The Annals of Probability
1995, Vol. 23, No. 1, 388–399

LAWS OF THE ITERATED LOGARITHM FOR LOCAL TIMES OF THE EMPIRICAL PROCESS[1]

BY RICHARD F. BASS AND DAVAR KHOSHNEVISAN

University of Washington

We give exact expansions for the upper and lower tails of the distribution of the maximum of local time of standard Brownian bridge on interval $[0,1]$. We use the above expansions to prove upper and lower laws of the iterated logarithm for the maximum of the local time of the uniform empirical process. This solves two open problems cited in the book of Shorack and Wellner.

1. Introduction. Let $X_1, X_2, \ldots$ be independent random variables, each distributed uniformly on the interval $[0,1]$. Define the corresponding empirical process,

$$\nu_n(t) = \sqrt{n}\,(F_n(t) - t), \qquad 0 \le t \le 1,$$

where F_n is the usual empirical distribution function given by

$$F_n(t) = \frac{1}{n}\sum_{j=1}^{n} 1_{[0,t]}(X_j).$$

It is well known (see, e.g., [5]) that ν_n converges weakly to a Brownian bridge, $\{\omega(t);\ 0 \le t \le 1\}$. Convergence here takes place in the space $D([0,1])$.

Suppose next that for some $\tau \in [0,\infty)$, $f\colon [0,\tau] \mapsto \mathbb{R}^1$ is a Borel measurable function. When it makes sense, by local times for f we mean a family of functions $L_t^x(f)$ such that for all $t \in [0,\tau]$ and all bounded Borel measurable $h\colon \mathbb{R}^1 \mapsto \mathbb{R}^1$,

$$\int_0^t h(f(s))\,ds = \int_{-\infty}^{\infty} h(x) L_t^x(f)\,dx.$$

The question of whether $\{L_t^x(\nu_n); t \in [0,1], x \in \mathbb{R}^1\}$ converges to $\{L_t^x(\omega); t \in [0,1], x \in \mathbb{R}^1\}$ was a problem of Smirnov cited in [24] which was recently solved in Khoshnevisan [17]. This problem arises from statistical applications, since (see [24], page 398):

$$L_t^x(\nu_n) = n^{-1/2} \sum_{s \le t} 1_{\{x\}}(\nu_n(s)). \tag{1.1}$$

Received March 1993; revised March 1994.
[1]Research supported by NSF Grant DMS-91-00224.
AMS 1991 subject classifications. Primary 60J55, 62G30; secondary 60J60, 60J75.
Key words and phrases. Empirical process, local times, Brownian bridge, laws of the iterated logarithm.

The above is, of course, a finite sum due to the sample path properties of $t \mapsto \nu_n(t)$. In particular, $\sqrt{n}\, L_1^0(\nu_n)$ is the total number of times the empirical distribution function, F_n, crosses the theoretical one. For history and references, see Shorack and Wellner [24] and Khoshnevisan [17].

This paper is concerned with the remaining open problems in [24] (see page 400) on this matter. Let $L_t^*(f) = \sup_x L_t^x(f)$. We shall prove upper and lower laws of the iterated logarithm for the quantity $L_1^*(\nu_n)$. More precisely, we prove the following theorem.

THEOREM 1.1. *With probability* 1,

$$\limsup_{n\to\infty} \frac{L_1^0(\nu_n)}{\sqrt{2\ln\ln n}} = \limsup_{n\to\infty} \frac{L_1^*(\nu_n)}{\sqrt{2\ln\ln n}} = 1.$$

THEOREM 1.2. *With probability* 1,

$$\liminf_{n\to\infty} \sqrt{\ln\ln n}\; L_1^*(\nu_n) = \sqrt{2}\,\pi.$$

It is interesting to note that the constant in Theorem 1.2 is quite different from the one in the analogous result for Brownian motion, which involves the first zero of a certain Bessel function (see [16] and Földes [9]).

In the next section, we review some known facts concerning the exact distribution of $L_1^*(\omega)$, where ω is a Brownian bridge on $[0,1]$. Section 3 applies the results of Section 2 and Khoshnevisan [17] to prove Theorems 1.1 and 1.2. In Section 4 we discuss how our methods can be used to study Kiefer processes.

2. Local times for the Brownian bridge. As in the Introduction, let ω denote the standard Brownian bridge on $[0,1]$, that is, a mean zero Gaussian process with $\mathbb{E}(\omega(s)\omega(t)) = (s \wedge t) - st$ for all $s,t \in [0,1]$. It is known that $L_t^x(\omega)$ exists and is jointly continuous in x and t; see [22]. The main theorem of this section is the following:

THEOREM 2.1. *For any* $\beta > 0$:
(a) $\mathbb{P}(L_1^*(\omega) < \beta) = 2(2\pi)^{5/2}\beta^{-3}\sum_{j=1}^{\infty} j^2 \exp(-2j^2\pi^2/\beta^2)$.
(b) $\mathbb{P}(L_1^*(\omega) \geq \beta) = 2\sum_{j=1}^{\infty}(j^2\beta^2 - 1)\exp(-j^2\beta^2/2)$.
(c) $\mathbb{P}(L_1^x(\omega) \geq \beta) = \exp(-(2|x| + \beta)^2/2)$.

