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A centred norm inequality for singular integral operators

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Abstract

Let K be a standard singular integral kernel on $\mathbb{R}$ satisfying the usual Hölder continuity condition of order δ, and define $w(x) = c(1+|x|)^{-(1+\alpha)}$ (where c is chosen so that the integral of w is 1), $Tf = K*f, \bar{g}$ the mean of g with respect to the measure $w(x)\,dx$, and $\|\cdot\|_p$ the L^p norm with respect to $w(x)\,dx$. Although the inequality $\|Tf\|_p \leqslant c_p\|f\|_p$ is not true in general, the centred norm inequality $\|Tf-\overline{Tf}\|_p \leqslant c_p\|f\|_p$ does hold for $1 < p < \infty$ if $\alpha < \delta$.

1. *Introduction*

Let $K\colon \mathbb{R}\to\mathbb{R}$ be a function satisfying the following conditions for singular integral kernels:

there exist $\delta\in(0,1]$ and $B>0$ such that

$$\left.\begin{array}{l} (a)\ |K(x)| \leqslant B|x|^{-1} \text{ if } x\in\mathbb{R}\backslash\{0\}, \\ (b)\ |K(x)-K(y)| \leqslant B|x-y|^{\delta}/|x|^{1+\delta} \text{ if } |x-y| < \tfrac{1}{2}|x|,\ x\in\mathbb{R}\backslash\{0\}, \\ (c)\ K \text{ is odd.} \end{array}\right\} \tag{1·1}$$

Define the singular integral operator T by $Tf(x) = \int K(x-y)f(y)\,dy$ for $f\in C_K^\infty$, the integral being taken in the principle value sense. It is very well known that T can be extended to a bounded operator on $L^p(dx)$, for $1<p<\infty$.

Let $\alpha\in(0,\delta)$ and let

$$w(x) = \zeta(1+|x|)^{-(1+\alpha)}, \tag{1·2}$$

where $\zeta = (\int_{\mathbb{R}}(1+|x|)^{-(1+\alpha)}\,dx)^{-1}$ is chosen so that $\int_{\mathbb{R}} w(x)\,dx = 1$. The function w is not in the class A_p for any $p\in[1,\infty]$ (see Remark 5·2), and so one does not expect an inequality of the form $\|Tf\|_p \leqslant c_p\|f\|_p$ to hold, where $\|\cdot\|_p$ denotes the L^p norm with respect to the measure $w(x)\,dx$; in fact, if T is the Hilbert transform, such an inequality is known to fail.

The main result of this paper is that such an inequality does hold, however, provided that one first centres Tf. More precisely, for $g\in L^1(w(x)\,dx)$, define

$$\bar{g} = \int g(x)\,w(x)\,dx, \tag{1·3}$$

the mean of g. Then we prove

THEOREM 1·1. *Suppose that* $1<p<\infty$. *There exists a constant* c_p, *depending only on* p, *such that*

$$\|Tf-\overline{Tf}\|_p \leqslant c_p\|f\|_p \quad (f\in C_K^\infty). \tag{1·4}$$

Here $\|f\|_p$ denotes $(\int |f(x)|^p w(x)\,dx)^{1/p}$. As usual, once this inequality is established for $f \in C_K^\infty$, there is no difficulty in extending it to $f \in L^p(w(x)\,dx)$.

The case $p = 2$ of Theorem 1·1 was first proved in [1], theorem 3·1 by means of Cotlar's lemma and some rather involved calculations (see Remark 5·3 below). In this paper we establish (1·4) for the full range $p \in (1,\infty)$. Our technique is to define a suitable substitute for BMO, show that both the operator $f \to Tf - \overline{Tf}$ and its adjoint are bounded operators from L^∞ to BMO, and then to use interpolation and duality.

The main part of the argument is given in Section 2. Since the measure $w(x)\,dx$ is a finite measure and does not satisfy the doubling condition (see Remark 5·1), some modifications are needed to the Fefferman–Stein results about interpolating between L^2 and BMO; these are given in Section 3. The L^∞-BMO boundedness argument of Section 2 depends heavily on some technical estimates, and these are proved in Section 4. Some assorted remarks make up Section 5.

The latter c, with or without subscripts, will denote constants whose value is unimportant and which may change from line to line. We will use $B(x,r)$ to denote the set $\{y: |y-x| < r\}$. To simplify the notation, we will write T both for the operator T and for the kernel K.

2. L^∞-BMO *boundedness*

In Section 3 we will need to use a type of Calderón–Zygmund decomposition, so let us begin by describing our 'cubes'. Define $\lambda(t)$ to be that real number such that

$$\int_{-\infty}^{\lambda(t)} w(x)\,dx = t \quad (t \in (0,1)). \tag{2·1}$$

Let $\lambda(0) = -\infty, \lambda(1) = +\infty$. Let $\Lambda_0 = \lambda(\tfrac{5}{8})$. Let $\mathcal{Q}_0 = \mathbb{R}$,

$$\mathcal{Q}_n = \{[\lambda(k/2^n), \lambda((k+1)/2^n)] : 0 \leqslant k \leqslant 2^n - 1\} \quad (n = 1, 2, \ldots).$$

If $Q = [\lambda(k/2^n), \lambda((k+1)/2^n)] \in \mathcal{Q}_n$, let $\tilde{Q} = [\lambda(((k-1)/2^n) \vee 0), \lambda(((k+2)/2^n) \wedge 1)]$ and $x_Q = \lambda((k/2^n) \vee (1/2^n))$.

Since $\lambda(t) \sim -ct^{-1/\alpha}, \lambda(1-t) \sim ct^{-1/\alpha}$, for t near 0, a simple calculation shows that there exists a constant $\eta > 0$ independent of n such that if Q is a bounded interval in $\mathcal{Q}_n$, then

$$\inf_{x \in Q,\, y \notin \tilde{Q}} |y-x| \geqslant \eta \operatorname{diam}(Q), \tag{2·2}$$

and if Q is an unbounded interval in $\mathcal{Q}_n$,

$$\inf_{x \in Q,\, y \notin \tilde{Q}} |y-x| \geqslant \eta |x_Q|. \tag{2·3}$$

If Q is unbounded, $y \notin \tilde{Q}$ and $x \in Q$, then either $|x| \leqslant 2|x_Q|$, in which case $|y-x| \geqslant \tfrac{1}{2}\eta|x|$; or else $|x| > 2|x_Q|$, in which case $|y-x| \geqslant \tfrac{1}{2}|x|$. So replacing η by $\tfrac{1}{2}(\eta \wedge 1)$, we have

$$|y-x| \geqslant \eta|x| \quad \text{if } x \in Q, y \in \tilde{Q}^c. \tag{2·4}$$

