}_n, \mu^n)$.

Remark 2.2 Note that under (A1), $\{Y_t^{(n)}\}$ is conservative. Indeed, define a symmetric Markov chain $\{X_m^{(n)}\}$ by

$$\mathbb{P}^x(X_1^{(n)} = y) = \frac{C^n(x,y)}{\nu_x^n} \quad \text{for all} \quad x,y\in\mathcal{S}_n.$$

Then the corresponding semigroup satisfies $P_1^{X,n}1(x) = \sum_{y\in\mathcal{S}_n} \mathbb{P}^x(X_1^{(n)} = y) = 1$ by (A1), so inductively we have $P_m^{X,n}1 = 1$ for all $m \in \mathbb{N}$, so that $\{X_m^{(n)}\}$ is conservative. But $\{Y_t^{(n)}\}$ is a time changed process of $\{X_m^{(n)}\}$. To see this, let $\{U_i^{x,n} : i \in \mathbb{N}, x \in \mathcal{S}_n\}$ be an independent sequence of exponential random variables, where the parameter for $U_i^{x,n}$ is ν_x^n, that is independent of $X_m^{(n)}$, and define $T_0^{(n)} = 0, T_m^{(n)} = \sum_{k=1}^m U_k^{X_{k-1}^{(n)},n}$. Set $\widetilde{Y}_t^{(n)} = X_m$ if $T_m^{(n)} \le t < T_{m+1}^{(n)}$; then the laws of $\widetilde{Y}^{(n)}$ and $Y^{(n)}$ are the same, and hence $\widetilde{Y}^{(n)}$ is a realization of the continuous time Markov chain corresponding to (a time change of) $X_m^{(n)}$. Note that by (A1), the mean exponential holding time at each point for $\widetilde{Y}^{(n)}$ can be controlled uniformly from above and below by a positive constant, so we conclude $P_t^n 1 = 1$ for all $t > 0$, where P_t^n is the semigroup corresponding to $\{Y_t^{(n)}\}$.

3 Heat kernel estimates

3.1 Nash inequality

For $f \in L^2(\mathcal{S}_n, \mu^n)$, let

$$\mathcal{E}_{NN}^n(f,f) = \frac{n^{2-d}}{2} \sum_{\substack{x,y\in\mathcal{S}_n \\ |x-y|=n^{-1}}} (f(x)-f(y))^2,$$

which is the Dirichlet form for the simple symmetric random walk in $\mathcal{S}_n$. By [BKu08, Proposition 3.1] there exists $c_1 > 0$ independent of n such that for any $f \in L^2(\mathcal{S}_n, \mu^n)$,

$$\|f\|_{2,n}^{2(1+2/d)} \le c_1 \mathcal{E}^n(f,f) \|f\|_{1,n}^{4/d}, \tag{3.1}$$

and

$$p^n(t,x,y) \le c_1 t^{-d/2} \qquad \text{for all} \ \ x,y \in \mathcal{S}_n, t > 0. \tag{3.2}$$

For $r \in (n^{-1}, 1]$, let $\mathcal{E}^{n,r}$ be the Dirichlet form corresponding to $\{Y_t^{(n),r} := r^{-1} Y_{r^2 t}^{(n)}, t \ge 0\}$. By simple computations, we have

$$\mathcal{E}^{n,r}(f,f) = \frac{(nr)^{2-d}}{2} \sum_{x,y\in\mathcal{S}_{nr}} (f(y)-f(x))^2 C^n(rx,ry),$$

where $\mathcal{S}_{nr} = \{x/r : x \in \mathcal{S}_n\} = (nr)^{-1}\mathbb{Z}^d$. Define

$$p^{n,r}(t,x,y) := r^d p^n(r^2 t, rx, ry). \tag{3.3}$$

Then $p^{n,r}(t,x,y)$ is the heat kernel for $\mathcal{E}^{n,r}$. By (3.2), we have

$$p^{n,r}(t,x,y) \le c_1 t^{-d/2} \qquad \text{for all} \ \ x,y \in \mathcal{S}_{nr}, t > 0. \tag{3.4}$$

For $\lambda \ge 1$, let $Y_t^{(n),r,\lambda}$ be a process on $\mathcal{S}_{nr}$ with the large jumps of $Y_t^{(n)}$ removed. More precisely, $Y_t^{(n),r,\lambda}$ is a process whose Dirichlet form is

$$\mathcal{E}^{n,r,\lambda}(f,f) = \frac{1}{2} \sum_{\substack{x,y\in\mathcal{S}_{nr} \\ |x-y|\le\lambda}} (f(x)-f(y))^2 (nr)^{2-d} C^n(rx,ry),$$

for each $f \in L^2(\mathcal{S}_{nr}, \mu^{nr})$. We denote the heat kernel for $Y_t^{(n),r,\lambda}$ by $p^{n,r,\lambda}(t,x,y)$, $x,y \in \mathcal{S}_{nr}$.

3.2 Exit time probability estimates

In this subsection, we will obtain some exit time estimates. Note that similar estimates are obtained in [Foo, Proposition 3.7] and [CK09].

Proposition 3.1 *For $A > 0$ and $0 < B < 1$, there exists $t_0 = t_0(A, B) \in (0, 1)$ such that for every $n \in \mathbb{N}$, $r \in (0, 1]$ and $x \in \mathcal{S}_n$,*

$$\mathbb{P}^x\left(\sup_{s\le r^2 t_0} |Y_t^{(n)} - Y_0^{(n)}| > rA\right) = \mathbb{P}^x\left(\sup_{s\le t_0} |Y_t^{(n),r} - Y_0^{(n),r}| > A\right) \le B. \tag{3.5}$$

Proof. Let $\lambda > 0$. Since we have (3.4) and $p^{n,r,\lambda}(t, x, y) \le p^{n,r}(t, x, y)$, by Theorem (3.25) of [CKS], we have

$$p^{n,r,\lambda}(t, x, y) \le c_1\, t^{-\frac{d}{2}}\, \exp\left(-E(2t, x, y)\right) \tag{3.6}$$

for all $t \le 1$ and $x, y \in \mathcal{S}_{nr}$, where

$$\begin{aligned} E(t,x,y) &= \sup\{|\psi(y) - \psi(x)| - t\,\Lambda(\psi)^2 : \Lambda(\psi) < \infty\}, \\ \Lambda(\psi)^2 &= \|e^{-2\psi}\Gamma_{\lambda,r}[e^{\psi}]\|_\infty \vee \|e^{2\psi}\Gamma_{\lambda,r}[e^{-\psi}]\|_\infty, \end{aligned}$$

and $\Gamma_{\lambda,r}$ is defined by

$$\Gamma_{\lambda,r}[v](\xi) = \sum_{\substack{\eta,\xi\in\mathcal{S}_{nr}\\ |\xi-\eta|\le\lambda}} (v(\eta) - v(\xi))^2 C^n(r\eta, r\xi)(nr)^2, \qquad \xi \in \mathcal{S}_{nr}. \tag{3.7}$$

Now let $R = |x - y|$ and let $\psi(\xi) = s(|\xi - x| \wedge R)$. Then, $|\psi(\eta) - \psi(\xi)| \le s|\eta - \xi|$, so that

$$(e^{\psi(\eta)-\psi(\xi)} - 1)^2 \le |\psi(\eta) - \psi(\xi)|^2 e^{2|\psi(\eta)-\psi(\xi)|} \le cs^2|\eta - \xi|^2 e^{2|\psi(\eta)-\psi(\xi)|}$$

for $\eta, \xi \in \mathcal{S}_{nr}$ where $|\eta - \xi| \le \lambda$. Hence

$$\begin{aligned} e^{-2\psi(\xi)}\Gamma_{\lambda,r}[e^{\psi}](\xi) &= \sum_{\substack{\eta\in\mathcal{S}_{nr}\\ |\xi-\eta|\le\lambda}} (e^{\psi(\eta)-\psi(\xi)} - 1)^2 C^n(r\eta, r\xi)(nr)^2 \\ &\le c_1 s^2 e^{2s\lambda} \sum_{\substack{\eta\in\mathcal{S}_{nr}\\ |\xi-\eta|\le\lambda}} |\eta - \xi|^2 C^n(r\eta, r\xi)(nr)^2 \\ &= c_1 s^2 e^{2s\lambda} \sum_{\substack{\eta'\in\mathcal{S}_{n}\\ |\xi'-\eta'|\le\lambda r}} |\eta' - \xi'|^2 C^n(\eta', \xi')n^2 \\ &\le c_1 s^2 e^{2s\lambda}\Big(\sum_{\substack{\eta'\in\mathcal{S}_{n}\\ |\xi'-\eta'|\le 1}} |\eta' - \xi'|^2 C^n(\eta', \xi')n^2 + ((\lambda r)^2 \vee 1) \sum_{\substack{\eta'\in\mathcal{S}_{n}\\ |\xi'-\eta'|\ge 1}} C^n(\eta', \xi')n^2\Big) \\ &\le c_2(\lambda^2 \vee 1)s^2 e^{2s\lambda} \le c_3 e^{3s\lambda}(1 + 1/\lambda^2) \end{aligned}$$

for all $\xi \in \mathcal{S}_{nr}$ where (A3) and $r \le 1$ are used in the third inequality. We have the same bound when ψ is replaced by $-\psi$, so $\Lambda(\psi)^2 \le c_3 e^{3s\lambda}(1 + 1/\lambda^2)$. Now, let $\lambda = A/(6d)$, $t_0 \le 1 \wedge \lambda^4 =$