While both (a) and (b) are valid for all $\beta > 0$, the former is useful for small β, whereas the latter is accurate for large values of β. Indeed the following corollary is an immediate consequence of Theorem 2.1:

COROLLARY 2.2. (a) *As* $\beta \to 0$, $\mathbb{P}(L_1^*(\omega) < \beta) \sim 2(2\pi)^{5/2}\beta^{-3}\exp(-2\pi^2/\beta^2)$.
(b) *As* $\beta \to \infty$, $\mathbb{P}(L_1^*(\omega) \geq \beta) \sim 2\beta^2\exp(-\beta^2/2)$.

Corollary 2.2 and [16] together show that for large values of β, $\mathbb{P}(L_1^*(\omega) \geq \beta)$ is (at least up to the exponential terms) of the same order as the corresponding quantity for Brownian motion. However, it is surprising that for small β, $\mathbb{P}(L_1^*(\omega) < \beta)$ is substantially different from the Brownian motion analogue. The latter involves the smallest positive zero of the modified Bessel function of the first kind. For this see [9].

An application of Khoshnevisan ([17], Theorem 2) shows that $L_1^*(\nu_n)$ converges in distribution to $L_1^*(\omega)$. Corollary 2.2(b) might then be of particular use for asymptotic confidence evaluations based on $L_1^*(\nu_n)$.

Theorem 2.1 is contained in some prior work in the literature which we would like to mention here: as a consequence of Biane ([3], Theorem 4), Vervaat [25] and Jeulin ([14], page 264), L_1^* has the same distribution as $2M$ and $2R$, where M is the maximum of a Brownian excursion and R is the range of a Brownian bridge. The distribution of M and R can then be read off by putting together the work of Gnedenko [13], Kiefer [18] and Gikhman [12]. (The relevance of Kiefer's results is due to Williams' identification ([23], pages 88–89), of the law of Brownian excursion with that of a Bessel(3) bridge.) See also [8], [15] and [21]. This is a nice argument, and we encourage the reader to read Zhan and Yor [27] for details. Theorem 2.1(c) is due to Borodin [6], [7]. The equivalence of Theorem 2.1(a) and (b) is due to Kiefer [18]. Using the equivalence in law of L_1^* and $2R$, the distribution of $L_1^*/2$ can be found and tabulated in [24], pages 39 and 144.

We present below a proof of Theorem 2.1 which does not use Brownian excursion. The proof is an adaptation of the method used by Borodin [7] to obtain the analogous results for Brownian motion. See also [9] and [10], Section 4, for a number of interesting extensions to Borodin's work.

Let $B(\cdot)$ be a standard Brownian motion and S_θ be an exponential time independent of B with mean $2\theta^{-2}$. We have the following proposition.

PROPOSITION 2.3. *For all θ, β and $s > 0$,*

$$\frac{d}{dx}\mathbb{P}(B(S_\theta) \leq x; L_{S_\theta}^*(B) < \beta)\Big|_{x=0} = \frac{\theta(e^{2\theta\beta} - 2\theta\beta e^{\theta\beta} - 1)}{2(e^{\theta\beta} - 1)^2}. \tag{2.1}$$

PROOF. This proposition is a consequence of the Ray–Knight theorems. We use the version given in Biane and Yor [4], Theorem 1. By letting $a \to 0$ in that theorem, we see that conditional on $\{L_{S_\theta}^0(B) = s, B(S_\theta) = 0\}$,

$$\{L_{S_\theta}^x(B); x \in \mathbb{R}\} \stackrel{\mathscr{D}}{=} \begin{cases} W(x), & \text{if } x \geq 0, \\ W'(-x), & \text{if } x \geq 0, \end{cases}$$

where $\stackrel{\mathscr{D}}{=}$ means equality in distribution and W and W' are independent copies of a one-dimensional diffusion starting at s, absorbing at zero, with infinitesimal generator: $\mathscr{L}f(x) = 2xf''(x) - 2\theta x f'(x)$.

