Hölder continuity of harmonic functions with respect to operators of variable order

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Abstract

We consider a class of integro-differential operators and their corresponding harmonic functions. Under mild assumptions on the family of jump measures we prove *a priori* estimates and establish Hölder continuity of bounded functions that are harmonic in a domain.

Keywords: Markov jump processes, harmonic functions, martingale problem, continuity estimates, regularity.

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1 Introduction

Continuity estimates for harmonic functions have a long history, both in analysis and probability theory. For second order elliptic partial differential operators in divergence form and under the assumption that the coeffcients are bounded they were proved in two dimensions by Morrey [Mor38] and in the higher dimensional case independently by DeGiorgi [DG57] and Nash [Nas58]. Another proof was given later by Moser [Mos61] and several extensions were made by Stampacchia [Sta66], Ladyzhenskaya/Uraltseva [LU68] and others. The corresponding result for equations in non-divergence form was obtained by Krylov/Safonov [KS79], see also Trudinger [Tru80].

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The aim of this work is to derive analogous results for a class of integro-differential operators. For a family of measures $n(x, dh)$ we study the operator $\mathcal{L}$ defined by

$$\mathcal{L}f(x) = \int\limits_{\mathbb{R}^d \setminus \{0\}} [f(x+h) - f(x) - 1_{(|h|\leq 1)} h \cdot \nabla f(x)]\, n(x, dh)\,. \qquad (1.1)$$

We prove continuity estimates under very mild assumptions on the jump measure $n(x, dh)$. We do not require the jump measure to have a density with respect to the Lebesgue measure; the assumptions are formulated in terms of the measure itself. More importantly, the corresponding integral operator is allowed to be of variable order.

In the above mentioned works the presence of a non-degenerate diffusion matrix is crucial. Perturbations of strongly elliptic operators by nonlocal operators can also be dealt with as shown in [MP88], [HY97] and [Kas03]. Here, we deal with purely nonlocal operators. Our method of proof is stochastic and is based on estimates for hitting probabilities of certain sets. Since the underlying stochastic process is a jump process and not a diffusion our techniques differ significantly from the ones in [KS79]. In addition, we allow the operator to be of variable order. Let us mention that so far, there seem to be no successful analytic approach leading to similar results for harmonic functions of purely nonlocal operators.

In [BL02a] the operator $\mathcal{L}$ is studied under the assumption $n(x, dh) = \bar{n}(x, h)\, dh$ and $\bar{n}(x, h)$ satisfies $c_1|h|^{-d-\alpha} \leq \bar{n}(x, h) \leq c_2|h|^{-d-\alpha}$, where $\alpha \in (0, 2)$ and c_1, c_2 are positive constants. Under these assumptions a Harnack inequality and Hölder continuity for harmonic functions are established in [BL02a]. For a study of symmetric jump processes under similar assumptions see [BL02b] and [CK03]. Some extensions have been made in [SV04]. Our results generalize these results considerably.

Under rather strong smoothness assumptions on the measure $n(x, dh)$ generators of jump processes that are anisotropic or of variable order have been studied in [JL93, Hoh94, Hoh00, Kol00]. For one spatial dimension and the symmetric case there are recent related results in [Uem02]. In [BK04] it was shown that the variable order case does not exclude pointwise estimates of harmonic functions. Under different assumptions from the ones in this paper, a scale-dependent Harnack inequality was proved.

The operator $\mathcal{L}$ is a natural object. Formally, one has $\mathcal{L} = -(-\Delta^{\alpha/2})$ for

$n(x,dh) = c|h|^{-n-\alpha}dh$, $\alpha \in (0,2)$ a constant and $\mathcal{L} = -(-\Delta^{\alpha(x)/2})$ for $n(x,dh) = c|h|^{-n-\alpha(x)}dh$ when $\alpha : \mathbb{R}^n \to (0,2)$ is a function. On the other hand, $\mathcal{L}$ generates a Markov process when certain conditions on $n(x,dh)$ are satisfied. One important application of continuity estimates for harmonic functions is that it allows one to prove existence of strong Markov solutions to a martingale problem when the coefficients need not be continuous.