The $\mathcal{Q}_n$ form a decomposition of $\mathbb{R}$ into 'cubes', each with mass 2^{-n} with respect to $w(x)\,dx$ measure, $\mathcal{Q}_n$ is a partition of $\mathcal{Q}_{n-1}$, and

$$\int_{\tilde{Q}} w(y)\,dy \leqslant c \int_Q w(y)\,dy, \quad \text{with } c \text{ independent of } n \text{ and } Q. \tag{2·5}$$

Let $\mathcal{Q} = \bigcup_n \mathcal{Q}_n$, and let the measure $w(x)\,dx$ be also denoted by $w(dx)$, so that

$$w(Q) = \int_Q w(x)\,dx. \tag{2·6}$$

Let us next turn to the singular integral operators. Define the operator U by

$$Uf(x) = Tf(x) - \overline{Tf}. \tag{2·7}$$

Note that

$$Uf(x) = \int T(x-y)f(y)\,dy - \int\int T(x-y)f(y)\,w(x)\,dx\,dy$$
$$= \int [T(x-y) - T^*w(y)]f(y)\,dy,$$

where T^* is the adjoint of T with respect to Lebesgue measure. We observe that clearly T^* also satisfies the estimates (1·1).

If we define

$$U(x,y) = \frac{T^*(y-x) - T^*w(y)}{w(y)}, \tag{2·8}$$

then $U(x,y)$ is the kernel for U with respect to $w(x)\,dx$ in the sense that

$$Uf(x) = \int U(x,y)f(y)\,w(y)\,dy.$$

Consequently, the adjoint of U with respect to the measure $w(x)\,dx$ has kernel

$$U^*(x,y) = \frac{T^*(x-y) - T^*w(x)}{w(x)}. \tag{2·9}$$

One of the main results of [1] (see theorem 3·1 of that paper) is that

$$\|Uf\|_2 \leqslant c\|f\|_2, \quad \|U^*f\|_2 \leqslant c\|f\|_2. \tag{2·10}$$

For $f \in L^1(w(dx))$ and $Q \in \mathcal{Q}_n$ for some n, let

$$f_Q = \frac{1}{w(Q)} \int_Q f(x)\,w(x)\,dx. \tag{2·11}$$

Define BMO_w to be set of all f such that

$$\|f\|_* = \sup_{Q\in\mathcal{Q}} \frac{1}{w(Q)} \int_Q |f(x) - f_Q|\,w(x)\,dx < \infty.$$

As usual, BMO_w is a Banach space provided that we identify functions that differ by a constant.

Our goal in this section is to prove

THEOREM 2·1. *There exists a constant* $c > 0$ *such that* (*a*) $\|Uf\|_* \leqslant c\|f\|_\infty$ *and* (*b*) $\|U^*f\|_* \leqslant c\|f\|_\infty$.

Before proceeding to the proof of Theorem 2·1, we need

PROPOSITION 2·2. (*a*) $U1 \equiv 0$ *and* (*b*) $U^*1 \equiv 0$.

Proof. (*a*) follows by the definition of U and the fact that (1·1)(*c*) implies $T1 \equiv 0$. For (*b*), we write

$$U^*1(x) = \frac{1}{w(x)} \int (T^*(x-y) - T^*w(x))\,w(y)\,dy = \frac{1}{w(x)}(T^*w(x) - T^*w(x)) = 0,$$

using the fact that $w(dx)$ has unit mass. ∎

The statements of the key estimates we need are given in the following two propositions.

PROPOSITION 2·3. *There exists $c > 0$ such that if $Q \in \mathcal{Q}$ is bounded and $x \in Q$, then*

$$\int_{\tilde{Q}^c} |U(x,y) - U(x_Q, y)|\, w(y)\, dy \leqslant c.$$

PROPOSITION 2·4. *There exists $c > 0$ such that*
(a) if $Q \in \mathcal{Q}$ is bounded with $|x_Q| < \Lambda_0$ and $w(Q) \leqslant 10^{-4}$, then

$$\int_{\tilde{Q}^c} |U^*(x,y) - U^*(x_Q,y)|\, w(y) dy \leqslant c \quad \text{for } x \in Q;$$

(b) if $Q \in \mathcal{Q}$ is bounded and $|x_Q| \geqslant \Lambda_0$, then

$$\int_{\tilde{Q}^c} |U^*(x,y) - U^*(x_Q,y)|\, w(y)\, dy \leqslant c \quad \text{for } x \in Q;$$

(c) if $Q \in \mathcal{Q}$ is unbounded, then

$$\int_{\tilde{Q}^c} |U^*(x,y)|\, w(y)\, dy \leqslant c \quad \text{for } x \in Q.$$

The proofs of Propositions 2·3 and 2·4 are a bit lengthy and we defer them until Section 4.

Proof of Theorem 2·1 (*a*). Let $f \in C_K^\infty$ and let $Q \in \mathcal{Q}$. Define $f_1 = f 1_{\tilde{Q}}, f_2 = f - f_1$. By the Cauchy–Schwarz inequality, (2·10), and (2·5),

$$\begin{aligned}\frac{1}{w(Q)} \int_Q |Uf_1(y)|\, w(y)\, dy &\leqslant \frac{1}{w(Q)} \left(\int_Q |Uf_1(y)|^2 w(y)\, dy \right)^{\frac{1}{2}} (w(Q))^{\frac{1}{2}} \\ &\leqslant \frac{c}{w(Q)^{\frac{1}{2}}} \left(\int |f_1(y)|^2 w(y)\, dy \right)^{\frac{1}{2}} \\ &\leqslant c \|f\|_\infty (w(\tilde{Q})/w(Q))^{\frac{1}{2}} \leqslant c\|f\|_\infty. \end{aligned} \tag{2·12}$$

To look at Uf_2, we consider two cases.

Case 1. Suppose that Q is bounded.

By Proposition 2·3

$$\begin{aligned}|Uf_2(x) - Uf_2(x_Q)| &\leqslant \int |U(x,y) - U(x_Q,y)|\, |f_2(y)|\, w(y)\, dy \\ &\leqslant \|f\|_\infty \int_{\tilde{Q}^c} |U(x,y) - U(x_Q,y)|\, w(y)\, dy \\ &\leqslant c\|f\|_\infty\end{aligned}$$

if $x \in Q$. So

$$\frac{1}{w(Q)} \int_Q |Uf_2(x) - Uf_2(x_Q)|\, w(x)\, dx \leqslant c\|f\|_\infty. \tag{2·13}$$

Case 2. Suppose that Q is unbounded.

If $Q = \mathbb{R}, f_2 = 0$; so suppose that $Q \neq \mathbb{R}$. Then Q will be of the form $[N, \infty)$ or $(-\infty, -N]$ for some $N > \Lambda_0$. Choose k_0 to be the smallest integer k larger than 4 such that $2^k \geqslant 4k^{2/\alpha}$.

