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On Domain Monotonicity of the Neumann Heat Kernel*

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Some examples are given of convex domains for which domain monotonicity of the Neumann heat kernel does not hold.

1. INTRODUCTION

Let $p^D(t, x, y)$ denote the Neumann heat kernel for D, that is, the fundamental solution to the heat equation with Neumann boundary conditions. Equivalently, $p^D(t, x, y)$ is the transition density of reflecting Brownian motion in D. A question that has been of considerable interest is the monotonicity of $p^D(t, x, y)$ as a function of D.

Problem 1. Under what conditions on $D_1 \subseteq D_2$ is it true that $p^{D_1}(t, x, y) \geqslant p^{D_2}(t, x, y)$ for all $(x, y, t) \in D_1 \times D_1 \times (0, \infty)$?

This question was raised by Chavel [1] who showed that the inequality holds if D_2 is a ball centered at x and D_1 is convex. Kendall [2] has shown that if D_1 and D_2 are convex and a sphere about x separates ∂D_1 and ∂D_2, then domain monotonicity holds. Carmona and Zheng [3] showed that it also holds when the domains are convex and $\bar{D}_1 \subset D_2$, provided t is sufficiently small. Hsu [4] has considered the case where D_2 is a parallelopiped and D_1 satisfies a certain condition relating its inward normals to D_2.

As these results indicate (see the introduction to Hsu [4]), in the above problem it is the case of convex domains that is of interest. The following special case of Problem 1 has received a great deal of attention.

Problem 2. It is true that $p^{D_1}(t, x, y) \geqslant p^{D_2}(t, x, y)$ for all $(x, y, t) \in D_1 \times D_1 \times (0, \infty)$ if $D_1 \subseteq D_2$ and both domains are convex?

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This paper is devoted to counterexamples showing that domain monotonicity need not hold for arbitrary convex domains D_1 and D_2.

THEOREM 1. *There exist convex domains* $D_1 \subseteq D_2 \subset \mathbb{R}^2$, *points* $x, y \in \bar{D}_1$, *and* $t > 0$ *such that*

$$p^{D_1}(t, x, y) < p^{D_2}(t, x, y).$$

In our example, the domains D_1 and D_2 are wedges and x and y lie on the boundary of D_1, but it is easy, using translation invariance, continuity of the heat kernel, and the idea of (3.2) below, to modify the example so that D_1 and D_2 are bounded, strictly convex, and have a smooth boundary, the closure of D_1 is contained in the interior of D_2, and x and y belong to the interior of D_1.

We also give an example of two bounded convex domains where domain monotonicity does not hold for arbitrarily small t. It is not hard to modify this example so that $\partial D_1 \cap \partial D_2$ consists of a single point. Thus, in the Carmona–Zheng results [3] it is crucial that the closure of D_1 be contained in the interior of D_2.

THEOREM 2. *There exist bounded convex domains* D_1 *and* D_2 *with* $D_1 \subseteq D_2 \subset \mathbb{R}^2$ *and triples* $(x_m, y_m, t_m) \in \bar{D}_1 \times \bar{D}_1 \times (0, \infty)$ *such that* $t_m \to 0$ *and* $p^{D_1}(t_m, x_m, y_m) < p^{D_2}(t_m, x_m, y_m)$ *for all* m.

In Section 2 we prove Theorem 1 and in Section 4 we prove Theorem 2. Section 3 gives the proofs of the main lemmas.

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2. First Example

In what follows, we use the probabilist's $p^D(t, x, y)$, that is, the kernel corresponding to $\frac{1}{2}\Delta$. Thus $p^{\mathbb{R}^2}(t, x, y) = (2\pi t)^{-1} \exp(-|x-y|^2/2t)$.

Fix $t > 0$. Let

$$W(0) = \{z = re^{i\theta} : r \in (0, \infty), \theta \in (0, 3\pi/4)\},$$

$$W(\varepsilon) = W(0) + (-\varepsilon, \varepsilon),$$

$$x_\varepsilon = (-\varepsilon, \varepsilon), \qquad y_\varepsilon = (1, \varepsilon), \qquad z_\varepsilon = (1+\varepsilon, 0).$$

The proof of Theorem 1 hinges on the following two lemmas. We prove them is Section 3.

