The measurability of hitting times (corrected version)

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Abstract

Under very general conditions the hitting time of a set by a stochastic process is a stopping time. We give a new simple proof of this fact. The section theorems for optional and predictable sets are easy corollaries of the proof.

1 Introduction

A fundamental theorem in the foundations of stochastic processes is the one that says that, under very general conditions, the first time a stochastic process enters a set is a stopping time. The proof uses capacities, analytic sets, and Choquet's capacibility theorem, and is considered hard. To the best of our knowledge, no more than a handful of books have an exposition that starts with the definition of capacity and proceeds to the hitting time theorem. (One that does is [5].)

The purpose of this paper is to give a short and elementary proof of this theorem. The proof is simple enough that it could easily be included in a first year graduate course in probability.

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In Section 2 we give a proof of the debut theorem, from which the measurability theorem follows. As easy corollaries we obtain the section theorems for optional and predictable sets. This argument is given in Section 3.

Note that this paper is a version of [1], revised to take into account the corrections in [2].

2 The debut theorem

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. The outer probability $\mathbb{P}^*$ associated with $\mathbb{P}$ is given by

$$\mathbb{P}^*(A) = \inf\{\mathbb{P}(B) : A \subset B, B \in \mathcal{F}\}.$$

A set A is a $\mathbb{P}$-null set if $\mathbb{P}^*(A) = 0$. Suppose $\{\mathcal{F}_t\}$ is a filtration satisfying the usual conditions: $\cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$ for all $t \geq 0$, and each $\mathcal{F}_t$ contains every $\mathbb{P}$-null set. Let $\pi : [0, \infty) \times \Omega \to \Omega$ be defined by $\pi(t, \omega) = \omega$.

Recall that a random variable taking values in $[0, \infty]$ is a stopping time if $(T \leq t) \in \mathcal{F}_t$ for all t; we allow our stopping times to take the value infinity. Since the filtration satisfies the usual conditions, T will be a stopping time if $(T < t) \in \mathcal{F}_t$ for all t. If T_i is a finite collection or countable collection of stopping times, then $\sup_i T_i$ and $\inf_i T_i$ are also stopping times.

Given a topological space $\mathcal{S}$, the Borel σ-field is the one generated by the open sets. Let $\mathcal{B}[0, t]$ denote the Borel σ-field on $[0, t]$ and $\mathcal{B}[0, t] \times \mathcal{F}_t$ the product σ-field. A process X taking values in a topological space $\mathcal{S}$ is progressively measurable if for each t the map $(s, \omega) \to X_s(\omega)$ from $[0, t] \times \Omega$ to $\mathcal{S}$ is measurable with respect to $\mathcal{B}[0, t] \times \mathcal{F}_t$, that is, the inverse image of Borel subsets of $\mathcal{S}$ are elements of $\mathcal{B}[0, t] \times \mathcal{F}_t$. If the paths of X are right continuous, then X is easily seen to be progressively measurable. The same is true if X has left continuous paths. A subset of $[0, \infty) \times \Omega$ is progressively measurable if its indicator is a progressively measurable process.

If $E \subset [0, \infty) \times \Omega$, let $D_E = \inf\{t \geq 0 : (t, \omega) \in E\}$, the debut of E. We will prove

Theorem 2.1 *If E is a progressively measurable set, then D_E is a stopping time.*

Fix t. Let $\mathcal{K}^0(t)$ be the collection of subsets of $[0,t]\times\Omega$ of the form $K\times C$, where K is a compact subset of $[0,t]$ and $C\in\mathcal{F}_t$. Let $\mathcal{K}(t)$ be the collection of finite unions of sets in $\mathcal{K}^0(t)$ and let $\mathcal{K}_\delta(t)$ be the collection of countable intersections of sets in $\mathcal{K}(t)$. We say $A\in\mathcal{B}[0,t]\times\mathcal{F}_t$ is t-approximable if given $\varepsilon>0$, there exists $B\in\mathcal{K}_\delta(t)$ with $B\subset A$ and

$$\mathbb{P}^*(\pi(A))\le\mathbb{P}^*(\pi(B))+\varepsilon. \tag{2.1}$$

Lemma 2.2 *If $B\in\mathcal{K}_\delta(t)$, then $\pi(B)\in\mathcal{F}_t$. If $B_n\in\mathcal{K}_\delta(t)$ and $B_n\downarrow B$, then $\pi(B)=\cap_n\pi(B_n)$.*

The hypothesis that the B_n be in $\mathcal{K}_\delta(t)$ is important. For example, if $B_n=[1-(1/n),1)\times\Omega$, then $\pi(B_n)=\Omega$ but $\pi(\cap_n B_n)=\emptyset$. This is why the proof given in [6, Lemma 6.18] is incorrect.