For $k \geqslant k_0$, define

$$g_k(x) = f_2(x)\, 1_{[2^k N,\, 2^{(k+1)}N)}(|x|)$$

and let

$$g_0(x) = f_2(x)\, 1_{[0,\, 2^{k_0}N)}(|x|).$$

So $f_2 = g_0 + \sum_{k=k_0}^{\infty} g_k$. Since f_2 has compact support, this is actually a finite sum.

We examine g_0 first. Let $a_0 = -\overline{Tg_0}$. So $Ug_0 - a_0 = Tg_0$. If $x \in Q, y \in \tilde{Q}^c$, then by (2·3), $|x-y| \geqslant \eta |x_Q| = \eta N$. Then by (1·1)(*a*),

$$|Tg_0(x)| = \left|\int T(x-y)\, g_0(y)\, dy\right| \leqslant \|f\|_\infty \int_{\tilde{Q}^c \cap B(0,\, 2^{k_0}N)} |T(x-y)|\, dy$$

$$\leqslant c\|f\|_\infty (\eta N)^{-1} \int_{B(0,\, 2^{k_0}N)} dy \leqslant c\|f\|_\infty.$$

So

$$\frac{1}{w(Q)} \int_Q |Ug_0(x) - a_0|\, w(x)\, dx \leqslant c\|f\|_\infty. \tag{2·14}$$

Now suppose that $k \geqslant k_0$. Let $a_k = Ug_k(x_Q)$. Of course,

$$Ug_k(x) - Ug_k(x_Q) = Tg_k(x) - Tg_k(x_Q).$$

If $N \leqslant |x| < k^{2/\alpha}N$ and $|y| \geqslant 2^k N > 4k^{2/\alpha}N$, then $|x| + |x_Q| \leqslant 2|x| \leqslant 2k^{2/\alpha}N \leqslant \frac{1}{2}|y|$

and

$$|Tg_k(x) - Tg_k(x_Q)| = \left|\int [T(x-y) - T(x_Q - y)]\, g_k(y)\, dy\right|$$

$$\leqslant \|f\|_\infty \int_{2^k N \leqslant |y| < 2^{(k+1)}N} |T(x-y) - T(x_Q - y)|\, dy$$

$$\leqslant c\|f\|_\infty (|x|^\delta + |x_Q|^\delta) \int_{2^k N \leqslant |y| \leqslant 2^{k+1}N} |y|^{-(1+\delta)}\, dy \leqslant c\|f\|_\infty\, k^{2\delta/\alpha}\, 2^{-k\delta},$$

using (1·1)(*b*).

Therefore

$$\frac{1}{w(Q)} \int_{Q \cap B(0,\, k^{2/\alpha}N)} |Ug_k(x) - a_k|\, w(x)\, dx \leqslant ck^{2\delta/\alpha}\, 2^{-k\delta} \|f\|_\infty. \tag{2·15}$$

On the other hand, suppose that $x \in Q, y \in \tilde{Q}^c$, $|x| \geqslant k^{2/\alpha}N$, and $|y| \geqslant 2^k N$. Of course, $|x_Q - y| \geqslant (2^k - 1)N \geqslant 2^k N/2$. If $|x| \leqslant 2^k N/2$, then $|x-y| \geqslant 2^k N/2$, and if $|x| \geqslant 2^k N/2$, then $|x-y| \geqslant \eta 2^k N/2$ by (2·6). Hence, using (1·1)(*a*),

$$|Tg_k(x) - Tg_k(x_Q)| \leqslant \|f\|_\infty \int_{\tilde{Q}^c \cap \{2^k N \leqslant |y| < 2^{(k+1)}N\}} [|T(x-y)| + |T(x_Q - y)|]\, dy$$

$$\leqslant c\|f\|_\infty (2^k N)^{-1} \int_{2^k N \leqslant |y| \leqslant 2^{k+1}N} dy \leqslant c\|f\|_\infty.$$

Therefore

$$\frac{1}{w(Q)}\int_{Q\cap\{|x|\geq k^{2/\alpha}N\}}|Ug_k(x)-a_k|\,w(x)\,dx\leq\frac{c\|f\|_\infty}{w(Q)}\int_{Q\cap\{|x|\geq k^{2/\alpha}N\}}w(dx)\leqslant c\|f\|_\infty k^{-2}. \tag{2·16}$$

Summing (2·15) and (2·16) gives

$$\frac{1}{w(Q)}\int_Q|Ug_k(x)-a_k|\,w(x)\,dx\leqslant c(k^{2\delta/\alpha}2^{-k\delta}+k^{-2})\,\|f\|_\infty\quad\text{for } k\geqslant k_0. \tag{2·17}$$

Letting $a=a_0+\sum_{k=k_0}^{\infty}a_k$ (recalling that the sum is actually finite), and summing (2·14) and (2·17) gives

$$\frac{1}{w(Q)}\int_Q|Uf_2(x)-a|\,w(x)\,dx\leqslant\sum_{k=0,\,k\geqslant k_0}\frac{1}{w(Q)}\int_Q|Ug_k(x)-a_k|\,w(x)\,dx\leqslant c\|f\|_\infty. \tag{2·18}$$

Finally, combining (2·12) with either (2·13) or (2·18), depending whether we are in Case 1 or Case 2, shows there exists $b_Q\in\mathbb{R}$ such that

$$\frac{1}{w(Q)}\int_Q|Uf(x)-b_Q|\,w(x)\,dx\leqslant c\|f\|_\infty.$$

Just as in the classical BMO case, this suffices to prove Theorem 2·1 (*a*). ∎

Proof of Theorem 2·1 (*b*). Let $f\in C_K^\infty$, let $Q\in\mathcal{Q}$, and let f_1, f_2 be defined as in the proof of Theorem 2·1 (*a*). Exactly as in (2·12), we get

$$\frac{1}{w(Q)}\int_Q|U^*f_1(y)|\,w(y)\,dy\leqslant c\|f\|_\infty. \tag{2·19}$$

Turning to U^*f_2, we look at four cases. (Again, the case when $Q=\mathbb{R}$ is trivial.)

Case 1. $|x_Q|<\Lambda_0$, $w(Q)>10^{-4}$.

As in the derivation of (2·12), we use the Cauchy–Schwarz inequality and (2·10) to get

$$\frac{1}{w(Q)}\int_Q|U^*f_2(x)|\,w(x)\,dx\leqslant w(Q)^{-\frac{1}{2}}\left(\int_Q|U^*f_2(x)|^2w(x)\,dx\right)^{\frac{1}{2}}$$

$$\leqslant cw(Q)^{-\frac{1}{2}}\left(\int|f_2(x)|^2w(y)\,dy\right)^{\frac{1}{2}}\leqslant c10^2\|f\|_\infty. \tag{2·20}$$

Case 2. $|x_Q|<\Lambda_0$, $w(Q)\leqslant 10^{-4}$.