Note $S(x) = \exp(\theta x)$ satisfies $\mathscr{L}S = 0$, hence S is the scale function for W and W'. Let $\tau_x = \inf\{s > 0: W(s) = x\}$. Since W gets absorbed upon hitting 0,

$$\begin{aligned}
&\mathbb{P}(L^*_{S_\theta}(B) < \beta \mid L^0_{S_\theta}(B) = s, B(S_\theta) = 0) \\
&\quad = \mathbb{P}(\sup_x W(x) < \beta, \sup_x W'(x) < \beta) \\
&\quad = (\mathbb{P}(\sup_x W(x) < \beta))^2 \\
&\quad = (\mathbb{P}(\tau_0 < \tau_\beta \mid W(0) = s))^2 \\
&\quad = \left(\frac{S(\beta) - S(s)}{S(\beta) - S(0)}\right)^2,
\end{aligned}$$

by the definition of scale function.

The proposition follows from the above, the fact that $L^0_{S_\theta}$ has an exponential distribution with parameter θ and is independent of $B(S_\theta)$, which has a bilateral exponential distribution with parameter θ ([4], Theorem 1(i)), and integration. □

By expanding the right-hand side of Proposition 2.3 in powers of $e^{-\theta\beta}$ and inverting the Laplace transform term by term (cf. [26], Example II, (67.4), and [21], Section 4]), we get the next lemma.

LEMMA 2.4. *For all positive t and β,*

$$\frac{d\mathbb{P}(B(t) \le x; L^*_t(B) < \beta)}{dx}\bigg|_{x=0} = \sum_{j=-\infty}^{\infty} a_j(t) \exp\left(\frac{-\beta^2 j^2}{2t}\right),$$

where $a_j(t) = (2\pi t)^{-1/2} - \beta^2 (2\pi)^{-1/2} t^{-3/2} j^2$.

PROOF OF THEOREM 2.1. By Khoshnevisan ([17], Proposition A.2) and Lemma 2.4,

$$\begin{aligned}
\mathbb{P}(L^*_1(\omega) < \beta) &= \mathbb{P}(L^*_1(B) < \beta \mid B(1) = 0) \\
&= (2\pi)^{1/2} \sum_{j=-\infty}^{\infty} a_j(1) \exp(-\beta^2 j^2/2).
\end{aligned} \tag{2.2}$$

Statement (b) is immediate from this; (a) follows by applying the Poisson summation formula (see [11], XIX, (5.3), or [2], Section 9). We omit the proof of Theorem 2.1(c); it may be found in [6], page 35, or [7], page 67. □

3. Empirical processes. We now begin the proofs of Theorems 1.1 and 1.2. The proofs rest on some preliminary lemmas which are presented below. Suppose $f: [0,1] \mapsto \mathbb{R}$ possesses local times. Then we shall define for all $\varepsilon > 0$,

$$\mu(f;\varepsilon) = \sup_{\substack{x,y\in\mathbb{R}\\ |x-y|\le\varepsilon}} |L_1^x(f) - L_1^y(f)|. \tag{3.1}$$

A key ingredient to the proofs of Theorems 1.1 and 1.2 is our next lemma. The ideas behind this lemma appeared earlier in [1].

LEMMA 3.1. *Suppose f_1 and f_2 are Borel measurable functions on $[0,1]$ which possess local times. Suppose further that for some $\delta,\eta,\varepsilon > 0$, $\|f_1 - f_2\|_\infty \le \delta$ and $\mu(f_1;\varepsilon) + \mu(f_2;\varepsilon) \le \eta$. Then $\sup_{x\in\mathbb{R}} |L_1^x(f_1) - L_1^x(f_2)| \le \varepsilon^{-2}\delta + \eta$.*

Lemma 3.1 shows that if two processes are uniformly close and if their respective local times are sufficiently smooth, then their local times are also uniformly close. We shall have need for this basic estimate on several occasions in the course of the proofs. Indeed, the major obstacle in proving Theorems 1.1 and 1.2 is a lack of maximal inequalities in the variable n [see the proof of (3.8) below]. Lemma 3.1 is used to overcome this difficulty.

PROOF. Define $\Psi(x) = [1 - x\,\mathrm{sgn}(x)]1_{[-1,1]}(x)$. Note that $\Psi \ge 0$, $|\Psi(x) - \Psi(y)| \le |x-y|$ for all $x,y \in \mathbb{R}$ and $\int_{-\infty}^{\infty} \Psi(x)\,dx = 1$. Define an approximation to the identity $(\Psi_\varepsilon)_{\varepsilon>0}$ by letting $\Psi_\varepsilon(x) = \varepsilon^{-1}\Psi(x/\varepsilon)$ for all $x \in \mathbb{R}$. It follows that for all $x,y \in \mathbb{R}$ and all $\varepsilon > 0$,

$$\Psi_\varepsilon(x) \ge 0, \qquad \int_{-\infty}^{\infty} \Psi_\varepsilon(x)\,dx = 1 \quad \text{and} \quad \Psi_\varepsilon(x) = 0 \quad \text{for all } |x| \ge \varepsilon, \tag{3.2a}$$

$$|\Psi_\varepsilon(x) - \Psi_\varepsilon(y)| \le \varepsilon^{-2}|x-y|. \tag{3.2b}$$