Proof. If $B=K\times C$, where K is a nonempty subset of $[0,t]$ and $C\in\mathcal{F}_t$, then $\pi(B)=C\in\mathcal{F}_t$. Therefore $\pi(B)\in\mathcal{F}_t$ if $B\in\mathcal{K}^0(t)$. If $B=\cup_{i=1}^m A_i$ with $A_i\in\mathcal{K}^0(t)$, then $\pi(B)=\cup_{i=1}^m\pi(A_i)\in\mathcal{F}_t$.

For each ω and each set C, let

$$S(C)(\omega)=\{s\le t:(s,\omega)\in C\}. \tag{2.2}$$

If $B\in\mathcal{K}_\delta(t)$ and $B_n\downarrow B$ with $B_n\in\mathcal{K}(t)$ for each t, then $S(B_n)(\omega)\downarrow S(B)(\omega)$, so $S(B)(\omega)$ is compact.

Now suppose $B\in\mathcal{K}_\delta(t)$ and take $B_n\downarrow B$ with $B_n\in\mathcal{K}_\delta(t)$. $S(B_n)(\omega)$ is a compact subset of $[0,t]$ for each n and $S(B_n)(\omega)\downarrow S(B)(\omega)$. One possibility is that $\cap_n S(B_n)(\omega)\ne\emptyset$; in this case, if $s\in\cap_n S(B_n)(\omega)$, then $(s,\omega)\in B_n$ for each n, and so $(s,\omega)\in B$. Therefore $\omega\in\pi(B_n)$ for each n and $\omega\in\pi(B)$. The other possibility is that $\cap_n S(B_n)(\omega)=\emptyset$. Since the sequence $S(B_n)(\omega)$ is a decreasing sequence of compact sets, $S(B_n)(\omega)=\emptyset$ for some n, for otherwise $\{S(B_n)(\omega)^c\}$ would be an open cover of $[0,t]$ with no finite subcover. Therefore $\omega\notin\pi(B_n)$ and $\omega\notin\pi(B)$. We conclude that $\pi(B)=\cap_n\pi(B_n)$.

Finally, suppose $B\in\mathcal{K}_\delta(t)$ and $B_n\downarrow B$ with $B_n\in\mathcal{K}(t)$. Then $\pi(B)=\cap_n\pi(B_n)\in\mathcal{F}_t$. □

Proposition 2.3 *Suppose A is t-approximable. Then $\pi(A) \in \mathcal{F}_t$. Moreover, given $\varepsilon > 0$ there exists $B \in \mathcal{K}_\delta(t)$ such that $\mathbb{P}(\pi(A) \setminus \pi(B)) < \varepsilon$.*

Proof. Choose $A_n \in \mathcal{K}_\delta(t)$ with $A_n \subset A$ and $\mathbb{P}(\pi(A_n)) \to \mathbb{P}^*(\pi(A))$. Let $B_n = A_1 \cup \cdots \cup A_n$ and let $B = \cup_n B_n$. Then $B_n \in \mathcal{K}_\delta(t)$, $B_n \uparrow B$, and $\mathbb{P}(\pi(B_n)) \geq \mathbb{P}(\pi(A_n)) \to \mathbb{P}^*(\pi(A))$. It follows that $\pi(B_n) \uparrow \pi(B)$, and so $\pi(B) \in \mathcal{F}_t$ and

$$\mathbb{P}(\pi(B)) = \lim \mathbb{P}(\pi(B_n)) = \mathbb{P}^*(\pi(A)).$$

For each n, there exists $C_n \in \mathcal{F}$ such that $\pi(A) \subset C_n$ and $\mathbb{P}(C_n) \leq \mathbb{P}^*(\pi(A)) + 1/n$. Setting $C = \cap_n C_n$, we have $\pi(A) \subset C$ and $\mathbb{P}^*(\pi(A)) = \mathbb{P}(C)$. Therefore $\pi(B) \subset \pi(A) \subset C$ and $\mathbb{P}(\pi(B)) = \mathbb{P}^*(\pi(A)) = \mathbb{P}(C)$. This implies that $\pi(A) \setminus \pi(B)$ is a $\mathbb{P}$-null set, and by the completeness assumption, $\pi(A) = (\pi(A) \setminus \pi(B)) \cup \pi(B) \in \mathcal{F}_t$. Finally,

$$\lim_n \mathbb{P}(\pi(A) \setminus \pi(B_n)) = \mathbb{P}(\pi(A)) \setminus \pi(B)) = 0.$$

$\square$

The following lemma is well known; see, e.g., [3, p. 94].