We get

$$\frac{1}{w(Q)}\int_Q|U^*f_2(x)-U^*f_2(x_Q)|\,w(x)\,dx\leqslant c\|f\|_\infty \tag{2·21}$$

by arguing exactly as in the derivation of (2·13) in the proof of Theorem 2·1 (*a*), using Proposition 2·4 (*a*) instead of Proposition 2·3.

Case 3. $|x_Q|\geqslant\Lambda_0$, Q bounded.

This is done exactly the same as Case 2 (using Proposition 2·4 (*b*) instead of Proposition 2·4 (*a*)), and we get (2·21) in this case also.

Case 4. $|x_Q| \geqslant \Lambda_0, Q$ unbounded.

Using Proposition 2·4 (*c*),

$$\frac{1}{w(Q)}\int_Q |U^*f_2(x)|\, w(x)\, dx = \frac{1}{w(Q)}\int_Q \left|\int U^*(x,y) f_2(y)\, w(y)\, dy\right| w(x)\, dx$$

$$\leqslant c\|f\|_\infty \frac{1}{w(Q)}\int_Q \left(\int_{\tilde{Q}^c} |U^*(x,y)|\, w(y)\, dy\right) w(x)\, dx$$

$$\leqslant c\|f\|_\infty. \tag{2·22}$$

Combining (2·19) with either (2·20), (2·21), or (2·22), we get the existence of $b_Q \in \mathbb{R}$ such that

$$\frac{1}{w(Q)}\int_Q |U^*f(x) - b_Q|\, w(x)\, dx \leqslant c\|f\|_\infty,$$

which proves Theorem 2·1 (*b*). ▌

Given Theorem 2·1, we can now extend the definition of Uf and U^*f from C_K^∞ to L^∞.

3. *Interpolation*

In this section, we give the interpolation argument that gives the L^p-boundedness of U. We follow [2], section 4 closely. But because the measure $w(x)\,dx$ is a finite measure and because this measure does not satisfy the doubling condition (see Remark 5·1), some care must be taken with the details.

Define $M_w f$, our substitute for the maximal function, by

$$M_w f(x) = \sup_{x\in Q\in\mathcal{Q}} \frac{1}{w(Q)}\int_Q |f(y)|\, w(y)\, dy. \tag{3·1}$$

PROPOSITION 3·1. (*a*) $w(\{x : M_w f(x) > \alpha\}) \leqslant \|f\|_1/\alpha$.

(*b*) *There exists* c_p *such that* $\|M_w f\|_p \leqslant c_p\|f\|_p$ *for* $1 < p < \infty$.

Proof. Suppose that E is a measurable set contained in $\bigcup_\beta Q_\beta$, where $\{Q_\beta\}$ is a subset of $\mathcal{Q}$. Let $\mathcal{C}_0 = \{Q_\beta\}$. If $Q_1', \ldots, Q_k'$ have been chosen, let

$$\mathcal{C}_k = \{Q_\beta\text{: the interior of } Q_\beta \text{ is disjoint from the interiors of } Q_1', \ldots, Q_k'\}.$$

Starting with $k = 0$, we choose Q_{k+1}' so that $w(Q_{k+1}') = \sup_{Q_\beta\in\mathcal{C}_k} w(Q_\beta)$. Since each $w(Q_\beta)$ equals 2^{-n} for some n, the supremum is actually attained. Since each $\mathcal{Q}_n$ is finite, $w(Q_k') \to 0$ as $k \to \infty$.

Suppose that Q is any of the Q_β. We will show that $Q \subseteq \bigcup_k Q_k'$, hence $E \subseteq \bigcup_\beta Q_\beta \subseteq \bigcup_k Q_k'$. If Q is one of the Q_k', we are done. So suppose not. Since $w(Q_k') \to 0$ and Q is not one of the Q_k', then the interior of Q must intersect the interior of one of the Q_k'; let Q_{j_0}' be the first such. By our construction, $w(Q_{j_0}') \geqslant w(Q)$. But this together with the fact that the interiors of Q_{j_0}' and Q intersect implies that $Q \subseteq Q_{j_0}'$.

At this point, we follow [3], section I·1·5 closely, using the above covering idea in place of his I·1·6. ▌

As in [3], we get that if $f \in L^1(w(dx))$ and $Q_{n,x}$ is an element of $\mathcal{Q}_n$ containing x, then

$$\lim_{n\to\infty} \frac{1}{w(Q_{n,x})} \int_{Q_{n,x}} f(y)\, w(y)\, dy = f(x) \quad \text{a.e.} \tag{3·2}$$

Define
$$f^{\#}(x) = \sup_{x \in Q \in \mathcal{Q}} \frac{1}{w(Q)} \int_Q |f(y) - f_Q|\, w(y)\, dy. \tag{3·3}$$

Since
$$\int_Q |f(y) - f_Q|\, w(y)\, dy \leqslant \int_Q |f(y)|\, w(y)\, dy + w(Q)\,|f_Q|,$$

we have
$$f^{\#}(x) \leqslant 2M_w f(x). \tag{3·4}$$

PROPOSITION 3·2. *If* $f \in L^1(w(dx))$ *and* $\bar{f} = 0$, *then* $\|f\|_p \leqslant c_p \|f^{\#}\| p$, *for* $1 < p < \infty$.

Proof. We begin by imitating the Calderón–Zygmund procedure. Let $\alpha > 0$. If it happens that $\int_{\mathbb{R}} |f(y)|\, w(y)\, dy > \alpha$, we let $Q_1 = \mathbb{R}$ and stop. Otherwise we split $\mathbb{R}$ into the 2 'cubes' of $\mathcal{Q}_1$ and proceed. We let $\{Q_j\}$ consist of those cubes Q in $\bigcup_{n=1}^{\infty} \mathcal{Q}_n$ such that $|f|_Q > \alpha$ and Q is not contained in any larger $Q' \in \mathcal{Q}$ with $|f|_{Q'} > \alpha$.

We then have a countable collection of cubes $\{Q_j\}$ with $|f|_{Q_j} > \alpha$. If $Q_j \neq \mathbb{R}$, and $Q_j \in \mathcal{Q}_n$, then Q_j is a subset of a cube Q' in $\mathcal{Q}_{n-1}$ that is not in $\{Q_j\}$. So

$$|f_{Q_j}| \leqslant |f|_{Q_j} \leqslant \frac{w(Q')}{w(Q_j)} \frac{1}{w(Q')} \int_{Q'} |f(y)|\, w(y)\, dy \leqslant 2\alpha.$$

If $Q_j = \mathbb{R}$, then $|f_{Q_j}| = 0$. In either case, $|f_{Q_j}| \leqslant 2\alpha$.

As usual, for almost every point in $(\bigcup_j Q_j)^c$, we have $f \leqslant \alpha$.

