Cutting Brownian paths

Richard F. Bass and Krzysztof Burdzy

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Dedicated to Professor S. James Taylor

CONTENTS

Abstract. Let Z_t be two-dimensional Brownian motion. We say that a straight line $\mathcal{L}$ is a cut line if there exists a time $t \in (0,1)$ such that the trace of $\{Z_s : 0 \leq s < t\}$ lies on one side of $\mathcal{L}$ and the trace of $\{Z_s : t < s < 1\}$ lies on the other side of $\mathcal{L}$. In this paper we prove that with probability one cut lines do not exist. This provides a solution to Problem 8 in Taylor (1986).

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0. Introduction.

Let Z_t denote two-dimensional Brownian motion and for a set $A \subseteq [0, \infty)$, let $Z(A) = \{Z_s : s \in A\}$. If there exists a straight line $\mathcal{L}$ and a time $t \in (0, 1)$ such that $Z_t \in \mathcal{L}$, $Z([0, t))$ lies on one side of $\mathcal{L}$, and $Z((t, 1])$ lies on the other side of $\mathcal{L}$, we say that $\mathcal{L}$ is a cut line for Z_t. Our main purpose is to present a proof of the following result.

Theorem 0.1. *Cut lines for 2-dimensional Brownian motion do not exist.*

This answers Problem 8 in Taylor (1986). Prof. Taylor has informed us that the problem is older than that.

A rigorous version of Theorem 0.1 is stated below as Theorem 0.7.

Taylor's problem has attracted much attention over the years and we will first review some related results. We start with the well known one dimensional counterpart of Theorem 0.1. Let X_t denote one dimensional Brownian motion.

Theorem 0.2. *(Dvoretzky, Erdös and Kakutani (1961)) One dimensional Brownian motion has no points of increase, a.s. In other words, with probability 1, there is no* $t \in (0, 1)$ *such that* $X([0, t)) \cap X((t, 1]) = \emptyset$.

The original proof of Theorem 0.2 is considered difficult, but a number of alternative proofs have been found. It may be instructive to see why all attempts at generalizing the proofs to higher dimensions have failed so far.

Knight (1981) and Berman (1983) (see also Karatzas and Shreve (1987)) proved the theorem using properties of Brownian local time. Local time is a function of time and space variables. Under some assumptions, local time, viewed as a function of the space variable, is a Markov process with known distribution and one can prove that this Markov process does not visit 0. It is intuitively clear (although it requires some argument) that the existence of a point of increase would imply vanishing of the local time process in the space variable. The existence of local time indexed by all straight lines in the plane for two dimensional Brownian motion has been proved in Bass (1984). The distribution of this local time process, which could conceivably be the key to a potential proof of Theorem 0.1, is elusive, though.

Short and elementary proofs of Theorem 0.2 were found by Adelman (1985) and by Burdzy (1989). Aldous (1989) included in his book a heuristic argument based on "Poisson clumping." Peres (1996) showed that one can first consider the discrete analogue of the problem and then deduce Theorem 0.2 from its discrete counterpart. Burdzy (1996) contains a proof based on a branching idea. Doney (1996) has some related results on Lévy processes. All of these proofs rely in some way on the order structure of the real line and do not apply directly to two dimensional processes.

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Bertoin (1991) has a beautiful proof using a covering theorem. So far, there is no two dimensional covering theorem which would give Theorem 0.1.

The proofs of Knight (1981) and Bertoin (1991) are very elegant applications of powerful techniques – local time and coverings. Both proofs are short and based on clever ways of looking at the problem. The proofs of Adelman (1985) and Burdzy (1989) seem to be the most elementary of all proofs.

For any fixed unit vector v and d-dimensional Brownian motion B_t, the one dimensional Brownian motion $B_t \cdot v$ does not have a point of increase, a.s., by Theorem 0.2. Fubini's theorem easily implies that the set of unit vectors v for which $B_t \cdot v$ has a point of increase has zero $d-1$-dimensional Lebesgue measure (as a subset of the unit sphere). A much stronger result is the following.

Theorem 0.3. *Let B_t denote d-dimensional Brownian motion. The set of $t \in (0,1)$ such that $B([0,t))$ and $B((t,1])$ lie on opposite sides of some $d-1$-dimensional hyperplane has Hausdorff dimension zero.*

Theorem 0.3 is well-known, but it seems never to have been published. It is not hard to see that it can be proved using the methods of Evans (1985).

There exist results closely related to Theorem 0.1 which go in both "positive" and "negative" directions. We start with a positive result.

Theorem 0.4. *(Burdzy (1989)) Two-dimensional Brownian motion has cut points: with probability 1, there exist times t such that $Z([0,1]-\{t\})$ is not a connected set.*

The original proof given in Burdzy (1989) contains a gap; see Burdzy (1995) for the correction. A routine modification of the proof in Burdzy (1989) shows that for each ε, with probability $p(\varepsilon) > 0$, a Lipschitz curve can be found that separates the two pieces $Z([0,t))$ and $Z((t,1])$, such that the curve has Lipschitz constant less than ε. Lawler (1996) has a new proof of Theorem 0.4 and an estimate for the Hausdorff dimension of cut points. The two-dimensional result easily implies the existence of cut points for three dimensional Brownian motion. In higher dimensions, all points on Brownian paths are cut points.

In order to state the next result, we need some notation. Let $H(z,\alpha)$ be the half-plane obtained by first rotating the right half-plane by an angle α and then translating it by the vector z. Let $A(\alpha_1,\alpha_2)$ denote the event that for some $t \in (0,1)$, we have $Z([0,t)) \subset H(Z_t,\alpha_1)$ and $Z((t,1]) \subset H(Z_t,\alpha_2)$.

Theorem 0.5. *(i) (Shimura (1988)) Fix any $\alpha_1 \neq \alpha_2$. Then $P(A(\alpha_1,\alpha_2)) = 0$.*

(ii) (Shimura (1992)) Fix any $\alpha > 0$. Then $\mathbb{P}\Big(\bigcup_{-\alpha<\alpha_1,\alpha_2<\alpha} A(\alpha_1+\pi,\alpha_2)\Big) > 0$.

Theorem 0.5 (ii) is of special interest to us as it shows that Theorem 0.1 is a very "sharp" result. This claim is reinforced by the following result, which asserts the existence of cut planes in three and higher dimensions.

Theorem 0.6. *Let B_t denote d-dimensional Brownian motion, where $d \geq 3$. With positive probability, there exists $t \in (0,1)$ such that $B([0,t))$ and $B((t,1])$ lie on the opposite sides of some $(d-1)$-dimensional hyperplane.*

We will provide a sketch of the proof in Section 9. The argument uses estimates which are similar to those derived in the proof of Theorem 0.1. We will skip the technical details in the hope that the reader will be able to reconstruct a rigorous proof from our sketch, using the arguments developed in the earlier sections.

We learned Theorem 0.6 from Robin Pemantle (private communication). We would like to thank him for allowing us to include his unpublished result in our paper together with our own proof.

A version of Theorem 0.1 stated below asserts that "local cut lines" do not exist. Let $Z_t = (X_t, Y_t)$ be two dimensional Brownian motion, and for $\theta \in [0, 2\pi)$, set

$$Z_t(\theta) = X_t \cos\theta + Y_t \sin\theta,$$

the component of Z_t in the θ direction. Let

$$\begin{aligned} \mathcal{S} = \{\exists t, h > 0, \theta \in [0,2\pi) \,:\, & Z_s(\theta) \leq Z_t(\theta) \text{ for } (t-h)\vee 0 \leq s \leq t, \\ & Z_u(\theta) \geq Z_t(\theta) \text{ for } t \leq u \leq t+h\}. \end{aligned} \tag{0.1}$$

Theorem 0.7. $\mathbb{P}(\mathcal{S}) = 0$.

We give a sketch of our argument. Let us say that a 1-dimensional Brownian motion starting at 0 has an approximate point of increase of size ε at level $x \in (0,1)$ if after hitting the level x, the Brownian motion reaches the level 2 before returning to $x - \varepsilon$. It is not hard to show (cf. Burdzy (1990)) that if we fix α then the probability that the one dimensional Brownian motion $Z_t(\alpha)$ has an approximate point of increase is of order $1/\log(1/\varepsilon)$. This immediately implies that the expected number of directions of the form $\alpha = m/\log(1/\varepsilon)$, m integer, with an approximate point of increase is finite. This estimate, however, is not good enough. We circumvent it by conditioning on having an approximate point of increase in a fixed direction. We calculate the probabilities for the occurrence of approximate points of increase in a second fixed direction and then for having such points in two other fixed directions. A second moment argument then gives a better bound for the probability of having an approximate point of increase in one of the directions $m/\log(1/\varepsilon)$. The argument is further complicated by the facts that the second moment argument must be iterated to get a sufficiently good bound and that we are only able to obtain the probabilities for an approximate point of increase in two or more directions for certain levels. This conditional second moment argument was used in a somewhat simpler setting in Bass and Burdzy (1996).

The first part of the proof shows that with high probability there are no approximate points of increase at angles $m/\log(1/\varepsilon)$, m integer. The second part of the proof is to show that there are no approximate points of increase of size ε at any level or any angle. We do this by repeated bisection of angles. Given that there are no approximate points of increase for any angle α that is a multiple of $1/2^k \log(1/\varepsilon)$, we show that there is very small probability that there is an approximate point of increase at any level x for any angle that is a multiple of $1/2^{k+1} \log(1/\varepsilon)$. We show, in fact, that these probabilities are summable in k.

The next section presents some preliminaries. Estimates for a Bessel process of order 3 are derived in Sections 2-4. The first part of the proof is presented in Sections 5 and 6, while Sections 7 and 8 complete the proof. Additional heuristic arguments are given in each section.

Section 2 is devoted to a decomposition of Bessel processes of order 3. These results are of independent interest; see Theorem 2.5, for example. We show that if L_i, M_i is a certain sequence of relative maxima and minima (see Figure 2.1), then $\log(L_i - M_i)$ has the same law as the sum of i.i.d. bilateral exponentials.

Section 3 then shows that h-path transforms of this i.i.d. sequence satisfy estimates akin to those of conditioned Brownian motion. Section 4 uses the results of Sections 2 and 3 to give upper and lower bounds for the existence of approximate points of increase for a Bessel process.

Section 5 begins the consideration of two-dimensional Brownian motion with some estimates for exit probabilities for wedges and then continues with the two and three angle estimates mentioned above. The second moment argument is given in Section 6.

Section 7 has more estimates on the exit distribution of a wedge and then proceeds with upper bounds on the probability of an approximate point of increase in a specified direction given that there are no approximate points of increase in nearby directions. These estimates are pulled together in Section 8, and the proofs of Theorems 0.1 and 0.7 are completed there.

Section 9 contains a sketch of the proof of Theorem 0.6. We also include a discussion of some open problems.

We have discussed Taylor's problem about cut lines with many people over several years. We are grateful for their interest and advice. We would like to especially thank Davar Khoshnevisan, Robin Pemantle, Yuval Peres, Michio Shimura, and James Taylor. We would like to thank the referee for a very careful reading of the whole manuscript and for many valuable suggestions.

1. Preliminaries.

The letter c with subscripts will denote positive finite constants whose values are unimportant. We begin numbering the constants anew in each proposition, theorem, lemma, and corollary. We use "Bes(3)" as an abbreviation for "Bessel process of index 3" and "BM $\times$ Bes(3)" for the planar process (X_t, Y_t) where X_t is a Brownian motion and Y_t is a Bes(3) independent of X_t. We will use the symbol $\mathbb{P}^x$ to denote the distribution of a process starting from x when it is clear from the context which process we are talking about. When this notation might be confusing, we will write $\mathbb{P}_{BM}, \mathbb{P}_{BES}, \mathbb{P}_{BMBS}$, and $\mathbb{P}_V$ to denote the distributions of a Brownian motion (one- or two-dimensional), a Bes(3), a BM $\times$ Bes(3), and the process V defined in (2.3) below, respectively. Similar subscript conventions will apply to expectations.

For Borel sets A we define

$$\tau_A = \tau(A) = \inf\{t : Z_t \notin A\}, \qquad \mathcal{T}_A = \mathcal{T}(A) = \inf\{t : Z_t \in A\},$$

where Z_t is a process such as planar Brownian motion or a BM $\times$ Bes(3). We will use $B(x, r)$ to denote the open ball of radius r about x. We use ∂A to denote the boundary of a Borel set A.

Throughout this paper we will make frequent use of h-path transforms. Recall that if $(\mathbb{P}^x, X_t)$ is a strong Markov process killed on exiting a domain D with transition densities $p(t, x, y)$ and h is positive and $\mathbb{P}^x$-harmonic in D, then $h(y)p(t, x, y)/h(x)$ are densities of a strong Markov process $(\mathbb{P}^x_h, X_t)$ in D. For details and results concerning h-path transforms, see Doob (1984) or Bass (1995). The result we will use most frequently is that

$$\mathbb{E}^x_h Y = \mathbb{E}^x[h(X_T)Y]/h(x) \tag{1.1}$$

if T is a stopping time and Y is $\mathcal{F}_T$-measurable and either bounded or nonnegative.

It is well known that if one takes one dimensional Brownian motion killed on hitting $(-\infty, 0]$ and h-path transforms it by the harmonic function $h(x) = x$, one gets a Bes(3).

We will use several times a bound on the probability that 2-dimensional Brownian motion exits a rectangle through the vertical sides. If $Z_t = (X_t, Y_t)$ is 2-dimensional Brownian motion and $D_R = \{(x, y) : 0 < y < 1, |x| < R\}$, then there exist constants c_1 and c_2 such that

$$\sup_y \mathbb{P}^{(0,y)}(|X_{\tau(D_R)}| = R) \le c_1 e^{-c_2 R}. \tag{1.2}$$

A proof of this may be found in Bass (1995), p. 180.

The estimate of (1.2) also holds when Z_t is a BM $\times$ Bes(3). To see this, we use the fact that Y_t is equal in law to the modulus of a 3-dimensional Brownian motion, and the probability that 4-dimensional Brownian motion exits the cylinder $\{(x_1, x_2, x_3, x_4) : |x_1| < R, x_2^2 + x_3^2 + x_4^2 < 1\}$ through the ends is bounded by the right hand side of (1.2) by Bass (1995), p. 180.

If Z_t is a 2-dimensional Brownian motion and H is the upper half plane, then there exists c_1 such that

$$\sup_{(x,y)\in B(0,1)\cap H} \mathbb{P}^{(x,y)}(Z_{\tau(B(0,2)\cap H)} \notin \partial H) \le c_1 y; \tag{1.3}$$

see Bass (1995), p. 313, for a proof.

We will use several times the boundary Harnack principle; see Bass(1995), p. 178.

Theorem 1.1. *Let D be a connected Lipschitz domain, $x_0 \in D$, V open, and K compact so that $K \subseteq V$. There exists c_1 depending on K, V, and D such that if u and v are two positive harmonic functions on D vanishing continuously on $V \cap \partial D$, then*

$$\frac{u(x)}{v(x)} \le c_1 \frac{u(x_0)}{v(x_0)}, \qquad x \in K \cap D.$$

Note that if $rA = \{rx : x \in A\}$ for any set A, then by scaling the constant c_1 for the triple (rK, rV, rD) can be taken to be independent of r.

2. Decomposition of a Bessel process.

After reviewing the first version of our paper, Jim Pitman pointed out to us that ideas similar to the path decomposition developed in this section have already appeared in literature; see, e.g., Neveu and Pitman (1989) or Imhof (1992).

We need some detailed estimates, which we obtain by means of a path decomposition of a Bes(3), Z_t. We will repeatedly use Williams' path decompositions (Williams (1979), III.79). See also Millar (1978) for an alternate proof.