LEMMA 1. *For all* $y \in \overline{W(0)}$ *and* $t > 0$, $p^{W(0)}(t, y, 0) = (8/3)\, p^{\mathbb{R}^2}(t, y, 0)$.

LEMMA 2. *For every fixed* $\beta \in (0, \frac{1}{3})$ *and* $t > 0$,

$$p^{W(0)}(t, y_\varepsilon, x_\varepsilon) \geqslant p^{W(0)}(t, y_\varepsilon, 0) - O(\varepsilon^{1+\beta}).$$

Proof of Theorem 1. Fix some $t > 0$ and let $a = p^{\mathbb{R}^2}(t, y_0, 0)$. Then there exists $c_1 > 0$ independent of ε such that

$$p^{\mathbb{R}^2}(t, z_\varepsilon, 0) = a - c_1 \varepsilon + O(\varepsilon^2),$$

and since $|y_\varepsilon - 0| = (1 + \varepsilon^2)^{1/2} = 1 + O(\varepsilon^2)$,

$$p^{\mathbb{R}^2}(t, y_\varepsilon, 0) = a + O(\varepsilon^2).$$

By Lemma 1 and translation invariance,

$$p^{W(\varepsilon)}(t, y_\varepsilon, x_\varepsilon) = p^{W(0)}(t, z_\varepsilon, 0) = \tfrac{8}{3}(a - c_1 \varepsilon + O(\varepsilon^2)),$$

and

$$p^{W(0)}(t, y_\varepsilon, 0) = \tfrac{8}{3}(a + O(\varepsilon^2)).$$

By Lemma 2,

$$\begin{aligned} p^{W(0)}(t, y_\varepsilon, x_\varepsilon) &\geqslant p^{W(0)}(t, y_\varepsilon, 0) - O(\varepsilon^{7/6}) \\ &\geqslant \tfrac{8}{3} a - O(\varepsilon^{7/6}) \\ &= p^{W(\varepsilon)}(t, y_\varepsilon, x_\varepsilon) + \tfrac{8}{3} c_1 \varepsilon - O(\varepsilon^{7/6}). \end{aligned}$$

Now take ε sufficiently small that $p^{W(0)}(t, y_\varepsilon, x_\varepsilon) > p^{W(\varepsilon)}(t, y_\varepsilon, x_\varepsilon)$, let $D_1 = W(\varepsilon)$, and $D_2 = W(0)$. ∎

3. PROOFS OF LEMMAS

Proof of Lemma 1. Note $p^{\mathbb{R}^2}(t, \cdot, 0)$ satisfies the heat equation in $\overline{W(0)} - \{0\}$, has a singularity of the right form at 0, and by symmetry considerations (or by direct calculation) has 0 normal derivative on $\partial W(0) - \{0\}$. This characterizes the heat kernel up to constants, so there exists κ such that

$$p^{W(0)}(t, y, 0) = \kappa p^{\mathbb{R}^2}(t, y, 0).$$

Integrating over $y \in W(0)$, we get $1 = (3/8)\kappa$, hence $\kappa = 8/3$. ∎

We present two proofs of Lemma 2, the first largely analytic and the second entirely probabilistic, as each is of interest in itself.

First proof of Lemma 2. Let $K=\varepsilon^{-b}$, where $b>0$ is chosen small later. Let $q^D(t, x, y)$ denote the transition density of reflecting Brownian motion in D killed on exiting $B(0, K)$, the ball of radius K about 0. A proof analogous to that of Lemma 1 shows that

$$q^{W(0)}(t, y, 0)=\tfrac{8}{3}q^{\mathbb{R}^2}(t, y, 0).$$

Trivially,

$$p^{W(0)}(t, y_\varepsilon, x_\varepsilon)\geqslant q^{W(0)}(t, y_\varepsilon, x_\varepsilon). \tag{3.1}$$

If $r(t, x, y)$ denotes the transition density of two-dimensional Brownian motion killed on exiting $[-K/\sqrt{2}, K/\sqrt{2}]^2$, then $p^2(t, y_\varepsilon, 0)$ and $r(t, y_\varepsilon, 0)$ factor into the product of transition densities of one-dimensional Brownian motion, on the line and killed on exiting $[-K/\sqrt{2}, K/\sqrt{2}]$, respectively. Exact formulas (see Feller [5, p. 341]) then show that there exist c_1, c_2 such that

$$|p^{\mathbb{R}^2}(t, y_\varepsilon, 0)-r(t, y_\varepsilon, 0)|\leqslant c_1\, e^{-c_2K^2/t}.$$