$1 \wedge (A^4/(6d)^4)$ and $s = (3\lambda)^{-1} \log(1/t^{1/2}) > 0$. Then, for each $t \le t_0$ and $R \ge A$,

$$\begin{aligned} p^{n,r,\lambda}(t,x,y) &\le c_4 t^{-\frac{d}{2}} \exp\left(-sR + c_3 t e^{3s\lambda}(1+1/\lambda^2)\right) \\ &\le c_5 \exp\Big((d - \frac{2Rd}{A}) \log(\frac{1}{t^{1/2}}) \Big) \le c_5 \exp\Big(-\frac{Rd}{A} \log(\frac{1}{t^{1/2}}) \Big). \end{aligned} \tag{3.8}$$

Thus,

$$\begin{aligned} \sum_{B_{nr}(x,A)^c} p^{n,r,\lambda}(t,x,y)\mu_y^{nr} &\le c \int_A^\infty R^{d-1} \exp\Big(-\frac{Rd}{A} \log(\frac{1}{t^{1/2}}) \Big) dR \\ &= cA^d \int_1^\infty R'^{d-1} \exp\Big(-R'd \log(\frac{1}{t^{1/2}}) \Big) dR' < B/4 \end{aligned} \tag{3.9}$$

for all $t \le t_0$ if we choose t_0 small, depending on A and B. Thus, applying [BBCK, Lemma 3.8], we obtain

$$\mathbb{P}^x \left(\sup_{s \le t_0} |Y_t^{(n),r,\lambda} - Y_0^{(n),r,\lambda}| > A \right) \le B/2. \tag{3.10}$$

We now use Meyer's argument to obtain the estimate for $Y^{(n),r}$. Note that for any $x \in \mathcal{S}_{nr}$,

$$\begin{aligned} \mathcal{J}(x) &:= \sum_{\substack{y \in \mathcal{S}_{nr} \\ |x-y| \ge \lambda}} C^n(rx,ry)(nr)^2 \le \sum_{\substack{y \in \mathcal{S}_{nr} \\ |x-y| \ge \lambda}} \frac{(r^2|x-y|^2) \wedge 1}{\lambda^2 r^2} C^n(rx,ry)(nr)^2 \\ &= \frac{1}{\lambda^2} \sum_{y' \in \mathcal{S}_n} \Big(|x'-y'|^2 \wedge 1 \Big) C^n(x',y') n^2 \le \frac{M}{\lambda^2} = \frac{(6d)^2 M}{A^2}, \end{aligned}$$

where (A3) is used in the last inequality. So, if we let $U_1 := \inf\{t > 0 : \int_0^t \mathcal{J}(Y_s^{(n),r}) ds > S_1\}$, where S_1 is the independent exponential distribution with mean 1, we have

$$P(U_1 \le t_0) \le 1 - e^{-(6d)^2 t_0/A^2} < B/2 \tag{3.11}$$

by taking t_0 small. Using Meyer's argument (see, for example, Section 4.1 in [CK08]), we obtain

$$\begin{aligned} \mathbb{P}^x \Big(\sup_{s \le t_0} |Y_t^{(n),r} - Y_0^{(n),r}| > A \Big) &= \mathbb{P}^x \Big(\sup_{s \le t_0} |Y_t^{(n),r} - Y_0^{(n),r}| > A, U_1 > t_0 \Big) \\ &\quad + \mathbb{P}^x \Big(\sup_{s \le t_0} |Y_t^{(n),r} - Y_0^{(n),r}| > A, U_1 \le t_0 \Big) \\ &\le \mathbb{P}^x \left(\sup_{s \le t_0} |Y_t^{(n),r,\lambda} - Y_0^{(n),r,\lambda}| > A \right) + \mathbb{P}^x (U_1 \le t_0) \\ &\le B/2 + B/2 = B, \end{aligned}$$

where (3.10) and (3.11) are used in the last inequality. □

Corollary 3.2 *For $0 < A', B' < 1$, there exists $R_0 = R_0(A', B') > 0$, such that for every $n \in \mathbb{N}$, $r \in (0,1]$ and $x \in \mathcal{S}_n$,*

$$\mathbb{P}^x\left(\sup_{s\le r^2A'} |Y_t^{(n)} - Y_0^{(n)}| > rR_0\right) = \mathbb{P}^x\left(\sup_{s\le A'} |Y_t^{(n),r} - Y_0^{(n),r}| > R_0\right) \le B'. \tag{3.12}$$

Proof. In the proof of Proposition 3.1, take $A \ge 1$, $\lambda = A^{1/2}/(6d)$ (instead of $\lambda = A/(6d)$) and $A \ge 1$. Then, since $A^{1/2} \le A \le R$, we have (3.8) by changing A to $A^{1/2}$. So as in (3.9), there exists R_0 large such that for $t \le t_0 =: A'$ and $A \ge R_0$, we have

$$\begin{aligned}\sum_{B_{nr}(x,A)^c} p^{n,r,\lambda}(t,x,y)\mu_y^{nr} &\le cA^{d/2}\int_{A^{1/2}}^\infty R'^{d-1}\exp\Big(-R'd\log(\frac{1}{t^{1/2}})\Big)dR' \\ &\le cA^{d/2}\exp\Big(-\frac{A^{1/2}}{2}d\log(\frac{1}{t^{1/2}})\Big)\int_{A^{1/2}}^\infty R'^{d-1}\exp\Big(-\frac{R'}{2}d\log(\frac{1}{t^{1/2}})\Big)dR' < B/4.\end{aligned}$$

Also, similarly to (3.11), we have

$$P(U_1 \le t_0) \le 1 - e^{-(6d)^2 t_0/A} < B/2$$

for all $A \ge R_0$, by taking R_0 large. With these changes, we can obtain the result similarly to the proof of Proposition 3.1. □

4 Lower bounds and regularity for the heat kernel

We now introduce the space-time process $Z_s^{(n)} := (U_s, Y_s^{(n)})$, where $U_s = U_0 + s$. The filtration generated by $Z^{(n)}$ satisfying the usual conditions will be denoted by $\{\widetilde{\mathcal{F}}_s;\ s \ge 0\}$. The law of the space-time process $s \mapsto Z_s^{(n)}$ starting from (t,x) will be denoted by $\mathbb{P}^{(t,x)}$. We say that a non-negative Borel measurable function $q(t,x)$ on $[0,\infty)\times\mathcal{S}_n$ is *parabolic* in a relatively open subset B of $[0,\infty)\times\mathcal{S}_n$ if for every relatively compact open subset B_1 of B, $q(t,x) = \mathbb{E}^{(t,x)}\left[q(Z^{(n)}_{\tau^n_{B_1}})\right]$ for every $(t,x) \in B_1$, where $\tau^n_{B_1} = \inf\{s > 0 :\ Z_s^{(n)} \notin B_1\}$.

We denote $T_0 := t_0(1/2, 1/2) < 1$ the constant in (3.5) corresponding to $A = B = 1/2$. For $t \ge 0$ and $r > 0$, we define

$$Q^n(t,x,r) := [t, t + T_0 r^2] \times B_n(x,r),$$

where $B_n(x,r) = \{y \in \mathcal{S}_n : |x-y| < r\}$.

It is easy to see the following (see, for example, Lemma 4.5 in [CK03] for the proof).

Lemma 4.1 *For each $t_0 > 0$ and $x_0 \in \mathcal{S}_n$, $q^n(t,x) := p^n(t_0 - t, x, x_0)$ is parabolic on $[0,t_0)\times\mathcal{S}_n$.*

For $A \subset \mathcal{S}_n$ and a process Z_t on $\mathcal{S}_n$, let

$$\tau^n = \tau^n_A(Z) := \inf\{t \ge 0 : Z_t \notin A\}, \qquad T^n_A = T^n_A(Z) := \inf\{t \ge 0 : Z_t \in A\}.$$

The next proposition provides a lower bound for the heat kernel and is the key step for the proof of the Hölder continuity of $p^n(t,x,y)$.

Proposition 4.2 *There exist $c_1 > 0$ and $\theta \in (0,1)$ such that for each $n \in \mathbb{N}$, if $|x - x_0|, |y - x_0| \le t^{1/2}$, $x, y, x_0 \in \mathcal{S}_n$, $t \in (n^{-1}, 1]$ and $r \ge t^{1/2}/\theta$, then*

$$\mathbb{P}^x(Y_t^{(n)} = y, \tau^n_{B(x_0,r)} > t) \ge c_1 t^{-d/2} n^{-d}.$$

To prove this we first need some preliminary lemmas. The proof of the following weighted Poincaré inequality can be found in [SZ, Lemma 1.19] and [BKu08, Lemma 4.3].

Lemma 4.3 *Let*

$$g_n(x) = c_1 \prod_{i=1}^d e^{-|x_i|} \qquad x \in \mathcal{S}_n,$$

where c_1 is determined by the equation $\sum_{l \in \mathcal{S}_n} g_n(x)\mu_x^n = n^d$. Then there exists $c_2 > 0$ such that

$$c_2 \Big\langle (f - \langle f \rangle_{g_n})^2 \Big\rangle_{g_n} \le n^{2-d} \sum_{l \in \mathcal{S}_n} g_n(l) \sum_{i=1}^d \Big(f(l + \frac{e^i}{n}) - f(l) \Big)^2, \qquad f \in L^2(\mathcal{S}_n),$$

where

$$\langle f \rangle_{g_n} = \sum_{l \in \mathcal{S}_n} f(l) g_n(l) \mu_l^n$$

and e^i is the element of $\mathbb{Z}^d$ whose j-th component is 1 if $j = i$ and 0 otherwise.

We now give a key lemma.