Proposition 2.1. *Let Z_t be a Bes(3) started at $L > 0$.*

(a) *If $M = \inf\{Z_t : t > 0\}$, then M is uniformly distributed on $[0, L]$.*

(b) *Let $T = \inf\{t : Z_t = M\}$. Given $\{M = m\}$, the law of $\{Z_t, t \leq T\}$ is the same as the law of a Brownian motion started at L and run until the first time it hits m.*

(c) *Given $\{M = m\}$, the law of $\{Z_{t+T}, t \geq 0\}$ is the same as the law of $\widehat{Z}_t + m$, where $\widehat{Z}_t$ is a Bes(3) started at 0.*

(d) *The processes $\{Z_t, t \leq T\}$ and $\{Z_{t+T} - Z_T, t \geq 0\}$ are independent.*

Proposition 2.1(a) can be rephrased as saying

$$\mathbb{P}^L_{BES}(Z_t \text{ hits } M) = \frac{M}{L}. \tag{2.1}$$

A related path decomposition is the following.

Proposition 2.2. *Let X_t be a Brownian motion started at $L > 0$ and killed on hitting* 0.

(a) *If $L' = \sup\{X_t : t > 0\}$, then $\mathbb{P}^L(L' > \lambda) = L/\lambda$.*

(b) *Let $S = \inf\{t : X_t = L'\}$. Given $\{L' = a\}$, the law of $\{X_t, t \leq S\}$ is the same as the law of a Bes(3) started at L and run until it hits a.*

(c) *Given $\{L' = a\}$, the law of $\{X_{t+S}, t \geq 0\}$ is the same as the law of $a - \overline{Z}_t$, where $\overline{Z}_t$ is a Bes(3) started at 0 and run until it hits a.*

(d) *The processes $\{X_t, t \leq S\}$ and $\{X_{t+S} - X_S, t \geq 0\}$ are independent.*

Proof. The proof is very similar to that of Proposition 2.1. We follow Millar's approach. To see (a),

$$\mathbb{P}^L(L' > \lambda) = \mathbb{P}^L(X_t \text{ hits } \lambda \text{ before hitting } 0) = L/\lambda.$$

If $M_t = \sup_{s \leq t} X_t$, then the pair (X_t, M_t) is a strong Markov process, and $S = \sup\{t : X_t = M_t\}$. For any strong Markov process Y_t, the law of Y_t up to the last time U it is in a set A and given $\{Y_U = y\}$ is the same as Y_t conditioned to hit y, while the law of Y_t after U is the same as for Y_t starting from Y_U and conditioned never to return to A; moreover, given $\{Y_U = y\}$, the processes $\{Y_t, t \leq U\}$ and $Y_{t+U}, t \geq 0\}$ are independent (see Meyer, Smythe, and Walsh (1972)). (b), (c), and (d) follow easily from this fact with $Y_t = (X_t, M_t)$ and $A = \{(x, m) : x = m\}$. $\square$

Suppose Z_t is a Bes(3) and $Z_0 \equiv L_1 > 0$. Let $S_1 = 0$. Define

$$\begin{aligned} M_1 &= \inf\{Z_t : t > S_1\},\\ T_1 &= \inf\{t > S_1 : Z_t = M_1\},\\ U_1 &= \inf\{t > T_1 : Z_t = L_1\}. \end{aligned}$$

Since $Z_t \to \infty$ a.s. as $t \to \infty$, then $U_1 < \infty$ a.s. We then define inductively

$$\begin{aligned} M_{i+1} &= \inf\{Z_t : t > U_i\},\\ T_{i+1} &= \inf\{t > U_i : Z_t = M_{i+1}\},\\ L_{i+1} &= \sup\{Z_t : T_i \le t \le T_{i+1}\},\\ S_{i+1} &= \inf\{t > U_i : Z_t = L_{i+1}\},\\ U_{i+1} &= \inf\{t > T_{i+1} : Z_t = L_{i+1}\}. \end{aligned} \tag{2.2}$$

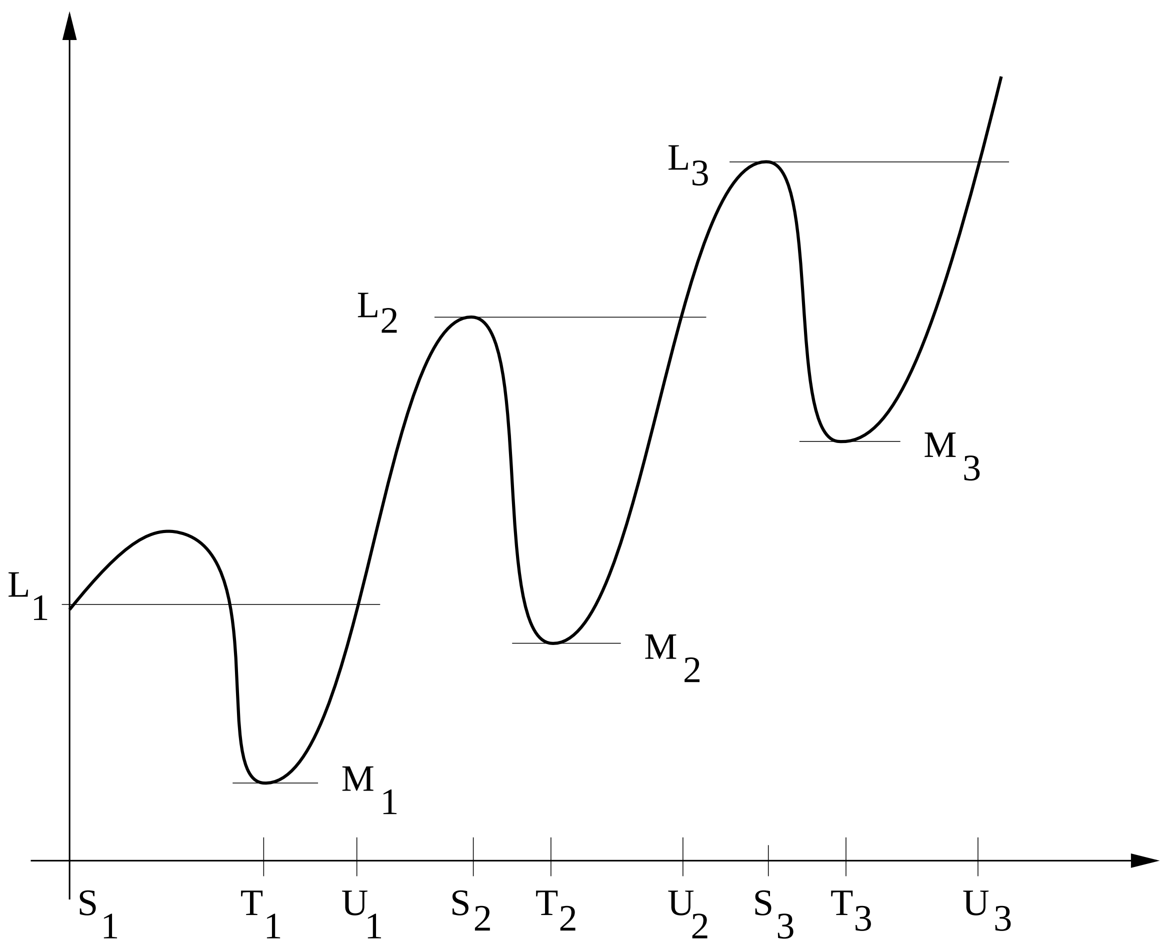

Figure 2.1.

We use Propositions 2.1 and 2.2 to decompose the path of a Bes(3) process. The M_i will be the future minima of Z_t after certain relative maxima, and the T_i will be the times they occur. The L_i are the maxima of the Bes(3) path up until time T_i and the S_i are the times these maxima occur. The U_i are the times Z_t returns to the level L_i after hitting a minimum. See Figure 2.1.

Remark. We will need the following property, which is analogous to the strong Markov property, but which can be applied at random times which are not stopping times. We call it the *pseudo-strong Markov property*. The justification is based on the repeated application of Proposition 2.1. Given L_1 and M_1, the processes $\{Z_t, 0 \leq t \leq U_1\}$ and $\{Z(U_1+t) - M_1, t \geq 0\}$ are independent and the second one is a Bes(3) starting from $L_1 - M_1$. The processes $\{Z_t, 0 \leq t \leq T_1\}$ and $\{Z(T_1+t) - M_1, t \geq 0\}$ are independent and the second one is a Bes(3) starting from 0. By induction, if L_i and M_i are given, then the processes $\{Z_t, 0 \leq t \leq U_i\}$ and $\{Z(U_i+t) - M_i, t \geq 0\}$ are independent and the second one is a Bes(3) starting from $L_i - M_i$. The processes $\{Z_t, 0 \leq t \leq T_i\}$ and $\{Z(T_i+t) - M_i, t \geq 0\}$ are independent and the second one is a Bes(3) starting from 0.

We have the following facts about the distribution of the M_i and L_i. These are straightforward applications of Propositions 2.1 and 2.2.

Proposition 2.3. *(a)* M_1 *is uniformly distributed on* $[0, L_1]$.

(b) Given M_i *and* L_i*, the distribution of* M_{i+1} *is uniform on* $[M_i, L_i]$.

(c) $\log[(L_i - M_i)/(M_{i+1} - M_i)]$ *is an exponential random variable with parameter 1.*

(d)

$$\mathbb{P}^{L_1}(L_{i+1} > \lambda \mid L_i, M_{i+1}) = \frac{L_i - M_{i+1}}{\lambda - M_{i+1}}.$$

(e) $\log[(L_{i+1} - M_{i+1})/(L_i - M_{i+1})]$ *is an exponential random variable with parameter 1.*

Proof. (a) This is immediate by Proposition 2.1(a).

(b) By Proposition 2.1(c) and the pseudo-strong Markov property,

$$\mathbb{P}(M_{i+1} < m \mid L_i, M_i) = \mathbb{P}(M_{i+1} - M_i < m - M_i \mid L_i, M_i) = \mathbb{P}(\widehat{Z}_t \text{ ever hits } m - M_i),$$

where $\widehat{Z}_t$ is a Bes(3) starting from $L_i - M_i$. As in (a), the distribution of the minimum of $\widehat{Z}_t$ is uniform on $[0, L_i - M_i]$.

(c) If U is uniform on $[0,1]$, then $-\log U$ is an exponential random variable with parameter 1. This and (b) show that that the conditional distribution of $\log[(L_i - M_i)/(M_{i+1} - M_i)]$ is exponential with parameter 1. Since the conditional distribution does not depend on L_i and M_i, we obtain (c).

(d) By the pseudo-strong Markov property, $\widetilde{Z}_t = Z_{t+U_i} - M_i$ is a Bes(3) started at $L_i - M_i$. By Proposition 2.1(b), the law of $\widetilde{Z}_t$ for $t \leq T_{i+1} - U_i$, given that the minimum of $\widetilde{Z}_t$ is M_{i+1}, is that of a Brownian motion killed on hitting M_{i+1}. Therefore by Proposition 2.2(a) and translation invariance,

$$\mathbb{P}^{L_1}\Big(\sup_{t \leq T_{i+1} - U_i} \widetilde{Z}_t > \lambda \mid L_i, M_{i+1}\Big) = \frac{L_i - M_{i+1}}{\lambda - M_{i+1}}.$$

(e) If $\mathbb{P}(V > \lambda') = \ell/\lambda'$, then $\log(V/\ell)$ is an exponential random variable with parameter 1. We condition on the values of L_i and M_{i+1} and apply (d) and the last remark with $V = L_{i+1} - M_{i+1}$, $\lambda' = \lambda - M_{i+1}$, and $\ell = L_i - M_{i+1}$, to conclude that the conditional distribution of $\log[(L_{i+1} - M_{i+1})/(L_i - M_{i+1})]$ is exponential with parameter 1. Since the conditional distribution does not depend on the values of L_i and M_{i+1}, we obtain (e). □

Let

$$V_i = \log(L_i - M_i). \tag{2.3}$$

The next two theorems say that V_i is the sum of i.i.d. bilateral exponentials.

Theorem 2.4. *Let A_i, B_i be two independent sequences of independent identically distributed exponential random variables with parameter 1. Then $\{V_m - V_1, m \geq 1\}$ has the same law as $\{\sum_{i=1}^{m-1}(A_i - B_i), m \geq 1\}$.*

Proof. Let

$$C_i = \log\Big(\frac{L_i - M_i}{M_{i+1} - M_i}\Big), \qquad A_i = \log\Big(\frac{L_{i+1} - M_{i+1}}{L_i - M_{i+1}}\Big). \tag{2.4}$$

By Proposition 2.3, C_i and A_i are exponentials with parameter 1. We next show independence. By Proposition 2.1(c) and the pseudo-strong Markov property at U_i, the process $Z_{t+U_i} - M_i$ is a Bes(3) started at $L_i - M_i$. Its law depends on $L_1, \dots, L_i, M_1, \dots, M_i$ only through the value of $L_i - M_i$. By scaling, $\widetilde{Z}_t = (Z_{t+U_i} - M_i)/(L_i - M_i)$ has the law of a Bes(3) started at 1, and is thus independent of $L_1, \dots, L_i, M_1, \dots, M_i$. Since e^{-C_i} is the minimum of $\widetilde{Z}_t$, then C_i is independent of $A_1, \dots, A_{i-1}, C_1, \dots, C_{i-1}$.

To see that A_i is independent of $A_1, \dots, A_{i-1}, C_1, \dots, C_i$, we use the pseudo-strong Markov property at time U_i. By Proposition 2.1(b), the law of $Z_{t+U_i} - M_i$, $t \leq T_{i+1} - U_i$, conditional on having minimum $M_{i+1} - M_i$ at time $T_{i+1} - U_i$, is that of a Brownian motion started at $L_i - M_i$ and killed on hitting $M_{i+1} - M_i$. By scaling, $\overline{Z}_t = (Z_{t+U_i} - M_{i+1})/(L_i - M_{i+1})$ has the law of a Brownian motion started at 1 and killed on hitting 0. Since e^{A_i} is the maximum of $\overline{Z}_t$, then A_i is independent of $L_1, \dots, L_i, M_1, \dots, M_{i+1}$, and hence of $A_1, \dots, A_{i-1}, C_1, \dots, C_i$.

We have

$$M_{i+1} = (L_i - M_i)e^{-C_i} + M_i,$$

and

$$L_{i+1} = (L_i - M_{i+1})e^{A_i} + M_{i+1}.$$

Let $B_i = -\log(1 - e^{-C_i})$. We then have

$$\begin{aligned} e^{V_{i+1}} &= L_{i+1} - M_{i+1} = (L_i - M_{i+1})e^{A_i} \\ &= (L_i - [(L_i - M_i)e^{-C_i} + M_i])e^{A_i} \\ &= (L_i - M_i)(1 - e^{-C_i})e^{A_i} \\ &= e^{V_i}e^{-B_i}e^{A_i}. \end{aligned} \tag{2.5}$$

Since C_i is an exponential with parameter 1, then e^{-C_i} is uniform on $[0,1]$, hence so is $1-e^{-C_i}$. This implies B_i is an exponential with parameter 1. Taking logarithms of (2.5) proves the proposition. $\square$

A distribution is called *bilateral exponential* if it has a density given by $e^{-|x|}/2$ for $x \in \mathbb{R}$. By Feller (1971), p. 49, $A_i - B_i$ is a bilateral exponential, and so we have

Theorem 2.5. *$V_m - V_1$ is the sum of i.i.d. bilateral exponentials.*

3. Random walk estimates.

We need a number of estimates on the sequence $\{V_n\}_{n\geq 0}$ which is the sum of independent identically distributed bilateral exponentials. We start by giving a modification of an argument of Feller (1971), p. 401, to show that the overshoots of V_n are exponentially distributed. This proof is very similar to the "lack of memory" of exponentially distributed random variables.