2 Notation and Main Result

Let $B(x,r)$ be the open ball in $\mathbb{R}^d$ centered at x with radius r. The letter c with subscripts will denote positive finite constants whose value is unimportant and which may change from place to place. For a Borel set A, we denote the Lebesgue measure of A by $|A|$, and write

$$T_A = \inf\{t > 0 : X_t \in A\}, \qquad \tau_A = \inf\{t > 0 : X_t \notin A\}$$

for the first entrance time and first exit time of A, resp. For a process whose paths are right continuous with left limits, we write $X_{t-} = \lim_{s\uparrow t, s<t} X_s$ and $\Delta X_t = X_t - X_{t-}$.

Let $n(x,dh)$ be a family of measures satisfying $\sup_x \int (|x|^2 \wedge 1)\, n(x,dh) < \infty$. We consider the operator $\mathcal{L}$ defined on $C^2(\mathbb{R}^d)$ by (1.1). Our method is probabilistic and we need to define the stochastic process associated to $\mathcal{L}$. We say a probability measure $\mathbb{P}^{x_0}$ is a solution to the martingale problem associated with $\mathcal{L}$ started at x_0 if $\mathbb{P}(X_0 = x_0) = 1$ and $f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$ is a $\mathbb{P}$-martingale whenever $f \in C^2$ with bounded first and second partial derivatives. We suppose that $(X_t, \mathbb{P}^x)$ is a strong Markov process such that for each x, $\mathbb{P}^x$ is a solution to the martingale problem for $\mathcal{L}$ started at x.

We remark that there is still no satisfactory answer to the problem of when there is a unique solution to the martingale problem for $\mathcal{L}$ for a given starting point. See [Bas04] for a survey on the current status of this problem, or see [Str75, Kom84b, Kom84a, Bas88, Tsu92, JL93, Neg94, Kom96, MP90, Hoh94, Hoh95, Hoh00, Kol02]. If $n(x,dh)$ is sufficiently smooth in x, then uniqueness holds: see [JL93, Hoh00]. Since our results do not depend on any smoothness of n other than Assumption 2.1, one may first assume n is smooth and then use a limit procedure to obtain results for nonsmooth n.

A function u is harmonic in a domain D with respect to $\mathcal{L}$ if $u(X_{t\wedge\tau_D})$ is a martingale. It is easy to see that if u is bounded and is in C^2 with bounded first and second partial derivatives and $\mathcal{L}u(x) = 0$ for $x \in D$, then h is harmonic in D.

We define

$$S(x,r) = \int_{|h|\geq r} n(x,dh),$$
$$L(x,r) = S(x,r) + r^{-1}\Big|\int_{1\geq|h|\geq r} hn(x,dh)\Big| + r^{-2}\int_{|h|<r} |h|^2 n(x,dh),$$
$$N(x,r) = \inf\{n(x,A-x) : A \subset B(x,2r), |A| \geq \frac{1}{3\cdot 2^d}|B(x,r)|\}.$$

An upper bound on L controls the number of jumps of various sizes, while a lower bound on N prevents there being too few jumps.

Assumption 2.1 *Suppose*

$$\sup_x L(x,1) < \infty.$$

Suppose also the following.

- *(a) There exist $\kappa_1 > 0$ and $\sigma > 0$ such that*

$$\frac{S(x,\lambda r)}{S(x,r)} \leq \kappa_1 \lambda^{-\sigma}, \qquad x \in \mathbb{R}^d, 1 < \lambda < 1/r,\ r < 1. \tag{2.1}$$

- *(b) There exists $\kappa_2 > 0$ such that if $x \in \mathbb{R}^d$, $r < 1$, $r/2 \leq s \leq 2r$, and $|x-y| \leq 2r$, then*

$$N(x,r) \geq \kappa_2 L(y,s). \tag{2.2}$$

Assumption 2.1(a) ensures that the behavior of the measure around the singularity $h = 0$ is uniform in x. Assumption 2.1(b) essentially says that the number of jumps at one point is not too much larger than the number of jumps at a nearby point and also imposes some symmetry.