Lemma 2.4 *(a) If $A \subset \Omega$, there exists $C \in \mathcal{F}$ such that $A \subset C$ and $\mathbb{P}^*(A) = \mathbb{P}(C)$.*

(b) Suppose $A_n \uparrow A$. Then $\mathbb{P}^*(A) = \lim_{n\to\infty} \mathbb{P}^*(A_n)$.

Proof. (a) By the definition of $\mathbb{P}^*(A)$, for each n there exists $C_n \in \mathcal{F}$ such that $A \subset C_n$ and $\mathbb{P}(C_n) \leq \mathbb{P}^*(A) + (1/n)$. Setting $C = \cap_n C_n$, we have $A \subset C$, $C \in \mathcal{F}$, and $\mathbb{P}(C) \leq \mathbb{P}(C_n) \leq \mathbb{P}^*(A) + (1/n)$ for each n, hence $\mathbb{P}(C) \leq \mathbb{P}^*(A)$.

(b) Choose $C_n \in \mathcal{F}$ with $A_n \subset C_n$ and $\mathbb{P}^*(A_n) = \mathbb{P}(C_n)$. Let $D_n = \cap_{k \geq n} C_k$ and $D = \cup_n D_n$. We see that $D_n \uparrow D$, $D \in \mathcal{F}$, and $A \subset D$. Then

$$\mathbb{P}^*(A) \geq \sup_n \mathbb{P}^*(A_n) = \sup_n \mathbb{P}(C_n) \geq \sup_n \mathbb{P}(D_n) = \mathbb{P}(D) \geq \mathbb{P}^*(A). \qquad \square$$

Let $\mathcal{T}_t = [0, t] \times \Omega$. Given a compact Hausdorff space X, let $\rho^X : X \times \mathcal{T}_t \to \mathcal{T}_t$ be defined by $\rho^X(x, (s, \omega)) = (s, \omega)$. Let

$$\mathcal{L}_0(X) = \{A \times B : A \subset X, A \text{ compact}, B \in \mathcal{K}(t)\},$$

$\mathcal{L}_1(X)$ the class of finite unions of sets in $\mathcal{L}_0(X)$, and $\mathcal{L}(X)$ the class of intersections of countable decreasing sequences in $\mathcal{L}_1(X)$. Let $\mathcal{L}_\sigma(X)$ be the class of unions of countable increasing sequences of sets in $\mathcal{L}(X)$ and $\mathcal{L}_{\sigma\delta}(X)$ the class of intersections of countable decreasing sequences of sets in $\mathcal{L}_\sigma(X)$.

Lemma 2.5 *If $A \in \mathcal{B}[0,t] \times \mathcal{F}_t$, there exists a compact Hausdorff space X and $B \in \mathcal{L}_{\sigma\delta}(X)$ such that $A = \rho^X(B)$.*

Proof. If $A \in \mathcal{K}(t)$, we take $X = [0,1]$, the unit interval with the usual topology and $B = X \times A$. Thus the collection $\mathcal{M}$ of subsets of $\mathcal{B}[0,t] \times \mathcal{F}_t$ for which the lemma is satisfied contains $\mathcal{K}(t)$. We will show that $\mathcal{M}$ is a monotone class.

Suppose $A_n \in \mathcal{M}$ with $A_n \downarrow A$. There exist compact Hausdorff spaces X_n and sets $B_n \in \mathcal{L}_{\sigma\delta}(X_n)$ such that $A_n = \rho^{X_n}(B_n)$. Let $X = \prod_{n=1}^\infty X_n$ be furnished with the product topology. Let $\tau_n : X \times \mathcal{T}_t \to X_n \times \mathcal{T}_t$ be defined by $\tau_n(x,(s,\omega)) = (x_n,(s,\omega))$ if $x = (x_1, x_2, \ldots)$. Let $C_n = \tau_n^{-1}(B_n)$ and let $C = \cap_n C_n$. It is easy to check that $\mathcal{L}(X)$ is closed under the operations of finite unions and intersections, from which it follows that $C \in \mathcal{L}_{\sigma\delta}(X)$. If $(s,\omega) \in A$, then for each n there exists $x_n \in X_n$ such that $(x_n,(s,\omega)) \in B_n$. Note that $((x_1,x_2,\ldots),(s,\omega)) \in C$ and therefore $(s,\omega) \in \rho^X(C)$. It is straightforward that $\rho^X(C) \subset A$, and we conclude $A \in \mathcal{M}$.