Proposition 3.1. *Let $a < b$, $I = [a, b]$, $N = \min\{n : V_n \notin I\}$, and $y \in I$. Then*

$$\mathbb{P}^y(V_N \in dx \mid V_N > b) = e^{-(x-b)}\, dx, \qquad x > b, \tag{3.1}$$

and

$$\mathbb{P}^y(V_N \in dx \mid V_N < a) = e^{-(a-x)}\, dx, \qquad x < a. \tag{3.2}$$

Proof. If $x \geq b$,

$$\begin{aligned}
\mathbb{P}^y(V_N \in dx) &= \sum_{n=1}^{\infty} \mathbb{P}^y(V_N \in dx, N = n) = \sum_{n=1}^{\infty} \mathbb{P}^y(V_1 \in I, \dots, V_{n-1} \in I, V_n \in dx) \\
&= \sum_{n=1}^{\infty} \mathbb{E}^y[\mathbb{P}^{V_{n-1}}(V_n \in dx); V_1 \in I, \dots, V_{n-1} \in I] \\
&= \sum_{n=1}^{\infty} \mathbb{E}^y\left[\frac{1}{2} e^{-(x-V_{n-1})}\, dx; V_1 \in I, \dots, V_{n-1} \in I\right] \\
&= e^{-(x-b)}\, dx \sum_{n=1}^{\infty} \mathbb{E}^y\left[\frac{1}{2} e^{-(b-V_{n-1})}; V_1 \in I, \dots, V_{n-1} \in I\right].
\end{aligned}$$

Note the infinite sum in the last line does not depend on x, so $\mathbb{P}^y(V_N \in dx) = c_1 e^{-(x-b)}\, dx$. Integrating over $x > b$, $\mathbb{P}^y(V_N \geq b) = c_1$, and then

$$\mathbb{P}^y(V_N \in dx \mid V_N \geq b) = \mathbb{P}^y(V_N \in dx)/\mathbb{P}^y(V_N \geq b) = e^{-(x-b)}\, dx.$$

The other inequality has a similar proof. $\square$

As a corollary we can get an exact solution to the gambler's ruin problem for V_n.

Proposition 3.2. *Let a, b, I, N, and y be as in Proposition 3.1. Then*

$$\mathbb{P}^y(V_N \geq b) = \frac{y-a+1}{b-a+2}, \qquad \mathbb{P}^y(V_N \leq a) = \frac{b+1-y}{b-a+2}.$$

Proof. Let $A = \mathbb{P}^y(V_N \leq a)$, $B = \mathbb{P}^y(V_N \geq b)$. Since bilateral exponentials have moments of all orders, the law of the iterated logarithm shows that $N < \infty$, a.s., or

$$A + B = 1. \tag{3.3}$$

Since $\mathbb{E}(V_{i+1} - V_i) = 0$, then V_n forms a martingale. By optional stopping,

$$y = \mathbb{E}^y V_{n\wedge N}. \tag{3.4}$$

Note

$$\mathbb{E}V_N^2 = \mathbb{E}[V_N^2; V_N \le a] + \mathbb{E}[V_N^2; V_N \ge b].$$

Since Proposition 3.1 gives the exact distributions of V_N given $V_N \le a$ and $V_N \ge b$, we see that $\mathbb{E}V_N^2$ is finite. The process $V_{n\wedge N}^2$ is a submartingale, and by Doob's inequality,

$$\mathbb{E}\sup_n V_{n\wedge N}^2 \le c_1 \sup_n \mathbb{E}V_{n\wedge N}^2 \le c_2(a^2 + b^2 + \mathbb{E}V_N^2) < \infty.$$

With this fact, we can let $n \to \infty$ in (3.4) and use dominated convergence to obtain

$$y = \mathbb{E}^y V_N = \mathbb{E}^y[V_N; V_N \le a] + \mathbb{E}^y[V_N; N \ge b].$$

By Proposition 3.1, $\mathbb{E}^y[V_N \mid V_N \le a] = a - 1$ and $\mathbb{E}^y[V_N; V_N \ge b] = b + 1$, so

$$y = (a-1)A + (b+1)B.$$

This and (3.3) give our result. □

Let $a = 0$ and $b = K$, where $K > 10$ is large, but fixed. None of the constants in what follows will depend on K, however. Let

$$h(y) = \mathbb{P}^y(V_N \ge K) = \frac{y+1}{K+2}, \qquad y \in (0, K). \tag{3.5}$$

Let $h(y) = 0$ if $y \le 0$ and $h(y) = 1$ if $y \ge K$. Then h is nonnegative, bounded above by 1, and there exist constants c_1, c_2 such that

$$c_1 y/K \le h(y) \le c_2 y/K, \qquad y \in (1, K). \tag{3.6}$$

The function h is not continuous; this is to be expected, since V_i is a random walk, not a diffusion.

Let $\mathbb{P}_h^y$ denote the h-path transform of $\mathbb{P}^y$ by the function h. We need an estimate on the overshoots of V_N under $\mathbb{P}_h^y$.

Proposition 3.3. *There exists c_1 independent of K such that if $y \in (0, K)$,then*

$$\mathbb{P}_h^y(V_N \ge K + \lambda) \le e^{-\lambda}.$$

Proof. We have by Proposition 3.1,

$$\mathbb{P}_h^y(V_N \ge K + \lambda) = \mathbb{P}^y(V_N \ge K + \lambda \mid V_N \ge K) = e^{-\lambda}.$$ □

We need some Green function estimates for $V_n, n \le N$. Let

$$D_j = \sum_{i=1}^N 1_{[j,j+1]}(V_i). \tag{3.7}$$

We use a renewal-type argument to say that the number of visits to $[j, j+1]$ by the random walk V_i goes down geometrically.

Proposition 3.4. *There exists c_1 independent of K such that for $0 \le j \le K$,*

$$\sup_y \mathbb{E}_h^y D_j \le c_1(j \wedge (K-j)).$$

Proof. Let $M_0 = \min\{n : V_n \le 0\}$, $M_K = \min\{n : V_n \ge K\}$, $I_j = \min\{n > 0 : V_n \in [j, j+1]\}$. Suppose $j \le K/2$. Starting at $z \in [j, j+1]$, there is probability at least $c_2 > 0$ that $V_1 \le j-1$. By Proposition 3.2 there exists c_3 such that starting at $z \le j-1$, there is probability at least c_3/j that $M_0 < I_j$. Hence, setting $c_4 = c_2 c_3$,

$$\inf_{z\in[j,j+1]} \mathbb{P}^z(M_0 \le I_j) \ge c_4/j.$$

Let ℓ_m be the m^{th} time that $V_i \in [j, j+1]$. Then

$$\sup_{z\in[j,j+1]} \mathbb{P}^z(\ell_1 < M_0) \le 1 - \frac{c_4}{j}.$$

By the strong Markov property,

$$\mathbb{P}^y(\ell_{r+1} < M_0) = \mathbb{E}^y[\mathbb{P}^{V(\ell_r)}(\ell_1 < M_0); \ell_r < M_0] \le \Big(1 - \frac{c_4}{j}\Big)\mathbb{P}^y(\ell_r < M_0),$$

and by induction,

$$\mathbb{P}^y(\ell_m < M_0) \le (1 - c_4/j)^m. \tag{3.8}$$

There exists an integer c_5 independent of j such that for all $j \ge 1$,

$$\Big(1 - \frac{c_4}{j}\Big)^{c_5 j} \le 1/2, \tag{3.9}$$

so for integers $r \ge 1$,

$$\sup_{y\in[j,j+1]} \mathbb{P}^y(\ell_{rc_5 j} < M_0) \le 2^{-r}.$$

Note that $h(x)/h(y) \le 2$ for $x, y \in (j, j+1]$. Thus

$$\begin{aligned}
\sup_y \mathbb{P}_h^y(D_j > rc_5 j) &= \sup_{y\in[j,j+1]} \mathbb{P}_h^y(D_j > rc_5 j)\\
&\le \sup_{y\in[j,j+1]} \mathbb{P}_h^y(\ell_{rc_5 j} < M_0)\\
&\le \sup_{y\in[j,j+1]} \frac{\mathbb{E}^y[h(V(\ell_{rc_5 j})); \ell_{rc_5 j} < M_0]}{h(y)}\\
&\le 2\mathbb{P}^y(\ell_{rc_5 j} < M_0) \le 2^{-r+1},
\end{aligned}$$

from which it follows that $\sup_y \mathbb{E}_h^y D_j \le c_6 j$.

The argument in the case $j > K/2$ is exactly similar, except that we use the bound

$$\mathbb{P}^y(D_j > m) \le \mathbb{P}^y(\ell_m < M_K). \qquad \square$$

As an application of Proposition 3.4, we have

Proposition 3.5. *There exist c_1 and c_2 independent of K such that if $y \in (0, K)$, then*

$$\mathbb{P}_h^y\Big(\sum_{i=1}^N e^{V_i} \geq \lambda e^K\Big) \leq c_1 e^{-c_2\lambda}.$$

Proof. Let $B_m = \sum_{i=1}^m e^{V_i}$. Using Proposition 3.4,

$$\sup_y \mathbb{E}_h^y B_N \leq \sum_{j=0}^K e^{j+1}\Big(\sup_y \mathbb{E}_h^y D_j\Big) \leq \sum_{j=0}^K e^{j+1} c_3(j \wedge (K-j)) \leq c_4 e^K.$$

Since B_m is an additive functional and hence subadditive (cf. Bass (1995) Corollary I.6.12),

$$\sup_y \mathbb{P}_h^y(B_N > r(2c_4 e^K)) \leq 2^{-r}.$$

The proposition follows easily from this. □

We also have

Proposition 3.6. *There exist c_1 and c_2 independent of K such that if $y \geq 1$, then*

$$\mathbb{P}^y\Big(\sum_{i=1}^N e^{V_i} \geq \lambda e^K\Big) \leq c_1 e^{-c_2\lambda}.$$

Proof. Recall the notation from the proof of Proposition 3.4 and constants c_4 and c_5. We obtain from (3.9),

$$\sup_y \mathbb{P}^y(D_j > c_5 j) = \sup_{y \in [j, j+1]} \mathbb{P}^y(D_j > c_5 j) \leq 1/2.$$

Since D_j is an additive functional of V_i, hence subadditive, then for integers $r \geq 1$,

$$\sup_y \mathbb{P}^y(D_j > rc_5 j) \leq 2^{-r}.$$

We obtain $\sup_y \mathbb{E}^y D_j \leq c_3 j$ for $j \leq K/2$. A similar argument applies to $j \geq K/2$ so

$$\sup_y \mathbb{E}^y D_j \leq c_3(j \wedge (K-j)).$$

We now proceed exactly as in the proof of Proposition 3.5. □

4. Estimates for approximate points of increase.

Recall the process V defined in (2.3). Recall that K is a fixed large number, $N = \min\{n : V_n \notin [0, K]\}$ and $h(x) = \mathbb{P}^x(V_N \geq K)$. We obtain an estimate for the rate of escape of V_i under $\mathbb{P}^y_h$ by a standard argument.

Proposition 4.1. *Given $\gamma > 0$, there exists $y_0 > 1$ independent of K such that*

$$\mathbb{P}^y_h(V_i \leq i^{1/16}, \text{ for some } i = 1, \ldots, N) \leq \gamma$$

if $y \geq y_0$.

Proof. Let $A \geq 1$. If $y \geq y_0 \geq 1$, by (3.6)

$$\mathbb{P}^y_h(V_{2^m} \leq A(2^m)^{1/8}) = \frac{\mathbb{E}^y[h(V_{2^m}); 0 \leq V_{2^m} \leq A2^{m/8}]}{h(y)}$$
$$\leq c_1 \frac{A2^{m/8}}{y_0} \mathbb{P}^y(0 \leq V_{2^m} \leq A2^{m/8}).$$

By the local central limit theorem (Stone (1965)),

$$\mathbb{P}^y(0 \leq V_{2^m} \leq A2^{m/8}) \leq c_2 \frac{A2^{m/8}}{2^{m/2}}.$$

Hence

$$\mathbb{P}^y_h(V_{2^m} \leq A2^{m/8}) \leq c_1 c_2 A^2 y_0^{-1} 2^{-m/4}.$$

This is summable in m, and, for any A, by taking y_0 large enough (depending on A), we have

$$\mathbb{P}^y_h(V_{2^m} \leq A2^{m/8} \text{ for some } m \leq \log_2 N) \leq \gamma/2. \tag{4.1}$$

Then

$$\begin{aligned}
&\mathbb{P}^y_h(V_i \leq i^{1/16} \text{ for some } i \leq N) \\
&\quad \leq \mathbb{P}^y_h(V_{2^m} \leq A2^{m/8} \text{ for some } m \leq \log_2 N) \\
&\qquad + \sum_{m=1}^{\log_2 N} \mathbb{P}^y_h\Big(V_{2^m} \geq A2^{m/8}, \\
&\qquad\qquad \text{and } V_i \leq 2 \cdot 2^{m/16} \text{ for some } i \in [2^m, 2^{m+1} \wedge N]\Big)
\end{aligned}$$

We bound the summands using the Markov property at time 2^m:

$$\sup_{z \geq A2^{m/8}} \mathbb{P}^z_h(V_i \leq 2 \cdot 2^{m/16} \text{ for some } i \leq 2^{m+1} \wedge N) \leq \frac{h(2 \cdot 2^{m/16})}{h(z)}$$
$$\leq c_3 \frac{2^{m/16}}{A2^{m/8}} = c_3 2^{-m/16} A^{-1}. \tag{4.2}$$

This is summable and we take A large enough so that the sum is less than $\gamma/2$. Then we choose y_0 large enough so that the estimate (4.1) holds for $y \geq y_0$. □

We show that the distribution of B_i under $\mathbb{P}^v_h$ has tails bounded by an exponential.

Lemma 4.2. *Suppose $v \in [1, K]$. Let A_i and B_i be defined as in the proof of Theorem 2.4. There exists c_1 independent of v and y such that*

$$\mathbb{P}^v_h(B_1 \in dy) \le c_1 e^{-y}\, dy.$$

Proof. If $V_1 = v$, $A_1 = x$, and $B_1 = y$, then $V_2 = v + x - y$. So for $x, y > 0$,

$$\mathbb{P}^v_h(A_1 \in dx, B_1 \in dy) = \frac{h(v+x-y)}{h(v)} e^{-x} e^{-y}\, dx\, dy.$$

Since $h(v) \ge c_2 v/K$ and $h(v + x - y) \le c_3((1 + x - y + v) \vee 1)/K$, then

$$\mathbb{P}^v_h(A_1 \in dx, B_1 \in dy) \le c_4(2 + x)e^{-x} e^{-y}\, dx\, dy.$$

We integrate over $x \ge 0$ to obtain our result. □

Proposition 4.1 and Lemma 4.2 combine to show that it is unlikely that $L_i - M_{i+1}$ is small.

Proposition 4.3. *Let $\gamma > 0$. There exists y_0 independent of K such that if $y \ge y_0$, then*

$$\mathbb{P}^y_h(L_i - M_{i+1} \le 1 \textit{ for some } i \le N - 1) < \gamma.$$

Proof. Recall the notation from the proof of Theorem 2.4. We have from (2.4),

$$L_i - M_{i+1} = (L_{i+1} - M_{i+1})e^{-A_i} = e^{V_{i+1}-A_i} = e^{V_i - B_i}.$$

So it suffices to bound the $\mathbb{P}^y_h$ probability that $V_i - B_i \le 0$ for some $i \le N - 1$.