We do this for each α and denote the cubes Q_j^α. Let $A = 4 \cdot 2^{2p}$ and let Q_0 be one of the $Q_j^{\alpha/4}$. We look at the $Q_j^\alpha \subseteq Q_0$.

If $Q_0 \subseteq \{x : f^{\#}(x) > \alpha/A\}$, then

$$\sum_{Q_j^\alpha \subseteq Q_0} w(Q_j^\alpha) \leqslant w(\{x : f^{\#}(x) > \alpha/A\} \cap Q_0). \tag{3·5}$$

If $Q_0 \nsubseteq \{x : f^{\#}(x) > \alpha/A\}$, then $|f_{Q_0}| \leqslant 2\alpha/4 = \alpha/2$, $|f|_{Q_j^\alpha} > \alpha$, and so

$$\int_{Q_j^\alpha} |f - f_{Q_0}|\, w(y)\, dy \geqslant \frac{\alpha}{2} w(Q_j^\alpha).$$

Summing over the Q_j^α contained in Q_0,

$$w(Q_0)\,\alpha/A \geqslant \int_{Q_0} |f - f_{Q_0}| \geqslant \frac{\alpha}{2} \sum w(Q_j^\alpha),$$

or
$$\sum w(Q_j^\alpha) \leqslant \frac{2}{A} w(Q_0). \tag{3·6}$$

Letting $\mu(\alpha) = \sum w(Q_j^\alpha)$ and summing over all such Q_0, we have

$$\mu(\alpha) \leqslant w(\{f^{\#} > \alpha/A\}) + \frac{2}{A} \mu(\alpha/4).$$

Let $I_n = p\int_0^N \alpha^{p-1}\mu(\alpha)\,d\alpha$. Since μ is bounded by $w(\mathbb{R}) = 1$, we have $I_N < \infty$. Then

$$I_N \leqslant p\int_0^\infty \alpha^{p-1} w(\{f^\# > \alpha/A\})\,d\alpha + \frac{2p}{A}\int_0^N \alpha^{p-1}\mu(\alpha/4)\,d\alpha \leqslant \|f^\#\|_p^p + \frac{2}{A}2^{2p} I_N.$$

Recalling our choice of A, we conclude that $I_N \leqslant c\|f^\#\|_p^p$.

Finally, note that $w(\{|f| > \alpha\}) \leqslant \mu(\alpha)$ since $f \leqslant \alpha$ a.e. on $(\bigcup_j Q_j^\alpha)^c$. So

$$\|f\|_p^p \leqslant \sup_{N\to\infty} I_N \leqslant c\|f^\#\|_p^p. \quad \blacksquare$$

Proof of Theorem 1·1. Let V denote either U or U^*. Define W by $Wf(x) = (Vf)^\#(x)$. By (2·10), V is bounded on $L^2(w(dx))$. So by (3·4) and Proposition 3·1,

$$\|Wf\|_2 \leqslant 2\|M_w(Vf)\|_2 \leqslant c\|Vf\|_2 \leqslant c\|f\|_2.$$

On the other hand, $(Vf)^\# \in L^\infty$ if and only if $Vf \in \mathrm{BMO}_w$. By Theorem 2·1,

$$\|Wf\|_\infty = \|Vf\|_* \leqslant c\|f\|_\infty.$$

V is a sublinear operator, and the $f \to f^\#$ operator clearly is. So by the Marcinkiewicz interpolation theorem,

$$\|(Vf)^\#\|_p = \|Wf\|_p \leqslant c_p\|f\|_p, \quad \text{for } 2 < p < \infty.$$

Of course, $\overline{Uf} = 0$. By Proposition 2·2 (a),

$$\overline{U^*f} = \int U^*f(x)\,w(x)\,dx = \int U1(x)f(x)\,w(x)\,dx = 0.$$

So by Proposition 3·2,

$$\|Vf\|_p \leqslant c\|(Vf)^\#\|_p \leqslant c_p\|f\|_p, \quad \text{for } 2 < p < \infty.$$

Taking $V = U$ gives the boundedness of U on $L^p(w(x)\,dx), 2 < p < \infty$. The case $p = 2$ is (2·10). Finally, taking $V = U^*$ gives the boundedness of U^* on L^p for $2 < p < \infty$. This gives the boundedness of U on L^p for $1 < p < 2$, by duality. $\blacksquare$

4. *Estimates*

In this section we prove Proposition 2·3 and 2·4. Let us start by observing some easy estimates. First, we have

$$\int_{|y|\leqslant b} |y|^\delta w(y)\,dy \leqslant cb^{\delta-\alpha} \tag{4·1}$$

and second,

$$\int_{|y|\geqslant b} w(y)\,dy \leqslant cb^{-\alpha}. \tag{4·2}$$

Third, using (1·1)(a),

$$\int_{r\leqslant|y|\leqslant s} |T(x)|\,dx \leqslant c(\ln s - \ln r) = c\ln\left(\frac{s}{r}\right), \tag{4·3}$$

and fourth,

$$\sup_{a\leqslant|y|\leqslant 4a} |w'(x)| \leqslant ca^{-(2+\alpha)}. \tag{4·4}$$

PROPOSITION 4·1. (*a*) *If* $|x| \geqslant \Lambda_0$ *and* $|y| \leqslant \frac{1}{2}|x|$, *then*

$$|T(x-y)-Tw(x)| \leqslant c|y|^\delta/|x|^{1+\delta}+c/|x|^{1+\alpha}.$$

(*b*) *If* $|x| \leqslant 2\Lambda_0$, *then* $|Tw(x)| \leqslant c$.

Proof. (*a*) We write

$$\begin{aligned} T(x-y)-Tw(x) &= \int [T(x-y)-T(x-z)]\, w(z)\, dz \\ &= \int_{|z|<\frac{1}{2}|x|} [T(x-y)-T(x-z)]\, w(z)\, dz + T(x-y)\int_{|z|\geqslant\frac{1}{2}|x|} w(z)\, dz \\ &\quad -\int_{|z|\geqslant\frac{1}{2}|x|} T(x-z)\, w(z)\, dz \\ &= I_1+I_2-I_3. \end{aligned}$$

If $|y|, |z| \leqslant \frac{1}{2}|x|$, then (1·1)(*b*) gives us

$$|T(x-y)-T(x-z)| \leqslant |T(x-y)-T(x)|+|T(x-z)-T(x)| \leqslant c|y|^\delta/|x|^{1+\delta}+c|z|^\delta/|x|^{1+\delta}.$$

Substituting, we get by (4·1) that

$$|I_1| \leqslant c|y|^\delta/|x|^{1+\delta}+\frac{c}{|x|^{1+\delta}}\int_{|z|\leqslant\frac{1}{2}|x|} |z|^\delta w(z)\, dz \leqslant c|y|^\delta/|x|^{1+\delta}+c/|x|^{1+\alpha}$$

as required.