Now suppose $A_n \in \mathcal{M}$ with $A_n \uparrow A$. Let X_n and B_n be as before. Let $X' = \cup_{n=1}^\infty (X_n \times \{n\})$ with the topology generated by $\{G \times \{n\} : G \text{ open in } X_n\}$. Let X be the one point compactification of X'. We can write $B_n = \cap_m B_{nm}$ with $B_{nm} \in \mathcal{L}_\sigma(X_n)$. Let

$$C_{nm} = \{((x,n),(s,\omega)) \in X \times \mathcal{T}_t : x \in X_n, (x,(s,\omega)) \in B_{nm}\},$$

$C_n = \cap_m C_{nm}$, and $C = \cup_n C_n$. Then $C_{nm} \in \mathcal{L}_\sigma(X)$ and so $C_n \in \mathcal{L}_{\sigma\delta}(X)$.

If $((x,p),(s,\omega)) \in \cap_m \cup_n C_{nm}$, then for each m there exists n_m such that $((x,p),(s,\omega)) \in C_{n_m m}$. This is only possible if $n_m = p$ for each m. Thus $((x,p),(s,\omega)) \in \cap_m C_{pm} = C_p \subset C$. The other inclusion is easier and we thus obtain $C = \cap_m \cup_n C_{nm}$, which implies $C \in \mathcal{L}_{\sigma\delta}(X)$. We check that $A = \rho^X(C)$ along the same lines, and therefore $A \in \mathcal{M}$.

If $\mathcal{I}^0(t)$ is the collection of sets of the form $[a,b) \times C$, where $a < b \leq t$ and $C \in \mathcal{F}_t$, and $\mathcal{I}(t)$ is the collection of finite unions of sets in $\mathcal{I}^0(t)$, then $\mathcal{I}(t)$

is an algebra of sets. We note that $\mathcal{I}(t)$ generates the σ-field $\mathcal{B}[0,t] \times \mathcal{F}_t$. A set in $\mathcal{I}^0(t)$ of the form $[a,b) \times C$ is the union of sets in $\mathcal{K}^0(t)$ of the form $[a, b-(1/m)] \times C$, and it follows that every set in $\mathcal{I}(t)$ is the increasing union of sets in $\mathcal{K}(t)$. Since $\mathcal{M}$ is a monotone class containing $\mathcal{K}(t)$, then $\mathcal{M}$ contains $\mathcal{I}(t)$. By the monotone class theorem, $\mathcal{M} = \mathcal{B}[0,t] \times \mathcal{F}_t$. $\square$

The works of Suslin and Lusin present a different approach to the idea of representing Borel sets as projections; see, e.g., [7, p. 88] or [4, p. 284].

Lemma 2.6 *If $A \in \mathcal{B}[0,t] \times \mathcal{F}_t$, then A is t-approximable.*

Proof. We first prove that if $H \in \mathcal{L}(X)$, then $\rho^X(H) \in \mathcal{K}_\delta$. If $H \in \mathcal{L}_1(X)$, this is clear. Suppose that $H_n \downarrow H$ with each $H_n \in \mathcal{L}_1(X)$. If $(s,\omega) \in \cap_n \rho^X(H_n)$, there exist $x_n \in X$ such that $(x_n,(s,\omega)) \in H_n$. If $\{x_n\}$ is an infinite set, then there exists a limit point x_∞ by the compactness of X. Since X is a Hausdorff space, x_∞ is also a limit point of $\{x_m, x_{m+1}, \ldots\}$ for each m. Now $(x_n,(s,\omega)) \in H_n \subset H_m$ for n larger than m. For fixed ω, $\{(x,s) : (x,(s,\omega)) \in H_m\}$ is compact, so $(x_\infty,(s,\omega)) \in H_m$ for all m. This implies $(x_\infty,(s,\omega)) \in H$. If $\{x_n\}$ is a finite set, there exists a point x_∞ such that $x_n = x_\infty$ for infinitely many n. Then $(x_\infty,(s,\omega)) \in H_n \subset H_m$ for infinitely many n. This is true for each m, so $(x_\infty,(s,\omega)) \in H$. We have thus shown $\cap_n \rho^X(H_n) \subset \rho^X(H)$. The other inclusion is easier and therefore $\cap_n \rho^X(H_n) = \rho^X(H)$. Since $\rho^X(H_n) \in \mathcal{K}_\delta(t)$, then $\rho^X(H) \in \mathcal{K}_\delta(t)$. We also observe that for fixed ω, $\{(x,s) : (x,(s,\omega)) \in H\}$ is compact.

Now suppose $A \in \mathcal{B}[0,t] \times \mathcal{F}_t$. Then by Lemma 2.5 there exists a compact Hausdorff space X and $B \in \mathcal{L}_{\sigma\delta}(X)$ such that $A = \rho^X(B)$. We can write $B = \cap_n B_n$ and $B_n = \cup_m B_{nm}$ with $B_n \downarrow B$, $B_{nm} \uparrow B_n$, and $B_{nm} \in \mathcal{L}(X)$.