From Theorem 2.4 and its proof, V_i may be represented using A_j's and B_j's for $j \le i - 1$. Hence, B_i is independent from V_i. By Lemma 4.2 and the Markov property,

$$\mathbb{P}^y_h(B_i > t, V_i \ge 1) = \mathbb{E}^y_h[\mathbb{P}^{V_i}_h(B_1 > t); V_i \ge 1] \le c_1 e^{-t}, \tag{4.3}$$

if $i < N$. Choose i_0 such that $\sum_{i=i_0}^{\infty} e^{-i^{1/16}/2} < \gamma/4c_1$ and $i_0^{1/16} > 4$. Choose $r > 4$ such that $e^{-r/2} < \gamma/4c_1 i_0$. By Proposition 4.1, provided that y_0 is sufficiently large and $y \ge y_0$,

$$\mathbb{P}^y_h(\exists i \le N \,:\, V_i < i^{1/16}) < \gamma/4.$$

By our bounds on h, if $y \ge y_0$ and y_0 is sufficiently large,

$$\mathbb{P}^y_h(\exists i \le i_0 \,:\, V_i < r) \le i_0 \sup_{i \le i_0} \mathbb{P}^y_h(V_i < r) \le i_0 \frac{h(r)}{h(y)} < \gamma/4.$$

So if $y \ge y_0$,

$$\begin{aligned}
&\mathbb{P}^y_h(\exists i \le N-1 \,:\, V_i - B_i \le 0) \\
&\quad \le \mathbb{P}^y_h(\exists i \le N \,:\, V_i < i^{1/16}) + \mathbb{P}^y_h(\exists i \le i_0 \,:\, V_i < r) \\
&\qquad + \mathbb{P}^y_h(B_i > r/2, V_i \ge r \text{ for some } i \le i_0) \\
&\qquad + \mathbb{P}^y_h(B_i > i^{1/16}/2, V_i \ge i^{1/16} \text{ for some } i_0 \le i \le N-1) \\
&\quad \le \gamma/4 + \gamma/4 + c_1 \sum_{i=1}^{i_0} e^{-r/2} + c_1 \sum_{i=i_0}^{\infty} e^{-i^{1/16}/2} < \gamma.
\end{aligned}$$

□

We define points of increase relative to two numbers $b \ge a \ge 2$. We say that a Bes(3) process Z_t has an approximate point of increase at level $j \le a$ if after first hitting j the process Z_t hits b before returning to $j-1$. The connection between our decomposition of a Bes(3) process and approximate points of increase is given in the next proposition. Let $y' \in [0,1)$ and let

$$\sigma_j = \inf\{t : Z_t = j + y'\}. \tag{4.4}$$

Proposition 4.4. *Let* $0 < \ell < a < b - y'$, $I \in \mathbb{Z}$,

$$A_1 = \{L_1 < \ell + y', L_I \in (a+y', b), L_i - M_{i+1} > 1 \text{ for } i = 1, 2, \dots, I-1\},$$

and

$$A_2 = \{\exists j \in [\ell, a] \cap \mathbb{Z} : \sigma_b \circ \theta_{\sigma_j} < \sigma_{j-1} \circ \theta_{\sigma_j}\}.$$

Then $A_1 \cap A_2 = \emptyset$.

Proof. Suppose $\omega \in A_1$. If j is an integer between ℓ and a, then $\sigma_j \in [L_{i-1}, L_i]$ for some i between 2 and I. Since $M_i < L_{i-1} - 1$, then $\sigma_{j-1} \circ \theta_{\sigma_j} < T_i$. Since $L_I < b$, then $\sigma_b \circ \theta_{\sigma_j} > T_I \ge T_i$, hence $\omega \notin A_2$. □

Propositions 4.3 and 4.4 will be used together to obtain lower bounds on the probability of no approximate points of increase. Proposition 4.3 says that $L_i - M_{i+1}$ is unlikely to be less than 1 and Proposition 4.4 says that this means there will not be approximate points of increase. The first and last pieces of the path, however, complicate the argument.

Proposition 4.5. *Let* Z_t *be a Bes(3) process,* $y' \in [0,1)$, $\varepsilon \in (0,1)$, *and*

$$B = \{\exists j \in [0, 1/\varepsilon] \cap \mathbb{Z} : \sigma_{3/2\varepsilon} \circ \theta_{\sigma_j} < \sigma_{j-1} \circ \theta_{\sigma_j}\}.$$

There exists c_1 *independent of* ε *such that*

$$\mathbb{P}^0(B^c) \ge c_1 / \log(1/\varepsilon).$$

Proof. Suppose ψ is a continuous function from $[0,1]$ to $[0,\infty)$. By the support theorem for 3-dimensional Brownian motion (see Bass (1995), p. 59) and the fact that a Bes(3) is equal in law to the modulus of 3-dimensional Brownian motion, given $\delta > 0$ there exists c_2 such that

$$\mathbb{P}^{\psi(0)}(\sup_{t \le 1} |Z_t - \psi(t)| < \delta) > c_2, \tag{4.5}$$

where c_2 depends only on δ and the modulus of continuity of ψ.

Let $\gamma = 1/4$, $t_0 > 0$, and let y_0 be defined as in Proposition 4.3. By (4.5), there is positive probability that started from 0, Z_t hits $2e^{y_0}$ at time t_0 and then returns to a level below $1/2$ before hitting $4e^{y_0}$. So by the Markov property at time t_0, it suffices to consider the approximate points of increase for Z_t started at $2e^{y_0}$ and conditioned by $\{M_1 \le 1/2\}$. Note that, given this condition, $V_1 \ge y_0$. It is enough to consider only levels j greater than $2e^{y_0}$.

Let $K = \log(1/\varepsilon) - \log(2\lambda)$, where $\lambda > 1$ will be chosen shortly. Let $N = \min\{i : V_i \notin [0, K]\}$. Recall that $h(y) = \mathbb{P}^y(V_N \ge K)$. Using Proposition 3.5, choose λ not depending on K or y_0 such that

$$\mathbb{P}_h^y(\sum_{i=1}^{N} e^{V_i} > \lambda e^K) \le 1/4, \tag{4.6}$$

for $y \ge y_0$. By Proposition 3.2, the probability that the process V hits K before 0 is at least $c_3 y_0/K \ge c_4/\log(1/\varepsilon)$. Using Proposition 4.3 and (4.6), the probability that $L_i - M_{i+1} > 1$ for $i \le N-1$ and $\sum_{i=1}^N e^{V_i} \le \lambda e^K$ is not less than $1 - \gamma - 1/4 = 1/2$ for the process conditioned by the function h. Since conditioning by the function h is the same as conditioning by the event $\{V_N \ge K\}$, the probability that $L_i - M_{i+1} > 1$ for $i \le N-1$ and $\sum_{i=1}^N e^{V_i} \le \lambda e^K$ is at least $c_4/2\log(1/\varepsilon)$. Recall from the proof of Proposition 4.3 that $L_N - M_{N+1} = \exp(V_N - B_N)$. The random variable B_N is exponential and independent of V_j's for $j < N$, so assuming $V_N > K$ we see that $\mathbb{P}(L_N - M_{N+1} > 1) > c_5$. Hence, we may strengthen our previous estimate to include $i = N$, i.e., the probability that $L_i - M_{i+1} > 1$ for $i \le N$ and $\sum_{i=1}^N e^{V_i} \le \lambda e^K$ is at least $c_4/2\log(1/\varepsilon)$. Since $L_i \ge M_{i+1}$ for all i,

$$L_N \le \sum_{i=1}^{N} (L_i - M_i) = \sum_{i=1}^{N} e^{V_i}. \tag{4.7}$$

On the set $\{\sum_{i=1}^N e^{V_i} \le \lambda e^K\}$, we have $L_N \le \lambda e^K \le 1/\varepsilon$. On the other hand, $L_N \ge L_N - M_N = e^{V_N} \ge e^K = 1/2\lambda\varepsilon$. Assuming that $L_i - M_{i+1} > 1$ for $i \le N$, Proposition 4.4 shows that for all $j \in [e^{2y_0}, 1/2\lambda\varepsilon] \cap \mathbb{Z}$ we have $\sigma_{3/2\varepsilon} \circ \theta_{\sigma_j} > \sigma_{j-1} \circ \theta_{\sigma_j}$. By the pseudo-strong Markov property, the process $\{Z(T_N + t) - Z(T_N), t \ge 0\}$ is a Bes(3) starting from 0 independent of $\{Z_t, t \le T_N\}$. We apply (4.5) and scaling to the first of these processes to see that there is probability greater than $c_6 > 0$ that there is no $j \in [1/2\lambda\varepsilon, 1/\varepsilon] \cap \mathbb{Z}$ for which $\sigma_{3/2\varepsilon} \circ \theta_{\sigma_j} > \sigma_{j-1} \circ \theta_{\sigma_j}$. This and the previous estimates show that with probability at least $c_7/\log(1/\varepsilon)$ there are no approximate points of increase at levels $j \in [2e^{y_0}, 1/\varepsilon]$. □

We need a more refined version of the preceding. Proposition 4.6 says that it is unlikely that there are approximate points of increase beyond level $\varepsilon^{\zeta-1}$ if ζ is close to 0.

Proposition 4.6. *Suppose $\varepsilon \in (0,1)$, $y' \in [0,1)$, and $\gamma \in (0,1)$. There exist $\zeta \in (0,1)$ and $c_1 > 1$ such that if ε is sufficiently small, $y \ge \varepsilon^{\zeta-1}$, and*

$$B = \{\exists j \in [y+2, \varepsilon^{-1}/c_1] \cap \mathbb{Z} : \sigma_{3/2\varepsilon} \circ \theta_{\sigma_j} < \sigma_{j-1} \circ \theta_{\sigma_j}\},$$

then $\mathbb{P}^y(B) < \gamma$.

Proof. The proof is very similar to the proof of Proposition 4.5 with the following changes: (1) We omit the use of (4.5). The inclusion of c_1 in the statement of the proposition means that we only need to worry about levels j up to $\varepsilon^{-1}/2\lambda$.
(2) By Proposition 3.2,

$$\mathbb{P}^{\log y}(V_i \text{ hits } K \text{ before } 0) \geq \frac{\log y + 1}{K+2}.$$

Since $\log y \geq (\zeta - 1)\log \varepsilon = (1-\zeta)\log(1/\varepsilon)$, by taking ζ sufficiently close to 0 we see that

$$\mathbb{P}^{\log y}(V_i \text{ hits } K \text{ before } 0) \geq 1 - \gamma/4$$

if ε is sufficiently small.
(3) By choosing λ large enough,

$$\mathbb{P}^y_h(\sum_{i=1}^N e^{V_i} > \lambda e^K) \leq \gamma/6.$$

(4) Since $\log y \geq (1-\zeta)\log(1/\varepsilon) \to \infty$ as $\varepsilon \to 0$,

$$\mathbb{P}^{y_0}_h(L_i - M_{i+1} \leq 1, i \leq N) \leq \gamma/6$$

for ε sufficiently small.

We now proceed as in the proof of Proposition 4.5 and conclude that

$$\mathbb{P}(B^c) \geq (1-\gamma/3)(1-\gamma/4) \geq 1-\gamma.$$

□

We now turn to upper bounds. The idea is to use a renewal-type argument. Each time V_i is close to 1, there is some chance that the next few M_i and L_i are such that there is an approximate point of increase. So the upper bound is given by an infinite series, the leading term of which is the probability that V_i is ever close to 1.

Proposition 4.7. *Let Z_t be a Bes(3) process started at ℓ, $4 \leq \ell < a < b/2$, and $y' \in [0,1)$. Let*

$$B = \{\exists j \in [\ell, a] \cap \mathbb{Z} : \sigma_b \circ \theta_{\sigma_j} < \sigma_{j-1} \circ \theta_{\sigma_j}\}.$$

Then there exists c_1 independent of a, b, and ℓ such that

$$\mathbb{P}^\ell(B^c) \leq c_1 \log \ell / \log a.$$

Proof. Let $I_1 = \min\{i : V_i \in [1.0, 1.01]\}$. For $i \geq 1$, define

$$J_i = \min\{i > I_i : V_i \geq 20\},$$
$$I_{i+1} = \min\{i > J_i : V_i \in [1.0, 1.01]\}.$$

Let

$$\begin{aligned} A_i = \{ & L_{I_i+1} - M_{I_i} \in [3.9, 3.91], M_{I_i+1} - M_{I_i} \in [.9, .91], \\ & L_{I_i+2} - M_{I_i} \in [4.7, 4.71], M_{I_i+2} - M_{I_i} \in [3.8, 3.81], \\ & L_{I_i+3} - M_{I_i} \in [5.5, 5.51], M_{I_i+3} - M_{I_i} \in [4.6, 4.61], \\ & L_{I_i+4} - M_{I_i} \in [6.3, 6.31], M_{I_i+4} - M_{I_i} \in [5.4, 5.41]\}. \end{aligned}$$

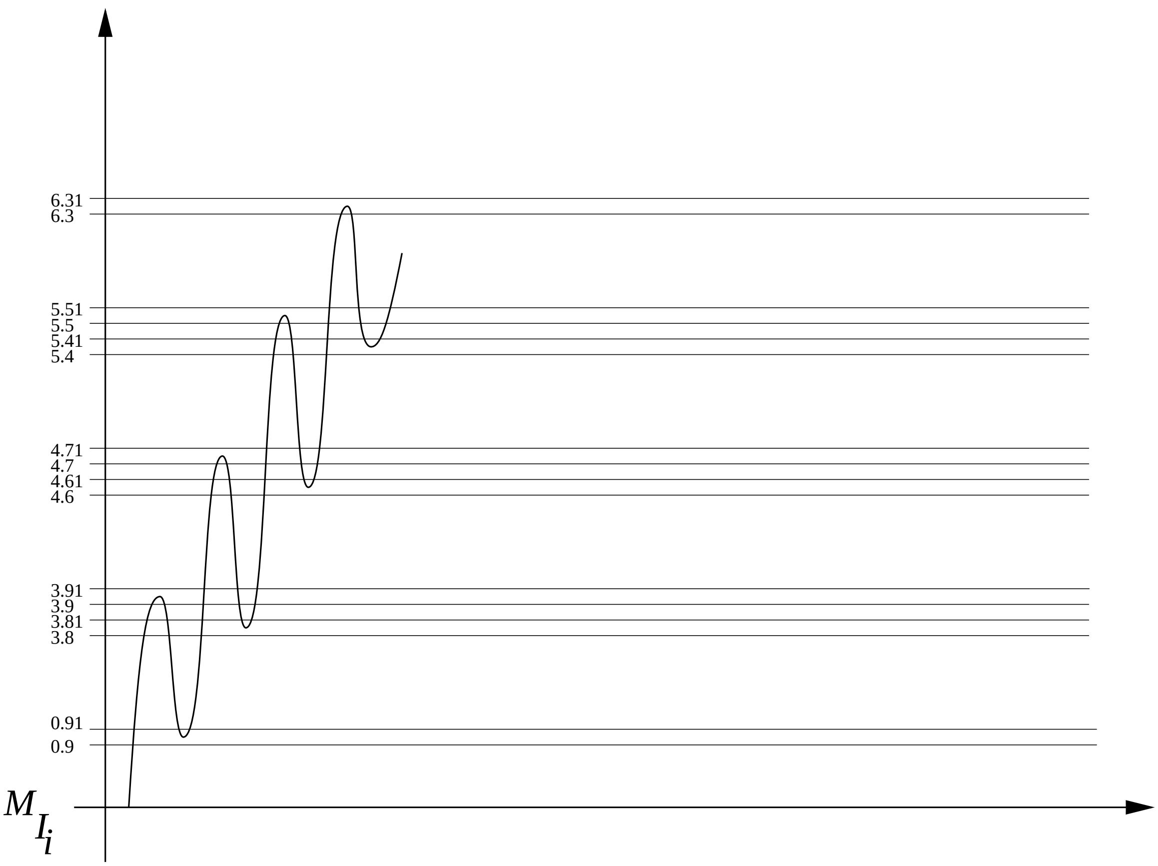

Figure 4.1.
The drawing is not to scale. The picture shows a sample path in the event A_i.
The ticks on the vertical axis show height above the level M_{I_i}.