For $j = 1,2,3$ and $\varepsilon > 0$ define $R_j(\varepsilon) = \sup_x |R_j(x;\varepsilon)|$, where for $j = 1,2$,

$$R_j(x;\varepsilon) = L_1^x(f_j) - \int L_1^a(f_j)\Psi_\varepsilon(a-x)\,da$$

and

$$R_3(x;\varepsilon) = \int (L_1^a(f_1) - L_1^a(f_2))\Psi_\varepsilon(a-x)\,da.$$

By (3.2a) for $j = 1,2$,

$$R_j(\varepsilon) = \sup_x \left| \int_{x-\varepsilon}^{x+\varepsilon} (L_1^x(f_j) - L_1^a(f_j))\Psi_\varepsilon(a-x)\,da \right| \le \mu(f_j;\varepsilon). \tag{3.3}$$

By the definition of local times together with (3.2b),

$$\begin{aligned} R_3(\varepsilon) &= \sup_x \left| \int_0^1 (\Psi_\varepsilon(f_1(s)-x) - \Psi_\varepsilon(f_2(s)-x))\,ds \right| \\ &\le \sup_x \sup_{0\le s\le 1} |\Psi_\varepsilon(f_1(s)-x) - \Psi_\varepsilon(f_2(s)-x)| \\ &\le \varepsilon^{-2}\|f_1 - f_2\|_\infty. \end{aligned} \tag{3.4}$$

Since by the triangle inequality, $\sup_x |L_1^x(f_1) - L_1^x(f_2)| \le \sum_{j=1}^3 R_j(\varepsilon)$, the lemma follows from (3.4) and (3.3). □

LEMMA 3.2. *Let ω_n be any sequence of Brownian bridges on* $[0,1]$. *Suppose $\varepsilon_k > 0$ is a nonrandom sequence which satisfies* $\liminf_{k\to\infty} \ln(1/\varepsilon_k)/\ln k > 0$. *Then for all $\gamma \in (0,1/2)$ and any subsequence (n_k), with probability* 1, $\mu(\omega_{n_k};\varepsilon_k) = O(\varepsilon_k^\gamma)$.

PROOF. Arguing as in the case of ordinary Brownian motion (see [22]), there exists a constant $c_0 > 0$ so that for all $\varepsilon, \lambda > 0$ and all $n \ge 1$,

$$\sup_{\substack{x,y\in\mathbb{R}\\|x-y|\le\varepsilon}} \mathbb{P}(|L_1^x(\omega_n) - L_1^y(\omega_n)| \ge \varepsilon^{1/2}\lambda) \le c_0 \exp(-\lambda^2/c_0).$$

Then by a metric entropy argument (this follows, e.g., by a minor modification of Lacey [20], Theorem 2.2), there exists c_1 so that for all $\varepsilon, \lambda > 0$ and all $n \ge 1$,

$$\mathbb{P}\Big(\mu(\omega_n;\varepsilon) \ge 2\sqrt{\varepsilon \ln(1/\varepsilon)} + \varepsilon^{1/2}\lambda\Big) \le c_1 \exp(-\lambda^2/c_1). \tag{3.5}$$

Substitute $n = n_k$, $\varepsilon = \varepsilon_k$ and $\lambda = (2c_1 \ln k)^{1/2}$ in (3.5) to obtain the following:

$$\mathbb{P}\Big(\mu(\omega_{n_k};\varepsilon_k) \ge 2\sqrt{\varepsilon_k \ln(1/\varepsilon_k)} + \sqrt{2c_1\varepsilon_k \ln k}\Big) \le c_1 k^{-2},$$

which sums. By the Borel–Cantelli lemma, with probability 1, as $k \to \infty$,

$$\mu(\omega_{n_k};\varepsilon_k) = O\Big(\sqrt{\varepsilon_k(\ln k \vee \ln(1/\varepsilon_k))}\Big).$$

The lemma follows from the assumption on the decay rate of ε_n. □

LEMMA 3.3. *Suppose $\zeta_n \to \infty$ is a sequence of positive numbers satisfying $\ln \zeta_n = O(\ln n)$ as $n \to \infty$. Then as $n \to \infty$,*

$$\ln \mathbb{P}(L_1^*(\nu_n) \ge \zeta_n) \sim \ln \mathbb{P}(L_1^0(\nu_n) \ge \zeta_n) \sim -\zeta_n^2/2.$$