Lemma 4.4 *There is an $\varepsilon > 0$ such that*

$$p^n(t, x, y) \ge \varepsilon t^{-d/2}, \tag{4.1}$$

for all $n \in \mathbb{N}$, $(t, x, y) \in (n^{-1}, 1] \times \mathcal{S}_n \times \mathcal{S}_n$ with $|x - y| \le 2t^{1/2}$.

Proof. It is enough to prove the following: there is an $\varepsilon > 0$ such that

$$(nr)^{-d} \sum_{l \in \mathcal{S}_{nr}} \log \Big(p^{n,r}(\tfrac{1}{2}, k, l + m) \Big) g_{nr}(l) \ge \tfrac{1}{2} \log \varepsilon, \tag{4.2}$$

for any $n \in \mathbb{N}$, $r \in (n^{-1}, 1]$ and $k, m \in \mathcal{S}_n$ with $|k - m| \le 2$. Indeed, by the Chapman-Kolmogorov equation, symmetry, and the fact $g_{nr}(j) \le 1$ for all $k, m \in \mathcal{S}_{nr}$,

$$p^{n,r}(1, k, m) \ge (nr)^{-d} \sum_{j \in \mathcal{S}_{nr}} p^{n,r}(\tfrac{1}{2}, k, j + k) p^{n,r}(\tfrac{1}{2}, m, j + k) g_{nr}(j).$$

Thus, by Jensen's inequality, (4.2) yields

$$r^d p^n(r^2, rk, rl) = p^{nr}(1, k, l) \ge \varepsilon \qquad D \ge 1, |k - l| \le 2.$$

Taking $t = r^2$, this gives (4.1).

So we will prove (4.2). Let $k, m \in \mathcal{S}_n$ satisfy $|k-m| \le 2$ and set $u_t(l) = p^{n,r}(t,k,l+m)$. Define

$$G(t) = (nr)^{-d} \sum_{l \in \mathcal{S}_{nr}} \log(u_t(l)) g_{nr}(l).$$

By Jensen's inequality, we see that $G(t) \le 0$. Further,

$$G'(t) = (nr)^{-d} \sum_{l \in \mathcal{S}_{nr}} \frac{\partial u}{\partial t}(l) \frac{g_{nr}(l)}{u_t(l)} = -\mathcal{E}^{(n),r}(u_t, \frac{g_{nr}}{u_t}).$$

Next, note that the following elementary inequality holds (see page 29 of [BBCK]).

$$\Big(\frac{d}{b} - \frac{c}{a}\Big)(b-a) \le -(c \wedge d)\Big(\log \frac{b}{d^{1/2}} - \log \frac{a}{c^{1/2}}\Big)^2 + (d^{1/2} - c^{1/2})^2, \qquad a,b,c,d > 0.$$

Applying this with $a = u_t(l)$, $b = u_t(l+m)$, $c = g_{nr}(l)$, $d = g_{nr}(l+m)$, we have

$$\begin{aligned}
&G'(t)\\
&= -(nr)^{2-d} \sum_{l \in \mathcal{S}_{nr}} \sum_{m \in \mathcal{S}_{nr}} \Big(\frac{g_{nr}(l+m)}{u_t(l+m)} - \frac{g_{nr}(l)}{u_t(l)}\Big)\Big(u_t(l+m) - u_t(l)\Big) C^n(rl, r(l+m))\\
&\ge (nr)^{2-d} \sum_{l \in \mathcal{S}_{nr}} \sum_{m \in \mathcal{S}_{nr}} (g_{nr}(l+m) \wedge g_{nr}(l))\Big(\log \frac{u_t(l+m)}{g_{nr}(l+m)^{1/2}} - \log \frac{u_t(l)}{g_{nr}(l)^{1/2}}\Big)^2 C^n(rl, r(l+m))\\
&\quad - (nr)^{2-d} \sum_{l \in \mathcal{S}_{nr}} \sum_{m \in \mathcal{S}_{nr}} (g_{nr}(l+m)^{1/2} - g_{nr}(l)^{1/2})^2 C^n(rl, r(l+m))\\
&\ge c(nr)^{2-d} \sum_{l \in \mathcal{S}_{nr}} \sum_{j=1}^{d} g_{nr}(l)\Big(\log u_t\Big(l + \frac{e^j}{nr}\Big) - \log u_t(l) + \frac{1}{2}\Big(\Big|l_j + \frac{1}{nr}\Big| - |l_j|\Big)\Big)^2\\
&\quad - (nr)^{2-d} \sum_{l \in \mathcal{S}_{nr}} \sum_{m \in \mathcal{S}_{nr}} (g_{nr}(l+m)^{1/2} - g_{nr}(l)^{1/2})^2 C^n(rl, r(l+m)) =: I - II,
\end{aligned}$$

where the last inequality is due to (A2) and the definition of g_{nr} (here e^j is the element of $\mathbb{Z}^d$ whose k-th component is 1 if $k = j$ and 0 otherwise). Note that

$$(g_{nr}(l+m)^{1/2} - g_{nr}(l)^{1/2})^2 \le c_1(|m|^2 \wedge 1)(g_{nr}(l+m) + g_{nr}(l)).$$

Thus

$$\begin{aligned}
II \le& c_2(nr)^{2-d} \sum_{l \in \mathcal{S}_{nr}} \sum_{m \in \mathcal{S}_{nr}} (g_{nr}(l+m) + g_{nr}(l))(|m|^2 \wedge 1) C^n(rl, r(l+m))\\
=& 2c_2(nr)^{2-d} \sum_{l \in \mathcal{S}_{nr}} \sum_{m \in \mathcal{S}_{nr}} g_{nr}(l)(|m|^2 \wedge 1) C^n(rl, r(l+m))\\
\le& c_3\Big(\sup_{l \in \mathcal{S}_{nr}} n^2 \sum_{m \in \mathcal{S}_{nr}} (r^2|m|^2 \wedge r^2) C^n(rl, r(l+m))\Big) \cdot (nr)^{-d} \sum_{l \in \mathcal{S}_{nr}} g_{nr}(l)\\
\le& c_4\Big(\sup_{l' \in \mathcal{S}_n} n^2 \sum_{m' \in \mathcal{S}_n} (|m'|^2 \wedge 1) C^n(l', l'+m')\Big) \le c_5,
\end{aligned}$$

where we used $r \le 1$ in the third inequality and (A3) in the last inequality. Further, since $(a+b)^2 \ge \frac{1}{2}a^2 - b^2$,

$$\begin{aligned} I \ge & c(nr)^{2-d} \sum_{l\in\mathcal{S}_{nr}} \sum_{j=1}^{d} g_{nr}(l) \Big\{ \frac{1}{2}\Big(\log u_t\Big(l+\frac{e^j}{nr}\Big) - \log u_t(l)\Big)^2 - \Big(\frac{1}{2}\Big(\Big|l_j + \frac{1}{nr}\Big| - |l_j|\Big)\Big)^2 \Big\} \\ \ge & \frac{c}{2}(nr)^{2-d} \sum_{l\in\mathcal{S}_{nr}} \sum_{j=1}^{d} g_{nr}(l) \Big(\log u_t\Big(l+\frac{e^j}{nr}\Big) - \log u_t(l)\Big)^2 - \frac{cd}{4}(nr)^{-d} \sum_{l\in\mathcal{S}_{nr}} g_{nr}(l) \\ \ge & \frac{c}{2}(nr)^{2-d} \sum_{l\in\mathcal{S}_{nr}} \sum_{j=1}^{d} g_{nr}(l) \Big(\log u_t\Big(l+\frac{e^j}{nr}\Big) - \log u_t(l)\Big)^2 - c' \end{aligned}$$

Combining these, we have

$$\begin{aligned} G'(t) \ \ge \ & c_6 (nr)^{2-d} \sum_{l\in\mathcal{S}_{nr}} \sum_{j=1}^{d} \Big(\log u_t\Big(l+\frac{e^j}{nr}\Big) - \log u_t(l)\Big)^2 g_{nr}(l) - c_5 \\ \ge \ & c_7 (nr)^{-d} \sum_{l\in\mathcal{S}_{nr}} (\log u_t(l) - G(t))^2 g_{nr}(l) - c_5, \end{aligned}$$

where we used Lemma 4.3 in the last inequality. Given these estimates, the rest of the proof is very similar to that of [BKu08, Lemma 4.4]. □

Remark 4.5 There is an error in the proof of [BKu08, Lemma 4.4]. The estimate $|g_D(l+e) - g_D(l)| \le c_1 D^{-1}|e|(g_D(l+e) \wedge g_D(l))$ in page 2051, line 23, is not true when $D \ll |e|$. However, one can easily fix the proof by arguing as in the proof here.

The next lemma can be proved exactly in the same way as [BKu08, Lemma 4.5].

Lemma 4.6 *Given $\delta > 0$ there exists κ such that for each $n \in \mathbb{N}$, if $x, y \in \mathcal{S}_n$ and $C \subset \mathcal{S}_n$ with* $\mathrm{dist}\,(x, C)$ *and* $\mathrm{dist}\,(y, C)$ *both larger than $\kappa t^{1/2}$ where $t \in (n^{-1}, 1]$, then*

$$\mathbb{P}^x(Y_t^{(n)} = y, T_C^n \le t) \le \delta t^{-d/2} n^{-d}.$$

Proof of Proposition 4.2. We have from Lemma 4.4 that there exists ε such that

$$\mathbb{P}^x(Y_t^{(n)} = y) = p^n(t, x, y)\mu_y^n \ge \varepsilon t^{-d/2} n^{-d}$$

if $|x - y| \le 2t^{1/2}$. If we take $\delta = \varepsilon/2$ in Lemma 4.6, then provided $r > (\kappa+1)t^{1/2}$, we have

$$\mathbb{P}^x(Y_t^{(n)} = y, \tau^n_{B_n(x_0,r)} \le t) \le \frac{\varepsilon}{2} t^{-d/2} n^{-d}.$$

Subtracting,

$$\mathbb{P}^x(Y_t^{(n)} = y, \tau^n_{B_n(x_0,r)} > t) \ge \frac{\varepsilon}{2} t^{-d/2} n^{-d}$$

if $|x-y| \le t^{1/2}$, which is equivalent to what we want. □

For $(t,x) \in [0,1] \times \mathcal{S}_n$ and $r > 0$ let $Q^n(t,x,r) := [t, t+\gamma r^2] \times B_n(x,r)$, where $\gamma := \gamma(1/2, 1/2) < 1$. Here $\gamma(1/2,1/2)$ is the constant in (3.5) corresponding to $A = B = 1/2$.