Suppose that $\omega \in A_i$. Let $r \in [0, 1)$ be such that $M_{I_i}(\omega) + r$ is an integer and let $j = M_{I_i} + r + 4$. Since $r + y' \in [0, 2)$, $j + y' - M_{I_i}$ must belong to one of the intervals $[4.0, 4.7]$, $[4.7, 5.5]$ or $[5.5, 6.31]$. If $j + y' - M_{I_i} \in [4.0, 4.7]$ then $U_{I_i+1} \leq \sigma_j \leq S_{I_i+2}$. Since $M_{I_i+2} - M_{I_i} \in [3.8, 3.81]$, we have $M_{I_i+2} - M_{I_i} > 3.75 > j + y' - M_{I_i} - 1$ which implies that $M_{I_i+2} > j + y' - 1$ and so Z_t does not hit $j + y' - 1$ after σ_j. A similar argument shows that Z_t does not hit $j + y' - 1$ after σ_j in the cases when $j + y' - M_{I_i}$ belongs to $[4.7, 5.5]$ or $[5.5, 6.31]$.

Recall the pseudo-strong Markov property discussed in the remark before Proposition 2.3. Given the values of L_{I_i} and M_{I_i} and assuming $V_{I_i} \in [1.0, 1.01]$, the process $Z(U_{I_i}+t)-M_{I_i}$ is a Bes(3) process starting from a point of $[e,e^{1.01}]$. Hence, (4.5) shows that $\mathbb{P}^y(A_i)\geq\delta$ and $\mathbb{P}^y(A_i^c)\geq\delta$. We apply the pseudo-strong Markov property at L_{I_i} and M_{I_i} to see that

$$\mathbb{P}^y(|V_{J_i}|\geq x \bigm| A_1^c\cap\ldots\cap A_i^c)=\mathbb{P}^y(|V_{J_i}|\geq x \bigm| A_i^c)\leq\delta^{-1}\mathbb{P}^y(|V_{J_i}|\geq x)\leq c_2e^{-c_3x},$$

independent of i.

Since Z_t starts at ℓ, we take $V_1=z=\log\ell$. Let $K=\log a-\log\log a$ and let $N=\min\{i:V_i\geq K\}$. For the process V starting at z,

$$\mathbb{P}^z(V \text{ hits } [K,\infty) \text{ before hitting } [1,1.01])\leq c_4(|z|\vee 1)/K$$

by Proposition 3.2, so we get

$$\begin{aligned}
&\mathbb{P}^y(V(J_i+\,\cdot\,) \text{ hits } [K,\infty) \text{ before hitting } [1,1.01] \bigm| A_1^c\cap\ldots\cap A_i^c)\\
&\quad\leq\int_0^\infty \mathbb{P}^y(|V_{J_i}|\in dx \bigm| A_1^c\cap\ldots\cap A_i^c)\frac{c_4(|x|\vee 1)}{K}\\
&\quad\leq\int_0^\infty \frac{c_4(|x|\vee 1)}{K}c_2e^{-c_3x}dx\leq c_5/K.
\end{aligned}\tag{4.8}$$

How is it possible that Z_t does not have an approximate point of increase for any $j\in[\ell,a]\cap(\mathbb{Z}+y')$? We will be mostly concerned with the case when $L_N<a$. We will later show that the probability of $\{L_N\geq a\}$ is small.

Consider two possible outcomes: (a) $N<I_1$ or (b) $I_1<N$ but the event A_1 does not happen. The probability of (a) is bounded by $c_4(|z|\vee 1)/K$ by Proposition 3.2. The probability of (b) is bounded by $1-\delta$. In case (b), we refine our partition of the probability space and consider the events: (ba) $N<I_2$ and (bb) $I_2<N$ but A_2 does not happen. The probability of (ba) is bounded by c_5/K, by (4.8). An application of the pseudo-strong Markov property at L_{I_2} and M_{I_2} shows that the probability of (bb) is bounded by $1-\delta$. Continuing we obtain the following estimate,

$$\begin{aligned}
&\mathbb{P}^z(\sigma_b\circ\theta_{\sigma_j}>\sigma_{j-1}\circ\theta_{\sigma_j} \text{ for } j\leq a)\\
&\quad\leq\frac{c_4(|z|\vee 1)}{K}+(1-\delta)\left[\frac{c_5}{K}+(1-\delta)\left[\frac{c_5}{K}+(1-\delta)[\ldots]\right]\right]\\
&\quad=\frac{c_4(|z|\vee 1)}{K}+\frac{c_5}{K}(1-\delta)+\frac{c_5}{K}(1-\delta)^2+\cdots\\
&\quad\leq\frac{c_6(|z|\vee 1)}{K}.
\end{aligned}\tag{4.9}$$

The last formula proves the proposition provided we can find a good bound for the probability of $\{L_N\geq a\}$. By Proposition 3.6 and (4.7),

$$\begin{aligned}
\mathbb{P}^z(L_N\geq a)&=\mathbb{P}^z(L_N\geq(\log a)e^K)\leq\mathbb{P}^z(\sum_{i=1}^N e^{V_i}\geq(\log a)e^K)\\
&\leq c_7e^{-c_8\log a}\leq c_7/a^{c_8}.
\end{aligned}$$

Since $a>3$, this proves our proposition. □

5. Two and three angle estimates.

Recall that BM × Bes(3) denotes a two-dimensional process with independent components; the first component is a Brownian motion and the second one is a Bes(3) process. Unless explicitly stated otherwise, we will take both components starting at 0. Two-dimensional Brownian motion conditioned to avoid the x-axis is a BM × Bes(3) process.

Remark. If $Z_t = (X_t, Y_t)$ is a two-dimensional Brownian motion conditioned to avoid the line $\mathcal{L} = \{(x, y) : y = -\varepsilon\}$ then $(X_t, Y_t + \varepsilon)$ is a BM × Bes(3) process starting from $(0, \varepsilon)$. We will prove a number of results for BM × Bes(3) processes and we will sometimes apply these results when ε is small to the process conditioned to avoid $\mathcal{L}$. The minor modifications of the proofs needed for such a generalization are omitted.

We have the following "support theorem" for a BM × Bes(3) process.

Proposition 5.1. *Let ψ be a continuous function from $[0, 1]$ into the closed upper half plane, let $\delta > 0$, and let Z_t be a BM× Bes(3). There exists $c_1 > 0$ depending only on δ and the modulus of continuity of ψ such that*

$$\mathbb{P}^{\psi(0)}_{BMBS}(\sup_{s\le 1} |Z_s - \psi(s)| \le \delta) \ge c_1.$$

Proof. This follows easily from the support theorem for 4-dimensional Brownian motion (see Bass (1995), p. 59) and the fact that a Bes(3) is equal in law to the modulus of a 3-dimensional Brownian motion. □

A straightforward argument using conformal invariance gives

Proposition 5.2. *Suppose $\alpha < \pi/4$ and let F be the wedge $\{re^{i\theta} : 0 < \theta < \pi - \alpha\}$. There exists c_1 such that if $y_0 < R < 1/2$ and $z_0 = (R, y_0)$, then*

$$R^\alpha y_0/4 \le \mathbb{P}^{z_0}_{BM}(\tau_F > \tau_{B(0,1)}) \le c_1 R^{\alpha/4} y_0.$$

Moreover, if $S = \{(x, y) \in \partial B(0, 1/2) : y > 0, |x| \le 1/8\}$, then there exists c_2 such that

$$\mathbb{P}^{z_0}_{BM}(\tau_F > \mathcal{T}_S) \ge c_2 R^\alpha y_0.$$

Proof. We map F onto the upper half plane by the conformal mapping $z \mapsto z^{\pi/(\pi-\alpha)}$. By the conformal invariance of Brownian motion, $\mathbb{P}^{z_0}_{BM}(\tau_F > \tau_{B(0,1)}) = \mathbb{P}^{z_1}_{BM}(\mathcal{T}_{\mathcal{L}_0} > \tau_{B(0,1)})$, where $\mathcal{L}_0$ is the x-axis and $z_1 = (x_1, y_1)$ is the image of z_0. Note $|z_1| = |z_0|^{\pi/(\pi-\alpha)}$, so

$$R^{\pi/(\pi-\alpha)} \le |z_1| \le 2R^{\pi/(\pi-\alpha)}.$$

Since $\arg z_1 = (\pi/(\pi-\alpha)) \arg z_0$, then

$$y_1 = |z_1| \sin\Big(\frac{\pi}{\pi-\alpha} \arctan(y_0/R)\Big).$$

Since $y_0 < R$, then

$$\frac{y_0}{2R} \le \arctan(y_0/R) \le \frac{y_0}{R}, \qquad \sin\Big(2\frac{y_0}{R}\Big) \le 2\frac{y_0}{R}$$

and

$$\frac{y_0}{4R} \le \sin(y_0/2R).$$

Hence

$$\begin{aligned} R^{\alpha/(\pi-\alpha)} y_0 &= R^{\pi/(\pi-\alpha)} \frac{y_0}{R} \le 4R^{\pi/(\pi-\alpha)} \sin\Big(\frac{y_0}{2R}\Big) \le 4R^{\pi/(\pi-\alpha)} \sin\Big(\frac{\pi}{\pi-\alpha}\frac{y_0}{2R}\Big) \\ &\le 4R^{\pi/(\pi-\alpha)} \sin\Big(\frac{\pi}{\pi-\alpha} \arctan(y_0/R)\Big) \le 4y_1, \end{aligned}$$

and

$$\begin{aligned} y_1 &\le 2R^{\pi/(\pi-\alpha)} \sin\Big(\frac{\pi}{\pi-\alpha}\frac{y_0}{R}\Big) \le 2R^{\pi/(\pi-\alpha)} \sin\Big(2\frac{y_0}{R}\Big) \\ &\le 4R^{\pi/(\pi-\alpha)} \frac{y_0}{R} = 4R^{\alpha/(\pi-\alpha)} y_0. \end{aligned}$$

If $\mathcal{L}_1$ is the line $y = 1$, then

$$\begin{aligned} \mathbb{P}^{z_0}_{BM}(\tau_F > \tau_{B(0,1)}) &= \mathbb{P}^{z_1}_{BM}(\mathcal{T}_{\mathcal{L}_0} > \tau_{B(0,1)}) \ge \mathbb{P}^{z_1}_{BM}(\mathcal{T}_{\mathcal{L}_0} > \mathcal{T}_{\mathcal{L}_1}) \\ &= y_1 \ge R^{\alpha/(\pi-\alpha)} y_0/4 \ge R^{\alpha} y_0/4. \end{aligned} \tag{5.1}$$

On the other hand, by (1.3) and scaling,

$$\begin{aligned} \mathbb{P}^{z_0}_{BM}(\tau_F > \tau_{B(0,1)}) &= \mathbb{P}^{z_1}_{BM}(\mathcal{T}_{\mathcal{L}_0} > \tau_{B(0,1)}) \\ &\le c_3 y_1 \le c_3 4R^{\alpha/(\pi-\alpha)} y_0 \le c_3 4R^{\alpha/4} y_0. \end{aligned} \tag{5.2}$$

This proves the first part of the proposition.

To prove the last assertion of the proposition, let S' be the image of S under the conformal mapping $z \mapsto z^{\pi/(\pi-\alpha)}$. Recall that two-dimensional Brownian motion conditioned to hit $\mathcal{L}_1$ before $\mathcal{L}_0$ is a BM $\times$ Bes(3). Note that $|z_1| < \sqrt{2}/2$. It is easy to see using Proposition 5.1 that $\mathbb{P}^{z_1}_{BMBS}(\mathcal{T}_{S'} \le \mathcal{T}_{\mathcal{L}_1}) > c_4 > 0$ provided $|z_1| < \sqrt{2}/2$. So

$$\begin{aligned} \mathbb{P}^{z_1}_{BM}(\mathcal{T}_{S'} < \mathcal{T}_{\mathcal{L}_0}) &= \mathbb{P}^{z_1}_{BM}(\mathcal{T}_{S'} < \mathcal{T}_{\mathcal{L}_0} \mid \mathcal{T}_{\mathcal{L}_1} \le \mathcal{T}_{\mathcal{L}_0}) \mathbb{P}^{z_1}_{BM}(\mathcal{T}_{\mathcal{L}_1} \le \mathcal{T}_{\mathcal{L}_0}) \\ &= \mathbb{P}^{z_1}_{BMBS}(\mathcal{T}_{S'} < \mathcal{T}_{\mathcal{L}_0}) \mathbb{P}^{z_1}_{BM}(\mathcal{T}_{\mathcal{L}_1} \le \mathcal{T}_{\mathcal{L}_0}) \ge c_4 y_1. \end{aligned}$$

Using the conformal invariance of Brownian motion, we see that $\mathbb{P}^{z_0}_{BM}(\mathcal{T}_S < \tau_F) = \mathbb{P}^{z_1}_{BM}(\mathcal{T}_{S'} < \mathcal{T}_{\mathcal{L}_0}) \ge c_4 y_1$. ∎

Corollary 5.3. *Let Z_t be a BM$\times$ Bes(3) process, y_0, R, z_0, and F as in Proposition 5.2. There exists c_1 such that $\mathbb{P}^{z_0}(\tau_F > \tau_{B(0,1)}) \geq c_1 R^\alpha$.*

Proof. Let S be as in Proposition 5.2. The process Z_t is two-dimensional Brownian motion h-path transformed by the function $h(x,y) = y$, so that

$$\mathbb{P}^{z_0}_{BMBS}(\tau_F > \mathcal{T}_S) = \frac{E^{z_0}_{BM}[h(Z_{\mathcal{T}_S}); \tau_F > \mathcal{T}_S]}{h(z_0)} \geq c_2 \frac{\mathbb{P}^{z_0}_{BM}(\tau_F > \mathcal{T}_S)}{y_0}.$$

By Proposition 5.2,

$$\mathbb{P}^{z_0}_{BMBS}(\tau_F > \mathcal{T}_S) \geq c_2 \frac{y_0 R^\alpha/4}{y_0} = c_2 R^\alpha/4.$$

By Proposition 5.1, there exists c_3 such that if $z \in S$, then $\mathbb{P}^{z}_{BMBS}(\tau_F > \tau_{B(0,1)}) \geq c_3$. So by the strong Markov property,

$$\mathbb{P}^{z_0}_{BMBS}(\tau_F > \tau_{B(0,1)}) \geq \mathbb{E}^{z_0}_{BMBS}[\mathbb{P}^{Z(\mathcal{T}_S)}_{BMBS}(\tau_F > \tau_{B(0,1)}); \tau_F > \mathcal{T}_S] \geq c_3 c_2 R^\alpha/4.$$

□

Throughout the rest of the paper we will use lines $\mathcal{L}(j,\alpha)$, which are approximately the lines $y = -\alpha x + j\varepsilon$ when α is small. We give a precise definition. Fix a point $\widetilde{w} \in \mathbb{R}^2$ and let $\mathbb{L}(\alpha)$ be the set of lines with slope $-\tan\alpha$ whose distance from $\widetilde{w}$ is an integer multiple of ε. Let $\mathcal{L}(j,\alpha)$ denote the element of $\mathbb{L}(\alpha)$ whose y-intercept is in the interval $[j\varepsilon/\cos\alpha, (j+1)\varepsilon/\cos\alpha)$. The distance from $\mathcal{L}(j,\alpha)$ to 0 is then in the interval $((|j|-1)\varepsilon, (|j|+1)\varepsilon)$. The constants in our estimates will not depend on $\widetilde{w}$. Actually, for the sake of convenience, we will assume that $\widetilde{w} = (0,0)$ for most of Sections 5 and 6. The only use of the arbitrary nature of $\widetilde{w}$ will be in Lemma 6.9 and Theorem 6.10.

Let

$$\sigma(j,\alpha) = \inf\{t : Z_t \in \mathcal{L}(j,\alpha)\}, \tag{5.3}$$

the hitting time to the line $\mathcal{L}(j,\alpha)$. Note that the "approximate points of increase" discussed in the next proposition have size ε; in Section 4 they had size 1.