We then have

$$\begin{aligned}
&|p^{W(0)}(t, y_\varepsilon, 0)-q^{W(0)}(t, y_\varepsilon, 0)|\\
&\quad=\tfrac{8}{3}|p^{\mathbb{R}^2}(t, y_\varepsilon, 0)-q^{\mathbb{R}^2}(t, y_\varepsilon, 0)|\\
&\quad\leqslant\tfrac{8}{3}|p^{\mathbb{R}^2}(t, y_\varepsilon, 0)-r(t, y_\varepsilon, 0)|\leqslant\tfrac{8}{3}c_1\, e^{-c_2K^2/t}.
\end{aligned}\tag{3.2}$$

By Bass and Hsu [6, Sect. 3] (with easy modifications to extend the results to the case of two dimensions), there exists c_3 such that $\sup_z p^{W(0)}(t/2, z, z)\leqslant c_3/t$. Since $W(0)\cap B(0, K)$ is a bounded region, we have an eigenvalue expansion for $q^{W(0)}$, namely

$$q^{W(0)}(t, y_\varepsilon, x)=\sum_{i=1}^{\infty} e^{-\lambda_i t}\varphi_i(x)\,\varphi_i(y_\varepsilon), \qquad x\in W(0),$$

where the φ_i are the eigenfunctions for one half the Laplacian with Neumann boundary conditions on $\partial W(0)$ and Dirichlet boundary conditions on $\partial B(0, K)$, and the eigenvalues λ_i are nonnegative. Fix an arbitrary $t>0$. Let

$$G(x)=G(t, x)=\sum_{i=0}^{\infty}\lambda_i e^{-\lambda_i t}\varphi_i(x)\,\varphi_i(y_\varepsilon).$$

Let $c_4 = \sup_{\lambda \geqslant 0} \lambda \exp(-\lambda t/2) < \infty$. Then by Cauchy–Schwarz,

$$\begin{aligned}
|G(x)| &\leqslant c_4 \sum_{i=1}^{\infty} e^{-\lambda_i t/2} |\varphi_i(x)| \, |\varphi_i(y_\varepsilon)| \\
&\leqslant c_4 \left(\sum e^{-\lambda_i t/2} \varphi_i(x)^2\right)^{1/2} \left(\sum e^{-\lambda_i t/2} \varphi_i(y_\varepsilon)^2\right)^{1/2} \\
&= c_4 (q^{W(0)}(t/2, x, x))^{1/2} (q^{W(0)}(t/2, y_\varepsilon, y_\varepsilon))^{1/2} \\
&\leqslant c_4 (p^{W(0)}(t/2, x, x))^{1/2} (p^{W(0)}(t/2, y_\varepsilon, y_\varepsilon))^{1/2} \leqslant c_4 c_3 / t.
\end{aligned}$$

But from the eigenvalue expansion,

$$G(x) = -\partial q^{W(0)}(t, y_\varepsilon, \cdot)/\partial t = -(1/2) \, \Delta q^{W(0)}(t, y_\varepsilon, \cdot),$$

so $q^{W(0)}(t, y_\varepsilon, x) = U_K G(x)$, where U_K is the potential kernel for X_s, reflecting Brownian motion in $W(0)$, killed on exiting $B(0, K)$.

Let $T = \inf\{s : |X_s| \geqslant K\}$. Let Z_s be standard Brownian motion in $\mathbb{R}^2$, $S = \inf\{s : |Z_s| \geqslant K^{4/3}\}$, V_K the potential kernel for Z_s killed on exiting $B(0, K^{4/3})$. Let $\gamma(z_1, z_2) = (z_1, |z_2|)$. So $\gamma(Z_s)$ is reflecting Brownian motion in the upper half plane.