We use (1·1)(*a*) to bound I_2:

$$|I_2| \leqslant c|x-y|^{-1}\int_{|z|\geqslant\frac{1}{2}|x|} w(z)\, dz \leqslant c/|x|^{1+\alpha},$$

since $|y| \leqslant \frac{1}{2}|x|$.

Finally, for I_3, define $t\colon \mathbb{R}\to\mathbb{R}$ so that $0 \leqslant t \leqslant w$, $t = w$ on $B(x, \frac{1}{4}|x|)$, the support of t is contained in $B(x, \frac{1}{2}|x|)$, $t\in C^\infty$, and $\|t'\|_\infty \leqslant c/|x|^{2+\alpha}$. One can see that the derivative estimate is possible by using a scaling argument.

Set $s = w-t$. Since $s = 0$ on $B(x, \frac{1}{4}|x|)$,

$$\begin{aligned} \left|\int_{|z|\geqslant\frac{1}{2}|x|} T(x-z)\, s(z)\, dz\right| &= \int_{\substack{|z|\geqslant\frac{1}{2}|x| \\ |z-x|>\frac{1}{4}|x|}} T(x-z)\, s(z)\, dz \Bigg| \\ &\leqslant c(\tfrac{1}{4}|x|)^{-1}\int_{|z|\geqslant\frac{1}{4}|x|} w(z)\, dz \leqslant c/|x|^{1+\alpha} \end{aligned}$$

by (1·1)(*a*) and (4·2).

Let $A = \{z\colon \frac{1}{2}|x| \leqslant |z| \leqslant \frac{3}{2}|x|\}$. By the fact that $t = 0$ on $B(0, \frac{1}{2}|x|)$, (1·1)(*c*), and the fact that $t(x-z) = 0$ if $z\notin A$, we have

$$\begin{aligned} \left|\int_{|z|\geqslant\frac{1}{2}|x|} T(x-z)\, t(z)\, dz\right| &= \left|\int T(x-z)\, t(z)\, dz\right| = \left|\int T(z)\, t(x-z)\, dz\right| \\ &= \left|\int T(z)\, [t(x-z)-t(x)\, 1_A(z)]\, dz\right| \\ &= \left|\int_A T(z)\, [t(x-z)-t(x)]\, dz\right|. \end{aligned}$$

We bound this by

$$c\int_A |z|^{-1}\,\|t'\|\,|z|\,dz \leqslant c|x|^{-(2+\alpha)}\int_A dz \leqslant c/|x|^{1+\alpha}.$$

Adding to (4·5) gives $|I_3| \leqslant c/|x|^{1+\alpha}$, and combining with the bounds for I_1 and I_2, (a) is proved.

The proof of (b) is similar. Let $v \in C^\infty$ be such that $0 \leqslant v \leqslant w, v = w$ on $B(0, 10\Lambda_0)$, the support of v is contained in $B(0, 15\Lambda_0)$, and $\|v'\| \leqslant c$. Set $u = w - v$. By (1·1)(a),

$$\left|\int T(x-z)\,u(z)\,dz\right| = \left|\int_{|z|\geqslant 10\Lambda_0} T(x-z)\,u(z)\,dz\right| \leqslant c\int_{|z|\geqslant 10\Lambda_0} |z|^{-1}w(z)\,dz \leqslant c. \quad (4·6)$$

We also have

$$\left|\int T(x-z)\,v(z)\,dz\right| = \left|\int T(z)\,v(x-z)\,dz\right| = \left|\int T(z)\,[v(x-z)-v(x)\,1_{B(0,\,20\Lambda_0)}(z)]\,dz\right|$$

$$= \left|\int_{B(0,\,20\Lambda_0)} T(z)\,[v(x-z)-v(x)]\,dz\right| \leqslant c\int_{|z|\leqslant 20\Lambda_0} |z|^{-1}\,\|v'\|_\infty\,|z|\,dz \leqslant c. \quad (4·7)$$

Adding (4·6) and (4·7) proves (b). ∎

Taking $y = 0$ in Proposition 4·1(a) and using (1·1)(a) we see that

$$|Tw(x)| \leqslant c/|x| \quad \text{if } |x| \geqslant \Lambda_0. \quad (4·8)$$

We note also that the conclusions of Proposition 4·1 and (4·8) hold when T is replaced by T^*.

Proof of Proposition 2·3. Substituting for U, we see that we must bound

$$\int_{\tilde{Q}^c} |T^*(x-y) - T^*(x_Q-y)|\,dy = \int_{\tilde{Q}^c} |T(y-x) - T(y-x_Q)|\,dy$$

$$\leqslant \int_{|y-x_Q|\geqslant 4\,\mathrm{diam}\,(Q)} + \int_{\substack{4\,\mathrm{diam}\,(Q) > |y-x_Q| \\ y\in\tilde{Q}^c}} \quad (4·9)$$

for $x \in Q$.

The first integral on the right of (4·9) is less than or equal to

$$\int_{|y|\geqslant 4\,\mathrm{diam}\,(Q)} |T(y-(x-x_Q)) - T(y)|\,dy \leqslant c$$

by a change of variables and (1·1)(b).

Since Q is bounded, then $x \in Q, y \in \tilde{Q}^c$, and $|y - x_Q| < 4\,\mathrm{diam}\,(Q)$ implies

$$4\,\mathrm{diam}\,(Q) \geqslant |y-x_Q| \geqslant \eta\,\mathrm{diam}\,(Q) \quad \text{and} \quad 5\,\mathrm{diam}\,(Q) \geqslant |y-x| \geqslant \eta\,\mathrm{diam}\,(Q).$$

The second integral on the right of (4·9) is bounded by

$$\int_{\substack{4\,\mathrm{diam}\,(Q) > |y-x_Q| \\ y\in\tilde{Q}^c}} (|T(y-x)| + |T(y-x_Q)|)\,dy \leqslant c,$$

using (4·3).

So altogether, the left-hand side of (4·9) is bounded by a constant, and this proves the proposition. ∎

We write

$$\frac{T^*(x-y)-T^*w(x)}{w(x)}-\frac{T^*(x_Q-y)-T^*w(x_Q)}{w(x_Q)}=A_1+A_2, \tag{4·10}$$

where
$$A_1=\frac{(T^*(x-y)-T^*w(x))-(T^*(x_Q-y)-T^*w(x_Q))}{w(x_Q)} \tag{4·11}$$

and
$$A_2=\frac{w(x_Q)-w(x)}{w(x)\,w(x_Q)}(T^*(x-y)-T^*w(x)).$$

Proof of Proposition 2·4 (*a*). Since $|x_Q|\leqslant\Lambda_0$, we have $w(x_Q)^{-1}\geqslant c$. Then since w is bounded above,

$$\left|\frac{1}{w(x_Q)}\int_{\tilde{Q}^c}(T^*(x-y)-T^*(x_Q-y))\,w(y)\,dy\right|\leqslant c\int_{\tilde{Q}^c}|T^*(x-y)-T^*(x_Q-y)|\,dy\leqslant c. \tag{4·12}$$

The last inequality follows by the proof of Proposition 2·3.