Let

$$A_1 = \{\forall j \in [1, 1/\varepsilon] \cap \mathbb{Z} : \sigma(3/(2\varepsilon), 0) \circ \theta_{\sigma(j,0)} > \sigma(j-1, 0) \circ \theta_{\sigma(j,0)}\}$$

and

$$A_2(\delta,\alpha) = \{\tau_{B(0,3)} < \sigma(-\varepsilon^{\delta-1}, -\alpha)\}.$$

We want a lower bound on the probability that a BM $\times$ Bes(3) process has no approximate points of increase and in addition does not hit the line $\mathcal{L}(-\varepsilon^{\delta-1}, -\alpha)$. The first portion of the path, up to time $\sigma(\varepsilon^{\zeta-1}, 0)$, is very unlikely to hit the line $\mathcal{L}(-\varepsilon^{\delta-1}, -\alpha)$ since that line is relatively far away. The portion of the path after this time has some chance of hitting this line and a smaller chance of having an approximate point of increase. The argument is further complicated by the necessity of also dealing with the chances of approximate points of increase near the level ε^ζ.

Proposition 5.4. *Let $Z_t = (X_t, Y_t)$ be a BM× Bes(3) process. There exist constants c_1, ε_0, and δ_0 with the following property. If $\varepsilon < \varepsilon_0$, $\delta \le \delta_1 \le \delta_0$, and $\alpha \le 3/(\delta_1 \log(1/\varepsilon))$, then*

$$\mathbb{P}^{(0,0)}(A_1 \cap A_2(\delta, \alpha)) \ge c_1/\log(1/\varepsilon).$$

Proof. Suppose ζ is a number in $(0,1)$ that will be chosen later. We first look at the portion of the path starting at the first return to $\mathcal{L}(\varepsilon^{\zeta-1}, 0)$ after hitting $\mathcal{L}(3\varepsilon^{\zeta-1}, 0)$.

Observe that $\delta \le \delta_1 \le 3/(\alpha \log(1/\varepsilon))$. Let $R = \min(\varepsilon^\delta/(4\tan\alpha), 1/2)$. Note that the line $\mathcal{L}(-\varepsilon^{\delta-1}, -\alpha)$ intersects the x-axis at the point which is at least R units away from $(0,0)$. If $z = (x, y)$ with $0 \le y \le R/2$ and $|x| \le R/2$, then z lies below the line $\{y = -x + R\}$. Corollary 5.3, scaling and translation invariance imply that there exists c_2 such that $\mathbb{P}^z(A_2(\delta, \alpha)) \ge c_2 R^\alpha$. Since $\tan\alpha \le \sqrt{2}\alpha$ for $\alpha \le \pi/4$ and α^α is bounded above and below by positive constants for $\alpha \in (0, \pi/4]$, then $R^\alpha \ge c_3 \varepsilon^{\alpha\delta}$. Since $\alpha\delta \le 3/\log(1/\varepsilon)$, then for $\varepsilon \le 1$,

$$\varepsilon^{\alpha\delta} \ge \varepsilon^{3/\log(1/\varepsilon)} = e^{3\log\varepsilon/\log(1/\varepsilon)} = e^{-3}.$$

Thus there exists c_4 such that

$$\mathbb{P}^z(A_2(\delta, \alpha)) \ge c_4 \tag{5.4}$$

if $z = (x, y)$, $|x| \le R/2$ and $0 \le y \le R/2$.

Proposition 4.6 shows that we can find $c_5 > 1$ and $\zeta \in (0,1)$ (independent of ε) so that

$$\mathbb{P}^z(B_1) \le c_4/4, \tag{5.5}$$

where $z = (x, y)$, $y \ge \varepsilon^\zeta$, and

$$B_1 = \{\exists j \in [2y, 1/(c_5\varepsilon)] \cap \mathbb{Z} : \sigma(3/(2\varepsilon), 0) \circ \theta_{\sigma(j,0)} < \sigma(j-1, 0) \circ \theta_{\sigma(j,0)}\}.$$

Recall that one-dimensional Brownian motion has no local points of increase. The trajectories of Bes(3) process have the same local path properties as Brownian motion so they have no local points of increase at any level above 0. It follows that when ε is sufficiently small, the probability that a Bes(3) process starting from $1/c_5$ and run until it hits $3/2$ will have an approximate point of increase of size ε at some level between $1/c_5$ and 1 is less than $c_4/4$. By the strong Markov property applied at the time $\sigma(1/(c_5\varepsilon)), 0)$,

$$\mathbb{P}^z(\exists j \in [1/(c_5\varepsilon), 1/\varepsilon] \cap \mathbb{Z} : \sigma(3/(2\varepsilon), 0) \circ \theta_{\sigma(j,0)} < \sigma(j-1, 0) \circ \theta_{\sigma(j,0)}) \le c_4/4,$$

and so, using (5.5),

$$\mathbb{P}^z(B_2) \le c_4/2, \tag{5.6}$$

where

$$B_2 = \{\exists j \in [2y, 1/\varepsilon] \cap \mathbb{Z} : \sigma(3/(2\varepsilon), 0) \circ \theta_{\sigma(j,0)} < \sigma(j-1, 0) \circ \theta_{\sigma(j,0)}\}.$$

Set $\delta_0 = \zeta/2$. If $\delta \le \delta_1 \le \delta_0$, then $\varepsilon^\zeta < \varepsilon^\delta$. Thus if $|x| \le \varepsilon^\delta/(4\tan\alpha)$ and $\varepsilon^\zeta \le y \le \varepsilon^\delta/(4\tan\alpha)$, then by (5.4) and (5.6),

$$\mathbb{P}^z(A_2(\delta, \alpha) \cap B_2^c) \ge c_4/2. \tag{5.7}$$

Now we look at the first part of the trajectory of Z_t. Let

$$B_3 = \{\exists j \in [0, 2\varepsilon^{\zeta-1}] \cap \mathbb{Z} : \sigma(3\varepsilon^{\zeta-1}, 0) \circ \theta_{\sigma(j,0)} < \sigma(j-1, 0) \circ \theta_{\sigma(j,0)}\}.$$

Recall that ζ is independent of ε. Proposition 4.5 and scaling show that for any $z = (x, 0)$,

$$\mathbb{P}^z(B_3^c) \ge c_5/\log(2\varepsilon^{\zeta-1}) \ge c_6(1-\zeta)/\log(1/\varepsilon). \tag{5.8}$$

Let $\eta = (\zeta - \delta)/2$. Since $\delta < \delta_0 = \zeta/2$, then $\eta \ge \zeta/4$ and $\zeta - \eta > \delta$. Then we have

$$\mathbb{P}^z(\sigma(3\varepsilon^{\zeta-1}, 0) > \varepsilon^{2\zeta-\eta}) \le c_7 \exp(-c_8 \varepsilon^{-\eta})$$

and

$$\mathbb{P}^z(\sup_{0 \le s \le \varepsilon^{2\zeta-\eta}} |Z_s - Z_0| \ge \varepsilon^{\zeta-\eta}) \le c_9 \exp(-c_{10}\varepsilon^{-\eta});$$

these two inequalities follow by standard estimates for Brownian motion and the fact that a Bes(3) has the same law as a three-dimensional Brownian motion. For $\alpha \le \pi/4$ and small ε we have $\varepsilon^\delta/(16\tan\alpha) > \varepsilon^{\zeta-\eta}$. Hence,

$$\mathbb{P}^z\Big(\sup_{0 \le s \le \sigma(3\varepsilon^{\zeta-1}, 0)} |Z_s - Z_0| \ge \varepsilon^\delta/(16\tan\alpha)\Big) \le c_{11} \exp(-c_{12}\varepsilon^{-c_{13}\eta}). \tag{5.9}$$

Let

$$B_4 = \{|Z(\sigma(3\varepsilon^{\zeta-1}, 0))| \le \varepsilon^\delta/(8\tan\alpha)\} \cap \{\sigma(3\varepsilon^{\zeta-1}, 0) < \sigma(-\varepsilon^{\delta-1}, -\alpha)\}.$$

Recall that $\mathcal{L}(-\varepsilon^{\delta-1}, -\alpha)$ intersects the x-axis at a distance larger than R from $(0,0)$. The estimates (5.8) and (5.9) show that, for small ε

$$\mathbb{P}^{(0,0)}(B_3^c \cap B_4) \ge c_{14}(1-\zeta)/\log(1/\varepsilon). \tag{5.10}$$

Let

$$T_1 = \sigma(3\varepsilon^{\zeta-1}, 0) + \sigma(\varepsilon^{\zeta-1}, 0) \circ \theta_{\sigma(3\varepsilon^{\zeta-1}, 0)}.$$

From Proposition 5.1 and the strong Markov property we see that given $\{|Z(\sigma(3\varepsilon^{\zeta-1}, 0))| \le \varepsilon^\delta/(8\tan\alpha)\}$, there is a positive probability c_{15} that $T_1 < \infty$ and

$$\sup_{\sigma(3\varepsilon^{\zeta-1}, 0) \le s \le T_1} |Z_s - Z_{\sigma(3\varepsilon^{\zeta-1}, 0)}| \le \varepsilon^\delta/(8\tan\alpha). \tag{5.11}$$

On the set where (5.11) holds, there cannot exist an integer j in $[2\varepsilon^{\zeta-1}, 3\varepsilon^{\zeta-1}]$ for which $\sigma(3/(2\varepsilon),0)\circ\theta_{\sigma(j,0)} < \sigma(j-1,0)\circ\theta_{\sigma(j,0)}$. This, the strong Markov property applied at T_1, (5.7), and (5.10) imply that for small ε

$$\mathbb{P}^{(0,0)}(A_1\cap A_2(\delta,\alpha)) \geq \Big(c_{14}(1-\zeta)/\log(1/\varepsilon)\Big)c_{15}(c_4/2) = c_{16}/\log(1/\varepsilon). \qquad \square$$

Next we will consider two-dimensional Brownian motion. We will obtain a lower bound for the probability that the process hits $\mathcal{L}(j,\alpha)$ before hitting $\mathcal{L}(-1,0)$ again; upon hitting $\mathcal{L}(j,\alpha)$ we have $|X_t| \leq j\varepsilon$; and the Brownian motion then exits $B(0,3)$ before returning to either $\mathcal{L}(j-1,\alpha)$ or $\mathcal{L}(-1,0)$. These estimates will follow from Proposition 5.2, standard estimates on Brownian motion, and the strong Markov property. Recall that we write $Z_t = (X_t, Y_t)$. Let

$$\begin{aligned} A_1 &= \{\sigma(j-1,\alpha)\circ\theta_{\sigma(0,0)} < (\sigma(-1,0)\wedge\tau_{B(0,3/2)})\circ\theta_{\sigma(0,0)}, |X_{\sigma(j,\alpha)}| \leq j\varepsilon\},\\ A_2 &= \{\tau_{B(0,3)}\circ\theta_{\sigma(j,\alpha)} < (\sigma(j-1,\alpha)\wedge\sigma(-1,0))\circ\theta_{\sigma(j,\alpha)}\}. \end{aligned} \tag{5.12}$$

Proposition 5.5. *Let Z_t be a Brownian motion, $\alpha\in(5/\log(1/\varepsilon),\pi/4)$, $w=(w_1,w_2)$, $|w_1|\leq \varepsilon^{3/2\alpha\log(1/\varepsilon)}$, and $0<w_2\leq\varepsilon^{3/2\alpha\log(1/\varepsilon)}$. Suppose $j\geq \varepsilon^{3/\alpha\log(1/\varepsilon)-1}$. There exists c_1 such that for ε sufficiently small,*

(a) $\mathbb{P}^{(0,0)}(A_1\cap A_2)\geq c_1\varepsilon/j$; and

(b) $\mathbb{P}^{(w_1,w_2)}(A_1\cap A_2)\geq c_1(w_2+\varepsilon)/j$.

Proof. First we show that (b) follows from (a). Let H be the half plane above $\mathcal{L}(-1,0)$. For small ε, $B(0,j\varepsilon/2)$ is disjoint from $\mathcal{L}(j-1,\alpha)$. Hence, the function $(w_1,w_2)\to\mathbb{P}^{(w_1,w_2+\varepsilon)}(A_1\cap A_2)$ is positive harmonic in $H\cap B(0,j\varepsilon/2)$, with boundary values 0 on ∂H. By applying the boundary Harnack principle (Theorem 1.1) to this function and the function $(w_1,w_2)\to w_2+\varepsilon$, we obtain

$$\frac{\mathbb{P}^{(w_1,w_2)}(A_1\cap A_2)}{\mathbb{P}^{(0,0)}(A_1\cap A_2)} \geq c_2\frac{w_2+\varepsilon}{\varepsilon},$$

for $(w_1,w_2)\in H\cap B(0,j\varepsilon/4)$. Note that $w\in H\cap B(0,j\varepsilon/4)$ if the assumptions of the proposition are satisfied. This and (a) easily imply (b).

Next we prove (a). Starting at $(0,0)$, there is probability at least $1/(2j+1)$ that Z_t hits $\mathcal{L}(2j,0)$ before hitting $\mathcal{L}(-1,0)$. If Z_t is conditioned to hit $\mathcal{L}(2j,0)$ before $\mathcal{L}(-1,0)$ then Y_t is a Bes(3) process and by scaling and Proposition 5.1,

$$\mathbb{P}^{(0,0)}_{BM}\Big(\sup_{s\leq\sigma(2j,0)}|X_s-X_0|\leq j\varepsilon/100 \;\Big|\; \sigma(2j,0)<\sigma(-1,0)\Big)\geq c_3.$$

So there is probability at least c_4/j that started at $(0,0)$, Z_t hits $\mathcal{L}(2j,0)$ before returning to $\mathcal{L}(-1,0)$, and

$$\sup_{s\leq\sigma(2j,0)}|X_s|\leq 2j\varepsilon/100.$$

If Z_t does this, then Z_t must have hit $\mathcal{L}(j,\alpha)$ before returning to $\mathcal{L}(-1,0)$ or exiting $B(0,3/2)$; moreover, $|X_{\sigma(j,\alpha)}| \le 2j\varepsilon/100$. With $\mathbb{P}^{(0,0)}$-probability 1 we have $\sigma(0,0)=0$, so

$$\mathbb{P}^{(0,0)}(A_1) \ge c_4/j. \tag{5.13}$$

To estimate A_2 we use Proposition 5.2. The line $\mathcal{L}(j-1,\alpha)$ crosses $\mathcal{L}(-1,0)$ at a point $(R,-\varepsilon)$ with $R \ge j\varepsilon/(2\tan\alpha)$. Let F be the open wedge contained in the upper half-plane bounded by the lines $\mathcal{L}(j-1,\alpha)$ and $\mathcal{L}(-1,0)$. Consider a point $z=(x,y)\in\mathcal{L}(j,\alpha)$ with $|x|\le j\varepsilon/25$. Then the distance of z from the vertex of F is greater than $R/2$ and its distance from ∂F is equal to ε. By Proposition 5.2 and scaling,

$$\mathbb{P}^z(\tau_{B(0,3)} < \sigma(j-1,\alpha)\wedge\sigma(-1,0)) \ge c_5\varepsilon R^\alpha.$$

Note $(\tan\alpha)^\alpha$ is bounded above by a positive constant when $\alpha\in(0,\pi/4)$. By our assumption on j, $j\varepsilon \ge \varepsilon^{3/\alpha\log(1/\varepsilon)}$, so

$$(j\varepsilon)^\alpha \ge \varepsilon^{3/\log(1/\varepsilon)} = e^{-3},$$

and, therefore, $R^\alpha > c_6$. Hence, for $z=(x,y)\in\mathcal{L}(j,\alpha)$ with $|x|\le j\varepsilon/25$, we have $\mathbb{P}^z(A_2)\ge c_7\varepsilon$. This, (5.13), the fact that $j\varepsilon/25 < j\varepsilon$, and the strong Markov property yield

$$\mathbb{P}^{(0,0)}(A_1\cap A_2) \ge c_8\varepsilon/j.$$

□

We now obtain an upper bound for the probability that Brownian motion has approximate points of increase at $\mathcal{L}(0,0)$, $\mathcal{L}(j,\alpha)$, and $\mathcal{L}(k,\beta)$. In the case where $\mathcal{L}(j,\alpha)$ and $\mathcal{L}(k,\beta)$ intersect at a point whose distance from the y-axis is greater than $j\varepsilon$, we again use Proposition 5.2, standard estimates on Brownian motion, and the strong Markov property. Let

$$\begin{aligned} A_1 &= \{\tau_{B(3/2,0)} < \sigma(-1,0)\},\\ A_2(j,\alpha) &= \{\tau_{B(3/2,0)}\circ\theta_{\sigma(j,\alpha)} < \sigma(j-1,\alpha)\circ\theta_{\sigma(j,\alpha)}, |X_{\sigma(j,\alpha)}|\le j\varepsilon\}. \end{aligned} \tag{5.14}$$

Proposition 5.6. *Let Z_t be a Brownian motion, $w=(w_1,w_2)$, $|w_1|\le\varepsilon^{3/2\alpha\log(1/\varepsilon)}$, and $0<w_2\le\varepsilon^{3/2\alpha\log(1/\varepsilon)}$. There exists c_1 such that if $\alpha,\beta\in[\log^{-1}(1/\varepsilon),\pi/100)$, $j,k\in[\varepsilon^{-1/2},\varepsilon^{-1}]$, $k\ge j+200j|\beta-\alpha|$, and ε is sufficiently small, then*

(a)

$$\mathbb{P}^{(0,0)}(A_1\cap A_2(j,\alpha)\cap A_2(k,\beta)) \le c_1\frac{((|k-j|+1)\varepsilon)^{|\beta-\alpha|/4}}{j(|k-j|+1)}\varepsilon,$$

and

(b)

$$\mathbb{P}^{w}(A_1\cap A_2(j,\alpha)\cap A_2(k,\beta)) \le c_1\frac{((|k-j|+1)\varepsilon)^{|\beta-\alpha|/4}}{j(|k-j|+1)}(w_2+\varepsilon).$$

Proof. Part (b) follows from (a) by the boundary Harnack principle, just as in Proposition 5.5, so it will suffice to prove (a).