Map $W(0) \cap B(0, K)$ onto the intersection of the upper half plane with $B(0, K^{4/3})$ by means of the conformal map $F(z) = z^{4/3}$. By Levy's theorem (see Durrett [7, pp. 79–80], for example), when X_s is away from the boundary, $F(X_s)$ is a certain time change of a Brownian motion. Since conformal maps preserve angles, the angle of reflection of $F(X_s)$ at the boundary of $F(W(0))$ is still norml. So $F(X_s)$ is a time change of $\gamma(Z_s)$. Since $|(F^{-1})'(z)| = 3|z|^{-1/4}/4$ and X_s spends zero time on the boundary, Lévy's theorem shows that

$$\begin{aligned}
U_K G(x) &= \mathbb{E}^x \int_0^T G(X_s) \, ds \\
&= \mathbb{E}^{F(x)} \int_0^S \tfrac{9}{16} |\gamma(Z_s)|^{-1/2} G(F^{-1}(\gamma(Z_s))) \, ds \\
&= V_K H(x^{4/3}),
\end{aligned}$$

where $H(z) = 9|z|^{-1/2} G(F^{-1}(\gamma(z))) \, 1_{B(0, K^{4/3})}(z)/16$. From the analyst's point of view, this formula is just the conformal invariance of the Green function and a change of variables.

Let $V(x, y) = -\log(|x-y|)/\pi$, the fundamental solution for one half the Laplacian on $\mathbb{R}^2$. Then $V_K H(x) - VH(x) = -\mathbb{E}^x VH(Z_S)$ is harmonic, and a crude estimate shows that it is bounded by $c_5 K$ on $B(0, 2)$ for some $c_5 > 0$ independent of K. By a standard gradient estimate for harmonic functions, $|\nabla(V_K H - VH)|$ is bounded by $c_6 K$ on $B(0, 1)$. Since $|\nabla V(x, y)| = 1/(\pi |x-y|)$, Hölder's inequality with exponents $\frac{3}{2}, 3$ shows that for some $c_7 > 0$, $\int |\nabla V(x-y)| \, |H(y)| \, dy$ is bounded by $c_7 K$ for $x \in B(0, 1)$. A similar argument shows that one can take the derivative under the integral sign, so that $|\nabla(VH)(x)| \leq \int |\nabla V(x, y)| \, |H(y)| \, dy$, $x \in B(0, 1)$. Hence $|\nabla V_K H|$ is bounded by $(c_6 + c_7) K$ on $B(0, 1)$, and therefore

$$|V_K H(x_\varepsilon^{4/3}) - V_K H(0)| \leq (c_6 + c_7) K \, |x_\varepsilon|^{4/3}.$$

We then get

$$|q^{W(0)}(t, y_\varepsilon, x_\varepsilon) - q^{W(0)}(t, y_\varepsilon, 0)| \leq (c_6 + c_7) \, K \, |x_\varepsilon|^{4/3}. \tag{3.3}$$

Combining (3.1), (3.2), and (3.3) and taking b sufficiently small proves the lemma. ▮

The next proof is based on a special construction of reflecting Brownian motion in $W(0)$ and a coupling argument.

Second proof of Lemma 2. Let X be a standard two-dimensional Brownian motion and assume that $X(0) \in W(0)$ a.s. One can easily modify the proof of the Skorohod lemma (Karatzas and Shreve ([8, 3.6.14]) to see that there exists a.s. a unique pair of continuous nondecreasing functions α_t^+ and α_t^- with $\alpha_0^+ = \alpha_0^- = 0$ and such that if

$$Y_t \stackrel{\mathrm{df}}{=} \exp(i(\alpha_t^+ - \alpha_t^-)) X_t,$$

then α_t^+ is flat off the set $\{t \geq 0 : \arg(Y_t) = 0\}$, α_t^- does not increase outside $\{t \geq 0 : \arg(Y_t) = 3\pi/4\}$ and $Y_t \in W(0)$ for all $t \geq 0$. The last property implies that we must have

$$\alpha_t^+ \geq -\min_{0 \leq s \leq t} \arg X(s) \tag{3.4}$$

and

$$\alpha_t^- \geq \max_{0 \leq s \leq t} \arg X(s) - 3\pi/4. \tag{3.5}$$

The process Y_t behaves like standard Brownian motion in the interior of $W(0)$ and is confined to $W(0)$, so it is a reflecting Brownian motion in $W(0)$. The angle of reflection is normal because the radial part of Y is the same as that of X, and therefore the same as that of standard Brownian motion. Hence, $p^{W(0)}(t, x, y)$ is the transition density for Y.