Since by Proposition 4·1 (*b*) and (4·8), T^*w is bounded above,

$$\int_{\tilde{Q}^c}(|T^*w(x)|+|T^*w(x_Q)|)\,w(y)\,dy\leqslant c. \tag{4·13}$$

The triangle inequality, (4·12), and (4·13) show that

$$\int_{\tilde{Q}^c}|A_1|\,w(y)\,dy\leqslant c. \tag{4·14}$$

Since w is bounded above and below on Q, and T^*w is bounded above, then

$$\frac{|w(x_Q)-w(x)|}{w(x_Q)\,w(x)}\int_{\tilde{Q}^c}|T^*w(x)|\,w(y)\,dy\leqslant c. \tag{4·15}$$

But by (1·1) (*a*) and (4·3),

$$\begin{aligned}\frac{|w(x_Q)-w(x)|}{w(x_Q)\,w(x)}\int_{\tilde{Q}^c}|T^*(x-y)|\,w(y)\,dy &\leqslant c\|w'\|_\infty|x-x_Q|\int_{|y-x|\geqslant\eta\,\mathrm{diam}\,(Q)}|x-y|^{-1}w(y)\,dy\\ &\leqslant c|x-x_Q|\int_{\substack{|y|\leqslant 10\Lambda_0\\ |y-x|\geqslant\eta\,\mathrm{diam}\,(Q)}}|x-y|^{-1}\,dy\\ &\quad+c\int_{|y|>10\Lambda_0}|x-y|^{-1}w(y)\,dy\\ &\leqslant c\,\mathrm{diam}\,(Q)\,(c+|\ln(\mathrm{diam}\,(Q))|)+c.\end{aligned} \tag{4·16}$$

However, $\mathrm{diam}\,(Q)\leqslant c_1$, independent of which $\mathcal{Q}_n$ that Q is an element of, and for each c_1 the function $z|\ln z|$ is bounded on $(0,c_1)$. Hence the left-hand side of (4·16) is bounded by a constant.

Combining (4·15) and (4·16) together,

$$\int_{\tilde{Q}^c}|A_2|\,w(y)\,dy\leqslant c. \tag{4·17}$$

Then, by the definition of U^*, (4·10), the triangle inequality, (4·14), and (4·17), Proposition 2·4 (a) is proved. ∎

Proof of Proposition 2·4 (b). As in the above proof, it suffices to bound

$$\int_{\tilde{Q}^c} |A_i|\, w(y)\, dy, \quad i = 1, 2.$$

We write

$$\int_{\tilde{Q}^c} |A_i|\, w(y)\, dy = \int_{\tilde{Q}^c \cap \{|y| \geqslant \frac{1}{2}|x_Q|\}} + \int_{\tilde{Q}^c \cap \{|y| \leqslant \frac{1}{2}|x_Q|\}} = I_{i1} + I_{i2}, \quad i = 1, 2.$$

First consider I_{11}. If $|y| \geqslant \frac{1}{2}|x_Q|$, then $w(y)/w(x_Q)$ is bounded. We bound

$$\int_{\tilde{Q}^c \cap \{|y| \geqslant \frac{1}{2}|x_Q|\}} |T^*(x-y) - T^*(x_Q - y)|\, dy$$

just as in the proof of Proposition 2·3. On the other hand, since $|x| \geqslant |x_Q|$,

$$\begin{aligned} |T^*w(x) - T^*w(x_Q)| &\leqslant |T^*w(x) - T^*(x)| + |T^*w(x_Q) - T^*(x_Q)| + |T^*(x)| + |T^*(x_Q)| \\ &\leqslant 2c|x_Q|^{-(1+\alpha)} + 2c|x_Q|^{-1} \leqslant c|x_Q|^{-1} \end{aligned}$$

by (1·1)(c) and Proposition 4·1. Hence by (4·2)

$$\frac{|T^*w(x) - T^*w(x_Q)|}{w(x_Q)} \int_{|y| \geqslant \frac{1}{2}|x_Q|} w(y)\, dy \leqslant c.$$

We have thus bounded $|I_{11}|$ by a constant.

Next consider I_{12}. Using Proposition 4·1 (a),

$$\begin{aligned} &\frac{1}{w(x_Q)} \int_{|y| \leqslant \frac{1}{2}|x_Q|} |T^*(x_Q - y) - T^*w(x_Q)|\, w(y)\, dy \\ &\qquad \leqslant \frac{c}{w(x_Q)\,|x_Q|^{1+\delta}} \int_{|y| \leqslant \frac{1}{2}|x_Q|} |y|^\delta w(y)\, dy + \frac{c}{w(x_Q)\,|x_Q|^{1+\alpha}} \leqslant c, \end{aligned}$$

by (4·1). Since $w(x)$ and $w(x_Q)$ are comparable and $|x| \geqslant |x_Q|$, we have similarly

$$\frac{1}{w(x_Q)} \int_{|y| \leqslant \frac{1}{2}|x_Q|} |T^*(x-y) - T^*w(x)|\, w(y)\, dy \leqslant c.$$

Combining, we have $|I_{12}| \leqslant c$, hence $\int_{\tilde{Q}^c} |A_1|\, w(y)\, dy \leqslant c$.

Next we look at I_{22}. The expression

$$\frac{1}{w(x_Q)} \int |T^*(x-y) - T^*w(x)|\, w(y)\, dy$$

is bounded exactly as in the argument for I_{12}. Since

$$|w(x_Q) - w(x)|/w(x) \leqslant 1 + w(x_Q)/w(x) \leqslant c,$$

$|I_{22}|$ is bounded.

Finally, consider I_{21}. First, note that since

$$|T^*w(x)| \leqslant |T^*w(x) - T^*(x)| + |T^*(x)| \leqslant c|x|^{-1}$$

by Proposition 4·1 (a) and (1·1) (a),

$$\frac{|w(x_Q)-w(x)|}{w(x)\,w(x_Q)}\int_{|y|\geqslant\frac{1}{2}|x_Q|}|T^*w(x)|\,w(y)\,dy\leqslant c\frac{w(x_Q)+w(x)}{w(x)\,w(x_Q)}|x|^{-1}\int_{|y|\geqslant\frac{1}{2}|x_Q|}w(y)\,dy\leqslant c, \tag{4·18}$$

recalling that $w(x)$ and $w(x_Q)$ are comparable.