Let $\gamma = |\beta - \alpha|$. Because $\gamma \le \pi/100$, then $\tan\gamma \le 2\gamma$. Since the angle between the lines $\mathcal{L}(j,\alpha)$ and $\mathcal{L}(k,\beta)$ is γ, the absolute value of the x-coordinate of their point of intersection is at least

$$(k-j)\varepsilon/2\tan\gamma \ge (k-j)\varepsilon/4\gamma.$$

Our assumptions on j and k imply that this is greater than or equal to $50j\varepsilon$. We also have $k/(1+200\gamma) \ge j$ and $1+200\gamma < 8$ so that

$$k-j \ge k - \frac{k}{1+200\gamma} = \frac{200\gamma k}{1+200\gamma} \ge 25\gamma k,$$

and hence $(k-j)\varepsilon/4\gamma \ge 6k\varepsilon$. Therefore, for the process Z_t started at $(0,0)$ to have an approximate point of increase at $\mathcal{L}(j,\alpha)$ with $|X_{\sigma(j,\alpha)}| \le j\varepsilon$ and to have an approximate point of increase at $\mathcal{L}(k,\beta)$ with $|X_{\sigma(k,\beta)}| \le k\varepsilon$, we must have the process hitting $\mathcal{L}(j,\alpha)$ before hitting $\mathcal{L}(k,\beta)$.

Starting at $(0,0)$, to hit $\mathcal{L}(j,\alpha)$ before hitting $\mathcal{L}(-1,0)$ with $|X_{\sigma(j,\alpha)}| \le j\varepsilon$, the process Z_t must hit the line $\mathcal{L}(j/2,0)$ before hitting $\mathcal{L}(-1,0)$, and the probability of that is bounded above by

$$1/(1+j/2) \le 2/j. \tag{5.15}$$

Suppose $z=(x,y) \in \mathcal{L}(j,\alpha)$ with $|x| \le j\varepsilon$. Since the absolute value of the x-coordinate of the point of intersection of $\mathcal{L}(j,\alpha)$ and $\mathcal{L}(k,\beta)$ is at least $6k\varepsilon$, for the process Z_t started at z to hit $\mathcal{L}(k,\beta)$ before hitting $\mathcal{L}(j-1,\alpha)$ and $|X_{\sigma(k,\beta)}| \le k\varepsilon$, it must hit $\mathcal{L}(j+(k-j)/2,\alpha)$ before hitting $\mathcal{L}(j-1,\alpha)$, and the probability of that is bounded by

$$1/(1+(k-j)/2) \le 2/((k-j)+1). \tag{5.16}$$

Finally, suppose $z=(x,y) \in \mathcal{L}(k,\beta)$ with $|x| \le k\varepsilon$. If $\beta=\alpha$, there exists c_2 such that

$$\mathbb{P}^z(\tau_{B(3/2,0)} < \sigma(k-1,\beta)) \le c_2\varepsilon$$

by (1.3) and scaling. Now consider the case $\beta \ne \alpha$ and let F be the wedge in the upper half plane bounded by $\mathcal{L}(j-1,\alpha)$ and $\mathcal{L}(k-1,\beta)$. Since $|x| \le k\varepsilon$, $\tan\gamma \ge \gamma/2$, and $k-j \ge 25\gamma k$, the distance from z to the vertex of F is no more than

$$2k\varepsilon + 2(k-j)\varepsilon/\tan\gamma + 2\varepsilon \le 5((k-j)+1)\varepsilon/\gamma.$$

Also the distance from z to ∂F is ε. So by Proposition 5.2,

$$\begin{aligned}\mathbb{P}^z(\tau_{B(3/2,0)} < \sigma(j-1,\alpha)\wedge\sigma(k-1,\beta)) &\le c_3\Big(\frac{((k-j)+1)\varepsilon}{\gamma}\Big)^{\gamma/4}\varepsilon \qquad (5.17)\\ &\le c_3((k-j)+1)^{|\beta-\alpha|/4}\varepsilon^{|\beta-\alpha|/4}\varepsilon.\end{aligned}$$

So both when $\beta = \alpha$ and when $\beta \neq \alpha$, the left hand side of (5.17) is bounded by

$$c_4((k-j)+1)^{|\beta-\alpha|/4}\varepsilon^{|\beta-\alpha|/4}\varepsilon.$$

Using this, (5.15), (5.16), and the strong Markov property at $\sigma(j,\alpha)$ and $\sigma(k,\beta)$, we obtain

$$\mathbb{P}^{(0,0)}(A_1 \cap A_2(j,\alpha) \cap A_2(k,\beta)) \leq \frac{2}{j}\frac{2}{(k-j)+1}c_4((k-j)+1)^{|\beta-\alpha|/4}\varepsilon^{|\beta-\alpha|/4}\varepsilon. \ \square$$

Let A_1 and $A_2(j,\alpha)$ continue to be defined by (5.14). We now need a bound for the probability of the event considered in Proposition 5.6 when $j \leq k \leq j + 200j|\beta-\alpha|$. Here we consider the possibility that $\mathcal{L}(j,\alpha)$ and $\mathcal{L}(k,\beta)$ intersect at a point within $j\varepsilon$ of the y-axis, and the proof is more complicated.

Proposition 5.7. *Let Z_t, w, α, and β be as in Proposition 5.6. There exists c_1 such that if $j,k \in [\varepsilon^{-1/2}, \varepsilon^{-1}]$,*

$$\log^{-1/2}(1/\varepsilon) \geq |\beta-\alpha| \geq \log^{-1}(1/\varepsilon), \qquad j \leq k \leq j + 200j|\beta-\alpha|,$$

and ε is sufficiently small, then

(a)

$$\mathbb{P}^{(0,0)}(A_1 \cap A_2(j,\alpha) \cap A_2(k,\beta)) \leq \frac{c_1\varepsilon}{j^2|\beta-\alpha|^2},$$

and

(b)

$$\mathbb{P}^{w}(A_1 \cap A_2(j,\alpha) \cap A_2(k,\beta)) \leq \frac{c_1(w_2+\varepsilon)}{j^2|\beta-\alpha|^2}.$$

Proof. Part (b) follows from (a) just as in Propositions 5.5 and 5.6.

We will now prove (a). Let $\gamma = |\beta-\alpha|$; our assumptions imply $\gamma > 0$. Let

$$B_1 = \{\sigma(j,\alpha) < \sigma(k,\beta)\}.$$

We derive a number of estimates for various types of paths started at various places, and then put the estimates together using the strong Markov property.

Let H be the upper half plane and let $K = \partial B(0, j\varepsilon/2) \cap H$. Observe that for Z_t started at $(0,0)$ to hit $\mathcal{L}(j,\alpha)$ before hitting $\mathcal{L}(-1,0)$ with $|X_{\sigma(j,\alpha)}| \leq j\varepsilon$, the process must hit K before hitting $\mathcal{L}(-1,0)$. So if

$$B_2 = \{\mathcal{T}_K < \sigma(-1,0)\},$$

then $A_1 \cap A_2(j,\alpha) \subseteq B_2$ and, using (1.3),

$$\mathbb{P}^{(0,0)}(B_2) \leq c_2/(1+j/2) \leq 2c_2/j. \tag{5.18}$$

Secondly, suppose $z \in K$. Let F_1 be the wedge containing the point $(0,0)$ that is bounded by the lines $\mathcal{L}(j,\alpha)$ and $\mathcal{L}(k,\beta)$, let v be its vertex, let $\psi_\alpha(s)$ be the point in $\partial F_1 \cap \mathcal{L}(j,\alpha)$ whose distance from v is s, and let $E(s_1,s_2) = \{\psi_\alpha(s) : s_1 \le s \le s_2\}$. The hitting distribution of $\mathcal{L}(j,\alpha)$ by the process Z_t is given by the Poisson kernel in a half plane (see, e.g., Bass (1995), p. 91), and so

$$\mathbb{P}^z(Z_{\sigma(j,\alpha)} \in E(s_1,s_2)) = \frac{1}{\pi}\int_{s_1}^{s_2} \frac{\operatorname{dist}(z,\mathcal{L}(j,\alpha))}{|z-\psi_\alpha(s)|^2}\, ds. \tag{5.19}$$

Observe that for any s, $\operatorname{dist}(z,\mathcal{L}(j,\alpha)) \le |z-\psi_\alpha(s)|$. Also, if $z \in K$ and the absolute value of the x-coordinate of $\psi_\alpha(s)$ is bounded by $j\varepsilon$, then $|z-\psi_\alpha(s)| \ge c_3 j\varepsilon$. So from (5.19) we conclude

$$\mathbb{P}^z(Z_{\sigma(j,\alpha)} \in E(s_1,s_2), |X_{\sigma(j,\alpha)}| \le j\varepsilon) \le \frac{c_4(s_2-s_1)}{j\varepsilon}, \tag{5.20}$$

for $z=(x,y)\in K$.

Let $R>0$, $D_1 = \{(x,y) : 0<y<1, |x|<R\}$ and $\partial D_1^- = \{(x,y) \in \partial D_1 : y=0\}$. Note that $u_1(z) = \mathbb{P}^z_{BM}(Z_{\tau(D_1)} \notin \partial D_1^-)$ is a nonnegative harmonic function in D_1 with boundary values 1 on the sides $\{|x|=R\}$ and on the top $\{y=1\}$ and boundary values 0 on the bottom $\{y=0\}$. The function $u_2(x,y)=y$ is also nonnegative and harmonic in D_1. By the boundary Harnack principle in $D_1 \cap B((0,1/2),2)$, there exists c_5 independent of R such that

$$\mathbb{P}^{(0,y)}_{BM}(Z_{\tau(D_1)} \notin \partial D_1^-) = u_1(0,y) \le c_5 u_1(0,1/2)\frac{u_2(0,y)}{u_2(0,1/2)}, \qquad 0<y<1/2.$$

Since $u_1(0,1/2)\le 1$, we obtain

$$\mathbb{P}^{(0,y)}_{BM}(Z_{\tau(D_1)} \notin \partial D_1^-) \le c_6 y. \tag{5.21}$$

Let $D(s)$ be the rectangle with two of its sides on the lines $\mathcal{L}(j-1,\alpha)$, $\mathcal{L}(j+s\gamma/(10\varepsilon),\alpha)$, and the other two sides containing the points $\psi_\alpha(s/2)$ and $\psi_\alpha(3s/2)$, respectively. If

$$B_3 = \{Z(\tau_{D(s)}) \notin \mathcal{L}(j-1,\alpha)\},$$

then by (5.21) and scaling

$$\mathbb{P}^{\psi_\alpha(s)}(B_3) \le c_6\Big(\frac{\varepsilon}{s\gamma}\wedge 1\Big). \tag{5.22}$$

Next suppose $z \in \partial D(s) - \mathcal{L}(j-1,\alpha)$. Then $|z-\psi_\alpha(s)|\le s$. We then see that the distance from z to $\mathcal{L}(k,\beta)$ is no more than $c_7\gamma s$. Let

$$B_4 = \{|Z_{\sigma(k,\beta)} - v| \ge 6|Z_{\sigma(j,\alpha)} - v|\}.$$

From the formula for the Poisson kernel in the upper half plane, there exists c_8 such that

$$\mathbb{P}^{(0,1)}(|X_{\sigma(0,0)}| \ge r) \le c_8/r.$$

Using this and scaling we obtain for $z \in \partial D(s) - \mathcal{L}(j-1,\alpha)$,

$$\mathbb{P}^z(B_4) \le c_9\gamma. \tag{5.23}$$

Let F_2 be the wedge bordered below by $\mathcal{L}(j,\alpha)$ and $\mathcal{L}(k,\beta)$, i.e., the wedge with obtuse angle and sides on the lines $\mathcal{L}(j,\alpha)$ and $\mathcal{L}(k,\beta)$ but different from F_1 (F_2 contains the point $(0,y)$ for all y sufficiently large). Note that F_1 and F_2 have the same vertex v. Let $\psi_\beta(t)$ be the point in $\mathcal{L}(k,\beta)\cap\partial F_2$ whose distance from the vertex v is t. The distance from $\psi_\beta(t)$ to the point of intersection of the lines $\mathcal{L}(j-1,\alpha)$ and $\mathcal{L}(k-1,\beta)$ is then less than $t+2\varepsilon$. The point $\psi_\beta(t)$ is ε units away from the boundary of the wedge with sides on the lines $\mathcal{L}(j-1,\alpha)$ and $\mathcal{L}(k-1,\beta)$ and containing F_2. So by Proposition 5.2,

$$\mathbb{P}^{\psi_\beta(t)}(\tau_{B(3/2,0)} < \sigma(j-1,\alpha)\wedge\sigma(k-1,\beta)) \le c_{10}(t+2\varepsilon)^{\gamma/4}\varepsilon. \tag{5.24}$$

We are now ready to put our estimates together. If the event $A_2(k,\beta)$ holds, then $|X_{\sigma(k,\beta)}| \le k\varepsilon$, which implies that the t in (5.24) is less than 2. By the strong Markov property at times $\mathcal{T}_K$, $\sigma(j,\alpha)$, $\tau_{D(s)}$, and $\sigma(k,\beta)$ and using (5.18), (5.20), (5.22), (5.23), and (5.24),

$$\begin{aligned}
&\mathbb{P}^{(0,0)}(A_1\cap A_2(j,\alpha)\cap A_2(k,\beta)\cap B_1\cap B_4) \qquad (5.25)\\
&\quad\le \frac{2c_2}{j}\frac{c_3}{j\varepsilon}\left(\int_0^2 c_6\Big(\frac{\varepsilon}{s\gamma}\wedge 1\Big)\,ds\right)(c_9\gamma)(c_{10}(2+2\varepsilon)^{\gamma/4}\varepsilon)\\
&\quad\le \frac{c_{11}\gamma}{j^2}\Big(\int_0^{\varepsilon/\gamma} ds + \frac{\varepsilon}{\gamma}\int_{\varepsilon/\gamma}^2 \frac{ds}{s}\Big)\\
&\quad\le \frac{c_{12}\gamma}{j^2}\Big(\frac{\varepsilon}{\gamma} + \frac{\varepsilon\log(1/\varepsilon)}{\gamma}\Big)\\
&\quad\le \frac{c_{13}\varepsilon}{j^2\gamma^2},
\end{aligned}$$

since $\gamma \le \log^{-1/2}(1/\varepsilon)$.