PROOF. Fix $\gamma \in (0,1/2)$. By Khoshnevisan ([17], Remark (1), page 338), on a suitable probability space there are versions ν_n' of ν_n and a sequence of Brownian bridges ω_n such that for all $c > 0$,

$$\limsup_{n\to\infty} n^c \mathbb{P}(\sup_x |L_1^x(\omega_n) - L_1^x(\nu_n')| \ge n^{-\gamma/2}) < \infty. \tag{3.6}$$

Therefore, for all $c > 0$, the following must be eventually true:

$$\begin{aligned}\mathbb{P}(L_1^*(\nu_n) \ge \zeta_n) &\le \mathbb{P}(L_1^*(\omega_n) \ge \zeta_n - n^{-\gamma/2}) + n^{-c}\\ &= \mathbb{P}(L_1^*(\omega) \ge \zeta_n - n^{-\gamma/2}) + n^{-c}\\ &\le 2\zeta_n^2(\exp(-\zeta_n^2/2)(1+o(1))) + n^{-c},\end{aligned}$$

by Corollary 2.2(b). A similar lower bound holds, using Theorem 2.1(c), with $L_1^*(\nu_n)$, $L_1^*(\omega_n)$ and n^{-c} replaced by $L_1^0(\nu_n)$, $L_1^0(\omega_n)$ and $-n^{-c}$, respectively. The growth condition on ζ_n finishes the result. □

LEMMA 3.4. *Suppose* $\varepsilon_k > 0$ *satisfies* $\liminf_{k\to\infty} \ln(1/\varepsilon_k)/\ln k > 0$. *Then for all* $\gamma \in (0, 1/2)$ *and all subsequences* (n_k), *with probability* 1, *as* $k \to \infty$,

$$\mu(\nu_{n_k}; \varepsilon_k) = O(\varepsilon_k^\gamma + n_k^{-\gamma/2}).$$

PROOF. Fix an arbitrary $\gamma \in (0, 1/2)$. By (3.6) and the Borel–Cantelli lemma, on a suitable probability space one can construct versions ν_n' of ν_n and a sequence of Brownian bridges ω_n such that

$$\sup_{x\in\mathbb{R}} |L_1^x(\omega_n) - L_1^x(\nu_n')| = o(n^{-\gamma/2}) \quad \text{a.s.}$$

An application of Lemma 3.2 completes the proof. □

We can now begin the proof of Theorem 1.1. As in the classical law of the iterated logarithm, the proof is divided into two halves: an upper bound and a lower bound. It suffices to prove that for all $\varepsilon > 0$, the following hold with probability 1:

$$L_1^*(\nu_n) \le (1+\varepsilon)\sqrt{2\ln\ln n} \text{ eventually,} \tag{3.7a}$$

$$L_1^0(\nu_n) \ge (1-\varepsilon)\sqrt{2\ln\ln n} \text{ i.o.} \tag{3.7b}$$

PROOF OF (3.7a). Fix $\varepsilon > 0$, $\gamma \in (0, 1/2)$. Define $\zeta_n = (1+\varepsilon)\sqrt{2\ln\ln n}$. Clearly, $\ln \zeta_n = O(\ln n)$ and hence Lemma 3.3 applies. Thus with probability 1 there exists N_ε such that for all $n \ge N_\varepsilon$,

$$\mathbb{P}\Big(L_1^*(\nu_n) \ge (1+\varepsilon)\sqrt{2\ln\ln n}\Big) \le (\ln n)^{-(1+\varepsilon)}.$$

Fix $\rho \in ((1+\varepsilon)^{-1}, 1)$ and define $n_k = n(k) = [\exp(k^\rho)]$. Then the above probability estimate implies the following:

$$\sum_{k\ge 10} \mathbb{P}\Big(L_1^*(\nu_{n(k)}) \ge (1+\varepsilon)\sqrt{2\ln\ln n_k}\Big) \le \sum_{k\ge 10} k^{-\rho(1+\varepsilon)} < \infty.$$

By the easy half of the Borel–Cantelli lemma, with probability 1,

$$\limsup_{k\to\infty} \frac{L_1^*(\nu_{n(k)})}{\sqrt{2\ln\ln n_k}} \le 1+\varepsilon.$$

Although the sequence, $\{n_k; k \ge 1\}$ depends on ε, since $\varepsilon > 0$ is arbitrary, it suffices to prove that with probability 1,

$$\max_{n_k \le m \le n_{k+1}} \sup_{x\in\mathbb{R}} |L_1^x(\nu_m) - L_1^x(\nu_{n(k)})| = o\Big(\sqrt{\ln\ln n_k}\Big). \tag{3.8}$$

This amounts to "filling in" the gaps between n_k and n_{k+1}. To do this, we start with the following elementary observation: for all $n_k \le m \le n_{k+1}$ and all $t \in [0, 1]$,

$$\nu_m(t) = \sqrt{n_k/m}\ \nu_{n(k)}(t) + \sqrt{1 - n_k/m}\ \nu_m^{n(k)}(t), \tag{3.9}$$

where for all integers $a < b$, ν_a^b is the empirical process based on $X_{a+1}, \dots, X_b$, that is,

$$\nu_a^b(t) = \frac{1}{\sqrt{b-a}} \sum_{j=a+1}^{b} (1_{[0,t]}(X_j) - t). \tag{3.10}$$

Furthermore, ν_a and ν_a^b are independent processes.