The following is our main result:

Theorem 2.2 *Suppose Assumption 2.1 holds. Suppose u is bounded in $\mathbb{R}^d$ and harmonic in $B(z_0, R)$ with respect to $\mathcal{L}$. Then there exist $\nu \in (0,1), C > 0$ depending only on κ_1, κ_2 such that*

$$|u(x) - u(y)| \le C\|u\|_\infty \Big(\frac{|x-y|}{R}\Big)^\nu, \qquad x, y \in B(z_0, R/2). \tag{2.3}$$

If instead of Assumption 2.1(a) we have that there exists $\kappa_3 > 0$ and $\gamma > 1$ such that

$$\frac{S(x, \lambda r)}{S(x,r)} \le \kappa_3 (\log \lambda)^{-\gamma}, \qquad x \in \mathbb{R}^d, 1 < \lambda < 1/r, r < 1, \tag{2.4}$$

we then have the following corollary.

Corollary 2.3 *Suppose Assumption 2.1 holds but with (2.4) instead of Assumption 2.1(a). (2.4) and Suppose u is bounded in $\mathbb{R}^d$ and harmonic in $B(z_0, R)$ with respect to $\mathcal{L}$. Then u is continuous in $B(z_0, R)$. Moreover, the modulus of continuity depends only on $\|u\|_\infty, \kappa_2, \kappa_3$, and γ.*

Let us give some examples that illustrate our assumptions.

Example 1: Suppose $n(x, dh) = \overline{n}(x,h)dh$ and there exist constants c_1, c_2, c_3 and $\alpha \in (0,2)$ such that

$$c_1 |h|^{-d-\alpha} \le \bar{n}(x,h) \le c_2 |h|^{-d-\alpha} \ \text{ for } 0 < |h| \le 2 \quad \text{and} \int\limits_{|h|>1} \bar{n}(x,h)\, dh \le c_3 \, .$$

Note that the function $\bar{n}(x,h)$ is assumed to be only measurable in x. We have $S(x,r) = c_4 r^{-\alpha}$, so Assumption 2.1(a) holds with $\sigma = \alpha$. We have $L(x,r) \le c_5 r^{-\alpha}$ and $N(x,r) \ge c_6 r^{-\alpha}$, hence Assumption 2.1(b) is satisfied.

Example 2: Suppose $n(x,dh) = \overline{n}(x,h)dh$ and there exist a continuous function $\alpha : \mathbb{R}^d \to (0,2)$ and constants c_1, c_2, and c_3 such that $0 < \inf \alpha(x) \le \sup \alpha(x) < 2$ and

(a) $\overline{n}$ satisfies

$$\frac{c_1}{|h|^{d+\alpha(x)}} \le \overline{n}(x,h) \le \frac{c_2}{|h|^{d+\alpha(x)}}, \qquad x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\}.$$

(b) For each x, either $\alpha(x) > 1$ or $n(x,h) = n(x,-h)$ for all h.

(c) For $|x-y| < 1/2$, α satisfies

$$|\alpha(x) - \alpha(y)| \le \frac{c_3}{\log(1/|x-y|)}. \tag{2.5}$$

We check that Example 2 satisfies Assumption 2.1 (a) and (b). We have $S(x,r) = c_4 r^{-\alpha(x)}$, so Assumption 2.1(a) holds with $\sigma = \inf \alpha(x)$. We have $L(x,r) \le c_5 r^{-\alpha(x)}$, and a short calculation shows that $N(x,r) \ge c_6 r^{-\alpha(x)}$. So with x, y, r, and s as in Assumption 2.1(b),

$$\frac{N(x,r)}{L(y,s)} \ge c_7 r^{\alpha(y)-\alpha(x)} = c_7 e^{(\alpha(y)-\alpha(x))\log r} \ge c_8$$

if $r \le 1/4$ by (2.5), and Assumption 2.1(b) is easy to check for $r \in (1/4, 1]$.