Let $a = \mathbb{P}^*(\pi(A)) = \mathbb{P}^*(\pi \circ \rho^X(B))$ and let $\varepsilon > 0$. By Lemma 2.4,

$$\mathbb{P}^*(\pi \circ \rho^X(B \cap B_{1m})) \uparrow \mathbb{P}^*(\pi \circ \rho^X(B \cap B_1)) = \mathbb{P}^*(\pi \circ \rho^X(B)) = a.$$

Take m large enough so that $\mathbb{P}^*(\pi \circ \rho^X(B \cap B_{1m})) > a - \varepsilon$, let $C_1 = B_{1m}$, and $D_1 = B \cap C_1$.

We proceed by induction. Suppose we are given sets $C_1, \ldots, C_{n-1}$ and sets $D_1, \ldots, D_{n-1}$ with $D_{n-1} = B \cap (\cap_{i=1}^{n-1} C_i)$, $\mathbb{P}^*(\pi \circ \rho^X(D_{n-1})) > a - \varepsilon$, and each $C_i = B_{im_i}$ for some m_i. Since $D_{n-1} \subset B \subset B_n$, by Lemma 2.4

$$\mathbb{P}^*(\pi \circ \rho^X(D_{n-1} \cap B_{nm})) \uparrow \mathbb{P}^*(\pi \circ \rho^X(D_{n-1} \cap B_n)) = \mathbb{P}^*(\pi \circ \rho^X(D_{n-1})).$$

We can take m large enough so that $\mathbb{P}^*(\pi \circ \rho^X(D_{n-1} \cap B_{nm})) > a - \varepsilon$, let $C_n = B_{nm}$, and $D_n = D_{n-1} \cap C_n$.

If we let $G_n = C_1 \cap \cdots \cap C_n$ and $G = \cap_n G_n = \cap_n C_n$, then each G_n is in $\mathcal{L}(X)$, hence $G \in \mathcal{L}(X)$. Since $C_n \subset B_n$, then $G \subset \cap_n B_n = B$. Each $G_n \in \mathcal{L}(X)$ and so by the first paragraph of this proof, for each fixed ω and n, $\{(x, s) : (x, (s, \omega)) \in G_n\}$ is compact. Hence by a proof very similar to that of Lemma 2.2, $\pi \circ \rho^X(G_n) \downarrow \pi \circ \rho^X(G)$. Using the first paragraph of this proof and Lemma 2.2, we see that

$$\mathbb{P}(\pi \circ \rho^X(G)) = \lim \mathbb{P}(\pi \circ \rho^X(G_n)) \geq \lim \mathbb{P}^*(\pi \circ \rho^X(D_n)) \geq a - \varepsilon.$$

Using the first paragraph of this proof once again, we see that A is t-approximable. $\square$

Proof of Theorem 2.1. Let E be a progressively measurable set and let $A_u = E \cap ([0, u] \times \Omega)$. By Lemma 2.6, A_u is u-approximable. By Proposition 2.3, $\pi(A_u) \in \mathcal{F}_u$. Now fix t. If $\omega \in (D_E \leq t)$, we see that $\omega \in \pi(A_u)$ for all $u > t$. Conversely, if $\omega \in \pi(A_u)$ for all $u > t$, note $\omega \in (D_E \leq t)$. If $u_1 < u_2$, then $A_{u_1} \subset A_{u_2}$ and hence $\pi(A_{u_1}) \subset \pi(A_{u_2})$. Therefore

$$(D_E \leq t) = \cap_{u>t} \pi(A_u) \in \cap_{u>t} \mathcal{F}_u = \mathcal{F}_t.$$

Because t was arbitrary, we conclude D_E is a stopping time. $\square$

If B is a Borel subset of a topological space $\mathcal{S}$, let

$$U_B = \inf\{t \geq 0 : X_t \in B\}$$

and

$$T_B = \inf\{t > 0 : X_t \in B\},$$

the first entry time and first hitting time of B, resp.

Here is the measurability theorem.

Theorem 2.7 *If X is a progressively measurable process taking values in $\mathcal{S}$ and B is a Borel subset of $\mathcal{S}$, then U_B and T_B are stopping times.*

Proof. Since B is a Borel subset of $\mathcal{S}$ and X is progressively measurable, then $1_B(X_t)$ is also progressively measurable. U_B is then the debut of the set $E = \{(s,\omega) : 1_B(X_s(\omega)) = 1\}$, and therefore is a stopping time.