Let $S_\varepsilon = \{x \in \mathbb{R}^2 : |x| = \varepsilon\}$ and $M_\varepsilon = S_\varepsilon \cap W(0)$. Since $|Y_t| = |X_t|$ a.s. for all $t \geq 0$, we have for all $x \in W(0)$ and $t > 0$

$$\int_{M_\varepsilon} p^{W(0)}(t, x, y)\, dy = \int_{S_\varepsilon} p^{\mathbb{R}^2}(t, x, y)\, dy.$$

In the last formula and in (3.6) below, we integrate with respect to the arc length measure on S_ε.

Fix some $t > 0$. Let $a = p^{\mathbb{R}^2}(t, y_0, 0)$. As in Section 2, $|y_\varepsilon - 0| = 1 + O(\varepsilon^2)$, and therefore

$$\int_{M_\varepsilon} p^{W(0)}(t, y_\varepsilon, z)\, dz = \int_{S_\varepsilon} p^{\mathbb{R}^2}(t, y_\varepsilon, z)\, dz = 2\pi\varepsilon a + O(\varepsilon^3). \tag{3.6}$$

Choose arbitrary points $v_1, v_2 \in M_\varepsilon$ and suppose that $X(0) = v_1$. Let $\tilde{X}_s = \exp(i\gamma) X_s$, where γ is chosen so that $\tilde{X}(0) = v_2$. Now construct reflecting Brownian motions Y_s and $\tilde{Y}_s$ from X_s and $\tilde{X}_s$ in the manner described at the beginning of the proof. Note that we always have $|Y_s| = |\tilde{Y}_s|$. Let $T = \inf\{s > 0 : Y_s = \tilde{Y}_s\}$ and $A = \{T \leq t\}$. Note that if $T < \infty$, then $Y_s = \tilde{Y}_s$ for all $s > T$, by the uniqueness of the processes α_s^+ and α_s^- in the definition of Y. We have

$$p^{W(0)}(t, y_\varepsilon, v_1)\, dy_\varepsilon = \mathbb{P}(Y_t \in dy_\varepsilon, A) + \mathbb{P}(Y_t \in dy_\varepsilon, A^c)$$

and

$$p^{W(0)}(t, y_\varepsilon, v_2)\, dy_\varepsilon = \mathbb{P}(\tilde{Y}_t \in dy_\varepsilon, A) + \mathbb{P}(\tilde{Y}_t \in dy_\varepsilon, A^c).$$

It follows that

$$p^{W(0)}(t, y_\varepsilon, v_1)\, dy_\varepsilon - p^{W(0)}(t, y_\varepsilon, v_2)\, dy_\varepsilon \leq \mathbb{P}(Y_t \in dy_\varepsilon, A^c). \tag{3.7}$$

Let $\tilde{\alpha}_t^+$ and $\tilde{\alpha}_t^-$ be defined relative to $\tilde{X}$ and $\tilde{Y}$ in the same way α_t^+ and α_t^- were defined relative to X and Y. Assume without loss of generality that $\arg v_1 < \arg v_2$. The process α^+ pushes Y towards $\tilde{Y}$ and $\tilde{\alpha}^-$ pushes $\tilde{Y}$ towards Y. The processes $\tilde{\alpha}^+$ and α^- cannot increase before T. If

$\alpha_t^+ + \tilde{\alpha}_t^- \geqslant \gamma$, then we must necessarily have $Y_t = \tilde{Y}_t$; i.e., A holds. Let $a = \min_{s \in [0,t]} \arg X_s$,

$$T_1 = \inf\{s > 0 : X_s \in S_{1/2}\},$$

$$B = \{ \max_{0 \leqslant s \leqslant T_1} \arg X(s) - \min_{0 \leqslant s \leqslant T_1} \arg X(s) < 3\pi/4 \},$$

$$B_1 \overset{\text{df}}{=} B \cap \{T_1 \leqslant t\}.$$

If $B^c \cap \{Y_t \in dy_\varepsilon\}$ holds, then $T_1 \leqslant t$ and

$$\max_{s \in [0,t]} \arg \tilde{X}_s \geqslant a + \gamma + 3\pi/4.$$

Then (3.4) and (3.5) imply that $\alpha_t^+ \geqslant -a$ and $\tilde{\alpha}_t^- \geqslant a + \gamma$. Hence $B^c \cap \{Y_t \in dy_\varepsilon\}$ implies that $\alpha_t^+ + \tilde{\alpha}_t^- \geqslant \gamma$ and therefore it implies $A \cap \{Y_t \in dy_\varepsilon\}$.