Given the above estimates, we can prove the uniform Hölder continuity of the heat kernel $p^n(t,x,y)$ similarly to [BKu08, Theorem 4.9].

Theorem 4.7 *There are constants $c > 0$ and $\beta > 0$ (independent of R, n) such that for every $0 < R \le 1$, every $n \ge 1$, and every bounded parabolic function q in $Q^n(0, x_0, 4R)$,*

$$|q(s,x) - q(t,y)| \le c\, \|q\|_{\infty,R}\, R^{-\beta} \left(|t-s|^{1/2} + |x-y|\right)^{\beta} \tag{4.3}$$

holds for (s,x), $(t,y) \in Q^n(0,x_0,R)$, where $\|q\|_{\infty,R} := \sup_{(t,y)\in[0,\gamma(4R)^2]\times\mathcal{S}_n} |q(t,y)|$. In particular, for the transition density function $p^n(t,x,y)$ of $Y^{(n)}$,

$$|p^n(s,x_1,y_1) - p^n(t,x_2,y_2)| \le c\, t_0^{-(d+\beta)/2} \left(|t-s|^{1/2} + |x_1-x_2| + |y_1-y_2|\right)^{\beta}, \tag{4.4}$$

for any $n^{-1} < t_0 < 1$, t, $s \in [t_0, 1]$ and $(x_i, y_i) \in \mathcal{S}_n \times \mathcal{S}_n$ with $i = 1, 2$.

Proof. Given the above estimates, we can prove the analogues of Corollary 4.6 and Lemma 4.7 in [BKu08] exactly in the same way as is done there. Thus the proof of Theorem 4.7 is almost the same as that of [BKu08, Theorem 4.9] except for the following small change.

The following computation is needed to obtain the first inequality of (4.13) in [BKu08]:

$$\sup_{z\in B_n(x,r)} n^2 \sum_{y\in\mathcal{S}_n\setminus\overline{B_n(x,s)}} C^n(z,y) \le \left(\frac{s}{2}\right)^{-2} \sup_{z\in B_n(x,r)} \sum_{y\in\mathcal{S}_n} \Big(|z-y|^2 \wedge 1\Big) C^n(z,y) n^2 \le \frac{c_2}{s^2}$$

where (A3) is used in the last inequality (note that $2r \le s \le 1$). □

5 Weak convergence of the process

Recall that $Y_t^{(n)}$ are the continuous time Markov chains on $\mathcal{S}_n$ corresponding to $(\mathcal{E}^n, \mathcal{F}^n)$ in (2.2) and (2.3). Since the state space of $Y^{(n)}$ is $\mathcal{S}_n$ while the limit process will have $\mathbb{R}^d$ as its state space, we need to exercise some care with the domains of the functions we deal with. First, if g is defined on $\mathbb{R}^d$, we define $R_n(g)$ to be the restriction of g to $\mathcal{S}_n$:

$$R_n(g)(x) = g(x), \qquad x \in \mathcal{S}_n.$$

If g is defined on $\mathcal{S}_n$, we define $E_n g$ to be the extension of g to $\mathbb{R}^d$ defined by

$$E_n g(x) = g([x]_n),$$

where $[x]_n = ([nx_1]/n, [nx_2]/n, \ldots, [nx_d]/n)$ for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$.

In order to consider the convergence of the processes and to identify the limit process, we need to show the convergence of the semigroups of the Dirichlet forms $(\mathcal{E}^n, \mathcal{F}^n)$ in an appropriate sense. To this end, we now prepare some notation to specify a condition under which the convergence holds. For $n \in \mathbb{N}$, set

$$|x-y|_n := n|x_1-y_1| + n|x_2-y_2| + \cdots + n|x_d-y_d| \ (\in \mathbb{N}), \quad \text{for } x, y \in \mathcal{S}_n.$$

Note that $1 \le |x-y|_n \le dn|x-y|$ holds for any $x, y \in \mathcal{S}_n$ with $x \neq y$, where $|x-y|$ is the Euclidean distance between x and y.

A shortest path σ from x to y is a sequence of points $p_i \in \mathcal{S}_n$ for $i = 0, 1, 2, \ldots, k = |x-y|_n$, which we denote by $\sigma = \sigma(p_0, \ldots, p_k)$, so that $p_0 = x, p_k = y$ and for any $\ell = 0, 1, \ldots, k-1$, there exists $j \in \{1, 2, \ldots, 2d\}$ such that

$$p_\ell = p_{\ell+1} + \frac{1}{n}\boldsymbol{\alpha}_j,$$

where $\boldsymbol{\alpha}_i = \boldsymbol{e}_i$ if $i = 1, 2, \ldots, d$ and $\boldsymbol{\alpha}_i = -\boldsymbol{e}_{i-d}$ if $i = d+1, \ldots, 2d$. Let $\mathcal{P}(x, y)$ be the set of all shortest paths σ from x to y. The number of all such shortest paths σ is

$$\Pi(x,y) := \frac{(|x-y|_n)!}{\big(n|x_1-y_1|\big)!\big(n|x_2-y_2|\big)!\cdots\big(n|x_d-y_d|\big)!}.$$

For $\sigma \in \mathcal{P}(x, y)$, define a function D_σ defined on $\mathcal{S}_n \times \mathcal{S}_n$ as follows:

$$D_\sigma(w,z) := \begin{cases} 1, & \text{if there exists } \ell \text{ such that } w = p_\ell \text{ and } z = p_{\ell+1}, \\ 0, & \text{otherwise.} \end{cases}$$

For any function u defined on $\mathcal{S}_n$ and for any $x, y \in \mathcal{S}_n$, we easily see that

$$u(x) - u(y) = \frac{1}{\Pi(x,y)} \sum_{\sigma \in \mathcal{P}(x,y)} \sum_{z,w \in \mathcal{S}_n} D_\sigma(w,z)\big(u(w) - u(z)\big).$$

Now let

$$P^{x,y}(w,z) = \frac{1}{\Pi(x,y)} \sum_{\sigma \in \mathcal{P}(x,y)} D_\sigma(w,z).$$

For $h \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $i = 1, 2, \ldots, d$, let

$$\nabla_h^i u(x) = \frac{u(x + h\boldsymbol{e}_i) - u(x)}{h}.$$

We then have the following.

Lemma 5.1

$$u(x) - u(y) = \frac{1}{n} \sum_{i=1}^{d} \sum_{z \in \mathcal{S}_n} \Big(P^{x,y}(z + \boldsymbol{e}_i/n, z) - P^{x,y}(z, z + \boldsymbol{e}_i/n)\Big) \nabla_{1/n}^i u(z).$$

Proof. We have

$$\begin{aligned}
\sum_{w\in\mathcal{S}_n} & D_\sigma(w,z)\big(u(w)-u(z)\big) \\
&= \sum_{i=1}^{2d} D_\sigma(z+\boldsymbol{\alpha}_i/n,z)\big(u(z+\boldsymbol{\alpha}_i/n)-u(z)\big) \\
&= \sum_{i=1}^{d}\Big\{D_\sigma(z+\boldsymbol{e}_i/n,z)\big(u(z+\boldsymbol{\alpha}_i/n)-u(z)\big) \\
&\qquad +D_\sigma(z-\boldsymbol{e}_i/n,z)\big(u(z-\boldsymbol{e}_i/n)-u(z)\big)\Big\} \\
&= \frac{1}{n}\sum_{i=1}^{d}\Big\{D_\sigma(z+\boldsymbol{e}_i/n,z)\nabla^i_{1/n}u(z)-D_\sigma(z-\boldsymbol{e}_i/n,z)\nabla^i_{-1/n}u(z)\Big\}.
\end{aligned}$$

So

$$\begin{aligned}
u(x)&-u(y) \\
&= \sum_{z\in\mathcal{S}_n}\frac{1}{\Pi(x,y)}\sum_{\sigma\in\mathcal{P}(x,y)}\sum_{w\in\mathcal{S}_n} D_\sigma(w,z)\big(u(w)-u(z)\big) \\
&= \frac{1}{n}\sum_{i=1}^{d}\sum_{z\in\mathcal{S}_n}\frac{1}{\Pi(x,y)}\sum_{\sigma\in\mathcal{P}(x,y)}\Big(D_\sigma(z+\boldsymbol{e}_i/n,z)\nabla^i_{1/n}u(z)-D_\sigma(z-\boldsymbol{e}_i/n,z)\nabla^i_{-1/n}u(z)\Big) \\
&= \frac{1}{n}\sum_{i=1}^{d}\sum_{z\in\mathcal{S}_n}\Big(P^{x,y}(z+\boldsymbol{e}_i/n,z)\nabla^i_{1/n}u(z)-P^{x,y}(z-\boldsymbol{e}_i/n,z)\nabla^i_{-1/n}u(z)\Big).
\end{aligned}$$