If we let $Y_t^\delta = X_{t+\delta}$ and $U_B^\delta = \inf\{t \geq 0 : Y_t^\delta \in B\}$, then by the above, U_B^δ is a stopping time with respect to the filtration $\{\mathcal{F}_t^\delta\}$, where $\mathcal{F}_t^\delta = \mathcal{F}_{t+\delta}$. It follows that $\delta + U_B^\delta$ is a stopping time with respect to the filtration $\{\mathcal{F}_t\}$. Since $(1/m) + U_B^{1/m} \downarrow T_B$, then T_B is a stopping time with respect to $\{\mathcal{F}_t\}$ as well. $\square$

We remark that in the theory of Markov processes, the notion of completion of a σ-field is a bit different. In that case, we suppose that $\mathcal{F}_t$ contains all sets N such that $\mathbb{P}^\mu(N) = 0$ for every starting measure μ. The proof in Proposition 2.3 shows that

$$(\mathbb{P}^\mu)^*(\pi(A) \setminus \pi(B)) = 0$$

for every starting measure μ, so $\pi(A) \setminus \pi(B)$ is a $\mathbb{P}^\mu$-null set for every starting measure μ. Therefore $\pi(A) = \pi(B) \cup (\pi(A) \setminus \pi(B)) \in \mathcal{F}_t$. With this modification, the rest of the proof of Theorem 2.1 goes through in the Markov process context.

3 The section theorems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions. The optional σ-field $\mathcal{O}$ is the σ-field of subsets of $[0,\infty) \times \Omega$ generated by the set of maps $X : [0,\infty) \times \Omega \to \mathbb{R}$ where X is bounded, adapted to the filtration $\{\mathcal{F}_t\}$, and has right continuous paths. The predictable σ-field $\mathcal{P}$ is the σ-field of subsets of $[0,\infty) \times \Omega$ generated by the set of maps $X : [0,\infty) \times \Omega \to \mathbb{R}$ where X is bounded, adapted to the filtration $\{\mathcal{F}_t\}$, and has left continuous paths.

Given a stopping time T, we define $[T,T] = \{(t,\omega) : t = T(\omega) < \infty\}$. A stopping time is predictable if there exist stopping times $T_1, T_2, \ldots$ with $T_1 \leq T_2 \leq \cdots$, $T_n \uparrow T$, and on the event $(T > 0)$, $T_n < T$ for all n. We say the stopping times T_n predict T. If T is a predictable stopping time and $S = T$ a.s., we also call S a predictable stopping time.

The optional section theorem is the following.

Theorem 3.1 *If E is an optional set and $\varepsilon > 0$, there exists a stopping time T such that $[T,T] \subset E$ and $\mathbb{P}(\pi(E)) \le \mathbb{P}(T < \infty) + \varepsilon$.*

The statement of the predictable section theorem is very similar.

Theorem 3.2 *If E is a predictable set and $\varepsilon > 0$, there exists a predictable stopping time T such that $[T,T] \subset E$ and $\mathbb{P}(\pi(E)) \le \mathbb{P}(T < \infty) + \varepsilon$.*

First we prove the following lemma.

Lemma 3.3 *(1) $\mathcal{O}$ is generated by the collection of processes $1_C(\omega)1_{[a,b)}(t)$ where $C \in \mathcal{F}_a$.*

(2) $\mathcal{P}$ is generated by the collection of processes $1_C(\omega)1_{[b,c)}(t)$ where $C \in \mathcal{F}_a$ and $a < b < c$.

Proof. (1) First of all, $1_C(\omega)1_{[a,b)}(t)$ is a bounded right continuous adapted process, so it is optional.

Let $\mathcal{O}'$ be the σ-field on $[0,\infty)\times\Omega$ generated by the collection of processes $1_C(\omega)1_{[a,b)}(t)$, where $C \in \mathcal{F}_a$. Letting $b \to \infty$, $\mathcal{O}'$ includes sets of the form $[a,\infty) \times C$ with $C \in \mathcal{F}_a$.

Let X_t be a right continuous, bounded, and adapted process and let $\varepsilon > 0$. Let $U_0 = 0$ and define $U_{i+1} = \inf\{t > U_i : |X_t - X_{U_i}| > \varepsilon\}$. Since $(U_1 < t) = \cup(|X_q - X_0| > \varepsilon)$, where the union is over all rational q less than t, U_1 is a stopping time, and an analogous argument shows that each U_i is also a stopping time. If S and T are stopping times, let $1_{[S,T)} = \{(t,\omega) \in [0,\infty)\times\Omega : S(\omega \le t < T(\omega)\}$. If we set

$$X_t^\varepsilon(\omega) = \sum_{i=0}^\infty X_{U_i}(\omega)1_{[U_i,U_{i+1})}(t),$$

then $\sup_{t\ge 0}|X_t - X_t^\varepsilon| \le \varepsilon$. Therefore we can approximate X by processes of the form

$$\sum_{i=0}^\infty X_{U_i}1_{[U_i,\infty)} - \sum_{i=0}^\infty X_{U_i}1_{[U_{i+1},\infty)}.$$

It therefore suffices to show that if V is a stopping time and $A \in \mathcal{F}_V$, then $1_A(\omega)1_{[V,\infty)}(t)$ is $\mathcal{O}'$ measurable.