Let $V = \{z = re^{i\theta} : r > 0,\ \theta \in (0, 7\pi/8)\}$ and consider a conformal mapping $f(z) = z^{8/7}$ of V onto the upper half plane. By conformal invariance of Brownian motion, the probability that X will hit $S_{1/2}$ before hitting the boundary of V is the same as the probability that a time-changed Brownian motion $f(X)$ starting from $f(v_1)$ will hit $S_{(1/2)^{8/7}}$ before hitting the real axis and, therefore, this probability is bounded by $O(|f(v_1)|) = O(\varepsilon^{8/7})$. Let $\{V_k\}_{k=0}^{15}$ be the family of wedges obtained from V by rotation by the angle $k\pi/8$. The same estimate applies to every wedge V_k. Since the family is finite, the probability that X will stay in any of V_k's before hitting $S_{1/2}$ is bounded by $O(\varepsilon^{8/7})$. Every wedge with angle $3\pi/4$ is covered by one of V_k's, so the probability of B is bounded by $O(\varepsilon^{8/7})$. The same bound holds for B_1.

It is elementary to check that $p(s, y_\varepsilon, x) \leqslant c_1 < \infty$ for all $x \in M_{1/2}$ and all $s \geqslant 0$. Thus

$$\begin{aligned} \mathbb{P}(Y_t \in dy_\varepsilon, A^c) &\leqslant \mathbb{P}(Y_t \in dy_\varepsilon, B) = \mathbb{P}(Y_t \in dy_\varepsilon, B_1) \\ &\leqslant dy_\varepsilon \int_{M_{1/2}} \mathbb{P}(Y(T(M_{1/2})) \in dx, B_1)\, p(t - T(M_{1/2}), y_\varepsilon, x) \\ &\leqslant c_1 O(\varepsilon^{8/7})\, dy_\varepsilon. \end{aligned}$$

This and (3.7) imply that for all $v_1, v_2 \in M_\varepsilon$

$$p^{W(0)}(t, y_\varepsilon, v_1)\, dy_\varepsilon - p^{W(0)}(t, y_\varepsilon, v_2)\, dy_\varepsilon \leqslant \mathbb{P}(Y_t \in dy_\varepsilon, A^c) \leqslant O(\varepsilon^{8/7})\, dy_\varepsilon.$$

It follows from (3.6) that

$$p^{W(0)}(t, y_\varepsilon, v_1) \geqslant \tfrac{8}{3}(a + O(\varepsilon^{8/7})).$$

Now combine this estimate, Lemma 1, and the fact that $p^{\mathbb{R}^2}(t, y_\varepsilon, 0) = p^{\mathbb{R}^2}(t, y_0, 0) + O(\varepsilon^2)$ to obtain

$$\begin{aligned} p^{W(0)}(t, y_\varepsilon, v_1) &\geqslant \tfrac{8}{3}(a + O(\varepsilon^{8/7})) = \tfrac{8}{3}(p^{\mathbb{R}^2}(t, y_0, 0) + O(\varepsilon^{8/7})) \\ &= \tfrac{8}{3}(p^{\mathbb{R}^2}(t, y_\varepsilon, 0) + O(\varepsilon^{8/7})) = p^{W(0)}(t, y_\varepsilon, 0) + O(\varepsilon^{8/7}). \end{aligned}$$

We can obtain the same result with 8/7 replaced by any exponent smaller than 4/3 by choosing a smaller angle for V. ∎

4. Second Example

Let $C(\alpha, h)$ be the cone $\{z = (z_1, z_2) : 0 < z_1 < h,\ |z_2| < \alpha z_1\}$. We say that a domain D satisfies the uniform interior cone condition with parameters α and h if, for every $z \in \partial D$, there exists a translation and rotation of $C(\alpha, h)$ that lies in D and has vertex at z.

Proof of Theorem 2. From the proof of Theorem 1 it should be clear that the aperture angle $3\pi/4$ can be replaced by any angle $\theta \in (\pi/2, \pi)$. Of course, ε depends on θ as well as t. For a particular θ, let us denote the domains $W(\varepsilon)$ and $W(0)$ that arise from Theorem 1 with $t = 1$ by $F_1(\theta)$ and $F_2(\theta)$, respectively.