Secondly, if $|y|\geqslant 4|x|$, then $|y-x|\geqslant 3|x|$. So by (1·1) (a),

$$\frac{|w(x_Q)-w(x)|}{w(x_Q)\,w(x)}\int_{|y|\geqslant 4|x|}|T^*(x-y)|\,w(y)\,dy\leqslant c\frac{w(x_Q)+w(x)}{w(x_Q)\,w(x)}|x|^{-1}\int_{|y|\geqslant 4|x|}w(y)\,dy\leqslant c. \tag{4·19}$$

Thirdly, if $|y|\geqslant\frac{1}{2}|x_Q|$, then $w(y)/w(x_Q)$ is bounded above, and

$$\begin{aligned}\frac{|w(x_Q)-w(x)|}{w(x_Q)\,w(x)}&\int_{\tilde{Q}^c\cap\{\frac{1}{2}|x_Q|\leqslant|y|\leqslant 4|x|\}}|T^*(x-y)|\,w(y)\,dy\\&\leqslant\frac{c|x_Q-x|}{w(x)}\sup_{|z|\geqslant|x_Q|}|w'(z)|\int_{5|x|\geqslant|y-x|\geqslant\eta\,\mathrm{diam}\,(Q)}|x-y|^{-1}\,dy\\&\leqslant c\frac{\mathrm{diam}\,(Q)}{|x|}\left(c+\left|\ln\left(\frac{|x|}{\mathrm{diam}\,(Q)}\right)\right|\right)\end{aligned} \tag{4·20}$$

by (4·4) and (4·3). But diam $(Q)/|x|$ is bounded above by some c_1, and $z|\ln z|$ is bounded on $(0,c_1)$; so the left-hand side of (4·20) is bounded by a constant.

Adding (4·18), (4·19), and (4·20) together, we get that $|I_{21}|$ is bounded. Combining this with the bound for $|I_{22}|$, we get $\int_{\tilde{Q}^c}|A_2|\,w(y)\,dy\leqslant c$, which completes the proof of part (b). ▌

Proof of Proposition 2·4 (c). Now we are looking at unbounded Q, and substituting for U^*, we need to bound

$$\int_{\tilde{Q}^c}\frac{T^*(x-y)-T^*w(x)}{w(x)}w(y)\,dy=\int_{\tilde{Q}^c\cap\{|y|\leqslant\frac{1}{2}|x|\}}+\int_{\tilde{Q}^c\cap\{|y|>\frac{1}{2}|x|\}}=I_3+I_4.$$

We get $|I_3|\leqslant c$ in the same way that we bounded $|I_{12}|$ in the proof of part (b).

So it remains to bound I_4. By (4·8),

$$\frac{|T^*w(x)|}{w(x)}\int_{|y|>\frac{1}{2}|x|}w(y)\,dy\leqslant c. \tag{4·21}$$

If $|y|>\frac{1}{2}|x|$, then $w(y)\leqslant cw(x)$, and if $y\in\tilde{Q}^c$ and $x\in Q$, then $|y-x|\geqslant\eta|x|$ by (2·4). So by (1·1) (a), (4·3), and (4·2) we have

$$\begin{aligned}\frac{1}{w(x)}\int_{\tilde{Q}^c\cap\{|y|>\frac{1}{2}|x|\}}|T^*(x-y)|\,w(y)\,dy&\leqslant c\int_{\eta|x|\leqslant|y-x|\leqslant 4|x|}|T^*(x-y)|\,dy\\&\quad+\frac{1}{w(x)}\int_{|y-x|>4|x|,\,|y|>\frac{1}{2}|x|}|T^*(x-y)|\,w(y)\,dy\\&\leqslant c+\frac{c}{w(x)}|x|^{-1}\int_{|y|>\frac{1}{2}|x|}w(y)\,dy\leqslant c.\end{aligned} \tag{4·22}$$

Adding (4·21) and (4·22) bounds $|I_4|$. ▌

5. *Remarks*

Remark 5·1. To see that $w(x)\,dx$ does not satisfy the doubling condition, consider the intervals $[N, 5N]$ for N large. Note that $w([N, 5N]) \leqslant c_1 N^{-\alpha}$ while $w([-N, 7N]) \geqslant w([-1, 1]) \geqslant c_2$ if $N \geqslant 1$.

Remark 5·2. The function w is not in A_p. To see this, recall the definition of A_p (see [**4**]) and consider the intervals $[1, N]$ for N large. On $[1, N]$, $w(x)$ is comparable to $x^{-(1+\alpha)}$.

On p. 21 in [**1**], it was incorrectly remarked that w was an A_2 weight. However, that comment was not used in the remainder of the paper and does not invalidate any of the results there.

Remark 5·3. Let $\mu_{r,y}(dx) = rw(rx+y)\,dx$ for $r > 0, y \in \mathbb{R}$. By scaling and translation, T satisfies a centred norm inequality with respect to the measures $\mu_{r,y}$ with constant independent of r and y. It is not too hard to show that $\|f\|_{\mathrm{BMO}(dx)}$ is comparable to $\sup_{r,y} \|f-\bar{f}\|_{L^2(\mu_{r,y})}$. Consequently our centred norm inequality implies that T maps $\mathrm{BMO}(dx)$ to $\mathrm{BMO}(dx)$.

One of the main results of [**1**] was a new proof of the $L^p(dx)$ boundedness of T. There were two steps. First, it was shown that T satisfies a centred L^2 inequality with respect to $\mu_{r,y}$ by using Cotlar's lemma. The second step, which was probabilistic, showed that any operator satisfying centred L^2 inequalities with respect to $\mu_{r,y}$ with constants independent of r, y must be bounded on $L^p(dx)$, for $1 < p < \infty$.

Remark 5·4. In view of Remark 5·3, one might wonder for which measures $w(dx)$ do centred norm inequalities hold. For example, could w be replaced by some other measure that characterizes BMO by its dilates and translates? Some positivity condition is clearly necessary. To see this, consider $v(dz) = 1_{[0,1]}(x)\,dx$ for T the Hilbert transform. If $f(x) = 1_{[2,3]}(x)$, the $L^p(v(dx))$ norm of F is 0, but Tf is not constant on $[0, 1]$; hence the centred norm inequality for Tf with w replaced by v fails.

Remark 5·5. If we replace the condition that K be odd by the condition $\int_{R_1<|x|<R_2} K(x)\,dx = 0$ whenever $0 < R_1 < R_2 < \infty$, then Theorem 1·1 holds for $p = 2$ for kernels K mapping $\mathbb{R}^d$ to $\mathbb{R}$, for $d > 1$: see remark 3 of §3 of [**1**]. However, we do not know what happens if $p \neq 2$ when $d > 1$. The difficulty in extending our proof lies in defining 'cubes' Q and $\tilde{Q}$ with respect to which T scales properly yet $w(\tilde{Q}) \leqslant cw(Q)$.

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