Remarks:
(1) The assumption (d) in Example 2 is even weaker than Dini continuity, and in particular holds when $\alpha(x)$ is Hölder continuous. Note that it is not known whether the martingale problem has a unique solution under the assumptions of Example 2. That is why we assume our stochastic process to be strong Markov.

(2) It is interesting to compare the conditions in Theorem 2.2 and Example 2 with the hypotheses for the Harnack inequality in [BK04]. Here much less is assumed about the structure of $n(x,dh)$. However, as assumption (d) in Example 2 shows, some continuity in x is required. In [BK04], no continuity in x appeared.

(3) We wanted to point out that there are two distinct definitions of *stable-like processes* in the literature. One, introduced in [Bas88], is that $n(x,dh) = \overline{n}(x,h)$ and that condition (b) in Example 2 holds with $c_1 = c_2$. The other definition, see, e.g., [CK03], is that condition (b) in Example 2 holds with $\alpha(x)$ identically constant.

3 Estimates

Define

$$\overline{L}(x_0,r) = \sup_{x\in B(x_0,r)} L(x,r), \qquad \underline{N}(x_0,r) = \inf_{x\in B(x_0,r)} N(x,r).$$

Proposition 3.1 *There exists c_1 such that*

$$\mathbb{P}^{x_0}(\tau_{B(x_0,r)} < t) \le c_1 t \overline{L}(x_0, r)\ .$$

Proof: Let u be a $C^2(\mathbb{R}^d)$ function with bounded first and second partial derivatives such that $u(x) = |x - x_0|^2/r^2$ for $|x - x_0| \le r$, u is greater than 1 if $|x - x_0| > r$, u is bounded by c_2, $|\nabla u|$ is bounded by c_3/r, and the second partial derivatives are bounded by c_4/r^2. Since $\mathbb{P}^{x_0}$ is a solution to the martingale problem for $\mathcal{L}$ started at x_0, then

$$\mathbb{E}^{x_0} u(X_{t\wedge\tau_{B(x_0,r)}}) - u(x_0) = \mathbb{E}^{x_0} \int_0^{t\wedge\tau_{B(x_0,r)}} \mathcal{L}u(X_s) ds.$$

The left hand side is larger than $\mathbb{P}^{x_0}(\tau_{B(x_0,r)} \le t)$, while the right hand side is bounded from above by

$$t \sup_{x\in B(x_0,r)} |\mathcal{L}u(x)| \le (c_2 + c_3 + c_4) t \overline{L}(x_0, r)\ .$$

$\square$

The assertion of the following proposition is the main step in the proof of [KS79]. It is remarkable that a similar result holds true for nonlocal operators of variable order.

Proposition 3.2 *Suppose Assumption 2.1 holds. Suppose $r < 1$, $A \subset B(x_0, r)$, $y \in B(x_0, r/2)$, and $|A|/|B(x_0, r)| \ge 1/3$. There exists κ_4 not depending on x_0, r, or A such that*

$$\mathbb{P}^y(T_A < \tau_{B(x_0,r)}) \ge \kappa_4.$$

Proof: Write τ for $\tau_{B(x_0,r)}$. If $\mathbb{P}^y(T_A < \tau) \ge \frac{1}{4}$ we are done, so we assume not. By Proposition 3.1 we can choose c_2 such that if $t_0 = c_2/\overline{L}(x_0, r)$, then $\mathbb{P}^y(\tau \le t_0) \le \frac{1}{2}$. If $x \in B(x_0, r)$, then $n(x, A - x) \ge \underline{N}(x_0, r)$. We have the identity

$$\mathbb{E}^y \sum_{s\le S} 1_{(X_{s-}\in B, X_s\in C)} = \mathbb{E}^y \int_0^S 1_B(X_s) n(X_s, C - X_s) ds \qquad (3.1)$$