Moreover, for each $i=1,2,\ldots,d$, and $x,y\in\mathcal{S}_n$,

$$\begin{aligned}
\sum_{z\in\mathcal{S}_n} P^{x,y}(z-\boldsymbol{e}_i/n,z)\nabla^i_{-1/n}u(z) &= \sum_{z\in\mathcal{S}_n} P^{x,y}(z,z+\boldsymbol{e}_i/n)\nabla^i_{-1/n}u(z+\boldsymbol{e}_i/n) \\
&= -n\sum_{z\in\mathcal{S}_n} P^{x,y}(z,z+\boldsymbol{e}_i/n)\big(u(z)-u(z+\boldsymbol{e}_i/n)\big) \\
&= \sum_{z\in\mathcal{S}_n} P^{x,y}(z,z+\boldsymbol{e}_i/n)\nabla^i_{1/n}u(z).
\end{aligned}$$

We thus obtain the desired equality. □

Remark 5.2 Here $P^{x,y}(\cdot,\cdot)$ is defined by averaging over the set of all shortest paths between x and y. However, we could take an average over other collections of paths. Let $x=(x_1,\cdots,x_d), y=(y_1,\cdots,y_d)$. Other possible collections of paths are the following:

(i) Let $H(x,y)$ be the d-dimensional cube whose vertices consist of $\{(z_1,\cdots,z_d): z_i \text{ is either } x_i \text{ or } y_i \text{ for } i=1,\cdots,d\}$. Let $\mathcal{P}(x,y)$ be the set of shortest paths between x and y that consist of a union of the edges of $H(x,y)$, and take the average over $\mathcal{P}(x,y)$. In this case $\Pi(x,y)$ in the definition of $P^{x,y}(\cdot,\cdot)$ is $d!$.
(ii) Let $L_{x,y}$ be the union of the line segment from x to $(y_1,x_2,\cdots,x_d)$, the line segment from $(y_1,x_2,\cdots,x_d)$ to $(y_1,y_2,x_3\cdots,x_d)$, $\cdots$, and the line segment from $(y_1,\cdots,y_{d-1},x_d)$ to y. Set $\mathcal{P}(x,y)=\{L_{xy}\}$ and $\Pi(x,y)=1$. This was used in [BKu08].

Next, let us fix a decreasing sequence $\{\varepsilon_n\}$ such that $1\ge\varepsilon_n\searrow 0$, and define functions $C^n_C(x,y)$, $C^n_J(x,y)$ on $\mathcal{S}_n\times\mathcal{S}_n$ as follows:

$$C^n_C(x,y) := \begin{cases} C^n(x,y), & \text{if} \quad |x-y|\le\varepsilon_n, \\ 0, & \text{otherwise,} \end{cases}$$

and $C^n_J(x,y) := C^n(x,y) - C^n_C(x,y), \quad x,y\in\mathcal{S}_n$.

Now define the following Dirichlet forms corresponding to the conductances $C^n_C(x,y)$ and $C^n_J(x,y)$, which we consider as the 'continuous part' and the 'jump part' of the Dirichlet form $(\mathcal{E}^n,\mathcal{F}^n)$; for $f\in L^2(\mathcal{S}_n,\mu^n)$,

$$\begin{cases} \mathcal{E}^n_C(f,g) & := & \dfrac{n^{2-d}}{2}\displaystyle\sum_{x,y\in\mathcal{S}_n}\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)C^n_C(x,y), \\ \mathcal{E}^n_J(f,g) & := & \dfrac{n^{2-d}}{2}\displaystyle\sum_{x,y\in\mathcal{S}_n}\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)C^n_J(x,y). \end{cases}$$

Then clearly $\mathcal{E}^n(f,g)=\mathcal{E}^n_C(f,g)+\mathcal{E}^n_J(f,g)$.

Using Lemma 5.1, we can write down $\mathcal{E}^n_C(u,v)$ as follows:

$$\begin{aligned} \mathcal{E}^n_C(u,v) &= \frac{n^{2-d}}{2}\sum_{x,y\in\mathcal{S}_n}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)C^n_C(x,y) \\ &= \frac{1}{2n^d}\sum_{x,y\in\mathcal{S}_n}\sum_{i,j=1}^{d}\sum_{z,w\in\mathcal{S}_n}\Big(P^{x,y}(z+\boldsymbol{e}_i/n,z)-P^{x,y}(z,z+\boldsymbol{e}_i/n)\Big) \\ &\qquad \times\Big(P^{x,y}(w+\boldsymbol{e}_j/n,w)-P^{x,y}(w,w+\boldsymbol{e}_j/n)\Big)\nabla^i_{1/n}u(z)\nabla^j_{1/n}v(w)C^n_C(x,y). \end{aligned} \tag{5.1}$$

For $i,j=1,2,\ldots,d$ and $w,z\in\mathcal{S}_n$, set

$$\begin{aligned} G^n_{ij}(w,z) &:= \sum_{x,y\in\mathcal{S}_n}\Big(P^{x,y}(z+\boldsymbol{e}_i/n,z)-P^{x,y}(z,z+\boldsymbol{e}_i/n)\Big) \\ &\qquad\qquad \times\Big(P^{x,y}(w+\boldsymbol{e}_j/n,w)-P^{x,y}(w,w+\boldsymbol{e}_j/n)\Big)C^n_C(x,y); \end{aligned}$$

then we see that

$$\mathcal{E}^n_C(u,v)=\frac{1}{2n^d}\sum_{i,j=1}^{d}\sum_{w,z\in\mathcal{S}_n}\nabla^i_{1/n}u(z)\nabla^j_{1/n}v(w)G^n_{ij}(w,z). \tag{5.2}$$

Let

$$F^n_{ij}(z) = \sum_{w\in\mathcal{S}_n} G^n_{ij}(w,z), \quad z\in\mathcal{S}_n,\ i,j=1,2,\dots,d. \tag{5.3}$$

Note that if (A4) below holds, then by the fact that $C^n_C(x,y)=0$ for $|x-y|>\varepsilon_n$, we have $F^n_{ij}\in L^1(\mathcal{S}_n,\mu^n)$.

From now on, we extend the conductances $C^n(x,y)$ to $\mathbb{R}^d\times\mathbb{R}^d$ as follows:

$$C^n(x,y) = C^n([x]_n,[y]_n) \quad \text{for } x,y\in\mathbb{R}^d.$$

We extend $C^n_C(\cdot,\cdot), C^n_J(\cdot,\cdot)$ to $\mathbb{R}^d\times\mathbb{R}^d$ and extend $F^n_{ij}(\cdot)$ to $\mathbb{R}^d$ similarly.

We now give an assumption needed to obtain weak convergence of the processes.

(A4) *There exist a decreasing sequence $\{\varepsilon_n\}$ satisfying $1/n\le\varepsilon_n\le 1$ and $\varepsilon_n\searrow 0$, symmetric matrix-valued functions $a(x)=(a_{ij}(x))$ on $\mathbb{R}^d$, and symmetric functions $j(x,y)$ on $\mathbb{R}^d\times\mathbb{R}^d\backslash D$ so that for any $i,j=1,2,\dots,d$, the functions $F^n_{ij}(x)$ are uniformly bounded and converge to $a_{ij}(x)$ locally in $L^1(\mathbb{R}^d)$, and*

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^d \xi_i\xi_j a_{ij}(x) \le \lambda|\xi|^2, \quad x,\xi\in\mathbb{R}^d,$$

for some $\lambda>0$. Further, for each $N>1$, the measures

$$n^{d+2}C^n(x,y)\mathbf{1}_{[N^{-1},N]}(|x-y|)dx\,dy \longrightarrow j(x,y)\mathbf{1}_{[N^{-1},N]}(|x-y|)dx\,dy \tag{5.4}$$

weakly as $n\to\infty$.

Remark 5.3 Here (5.4) refers to the weak convergence of the measures on the left to the measures on the right. Saying that the F^n_{ij} are uniformly bounded and converge locally in L^1 means that $\sup_{i,j,n}||F^n_{ij}||_\infty<\infty$ and for every compact set B,

$$\int_B |F^n_{ij}(x)-a_{ij}(x)|\,dx \to 0.$$

Since the F^n_{ij} are uniformly bounded, the convergence locally in L^1 is equivalent to the convergence in measure on each compact set. In particular, a subsequence will converge almost everywhere.

From (A3) and (A4), we have

$$\sup_x \int_{y\neq x}\big(1\wedge|x-y|^2\big)j(x,y)dy \le \int_{y\neq x}\big(1\wedge|x-y|^2\big)\varphi(|x-y|)dy = \int_{h\neq 0}\big(1\wedge|h|^2\big)\varphi(|h|)dh<\infty.$$

Since a is uniformly elliptic, if we define

$$\begin{aligned}\mathcal{E}(f,g) &:= \mathcal{E}_C(f,g)+\mathcal{E}_J(f,g)\\ &:= \frac12\int_{\mathbb{R}^d}\nabla f(x)\cdot a(x)\nabla g(x)dx + \frac12\iint_{x\neq y}\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)j(x,y)dx\,dy,\end{aligned}$$

then $(\mathcal{E},C^1_c(\mathbb{R}^d))$ is a closable Markovian form on $L^2(\mathbb{R}^d,dx)$. Denote the closure by $(\mathcal{E},\mathcal{F})$.

Lemma 5.4 *Let* $W^{1,2}(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d, dx) : \nabla f \in L^2(\mathbb{R}^d, dx)\}$. *Then,*

$$\{f \in L^2(\mathbb{R}^d, dx) : \mathcal{E}(f,f) < \infty\} = W^{1,2}(\mathbb{R}^d) = \mathcal{F}. \tag{5.5}$$

Further, if $(\mathcal{E}, \mathcal{F}')$ *is a regular Dirichlet form on* $L^2(\mathbb{R}^d, dx)$*, then* $\mathcal{F}' = W^{1,2}(\mathbb{R}^d)$.