Letting $V_n = (k+1)/2^n$ when $k/2^n \le V < (k+1)/2^n$,

$$\begin{aligned} 1_A(\omega)1_{[V(\omega),\infty)}(t) &= \lim_{n\to\infty} 1_A(\omega)1_{[V_n(\omega),\infty)}(t) \\ &= \lim_{n\to\infty} \sum_{k=0}^{\infty} 1_{A\cap(V_n=(k+1)/2^n)}1_{[(k+1)/2^n,\infty)}(t), \end{aligned}$$

which is $\mathcal{O}'$ measruable.

(2) As long as $a + (1/n) < b$, the processes $1_C(\omega)1_{(b-(1/n),c-(1/n)]}(t)$ are left continuous, bounded, and adapted, hence predictable. The process $1_C(\omega)1_{[b,c)}(t)$ is the limit of these processes as $n \to \infty$, so is predictable. On the other hand, if X_t is a bounded adapted left continuous process, it can be approximated by

$$\sum_{k=1}^{\infty} X_{(k-1)/2^n}(\omega)1_{(k/2^n,(k+1)/2^n]}(t).$$

Each summand can be approximated by linear combinations of processes of the form $1_C(\omega)1_{(b,c]}(t)$, where $C \in \mathcal{F}_a$ and $a < b < c$. Finally, $1_C 1_{(b,c]}$ is the limit of $1_C(\omega)1_{[b+(1/n),c+(1/n))}(t)$ as $n \to \infty$. $\square$

A consequence of this lemma is that $\mathcal{P} \subset \mathcal{O}$. Since $\mathcal{O}$ is generated by the class of right continuous processes and right continuous processes are progressively measurable, we have from Theorem 2.1 that the debut of a predictable or optional set is a stopping time.

Fix t and define

$$\mathcal{O}(t) = \{A \cap ([0,t] \times \Omega) : A \in \mathcal{O}\}.$$

Let $\overline{\mathcal{K}}^0(t)$ be the collection of subsets of $\mathcal{O}(t)$ of the form $K \times C$, where K is a compact subset of $[0,t]$ and $C \in \mathcal{F}_a$ with $a \le \inf\{s : s \in K\}$. Let $\overline{\mathcal{K}}(t)$ be the collection of finite unions of sets in $\overline{\mathcal{K}}^0(t)$ and $\overline{\mathcal{K}}_\delta(t)$ the collection of countable intersections of sets in $\overline{\mathcal{K}}(t)$. Define $\overline{\mathcal{I}}^0(t)$ to be the collection of sets of the form $[a,b) \times C$, where $a < b \le t$ and $C \in \mathcal{F}_a$, and let $\overline{\mathcal{I}}(t)$ be the collection of finite unions of sets in $\overline{\mathcal{I}}^0(t)$.

The proof of the following proposition is almost identical to the proof of Theorem 2.1. Because the debut of optional sets is now known to be a stopping time, it is not nececessary to work with $\mathbb{P}^*$.

Proposition 3.4 *Suppose* $A \in \mathcal{O}(t)$. *Then given* $\varepsilon > 0$, *there exists* $B \in \overline{\mathcal{K}}_\delta(t)$ *such that* $\mathbb{P}(\pi(A) \setminus \pi(B)) < \varepsilon$.

We now prove Theorem 3.1.

Proof of Theorem 3.1. If E is an optional set, choose t large enough so that if $A_t = E \cap ([0,t] \times \Omega)$, then $\mathbb{P}(\pi(A_t)) > \mathbb{P}(\pi(E)) - \varepsilon/2$. This is possible because $A_t \uparrow E$ and so $\pi(A_t) \uparrow \pi(E)$. With this value of t, choose $B \in \overline{\mathcal{K}}_\delta(t)$ such that $B \subset A_t$ and $\mathbb{P}(\pi(B)) > \mathbb{P}(\pi(A_t)) - \varepsilon/2$. We will show $[D_B, D_B] \subset B$. Since $(D_B < \infty) = \pi([D_B, D_B]) = \pi(B)$, we have $[D_B, D_B] \subset E$ and

$$\mathbb{P}(\pi(E)) < \mathbb{P}(\pi(A_t)) + \varepsilon/2 < \mathbb{P}(\pi(B)) + \varepsilon = \mathbb{P}(\pi([D_B, D_B])) + \varepsilon.$$