whenever B and C are disjoint Borel sets and S is a bounded stopping time; this is called the Lévy system identity and may be proved in a fashion almost identical to the proof of Proposition 2.3 in [BL02a]. We then have

$$\begin{aligned}\mathbb{P}^y(T_A < \tau) &\geq \mathbb{E}^y \sum_{s \leq T_A \wedge \tau \wedge t_0} 1_{(X_{s-} \neq X_s, X_s \in A)} \\ &= \mathbb{E}^y \int_0^{T_A \wedge \tau \wedge t_0} n(X_s, A - X_s) ds \\ &\geq \underline{N}(x_0, r) \mathbb{E}^y (T_A \wedge \tau \wedge t_0).\end{aligned}$$

Since

$$\begin{aligned}\mathbb{E}^y(T_A \wedge \tau \wedge t_0) &\geq \mathbb{E}^y(t_0; T_A \geq \tau \geq t_0) = t_0 \mathbb{P}^y(T_A \geq \tau \geq t_0) \\ &\geq t_0[1 - \mathbb{P}^y(T_A < \tau) - \mathbb{P}^y(\tau < t_0)] \geq t_0/4,\end{aligned}$$

then we obtain

$$\mathbb{P}^y(T_A < \tau) \geq c_3 \underline{N}(x_0, r)/\overline{L}(x_0, r),$$

and by Assumption 2.1(b), the right hand side is greater than $c_3 \kappa_2$. □

Proposition 3.3 *Let Q be a Borel set. For $r \leq 1$ define $U := \inf\{t : |\Delta X_t| \geq r\}$. Suppose that Assumption 2.1 holds. Then if $1 < \lambda < 1/r$,*

$$\mathbb{P}^x(|\Delta X_{U \wedge \tau_Q}| \geq \lambda r) \leq \kappa_1 \lambda^{-\sigma} . \tag{3.2}$$

Proof: Using the Lévy system identity (3.1) and Assumption 2.1(a) we have

$$\begin{aligned}\mathbb{P}^x(|\Delta X_{U \wedge \tau_Q \wedge t}| \geq \lambda r) &= \mathbb{E}^x \sum_{s \leq U \wedge \tau_Q \wedge t} 1_{(|\Delta X_s| \geq \lambda r)} = \mathbb{E}^x \int_0^{U \wedge \tau_Q \wedge t} \int_{|h| \geq \lambda r} n(X_s, dh)\, ds \\ &= \mathbb{E}^x \int_0^{U \wedge \tau_Q \wedge t} S(X_s, \lambda r) ds \leq \kappa_1 \lambda^{-\sigma} \mathbb{E}^x \int_0^{U \wedge \tau_Q \wedge t} S(X_s, r) ds \\ &= \kappa_1 \lambda^{-\sigma} \mathbb{E}^x \int_0^{U \wedge \tau_Q \wedge t} \int_{|h| \geq r} n(X_s, dh)\, ds \\ &= \kappa_1 \lambda^{-\sigma} \mathbb{E}^x \sum_{s \leq U \wedge \tau_Q \wedge t} 1_{(|\Delta X_s| > r)} = \kappa_1 \lambda^{-\sigma} \mathbb{P}^x(|\Delta X_{U \wedge \tau_Q \wedge t}| \geq r) \\ &\leq \kappa_1 \lambda^{-\sigma}.\end{aligned}$$

Now let $t \to \infty$ and use dominated convergence. □

4 Proof of Theorem 2.2

Proof of Theorem 2.2: Let us suppose u is bounded by K in $\mathbb{R}^d$ and $z_1 \in B(z_0, R/2)$. Set

$$r_n = \theta_2 4^{-n},$$

where θ_2 is chosen small enough that $B(z_1, 2r_1) \subset B(z_0, R/2)$. Write $B_n = B(z_1, r_n)$, $\tau_n = \tau_{B_n}$, and set

$$M_n = \sup_{x \in B_n} u(x), \qquad m_n = \inf_{x \in B_n} u(x).$$

Let $a > 1$, $n_0 \in \mathbb{N}$, and $\theta_1 > 2K$ be constants to be chosen later. Let

$$s_n = \theta_1 a^{-n}.$$

We will prove by induction that $M_n - m_n \le s_n$ for all n. It is standard that the Hölder continuity of u at z_1 follows from this; cf. [Mos61].