PROOF. Let $f \in L^2$ be such that $\mathcal{E}(f,f) < \infty$. Then, $\mathcal{E}_C(f,f) < \infty$ and $\mathcal{E}_C(f,f)$ is comparable to $\|\nabla f\|_2^2$, so $f \in W^{1,2}(\mathbb{R}^d)$. On the other hand, suppose $f \in W^{1,2}(\mathbb{R}^d)$. Then, $\mathcal{E}_J(f,f) \le \mathcal{E}_\varphi(f,f)$, where φ is given in (A3) and $\mathcal{E}_\varphi$ is the Dirichlet form for the symmetric Lévy process with Lévy measure $\varphi(|h|)dh$. By the Lévy-Khintchine formula (see *e.g.* (1.4.21) in [FOT]), the characteristic function ψ of the process is given by

$$\psi(u) = \int_{\mathbb{R}^d} \Big(1 - \cos\big(u \cdot h\big)\Big) \varphi(|h|) dh, \quad u \in \mathbb{R}^d.$$

According to (A3), we have,

$$\begin{aligned} \psi(u) &= \int [1 - \cos(u \cdot h)] \varphi(|h|)\, dh \\ &\le c_1 \int [|u|^2 |h|^2 \wedge 1] \varphi(|h|)\, dh \\ &\le c_2(|u|^2 + 1). \end{aligned}$$

Using Plancherel's theorem, for $f \in C_c^2(\mathbb{R}^d)$,

$$\begin{aligned} \mathcal{E}_\varphi(f,f) &= \frac{1}{2} \iint_{y \neq x} \big(f(x+h) - f(x)\big)^2 \varphi(|h|) dh dx \\ &= \int \big|\widehat{f}(u)\big|^2 \psi(u) du \\ &\le c_2 \int (1 + |u|^2) |\widehat{f}(u)|^2\, du = c_3(\|f\|_2^2 + \|\nabla f\|_2^2). \end{aligned}$$

Here $\hat{f}$ is the Fourier transform of f. A limit argument shows that

$$\mathcal{E}_\varphi(f,f) \le c_4(\|f\|_2^2 + \|\nabla f\|_2^2) \tag{5.6}$$

for $f \in W^{1,2}(\mathbb{R}^d)$. Since $\mathcal{E}_C(f,f)$ is comparable to $\|\nabla f\|_2^2$, adding shows that $\mathcal{E}(f,f) < \infty$, and the first equality in (5.5) is proved. Now suppose $(\mathcal{E}, \mathcal{F}')$ is a regular Dirichlet form on $L^2(\mathbb{R}^d, dx)$; then since $W^{1,2}(\mathbb{R}^d)$ is the maximal domain (due to the first equality in (5.5)), we have $\mathcal{F}' \subset W^{1,2}(\mathbb{R}^d)$. From the above results, we know that the $(\mathcal{E}(\cdot,\cdot) + \|\cdot\|_2^2)$-norm is comparable to the $W^{1,2}$-norm on $W^{1,2}(\mathbb{R}^d)$. Using this, we see that $(\|\nabla \cdot\|_2^2, \mathcal{F}')$ is a regular Dirichlet form. This implies $\mathcal{F}' = W^{1,2}(\mathbb{R}^d)$ (so $W^{1,2}(\mathbb{R}^d) = \mathcal{F}$ as well) and the proof is complete. □

Under the above set-up we have the following, which is the main theorem of this paper.

Theorem 5.5 *Suppose (A1)-(A4) hold. Then for each x and each t_0 the $\mathbb{P}^{[x]_n}$-laws of $\{Y_t^{(n)}; 0 \leq t \leq t_0\}$ converge weakly with respect to the topology of the space $D([0,t_0],\mathbb{R}^d)$. If Z_t is the canonical process on $D([0,t_0),\mathbb{R}^d)$ and $\mathbb{P}^x$ is the weak limit of the $\mathbb{P}^{[x]_n}$-laws of $Y^{(n)}$, then the process $\{Z_t, \mathbb{P}^x\}$ is the symmetric Markov process corresponding to the Dirichlet form $\mathcal{E}$ with domain $W^{1,2}(\mathbb{R}^d)$.*

6 Proof of Theorem 5.5

In this section, we will prove Theorem 5.5. We first extend $\mathcal{E}^n$ and define a quadratic form on $L^2(\mathbb{R}^d, dx)$. Define

$$\mathcal{H}_n := \Big\{ E_n u : \ u \text{ is a function on } \mathcal{S}_n \Big\} \cap L^2(\mathbb{R}^d, dx).$$

For $f = E_n u \in \mathcal{H}_n$, define

$$\tilde{\mathcal{E}}^n(f,f) = \frac{n^{2+d}}{2} \iint_{x \neq y} \big(f(x) - f(y)\big)^2 C^n(x,y) dx\, dy.$$

Then we see

$$\begin{aligned} \tilde{\mathcal{E}}^n(f,f) &= \frac{n^{2+d}}{2} \sum_{w_1,w_2 \in \mathcal{S}_n} \big(u(w_1) - u(w_2)\big)^2 C^n(w_1,w_2) \big(n^{-d}\big)^2 \\ &= \frac{n^{2-d}}{2} \sum_{w_1,w_2 \in \mathcal{S}_n} \big(u(w_1) - u(w_2)\big)^2 C^n(w_1,w_2) = \mathcal{E}^n(u,u). \end{aligned} \tag{6.1}$$

Proof of Theorem 5.5. Let U_n^λ be the λ-resolvent for $Y^{(n)}$; this means that

$$U_n^\lambda h(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} h(Y_t^{(n)})\, dt$$

for $x \in \mathcal{S}_n$ and $h : \mathcal{S}_n \to \mathbb{R}$. The first step is to show that any subsequence $\{n_j\}$ has a further subsequence $\{n_{j_k}\}$ such that $U_{n_{j_k}}^\lambda (R_{n_{j_k}} f)$ converges uniformly on compacts whenever $f \in C_c(\mathbb{R}^d)$, that is, f is continuous with compact support. Given Proposition 3.1 and Theorem 4.7, the proof of this is very similar to that of [BKu08, Proposition 6.2], and we refer the reader to that paper.

Now suppose we have a subsequence $\{n'\}$ such that the $U_{n'}^\lambda(R_{n'} f)$ are equicontinuous and converge uniformly on compacts whenever $f \in C_c(\mathbb{R}^d)$. Fix such an f and let H be the limit of $U_{n'}^\lambda(R_{n'} f)$. Let $g \in C_c^2(\mathbb{R}^d)$ and write $\langle f, g\rangle := \int_{\mathbb{R}^d} f(x) g(x) dx$.

In the following, we drop the primes for legibility. Set $u_n = U_n^\lambda(R_n f)$ for $\lambda > 0$. We will prove that

$$H \in W^{1,2}(\mathbb{R}^d) \qquad \text{and} \qquad \mathcal{E}^n(u_n, g) \to \mathcal{E}(H, g) \tag{6.2}$$

along some subsequence. Once we have (6.2), then

$$\begin{aligned}\mathcal{E}(H,g) &= \lim \mathcal{E}^n(u_n,g) = \lim(\langle f,g\rangle_n - \lambda\langle u_n,g\rangle_n)\\ &= \langle f,g\rangle - \lambda\langle H,g\rangle,\end{aligned}$$

the limit being taken along the subsequence and where $\langle h_1,h_2\rangle_n = n^{-d}\sum_{x\in\mathcal{S}_n} h_1(x)h_2(x)$ for $h_1,h_2:\mathcal{S}_n\to\mathbb{R}$. By (6.2), $H\in W^{1,2}(\mathbb{R}^d)$, and the equality

$$\mathcal{E}(H,g) = \langle f,g\rangle - \lambda\langle H,g\rangle \tag{6.3}$$

holds for all $g\in C_c^2(\mathbb{R}^d)$. By Lemma 5.4, $C_c^2(\mathbb{R}^d)$ is dense in $W^{1,2}(\mathbb{R}^d)$ with respect to the norm $(\mathcal{E}(\cdot,\cdot)+\|\cdot\|_2^2)^{1/2}$, and so (6.3) holds for all $g\in W^{1,2}(\mathbb{R}^d)$. Since $W^{1,2}(\mathbb{R}^d)$ is the maximal domain due to (5.5), this implies that H is the λ-resolvent of f for the process corresponding to $(\mathcal{E},W^{1,2}(\mathbb{R}^d))$, that is, $H=U^\lambda f$. We can then conclude that the full sequence $U_n^\lambda(R_nf)$ (without the primes) converges to $U^\lambda f$ whenever $f\in C_c(\mathbb{R}^d)$. The assertions about the convergence of $\mathbb{P}^{[x]_n}$ then follow as in [BKu08, Proposition 6.2]. The rest of the proof will be devoted to proving (6.2).

The jump part.

This part of the proof is similar to that of [BKK, Theorem 4.1]. We know

$$\mathcal{E}^n(u_n,u_n) = \langle R_nf,u_n\rangle_n - \lambda\|u_n\|_{2,n}^2. \tag{6.4}$$

Since $||\lambda u_n||_{2,n}^2 = ||\lambda U_n^\lambda R_nf||_{2,n} \le ||R_nf||_{2,n} \le \sup_n ||R_nf||_{2,n}$ (note that $\sup_n ||R_nf||_{2,n}<\infty$ because $\lim_{n\to\infty}||R_nf||_{2,n} = \|f\|_2$ for $f\in C_c(\mathbb{R}^d)$), the right hand side of (6.4) is bounded by

$$\left|\langle R_nf,u_n\rangle_n\right| + \lambda||u_n||_{2,n}^2 \le \frac{1}{\lambda}||R_nf||_{2,n}||\lambda u_n||_{2,n} + \frac{1}{\lambda}||\lambda u_n||_{2,n}^2 \le \frac{2}{\lambda}\sup_n ||R_nf||_{2,n}^2.$$

This tells us that $\{\mathcal{E}^n(u_n,u_n)\}_n$ is uniformly bounded.