It is easy to check that for all k large, $n_{k+1} - n_k \le \rho n_k k^{\rho-1}$. Therefore for k large,

$$\begin{aligned} \max_{n_k \le m \le n_{k+1}} \|\nu_m - \nu_{n(k)}\|_\infty &\le \left(1 - \sqrt{n_k/n_{k+1}}\right)\|\nu_{n(k)}\|_\infty \\ &\quad + \sqrt{1 - n_k/n_{k+1}} \max_{n_k \le m \le n_{k+1}} \|\nu_m^{n(k)}\|_\infty \\ &\le 2\rho k^{(\rho-1)/2}\big[\|\nu_{n(k)}\|_\infty + \max_{n_k \le m \le n_{k+1}} \|\nu_m^{n(k)}\|_\infty\big]. \end{aligned}$$

It follows from the functional law of the iterated logarithm for ν_n (see [24], page 504) that with probability 1, as $k \to \infty$,

$$\|\nu_{n(k)}\|_\infty \vee \max_{n_k \le m \le n_{k+1}} \|\nu_m^{n(k)}\|_\infty = O\left(\sqrt{\ln\ln n_k}\right) = O\left(\sqrt{\ln k}\right).$$

Since $\rho < 1$, by Kolmogorov's 0–1 law with probability 1 there exists $c_3, K_\rho \in (0, \infty)$ such that for all $k \ge K_\rho$,

$$\max_{n_k \le m \le n_{k+1}} \|\nu_m - \nu_{n(k)}\|_\infty \le c_3 k^{(\rho-1)/2}\sqrt{\ln k}. \tag{3.11}$$

Define $\delta_k = k^{(\rho-1)/2}(\ln k)^{1/2}$, $\varepsilon_k = (2\delta_k/\gamma)^{1/(2+\gamma)}$ and $\eta_k = \varepsilon_k^\gamma + n_k^{-\gamma/2}$. Since as $k \to \infty$, $\ln(1/\varepsilon_k)/\ln k \sim (1-\rho)/2 > 0$, Lemma 3.4 can be used to argue that with probability 1, as $k \to \infty$,

$$\max_{n_k \le j \le n_{k+1}} \mu(\nu_j; \varepsilon_k) = O(\eta_k). \tag{3.12}$$

Therefore, we can apply (3.11), (3.12) and Lemma 3.3 with $\varepsilon = \varepsilon_k$, $\eta = \eta_k$, $\delta = \delta_k$, $f_1 = \nu_{n(k)}$ and $f_2 = \nu_m$, (uniformly over $n_k \le m \le n_{k+1}$) to see that

$$\max_{n_k \le m \le n_{k+1}} \sup_{x \in \mathbb{R}} |L_1^x(\nu_{n(k)}) - L_1^x(\nu_m)| = O(\varepsilon_k^{-2}\delta_k + \eta_k) = o(k^{-\gamma/2(2+\gamma)}) = o(1).$$

This easily implies (3.8), thus completing the proof of (3.7a). □

PROOF OF (3.7b). For all $k \geq 1$ define $n(k) = n_k = k^k$ and $d(k) = d_k = n_{k+1} - n_k$. Notice that as $k \to \infty$, $d_k \sim n_{k+1}$. It is sufficient to prove that with probability 1,

$$\limsup_{k\to\infty} \frac{L_1^0(\nu_{n(k+1)})}{\sqrt{2\ln\ln n_{k+1}}} \geq 1. \tag{3.13}$$

Recall the definition of ν_b^a from (3.10). Since $\nu_{n(k+1)}^{n(k)}$ has the same distribution as $\nu_{d(k)}$, by Lemma 3.3 for all $\varepsilon > 0$, there exists N_ε such that for all $k \geq N_\varepsilon$,

$$\mathbb{P}\Big(L_1^0(\nu_{n(k+1)}^{n(k)}) \geq (1-\varepsilon)\sqrt{2\ln\ln d_k}\Big) \geq (\ln d_k)^{-(1-\varepsilon)} \sim (k\ln k)^{-(1-\varepsilon)}.$$

In other words,

$$\sum_{k\geq 10} \mathbb{P}\Big(L_1^0(\nu_{n(k+1)}^{n(k)}) \geq (1-\varepsilon)\sqrt{2\ln\ln d_k}\Big) = \infty.$$