Suppose we have the induction hypothesis for $1, 2, \ldots, n$, with $n \ge n_0$. We will prove the induction hypothesis for $n + 1$.

Let $\varepsilon > 0$ and pick $y, z \in B_{n+1}$ such that $u(y) \le m_{n+1} + \varepsilon$ and $u(z) \ge M_{n+1} - \varepsilon$. We wish to show

$$u(z) - u(y) \le s_{n+1}. \tag{4.1}$$

If we show (4.1), then since $\varepsilon > 0$ is arbitrary, we have $M_{n+1} - m_{n+1} \le s_{n+1}$ as desired.

Let $A_n = \{x \in B_n : u(x) \le (M_n + m_n)/2\}$. Let us assume that $|A_n|/|B_n| \ge \frac{1}{2}$; if not, we can look at the function $K - u$ instead. Let D be a compact subset of A_n with $|D|/|B_n| > \frac{1}{3}$. We write

$$\begin{aligned} u(z) - u(y) = \mathbb{E}^z&[u(X_{\tau_n \wedge T_D}) - u(y); T_D < \tau_n, X_{\tau_n} \in B_{n-1} \setminus B_n] \\ &+ \mathbb{E}^z[u(X_{\tau_n \wedge T_D}) - u(y); \tau_n > T_D; X_{\tau_n} \in B_{n-1} \setminus B_n] \\ &+ \sum_{i=1}^{n-2} \mathbb{E}^z[u(X_{\tau_n \wedge T_D}) - u(y); X_{\tau_n} \in B_{n-i-1} \setminus B_{n-i}] \\ &+ \mathbb{E}^z[u(X_{\tau_n \wedge T_D}) - u(y); X_{\tau_n} \notin B_1] \\ =:& \, I_1 + I_2 + I_3 + I_4 \, . \end{aligned} \tag{4.2}$$

Let

$$p_n = \mathbb{P}^z(T_D < \tau_n).$$

By Proposition 3.2, there exists c_1 such that

$$p_n \geq c_1, \qquad n = 1, 2, \ldots$$

Next we estimate

$$F_j := \mathbb{P}^z(X_{\tau_n} \notin B_{n-j}).$$

In order for X_{τ_n} not to be in B_{n-j}, the process must have jumped at least $r_{n-j} - r_n$ at time τ_n and the process could not have jumped more than $2r_n$ at any time t strictly less than τ_n. Since $2r_n < r_{n-j} - r_n$, by Proposition 3.3 we have

$$F_j = \mathbb{P}^z(X_{\tau_n} \notin B_{n-j}) \leq \mathbb{P}^z(|\Delta X_{U_n \wedge \tau_{B_n}}| \geq r_{n-j} - r_n) \leq \kappa_1 \Big(\frac{2r_n}{r_{n-j} - r_n} \Big)^\sigma,$$

where $U_n = \inf\{t : |\Delta X_t| \geq 2r_n\}$. Without loss of generality we may assume $\sigma < 1$. Let $c_2 = 2\kappa_1((1 - 4^{-\sigma}) \vee 1)$. Putting in the values for r_n, r_{n-1}, and r_{n-j}, we see

$$F_j \leq \frac{2\kappa_1}{4^{j\sigma} - 1} \leq c_2 4^{-j\sigma}. \tag{4.3}$$

Then

$$I_1 \leq \Big(\frac{M_n + m_n}{2} - m_n \Big) \mathbb{P}^z(T_D < \tau_n) = \frac{M_n - m_n}{2} p_n \leq \tfrac{1}{2} s_n p_n. \tag{4.4}$$