Since the u_n are equicontinuous and converge uniformly to H on $\overline{B(0,N)}$ for $N>0$, using (5.4), we have

$$\begin{aligned}\int\int_{N^{-1}\le|y-x|\le N} &(H(y)-H(x))^2 j(x,y)\,dy\,dx\\ &\le \limsup_{n\to\infty} n^{2-d}\sum_{\substack{x,y\in\mathcal{S}_n\\ N^{-1}<|y-x|\le N}} (u_n(y)-u_n(x))^2 C^n(x,y)\\ &\le \limsup_n \mathcal{E}^n(u_n,u_n) \le c<\infty.\end{aligned}$$

Letting $N\to\infty$, we have

$$\mathcal{E}_J(H,H)<\infty. \tag{6.5}$$

Fix a function g on $\mathcal{S}_n$ with compact support and choose M large enough so that the support of g is contained in $B(0,M)$. Then

$$\begin{aligned}&\Big|n^{2-d}\sum_{\substack{x,y\in\mathcal{S}_n\\ |y-x|>N}} (u_n(y)-u_n(x))(g(y)-g(x))C^n(x,y)\Big|\\ &\le \Big(n^{2-d}\sum_{x,y\in\mathcal{S}_n}(u_n(y)-u_n(x))^2C^n(x,y)\Big)^{1/2}\Big(n^{2-d}\sum_{\substack{x,y\in\mathcal{S}_n\\ |y-x|>N}}(g(y)-g(x))^2C^n(x,y)\Big)^{1/2}.\end{aligned}$$

The first factor is $(\mathcal{E}^n(u_n,u_n))^{1/2}$, while the second factor is bounded by

$$2\|g\|_\infty\Big(n^{2-d}\sum_{x\in B(0,M)\cap\mathcal{S}_n}\sum_{|y-x|>N}C^n(x,y)\Big)^{1/2},$$

which, in view of (2.1), will be small if N is large. Similarly,

$$\begin{aligned}&\Big|n^{2-d}\sum_{\substack{x,y\in\mathcal{S}_n\\|y-x|<N^{-1}}}(u_n(y)-u_n(x))(g(y)-g(x))C^n(x,y)\Big|\\&\le\Big(n^{2-d}\sum_{x,y\in\mathcal{S}_n}(u_n(y)-u_n(x))^2C^n(x,y)\Big)^{1/2}\cdot\Big(n^{2-d}\sum_{\substack{x,y\in\mathcal{S}_n\\|y-x|<N^{-1}}}(g(y)-g(x))^2C^n(x,y)\Big)^{1/2}.\end{aligned}$$

The first factor is as before, while the second is bounded by

$$\|\nabla g\|_\infty\Big(n^{2-d}\sum_{x\in B(0,M)\cap\mathcal{S}_n}\sum_{|y-x|<N^{-1}}|y-x|^2C^n(x,y)\Big)^{1/2}.$$

In view of (2.1), the second factor will be small if N is large.

Using (6.5), we have that

$$\Big|\int\int_{|y-x|\notin[N^{-1},N]}(H(y)-H(x))(g(y)-g(x))j(x,y)\,dy\,dx\Big|$$

will be small if N is taken large enough.

By (5.4) and the fact that the $U_n^\lambda f$ are equicontinuous and converge to H uniformly on compacts, if we take n large enough so that $\varepsilon_n\le N^{-1}$, we have

$$\begin{aligned}&n^{2-d}\sum_{\substack{x,y\in\mathcal{S}_n\\N^{-1}\le|y-x|\le N}}(u_n(y)-u_n(x))(g(y)-g(x))C_J^n(x,y)\\&=n^{2-d}\sum_{\substack{x,y\in\mathcal{S}_n\\N^{-1}\le|y-x|\le N}}(u_n(y)-u_n(x))(g(y)-g(x))C^n(x,y)\\&\to\int\int_{N^{-1}\le|y-x|\le N}(H(y)-H(x))(g(y)-g(x))j(x,y)\,dy\,dx.\end{aligned}$$

It follows that

$$\mathcal{E}_J^n(u_n,g)\to\mathcal{E}_J(H,g),\tag{6.6}$$

which takes care of the jump part of (6.2).

The continuous part.

Step 1. First we show that $H\in W^{1,2}(\mathbb{R}^d)$.

As in the discussion of the jump part, we know $\{\mathcal{E}^n(u_n,u_n)\}_n$ is uniformly bounded. On the other hand, making use of the assumption (A2), we see

$$\tilde{\mathcal{E}}^n(E_nu_n,E_nu_n)=\mathcal{E}^n(u_n,u_n)\ge c\mathcal{E}_{NN}^n(u_n,u_n)=c\tilde{\mathcal{E}}_{NN}^n(E_nu_n,E_nu_n).$$

Therefore, for $f \in C_c^1(\mathbb{R}^d)$, the sequence $\{\tilde{\mathcal{E}}_{NN}^n(E_nu_n, E_nu_n)\}_n$ is uniformly bounded with respect to n. Letting $Q_n(w) = \prod_{i=1}^d [w_i, w_i + 1/n)$, we see that for any $i = 1, 2, \ldots, d$,

$$\begin{aligned}
\tilde{\mathcal{E}}_{NN}^n(E_nu_n, E_nu_n) &= \frac{n^{2+d}}{2} \iint_{x\neq y} \big(E_nu_n(x) - E_nu_n(y)\big)^2 C_{NN}^n(x,y)dxdy \\
&= \frac{n^{2+d}}{2} \sum_{w\in\mathcal{S}_n} \int_{Q_n(w)} \Big(\int_{y\neq x} \big(E_nu_n(x) - E_nu_n(y)\big)^2 C_{NN}^n(w,[y]_n)dy\Big)dx \\
&\geq \frac{n^{2+d}}{2} \sum_{w\in\mathcal{S}_n} \int_{Q_n(w)} \big(E_nu_n(x) - E_nu_n(x+\boldsymbol{e}_i/n)\big)^2 \Big(\int_{Q_n(w+\boldsymbol{e}_i/n)} dy\Big)dx \\
&= \frac{n^2}{2} \sum_{w\in\mathcal{S}_n} \int_{Q_n(w)} \big(E_nu_n(x) - E_nu_n(x+\boldsymbol{e}_i/n)\big)^2 dx \\
&= \frac{n^2}{2} \int_{\mathbb{R}^d} \big(E_nu_n(x) - E_nu_n(x+\boldsymbol{e}_i/n)\big)^2 dx.
\end{aligned}$$

In other words, $\{n\big(E_nu_n(\cdot) - E_nu_n(\cdot + \boldsymbol{e}_i/n)\big)\}_n$ is a bounded sequence in $L^2(\mathbb{R}^d, dx)$. So there exists a subsequence $\{n'\}$ and a unique $v_i \in L^2(\mathbb{R}^d, dx)$ so that $n'\big(E_{n'}u_{n'}(\cdot) - E_{n'}u_{n'}(\cdot + \boldsymbol{e}_i/n')\big)$ converges to v_i weakly in $L^2(\mathbb{R}^d, dx)$. On the other hand, if $\varphi \in C_c^2(\mathbb{R}^d)$, it follows that

$$\langle E_{n'}u_{n'}(\cdot + \boldsymbol{e}_i/n'), \varphi\rangle = \langle E_{n'}u_{n'}, \varphi(\cdot - \boldsymbol{e}_i/n')\rangle$$

by a change of variables, and then

$$n'\langle E_{n'}u_{n'}(\cdot + \boldsymbol{e}_i/n'), \varphi\rangle - n'\langle E_{n'}u_{n'}, \varphi\rangle = n'\langle E_{n'}u_{n'}, \varphi(\cdot - \boldsymbol{e}_i/n') - \varphi\rangle.$$

Since $\varphi \in C_c^2(\mathbb{R}^d)$, we see that $n'\big(\varphi(\cdot - \boldsymbol{e}_i/n') - \varphi\big)$ converges to $-\partial\varphi/\partial x_i$ uniformly and in $L^2(\mathbb{R}^d, dx)$. So we have, letting $n' \to \infty$,

$$\langle v_i, \varphi\rangle = -\langle H, \partial\varphi/\partial x_i\rangle,$$

since u_n converges to H uniformly on compact sets. This shows that $v_i = \partial H/\partial x_i$ and so $H \in W^{1,2}(\mathbb{R}^d)$.