Let E be any convex domain, bounded or not, containing the point 0 and satisfying the uniform interior cone condition with parameters α and h. Let X_s be reflecting Brownian motion in E, let $q(s, x, y)$ be the transition density of X_s killed on exiting $B(0, K)$, and let $T = \inf\{s : |X_s| > K\}$. Convex domains are Lipschitz domains. By the estimates of Section 3 of Bass and Hsu [6], there exist c_1, c_2, and c_3 (depending on α and h) such that $p^E(t, x, y) \leqslant c_1/t$ and $\mathbb{P}^x(\sup_{s \leqslant t} |X_s - x| > \lambda) \leqslant c_2 \exp(-c_3 \lambda^2/t)$. By standard techniques (cf. the proof of Theorem 6.2 of Barlow and Bass [9]), it follows that there exist c_4 and $c_5 > 0$ (depending only on α and h) such that

$$0 \leqslant p^E(1, x, y) - q(1, x, y) = \mathbb{P}^x(X_1 \in dy, T \leqslant 1) \leqslant c_4 e^{-c_5 K^2}. \tag{4.1}$$

Suppose $E_1 \subseteq E_2$ are two convex domains satisfying the uniform interior cone condition with parameters α and h such that for some $\theta \in (\pi/2, \pi)$ and some $K = K(\theta)$ suitably large, $E_i \cap B(0, K) = F_i(\theta) \cap B(0, K)$, $i = 1, 2$. Applying the estimate (4.1) successively with $E = E_1$, E_2, $F_1(\theta)$, and $F_2(\theta)$, it follows that if $K(\theta)$ is sufficiently large, then there exist $x, y \in \bar{E}_1$ with $p^{E_1}(1, x, y) < p^{E_2}(1, x, y)$.

Let us say that a pair of domains $D_1 \subseteq D_2$ have a (θ, s)-*suitable pair of corners* if some translation and rotation of the pair (D_1, D_2) followed by a dilation by the factor $1/s$ result in a pair of domains whose intersections with $B(0, K(\theta))$ are the same as the intersections of $F_1(\theta)$ and $F_2(\theta)$ with $B(0, K(\theta))$, respectively. By scaling and the above, if D_1, D_2 have a (θ, s)-suitable pair of corners, then there exist $x, y \in \bar{D}_1$ such that $p^{D_1}(s^2, x, y) < p^{D_2}(s^2, x, y)$.

What we do now is take a sequence $\theta_m \in (\pi/2, \pi)$ increasing rapidly to π. We also take a sequence s_m decreasing to 0 sufficiently rapidly. We then form a pair of domains D_1, D_2 as follows. Define

$$\beta_1 = 0, \qquad \beta_j = \sum_{m=1}^{j} (\pi - \theta_m), \qquad j = 1, 2, ...,$$

$$z_1 = 0, \qquad z_{j+1} = z_j + 2s_j\, K(\theta_j)\, e^{i(\pi - \beta_j)}, \qquad j = 0, 1, 2, ...,$$

$$G_2^{j+1} = z_{j+1} + e^{-i\beta_j} F_2(\theta_{j+1}), \qquad j = 0, 1, 2, ...,$$

$$G_1^{j+1} = z_{j+1} + e^{-i\beta_j} F_1(\theta_{j+1}), \qquad j = 0, 2, 4, 6, ...,$$

$$G_1^{j+1} = z_{j+1} + e^{i\beta_j} F_2(\theta_{j+1}), \qquad j = 1, 3, 5, 7, ...,$$

$$D_l = B(0, M) \cap \left(\bigcap_{j=1}^{\infty} G_l^j \right), \qquad l = 1, 2.$$

It is easy to check that if we choose s_m tending to 0 sufficiently rapidly and M sufficiently large, then for each m odd, (D_1, D_2) contains (θ_m, s_m)-suitable pairs of corners (in the neighborhood of z_m). Also, D_1 and D_2 each satisfy the uniform interior cone condition with $h = 1$ and $\alpha = \frac{1}{4}$. Letting $t_m = s_m^2$ and using the results of the previous paragraph completes the proof.

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