We have

$$I_2 \leq (M_{n-1} - m_{n-1})(1 - p_n) \leq s_{n-1}(1 - p_n). \tag{4.5}$$

Note that

$$I_4 \leq 2K\mathbb{P}^z(X_{\tau_n} \notin B_1) \leq 2\theta_1 F_{n-1} \leq 8\theta_1 c_2 4^{-n\sigma}. \tag{4.6}$$

The most complicated part is estimating I_3. If $X_{\tau_n} \in B_{n-i-1}$, then

$$u(X_{\tau_n}) - u(y) \leq M_{n-i-1} - m_{n-i-1} \leq s_{n-i-1}.$$

Using summation by parts,

$$I_3 \leq \sum_{i=1}^{n-2} s_{n-i-1}(F_i - F_{i-1}) = s_1 F_{n-2} - s_{n-2} F_0 + \sum_{i=1}^{n-3} (s_{n-i-1} - s_{n-i}) F_i.$$

Note that $s_{n-2}F_0 \geq 0$, while $s_{n-i-1} - s_{n-i} = \theta_1(a-1)a^{i-n}$. Also $s_1 F_{n-2} \leq 16\theta_1 c_2 4^{-n\sigma}$. We will require $a \in (1, 4^{\sigma/2})$ so that $a/4^\sigma < 1$ and $1-(a/4^\sigma) > 1 - 4^{-\sigma/2}$. Substituting in (4.2) we have

$$\begin{aligned} u(z) - u(y) &\leq \tfrac{1}{2} s_n p_n + s_{n-1}(1-p_n) + 24\theta_1 c_2 4^{-\sigma n} \\ &\qquad + \theta_1 c_2 (a-1) \sum_{i=1}^{n-2} a^{-n+i} 4^{-i\sigma} \\ &\leq \tfrac{1}{2} s_n p_n + s_{n-1}(1-p_n) + 24\theta_1 c_2 4^{-\sigma n} + \theta_1 c_2 (a-1) a^{-n} (1-4^{-\sigma/2})^{-1} \\ &\leq s_n\left(\tfrac{1}{2} p_n + a(1-p_n) + c_3 4^{-\sigma n/2} + c_4(a-1)\right), \end{aligned} \tag{4.7}$$

where c_3 and c_4 are independent of θ_1. Recall $p_n \geq c_1$. If we take $a \in (1, 4^{\sigma/2})$ sufficiently close to 1, n_0 sufficiently large, and $n \geq n_0$, then the last term of (4.7) will be less than

$$s_n(1 - (p_n/4)) \leq s_n/a = s_{n+1},$$

as required. Once we have chosen a and n_0, we choose $\theta_1 > 2K$ sufficiently large so that $\theta_1 a^{-n_0} \geq 2K$, and hence the induction hypothesis is satisfied for $1, 2, \ldots, n_0$. (2.3) follows from the fact that all constants depend only on K, κ_1, and κ_2. □

Proof of Corollary 2.3: If (2.4) holds instead of Assumption 2.1(a), the same proof as that of (3.2) in Proposition 3.3 shows that

$$\mathbb{P}^x(|\Delta X_{U \wedge \tau_Q}| \geq \lambda r) \leq \kappa_2 (\log \lambda)^{-\gamma}.$$

The proof of the corollary is almost exactly the same as the proof of Theorem 2.2, except that now we take $s_n = \theta_1 n^{-\rho}$ for a suitable $\rho > 0$. We now have

$$F_j \leq c_5 j^{-\sigma}$$

and

$$\begin{aligned} I_3 &\leq 16\theta_1 c_2 4^{-n\sigma} + c_6 \theta_1 \sum_{i=1}^{n-3} (n-i)^{-\rho-1} i^{-\sigma} \\ &\leq 16\theta_1 c_2 4^{-n\sigma} + c_7 \theta_1 n^{-((\rho+1)\wedge\sigma)}. \end{aligned}$$

Using the fact that $\lim_{n\to\infty}(s_n/s_{n+1}) = 1$ we obtain $u(z) - u(y) \leq s_{n+1}$ from the analogue of (4.7), provided we choose ρ, θ_1, and n_0 suitably. □

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