Step 2. We next show that for some subsequence $\{n'\}$,

$$\mathcal{E}_C^{n'}(u_{n'}, g) \longrightarrow \frac{1}{2} \int_{\mathbb{R}^d} \nabla H(x) \cdot a(x) \nabla g(x) dx = \mathcal{E}_C(H, g)$$

for any $g \in C_c^2(\mathbb{R}^d)$. Recall (5.2); since $C_C^n(x,y) = 0$ if $|x-y| > \varepsilon_n$ and the w, z are on the shortest paths from x and y, it is enough to consider w's only for $|w-z| \leq \varepsilon_n$ in the sum of the right hand side of (5.2). So

$$\begin{aligned}
\mathcal{E}_C^n(u_n,g) &= \frac{1}{2n^d}\sum_{i,j=1}^d\sum_{z\in\mathcal{S}_n}\nabla_{1/n}^i u_n(z)\sum_{\substack{w\in\mathcal{S}_n\\|w-z|\le\varepsilon_n}}\nabla_{1/n}^j g(w)G_{ij}^n(w,z)\\
&= \frac{1}{2n^d}\sum_{i,j=1}^d\sum_{z\in\mathcal{S}_n}\nabla_{1/n}^i u_n(z)\nabla_{1/n}^j g(z)\sum_{\substack{w\in\mathcal{S}_n\\|w-z|\le\varepsilon_n}}G_{ij}^n(w,z)\\
&\quad+\frac{1}{2n^d}\sum_{i,j=1}^d\sum_{z\in\mathcal{S}_n}\nabla_{1/n}^i u_n(z)\sum_{\substack{w\in\mathcal{S}_n\\|w-z|\le\varepsilon_n}}\Big(\nabla_{1/n}^j g(w)-\nabla_{1/n}^j g(z)\Big)G_{ij}^n(w,z)\\
&=: I_1^n+I_2^n.
\end{aligned}$$

Let K be the support of $g\in C_c^2(\mathbb{R}^d)$. Since $1/n\le\varepsilon_n\le 1$ and $|w-z|\le\varepsilon_n$ in the summation defining I_2^n, the z's must lie in the set $K_1\cap\mathcal{S}_n$, where $K_1=\{x\in\mathbb{R}^d:\ d(K,x)\le 1\}$. By using the mean value theorem for g and the definition of $\nabla_{1/n}^i u_n$, we see that for some $0<\theta,\tilde\theta<1$ depending on z and w,

$$\begin{aligned}
2|I_2^n| &= \Big|n^{-d}\sum_{i,j=1}^d\sum_{z\in\mathcal{S}_n}\nabla_{1/n}^i u_n(z)\sum_{\substack{w\in K_1\cap\mathcal{S}_n\\|w-z|\le\varepsilon_n}}\Big(\nabla_{1/n}^j g(w)-\nabla_{1/n}^j g(z)\Big)G_{ij}^n(w,z)\Big|\\
&= \Big|n^{1-d}\sum_{i,j=1}^d\sum_{z\in\mathcal{S}_n}\Big(u_n(z+\boldsymbol{e}_i/n)-u_n(z)\Big)\\
&\qquad\times\sum_{\substack{w\in K_1\cap\mathcal{S}_n\\|w-z|\le\varepsilon_n}}\Big(\partial_j g(w+\theta\boldsymbol{e}_j/n)-\partial_j g(w+\tilde\theta\boldsymbol{e}_j/n)\Big)G_{ij}^n(w,z)\Big|\\
&\le \Big(\sup_{|z-z'|\le 1/n}\big|u_n(z)-u_n(z')\big|\Big)\cdot\sup_j||\partial_{jj}g||_\infty\times\Big(n^{-d}\sum_{i,j=1}^d\sum_{z\in\mathcal{S}_n}\sum_{\substack{w\in K_1\cap\mathcal{S}_n\\|w-z|\le\varepsilon_n}}\big|G_{ij}^n(w,z)\big|\Big)\\
&=: \Big(\sup_{|z-z'|\le 1/n}\big|u_n(z)-u_n(z')\big|\Big)\cdot\sup_j||\partial_{jj}g||_\infty\times I_3^n.
\end{aligned}$$

We now estimate I_3^n. Let $K_2 = \{x \in \mathbb{R}^d : \ d(K_1, x) \le 1\}$. Then,

$$
\begin{aligned}
I_3^n &= n^{-d} \sum_{i,j=1}^{d} \sum_{z \in \mathcal{S}_n} \sum_{\substack{w \in K_1 \cap \mathcal{S}_n \\ |w-z| \le \varepsilon_n}} \Big| \sum_{\substack{x,y \in \mathcal{S}_n \\ |x-y| \le \varepsilon_n}} \Big(P^{x,y}(z + \boldsymbol{e}_i/n, z) - P^{x,y}(z, z + \boldsymbol{e}_i/n) \Big) \\
&\qquad \times \Big(P^{x,y}(w + \boldsymbol{e}_j/n, w) - P^{x,y}(w, w + \boldsymbol{e}_j/n) \Big) C_C^n(x,y) \Big| \\
&\le n^{-d} \sum_{\substack{x,y \in \mathcal{S}_n \\ |x-y| \le \varepsilon_n}} C_C^n(x,y) \sum_{i=1}^{d} \sum_{z \in \mathcal{S}_n} \Big(P^{x,y}(z + \boldsymbol{e}_i/n, z) + P^{x,y}(z, z + \boldsymbol{e}_i/n) \Big) \\
&\qquad \times \sum_{j=1}^{d} \sum_{\substack{w \in K_1 \cap \mathcal{S}_n \\ |w-z| \le \varepsilon_n}} \Big(P^{x,y}(w + \boldsymbol{e}_j/n, w) + P^{x,y}(w, w + \boldsymbol{e}_j/n) \Big). \\
&= n^{-d} \sum_{\substack{x \in K_2 \cap \mathcal{S}_n, y \in \mathcal{S}_n \\ |x-y| \le \varepsilon_n}} C_C^n(x,y) \sum_{i=1}^{d} \sum_{z \in \mathcal{S}_n} \Big(P^{x,y}(z + \boldsymbol{e}_i/n, z) + P^{x,y}(z, z + \boldsymbol{e}_i/n) \Big) \\
&\qquad \times \sum_{j=1}^{d} \sum_{\substack{w \in K_1 \cap \mathcal{S}_n \\ |w-z| \le \varepsilon_n}} \Big(P^{x,y}(w + \boldsymbol{e}_j/n, w) + P^{x,y}(w, w + \boldsymbol{e}_j/n) \Big).
\end{aligned}
$$

The last equality holds since the w's (belonging to K_1) lie on some shortest path between x and y in the summations for some $x, y \in \mathcal{S}_n$ with $|x-y| \le \varepsilon_n$. Noting now that

$$
\begin{aligned}
&\sum_{j=1}^{d} \sum_{\substack{w \in K_1 \cap \mathcal{S}_n \\ |w-z| \le \varepsilon_n}} \Big(P^{x,y}(w + \boldsymbol{e}_j/n, w) + P^{x,y}(w, w + \boldsymbol{e}_j/n) \Big) \\
&\qquad \le \sum_{j=1}^{d} \sum_{w \in \mathcal{S}_n} \Big(P^{x,y}(w + \boldsymbol{e}_j/n, w) + P^{x,y}(w, w + \boldsymbol{e}_j/n) \Big) = n|x-y|
\end{aligned}
$$

and similarly

$$
\sum_{i=1}^{d} \sum_{z \in \mathcal{S}_n} \Big(P^{x,y}(z + \boldsymbol{e}_i/n, z) + P^{x,y}(z, z + \boldsymbol{e}_i/n) \Big) = n|x-y|,
$$

we see that, using (2.1),

$$
I_3^n \le n^{-d} \sum_{x \in K_2 \cap \mathcal{S}_n} \sum_{\substack{y \in \mathcal{S}_n \\ |x-y| \le \varepsilon_n}} n^2 |x-y|^2 C_C^n(x,y) \le M \mu^n(K_2),
$$

where M is the constant in the assumption (A3) (see (2.1)). So, I_3^n is uniformly bounded in n and hence I_2^n converges to 0 as n tends to ∞ since the $\{u_n\}$ are equicontinuous.

Finally we consider the term I_1^n:

$$\begin{aligned} I_1^n &= \frac{1}{2n^d}\sum_{i,j=1}^d \sum_{z\in\mathcal{S}_n} \nabla^i_{1/n} u_n(z)\nabla^j_{1/n} g(z) F^n_{ij}(z) \\ &= \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{R}^d} \nabla^i_{1/n} E_n u_n(x)\nabla^j_{1/n} E_n g(x) F^n_{ij}(x)dx. \end{aligned}$$

Observe that if f_n converges to f weakly in L^2 and g_n converges to g boundedly and almost everywhere, then $f_n g_n$ converges to fg weakly. To see this, if $h \in L^2$,

$$\int (f_n g_n)h - \int (fg)h = \int f_n(g_n - g)h + \Big[\int f_n g h - \int f g h\Big].$$

The term inside the brackets on the right hand side goes to 0 since f_n converges to f weakly and the boundedness of g implies that gh is in L^2. The first term on the right hand side is bounded, using Cauchy-Schwarz, by $\|f_n\|_2\,\|(g_n-g)h\|_2$. The factor $\|f_n\|_2$ is uniformly bounded since f_n converges weakly in L^2, while $\|(g_n-g)h\|_2$ converges to 0 by dominated convergence.

Since some subsequence of $\nabla^i_{1/n}E_n u_n$ converges to $v_i = \partial_i H$ weakly in L^2 (as proved in Step 1), and for some further subsequence F^n_{ij} converges to a_{ij} boundedly and almost everywhere (by (A4) and Remark 5.3) and $\nabla^j_{1/n} g$ converges to $\partial_j g$ uniformly on compact sets (because $g \in C^2_c(\mathbb{R}^d)$), we see that, along this further subsequence, the right hand side goes to

$$\frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{R}^d} \partial_j H\, \partial_j g\, a_{ij}\, dx = \frac{1}{2}\int_{\mathbb{R}^d} \nabla H(x)\cdot a(x)\nabla g(x) dx.$$

Hence

$$\mathcal{E}^{n'}_C(u_{n'}, g) \to \mathcal{E}_C(H, g).$$

This completes the proof of (6.2) and hence the theorem. □

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