Since $\{\nu_{n(k+1)}^{n(k)}; k \geq 1\}$ are independent processes and $\varepsilon > 0$ is arbitrary, the independence half of the Borel–Cantelli lemma shows that almost surely

$$\limsup_{k\to\infty} \frac{L_1^0(\nu_{n(k+1)}^{n(k)})}{\sqrt{2\ln\ln n_{k+1}}} = \limsup_{k\to\infty} \frac{L_1^0(\nu_{n(k+1)}^{n(k)})}{\sqrt{2\ln\ln d_k}} \geq 1. \tag{3.14}$$

By (3.9), for all $t \in [0,1]$ and all $k \geq 1$,

$$\nu_{n(k+1)}(t) = \sqrt{n_k/n_{k+1}}\; \nu_{n(k)}(t) + \sqrt{1-(n_k/n_{k+1})}\; \nu_{n(k+1)}^{n(k)}(t).$$

Hence,

$$\begin{aligned} &\|\nu_{n(k+1)} - \nu_{n(k+1)}^{n(k)}\|_\infty \\ &\quad \leq \sqrt{n_k/n_{k+1}}\; \|\nu_{n(k)}\|_\infty + \Big(1 - \sqrt{1-(n_k/n_{k+1})}\Big)\; \|\nu_{n(k+1)}^{n(k)}\|_\infty \\ &\quad \leq (ek)^{-1/2}\|\nu_{n(k)}\|_\infty(1+o(1)) + (4ek)^{-1/2}\|\nu_{n(k+1)}^{n(k)}\|_\infty(1+o(1)). \end{aligned}$$

However, by the law of the iterated logarithm for empirical processes,

$$\|\nu_{n(k)}\|_\infty \vee \|\nu_{n(k+1)}^{n(k)}\|_\infty = O\Big(\sqrt{\ln\ln n_k}\Big) = O\Big(\sqrt{\ln k}\Big).$$

Therefore,

$$\|\nu_{n(k+1)} - \nu_{n(k+1)}^{n(k)}\|_\infty = O((\ln k/k)^{1/2}). \tag{3.15}$$

Let us fix some $\gamma \in (0, 1/2)$. Define, $\delta_k = (\ln k/k)^{1/2}$, $\varepsilon_k = (2\delta_k/\gamma)^{1/(2+\gamma)}$ and $\eta_k = \varepsilon_k^\gamma + n_{k+1}^{-\gamma/2}$. Then the same argument as that leading to (3.12) shows that with probability 1,

$$\mu(\nu_{n(k+1)}; \varepsilon_k) + \mu(\nu^{n(k)}_{n(k+1)}; \varepsilon_k) = O(\eta_k).$$

This together with (3.15) and Lemma 3.1 with $\varepsilon = \varepsilon_k$, $\eta = \eta_k$, $\delta = \delta_k$, $f_1 = \nu_{n(k+1)}$ and $f_2 = \nu^{n(k)}_{n(k+1)}$ gives the following almost sure estimate:

$$\sup_{x\in\mathbb{R}} \left| L_1^x(\nu_{n(k+1)}) - L_1^x(\nu^{n(k)}_{n(k+1)}) \right| = O(\varepsilon_k^{-2}\delta_k + \eta_k) = o(1).$$

The above coupled with (3.14) together imply (3.13) and hence 3.7(b).

Starting from Theorem 2.1(c), the proof of the lim sup behavior of $L_1^0(\nu_n)$ is similar. □

This completes the proof of Theorem 1.1. The proof of Theorem 1.2 is virtually identical.

4. Kiefer processes. A Kiefer process $\{K(s,t); s \geq 0, t \in [0,1]\}$ is a centered Gaussian process with covariance given by

$$\mathbb{E}\ K(s,u)K(t,v) = (s \wedge t)(u \wedge v - uv).$$

See [24] for further information. Let $K_n(\cdot) = K(n,\cdot)$. We recall the following result of Komlós, Major and Tusnády [19]: on an appropriate probability space: there are versions ν'_n of ν_n, such that, almost surely as $n \to \infty$,

$$\sup_{0\leq t\leq 1} |K_n(t) - \nu'_n(t)| = O(n^{-1/2}(\ln n)^2).$$

The arguments of the previous section can be used to show that on the aforementioned probability space, for any $\varepsilon > 0$, almost surely,

$$\sup_{0\leq t\leq 1} \sup_{x\in\mathbb{R}} |L_t^x(K_n) - L_t^x(\nu'_n)| = o(n^{-1/10+\varepsilon}). \tag{4.1}$$

In particular, applying Theorems 1.1 and 1.2 to ν'_n, we obtain the following results:

$$\limsup_{n\to\infty} \frac{L_1^0(K_n)}{\sqrt{2\ln\ln n}} = \limsup_{n\to\infty} \frac{L_1^*(K_n)}{\sqrt{2\ln\ln n}} = 1 \quad \text{a.s.}, \tag{4.2}$$

$$\liminf_{n\to\infty} \sqrt{\ln\ln n}\ L_1^*(K_n) = \sqrt{2}\ \pi, \quad \text{a.s.} \tag{4.3}$$

Furthermore, the argument leading to (3.8) can be modified to show that the variable n in (4.2) and (4.3) need not be integer–valued.