By the argument of the proof of Lemma 2.2, $S(B)(\omega)$ is a compact set if $B \in \overline{\mathcal{K}}_\delta(t)$. Therefore $D_B(\omega) = \inf\{s : s \in S(B)(\omega)\}$ is in $S(B)(\omega)$, which implies $[D_B, D_B] \subset B$. □

To prove Theorem 3.2 we follow along the same lines. Define

$$\mathcal{P}(t) = \{A \cap ([0,t] \times \Omega) : A \in \mathcal{P}\}$$

and define $\widetilde{\mathcal{K}}^0(t)$ to be the collection of subsets of $\mathcal{P}(t)$ of the form $K \times C$, where K is a compact subset of $[0,t]$ and $C \in \mathcal{F}_a$ with $a < \inf\{s : s \in K\}$, let $\widetilde{\mathcal{K}}(t)$ be the collection of finite unions of sets in $\widetilde{\mathcal{K}}^0(t)$, and $\widetilde{\mathcal{K}}_\delta(t)$ the collection of countable intersections of sets in $\widetilde{\mathcal{K}}(t)$. Define $\widetilde{\mathcal{I}}^0(t)$ to be the collection of sets of the form $[b,c) \times C$, where $C \in \mathcal{F}_a$ and $a < b < c \le t$, and let $\widetilde{\mathcal{I}}(t)$ be the collection of finite unions of sets in $\widetilde{\mathcal{I}}^0(t)$. Following the proof of Theorem 3.1, we will be done once we show D_B is a predictable stopping time when $B \in \widetilde{\mathcal{K}}_\delta(t)$.

Proof of Theorem 3.2. Fix t. Suppose $B \in \widetilde{\mathcal{K}}^0(t)$ is of the form $B = K \times C$ with $C \in \mathcal{F}_a$ and $a < b = \inf\{s : s \in K\}$. Note that this implies $b > 0$. Then D_B equals b if $\omega \in C$ and equals infinity otherwise. As long as $a + (1/m) < b$, we see that D_A is predicted by the stopping times V_m, where V_m equals $b-(1/m)$ if $\omega \in C$ and equals m otherwise. Note also that $[D_B, D_B] \subset B$. If $B = \cup_{i=1}^m B_i$ with $B_i \in \widetilde{\mathcal{K}}^0(t)$, then $D_B = D_{B_1} \wedge \cdots \wedge D_{B_m}$, and it is easy to see that D_B is predictable because each D_{B_i} is, and also that $[D_B, D_B] \subset B$.

Now let $B \in \widetilde{\mathcal{K}}_\delta(t)$ with $B_n \downarrow B$ and $B_n \in \widetilde{\mathcal{K}}(t)$. We have $D_{B_n} \uparrow$, and the limit, which we call T, will be a stopping time. Since $B \subset B_n$, then $D_{B_n} \le D_B$, and therefore $T \le D_B$. Each D_{B_n} is a predictable stopping time. Let R_{nm} be stopping times predicting D_{B_n} and choose m_n large so that

$$\mathbb{P}(R_{nm_n} + 2^{-n} < D_{B_n} < \infty) < 2^{-n}, \qquad \mathbb{P}(R_{nm_n} < n, D_{B_n} = \infty) < 2^{-n}.$$

By the Borel-Cantelli lemma,

$$\mathbb{P}(\sup_n R_{nm_n} < T < \infty) = 0 \qquad \text{and} \qquad \mathbb{P}(\sup_n R_{nm_n} < T = \infty) = 0,$$

so if we set $Q_n = n \wedge (R_{1m_1} \vee \cdots \vee R_{nm_n})$, we see that $\{Q_n\}$ is a sequence of stopping times predicting T, except for a set of probability zero. Hence T is a predictable stopping time.

If $n > m$, then $[D_{B_n}, D_{B_n}] \subset B_n \subset B_m$. Since $S(B_m)(\omega)$ is a closed subset of t, the facts that $D_{B_n}(\omega) \in S(B_m)(\omega)$ for $n > m$ and $D_{B_n}(\omega) \to T(\omega)$ for each ω shows that $T(\omega) \in S(B_m)(\omega)$ for each ω. Thus $[T, T] \subset B_m$. This is true for all m, so $[T, T] \subset B$. In particular, $T \ge D_B$, so $T = D_B$. Therefore $\pi(B) = (D_B < \infty) = \pi([T, T])$.

This and the argument of the first paragraph of the proof of Theorem 3.1 proves Theorem 3.2. □

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