Acknowledgments. We would like to acknowledge helpful discussions and comments concerning Section 2 by N. Bingham, C. Goldie and an anonymous referee.

REFERENCES

[1] BASS, R. F. and KHOSHNEVISAN, D. (1993). Strong approximations to Brownian local time. In *Seminar on Stochastic Processes 1992* 43–65. Birkhäuser, Boston.

[2] BELLMAN, R. (1961). *A Brief Introduction to Theta Functions*. Holt, Rinehart and Winston, New York.

[3] BIANE, PH. (1986). Relations entre pont et excursion du mouvement Brownien réel. *Ann. Inst. H. Poincaré* **22** 1–7.

[4] BIANE, PH. and YOR, M. (1988). Sur la loi des temps locaux browniens pris en un temps exponentiel. *Séminaire de Probabilité XXII. Lecture Notes in Math.* **1321** 454–466. Springer, Berlin.

[5] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

[6] BORODIN, A. N. (1982). Distribution of integral functionals of Brownian motion. *Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov. (LOMI)* **119** 19–38 (in Russian).

[7] BORODIN, A. N. (1989). Brownian local time. *Usp. Mat. Nauk*, **44** 7–48.

[8] CHUNG, K. L. (1976). Excursions in Brownian motion. *Ark. Mat.* **14** 155–177.

[9] CSÁKI, E. and FÖLDES, A. (1986). How small are the increments of the local time of a Wiener process? *Ann. Probab.* **14** 533–546.

[10] CSÁKI, E., FÖLDES, A. and SALMINEN, P. (1987). On the joint distribution of the maximum and its location for a linear diffusion. *Ann. Inst. H. Poincaré Probab. Statist.* **23** 179–194.

[11] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**, 2nd ed. Wiley, New York.

[12] GIKHMAN, I. I. (1957). On a nonparametric criterion of homogeneity for k samples. *Theory Probab. Appl.* **2** 369–373.

[13] GNEDENKO, B. V. (1954). Kriterien für die Unveränderlichkeit der Wahrscheinlichkeitsverteilung von zwei unabhängigen Stichprobenreihen. *Mat. Nach.* **12** 29–66.

[14] JEULIN, TH. (1985). Temps local et théorie du grossissement: application de la theorie á l'étude des temps locaux brownien. *Grossissements de filtrations: exemples et applications. Séminaire de Calcul Stochastique, Paris 1982/83. Lecture Notes in Math.* **1118** 197–304. Springer, Berlin.

[15] KENNEDY, D. P. (1976). The distribution of the maximum Brownian excursion. *J. Appl. Probab.* **13** 371–376.

[16] KESTEN, H. (1965). An iterated logarithm law for local time. *Duke Math. J.* **32** 447–456.

[17] KHOSHNEVISAN, D. (1992). Level crossings of the empirical process. *Stochastic. Process Appl.* **43** 331–343.

[18] KIEFER, J. (1959). K-sample analogues of the Kolmogorov–Smirnov and Cramér–von Mises test. *Ann. Math. Statist.* **30** 420–447.

[19] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1975). An approximation of partial sums of independent random variables and the sample distribution function, I. *Z. Wahrsch. Verw. Gebiete* **32** 111–131.

[20] LACEY, M. (1990). Large deviations for the maximum local time of stable Lévy processes. *Ann. Probab.* **18** 1669–1675.

[21] LOUCHARD, G. (1984). Kac's formula, Lévy's local time and Brownian excursion. *J. Appl. Probab.* **21** 479–499.

[22] REVUZ, D. and YOR, M. (1991). *Continuous Martingales and Brownian Motion*. Springer, Heidelberg.

[23] ROGERS, L. C. G. and WILLIAMS, D. (1987). *Diffusions, Markov Processes, and Martingales* **2**: *Itô Calculus*. Wiley, New York.

[24] SHORACK, G. R. and WELLNER, J. A. (1986). *Empirical Processes with Applications in Statistics.* Wiley, New York.

[25] VERVAAT, W. (1979). A relation between Brownian bridge and Brownian excursion. *Ann. Probab.* **7** 464–474.

[26] WILLIAMS, D. (1979). *Diffusions, Markov Processes, and Martingales Foundations.* Wiley, New York.

[27] ZHAN, S. and YOR, M. (1993). Sur la loi du supremum des temps locaux d'un pont de Bessel. Preprint.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195