If the event B_4^c holds, then $|Z_{\sigma(k,\beta)} - v| < 6|Z_{\sigma(j,\alpha)} - v|$, which implies that the t in (5.24) is less than $6s$. So using (5.18), (5.20), (5.22), and (5.24),

$$\begin{aligned}
&\mathbb{P}^{(0,0)}(A_1\cap A_2(j,\alpha)\cap A_2(k,\beta)\cap B_1\cap B_4^c) \qquad (5.26)\\
&\quad\le \frac{2c_2}{j}\frac{c_4}{j\varepsilon}\int_0^2 c_6\Big(\frac{\varepsilon}{s\gamma}\wedge 1\Big)c_{10}(6s+2\varepsilon)^{\gamma/4}\varepsilon\,ds\\
&\quad\le \frac{c_{14}}{j^2}\Big(\int_0^{\varepsilon/\gamma} ds + \frac{\varepsilon}{\gamma}\int_{\varepsilon/\gamma}^2 s^{\gamma/4-1}\,ds\Big)\\
&\quad\le \frac{c_{15}}{j^2}\Big(\frac{\varepsilon}{\gamma} + \frac{\varepsilon}{\gamma^2}(2^{\gamma/4} - (\varepsilon/\gamma)^{\gamma/4})\Big)\\
&\quad\le \frac{c_{16}\varepsilon}{j^2\gamma^2}.
\end{aligned}$$

Combining (5.25) and (5.26),

$$\mathbb{P}^{(0,0)}(A_1 \cap A_2(j,\alpha) \cap A_2(k,\beta) \cap B_1) \le \frac{c_{17}\varepsilon}{j^2\gamma^2}.$$

The case when $\sigma(k,\beta) < \sigma(j,\alpha)$, i.e., when B_1^c holds, is treated entirely analogously. $\square$

Remark. We get similar bounds to those in Propositions 5.6 and 5.7 and in Proposition 5.11 below if we reverse the roles of j and k.

In the next proposition we show that a BM $\times$ Bes(3) process conditioned to go to a point not in $B(0,4)$ behaves essentially like a BM $\times$ Bes(3) up until first exiting $B(0,2)$.

Proposition 5.8. *Let $g(\,\cdot\,, z_0)$ be the Green function for BM$\times$ Bes(3) process with pole at z_0. There exists c_1 such that if z_0 is in the upper half plane H with $|z_0| > 4$ and Z_t is a BM$\times$ Bes(3) process, then*

$$c_1^{-1}\mathbb{P}^z(A) \le \mathbb{P}^z_{g(\cdot,z_0)}(A) \le c_1\mathbb{P}^z(A), \qquad A \in \mathcal{F}_{\tau(B(0,2))}, \quad z \in B(0,2). \tag{5.27}$$

Proof. The Green function for Brownian motion killed on exiting the upper half-plane H is

$$g_1(z_1,z_2) = \log\Big(\frac{|z_2+z_1|}{|z_2-z_1|}\Big), \qquad z_1,z_2 \in H.$$

Since a BM $\times$ Bes(3) process is the h-path transform by the function $h(x,y)=y$ of a two-dimensional Brownian motion killed on hitting ∂H, the Green function for a BM $\times$ Bes(3) process is given by

$$g(z_1,z_2) = \frac{y_2}{y_1}g_1(z_1,z_2), \qquad z_1=(x_1,y_1), \quad z_2=(x_2,y_2),$$

for $z_1,z_2 \in H$. Recall the boundary Harnack principle, Theorem 1.1. If $z_0 \in H - B(0,2)$ then the function $z_1 \to g_1(z_1,z_0)$ is positive and harmonic in $H \cap B(0,2)$ and has zero boundary values on ∂H. The same is true of the function $h(x,y)=y$. There exists c_2 independent of z_0 such that

$$\frac{g_1(z_1,z_0)}{y_1} \le c_2\frac{g_1(z_2,z_0)}{y_2} \tag{5.28}$$

for all $z_1,z_2 \in B(0,2)\cap H$, $z_0=(x_0,y_0)$, $z_1=(x_1,y_1)$, $z_2=(x_2,y_2)$. It follows from (5.28) that

$$\frac{g(z_1,z_0)}{g(z_2,z_0)} = \frac{y_2}{y_1}\frac{g_1(z_1,z_0)}{g_1(z_2,z_0)} \le c_2$$

for all $z_1,z_2 \in B(0,2)\cap H$. The inequality extends to z_1 and z_2 in the closure of $B(0,2)\cap H$ by continuity. We apply (1.1) to see that

$$\mathbb{P}^z_{g(\cdot,z_0)}(A) = \mathbb{E}^z_{g(\cdot,z_0)}(g(Z_{\tau(B(0,2))},z_0)1_A)/g(z,z_0) \le c_2\mathbb{E}^z(1_A) = \mathbb{P}^z(A).$$

The proof of the other inequality is similar. □

Let

$$A_n = \{\tau_{B(0,2)} \circ \theta_{\sigma(n,0)} < \sigma(n-1,0) \circ \theta_{\sigma(n,0)}, \sigma(n,0) < \infty\}. \tag{5.29}$$

Our definition implies that A_n^c holds if $\sigma(n,0) = \infty$. In other words, if the process does not hit the line $\mathcal{L}(n,0)$ then there is no approximate point of increase at $\mathcal{L}(n,0)$.

Let $\widetilde{Z}_t = (\widetilde{X}_t, \widetilde{Y}_t) = (-X_{\sigma(0,0)-t}, -Y_{\sigma(0,0)-t})$. If Z_t is a two-dimensional Brownian motion started at $z_0 = (x_0, y_0)$ with $y_0 < 0$ and and we set $z_1 = (-x_0, -y_0)$, then $\widetilde{Z}$ is a BM $\times$ Bes(3) process conditioned to go to z_1, that is, a BM $\times$ Bes(3) process conditioned by the Green function $g(\,\cdot\,, z_1)$ for BM $\times$ Bes(3) with pole at z_1. This result on time reversal follows easily from Chung and Walsh (1969), for example. We will often apply Proposition 5.8 to the process $\widetilde{Z}$.

Let $\widetilde{A}_n$ be the event A_n defined relative to $\widetilde{Z}_t$. We set

$$B = A_0 \cap \widetilde{A}_1^c \cap \cdots \cap \widetilde{A}_{1/\varepsilon}^c. \tag{5.30}$$

Let

$$C(j,\alpha) = \{\sigma(j,\alpha) < \sigma(0,0) + \tau_{B(0,2)} \circ \theta_{\sigma(0,0)}, \tau_{B(0,2)} \circ \theta_{\sigma(j,\alpha)} < \sigma(j-1,\alpha) \circ \theta_{\sigma(j,\alpha)}, \\ |X_{\sigma(j,\alpha)}| \le j\varepsilon\}. \tag{5.31}$$

The event $C(j,\alpha)$ holds if the process Z_t hits the line $\mathcal{L}(j,\alpha)$ so that $|X_{\sigma(j,\alpha)}| \le j\varepsilon$ (which implies that the line $\mathcal{L}(j,\alpha)$ is first hit in the upper half plane), the line $\mathcal{L}(j,\alpha)$ is hit before the process Z_t exits $B(0,2)$ after hitting $\mathcal{L}(0,0)$, and there is an approximate point of increase at $\mathcal{L}(j,\alpha)$. Let

$$D = \{X_{\sigma(0,0)} = 0\}. \tag{5.32}$$

Many propositions will involve conditioning Brownian motion by the event D, which has probability 0. Suppose that Z_t is a Brownian motion starting from $z = (x,y)$ with $y < 0$. Then, given D, the process $\widetilde{Z}$ is a BM $\times$ Bes(3) process starting from $(0,0)$ and conditioned to go to $(-x,-y)$.

The events, $A_n, \widetilde{A}_n, B, C(j,\alpha)$, and D will be referred to throughout the remainder of this section and in all of Section 6.

We put together Propositions 5.4 and 5.5 to obtain a lower bound for there being an approximate cut line at $\mathcal{L}(j,\alpha)$, an approximate cut line at $\mathcal{L}(0,0)$, but no approximate cut line at $\mathcal{L}(n,0)$ for $n < 0$.

Proposition 5.9. *Suppose $\alpha = 3/(\delta_1 \log(1/\varepsilon))$, where $3/\delta_1 > 5$ and δ_1 is less than the constant δ_0 defined in the statement of Proposition 5.4. Assume that*

$$\varepsilon^{(3/\alpha\log(1/\varepsilon))-1} \le j \le \varepsilon^{(1/\alpha\log(1/\varepsilon))-1}.$$

There exists c_1 independent of j, ε, and α such that if $z = (x, y)$, $y < -1$, and $|z| > 4$, then

$$\mathbb{P}^z(B \cap C(j, \alpha) \mid D) \geq \frac{c_1 \varepsilon}{j \log(1/\varepsilon)}.$$

Proof. We split the path at time $\sigma(0,0)$. An estimate for our probability may be obtained from the product of two factors, the first from the portion of the path after time $\sigma(0,0)$ and the second from the portion of the path before this time.

We use the strong Markov property and Proposition 5.5 (a) to give the bound $c_2\varepsilon/j$ for the first portion, recalling that we are conditioning $X_{\sigma(0,0)}$ to be 0.

To bound the second portion of the path, that is, the portion before time $\sigma(0,0)$, we use time reversal. Recall that $\widetilde{Z} = (-X_{\sigma(0,0)-t}, -Y_{\sigma(0,0)-t})$ is a BM $\times$ Bes(3) process conditioned to go to $(-x, -y)$. Note that if we write $j = \varepsilon^{\delta-1}$, then

$$\frac{1}{\alpha \log(1/\varepsilon)} \leq \delta \leq \frac{3}{\alpha \log(1/\varepsilon)},$$

and so $\delta \leq \delta_1 \leq \delta_0$. Hence we can apply Proposition 5.4 and Proposition 5.8 to get a factor $c_3/\log(1/\varepsilon)$. Taking the product of the two factors gives the desired estimate. □

Since there is always the possibility that a BM $\times$ Bes(3) process may exit $B(0,2)$ before hitting a line $\mathcal{L}(n, 0)$, $n \leq 1/\varepsilon$, we derive the following adaptation of Proposition 4.7.

Proposition 5.10. *Let Z_t be a BM$\times$ Bes(3) process and let the events A_n be defined as in (5.29). There exists c_1 such that*

$$\mathbb{P}^{(0,0)}(A_1^c \cap \cdots \cap A_{1/\varepsilon}^c) \leq \frac{c_1}{\log(1/\varepsilon)}.$$

Proof. Let $\widehat{A}$ be the event that Z_t escapes from $B(0, 1/4)$ before hitting $\mathcal{L}(2\varepsilon^{-1/2}, 0)$, and let

$$A'_j = \{\sigma(2\varepsilon^{-1/2}, 0) \circ \theta_{\sigma(j,0)} < \sigma(j-1, 0) \circ \theta_{\sigma(j,0)}\}.$$

We have

$$\mathbb{P}^{(0,0)}(A_2^c \cap \cdots \cap A_{1/\varepsilon}^c) \leq \mathbb{P}^{(0,0)}(\widehat{A}) + \mathbb{P}^{(0,0)}((A'_4)^c \cap \cdots \cap (A'_{\sqrt{1/\varepsilon}})^c).$$

We deduce from (1.2) and the remarks following it that

$$\mathbb{P}^{(0,0)}(\widehat{A}) \leq c_2 \exp(-c_3\sqrt{1/\varepsilon}).$$

By Proposition 4.7 and scaling,

$$\mathbb{P}^{(0,0)}((A'_4)^c \cap \cdots \cap (A'_{\sqrt{1/\varepsilon}})^c) \leq c_4/\log\sqrt{1/\varepsilon} = c_5/\log(1/\varepsilon).$$

It follows that

$$\mathbb{P}^{(0,0)}(A_2^c \cap \cdots \cap A_{1/\varepsilon}^c) \leq c_2 \exp(-c_3\sqrt{1/\varepsilon}) + c_5/\log(1/\varepsilon) \leq c_6/\log(1/\varepsilon). \qquad \square$$

We find an upper bound on the probability of there being approximate cut lines at $\mathcal{L}(0,0)$, $\mathcal{L}(j, \alpha)$, and $\mathcal{L}(k, \beta)$ and no approximate cut line at any $\mathcal{L}(n, 0)$, $n < 0$.

Proposition 5.11. *Suppose* $\alpha, \beta \in [6\log^{-1}(1/\varepsilon), \pi/100)$,

$$\varepsilon^{(3/\alpha\log(1/\varepsilon))-1} \le j \le \varepsilon^{(1/\alpha\log(1/\varepsilon))-1},$$
$$\varepsilon^{(3/\beta\log(1/\varepsilon))-1} \le k \le \varepsilon^{(1/\beta\log(1/\varepsilon))-1}.$$

There exists c_1 *independent of* $j, k, \varepsilon, \alpha$, *and* β *such that*

(a) *if* $|z| > 4$, $z = (x, y)$, $y < -1$, *and* $k \ge j + 200j|\beta - \alpha|$, *then*

$$\mathbb{P}^z(B \cap C(j,\alpha) \cap C(k,\beta) \mid D) \le \frac{c_1\varepsilon((|k-j|+1)\varepsilon)^{|\beta-\alpha|/4}}{j(|k-j|+1)\log(1/\varepsilon)}$$

for ε *sufficiently small.*

(b) *if* $|z| > 4$, $z = (x, y)$, $y < -1$, $j \le k \le j + 200j|\beta - \alpha|$, *and* $\log^{-1/2}(1/\varepsilon) \ge |\beta - \alpha| \ge \log^{-1}(1/\varepsilon)$, *then*

$$\mathbb{P}^z(B \cap C(j,\alpha) \cap C(k,\beta) \mid D) \le \frac{c_1\varepsilon}{j^2|\beta-\alpha|^2\log(1/\varepsilon)}$$

for ε *sufficiently small.*

Proof. The proof is analogous to the proof of Proposition 5.9. We split the path at time $\sigma(0,0)$. Our estimate is the product of two factors, the first from the portion of the path after time $\sigma(0,0)$ and the second from the portion of the path before this time.

The first factor comes from Propositions 5.6 (a) and 5.7 (a).

The second factor comes from Propositions 5.10 and 5.8 and is equal to $c_2/\log(1/\varepsilon)$. Note that in the proof of Proposition 5.9 we had to use the rather complicated Proposition 5.4. We can replace it with the simpler Proposition 5.10 here since we need an upper bound, and we can ignore the possibility that the path crosses the lines $\mathcal{L}(j,\alpha)$ and $\mathcal{L}(k,\beta)$ before time $\sigma(0,0)$. □

We similarly derive upper and lower bounds on the probability of there being an approximate cut line at $\mathcal{L}(0,0)$ but no approximate cut line at any $\mathcal{L}(n,0)$ for $n < 0$.

Proposition 5.12. *There exist* c_1 *and* c_2 *such that if* $|z| > 4$, $z = (x, y)$, *and* $y < -1$, *then*

$$c_1\varepsilon/\log(1/\varepsilon) \le \mathbb{P}^z(B \mid D) \le c_2\varepsilon/\log(1/\varepsilon).$$

Proof. We split the path at $\sigma(0,0)$ as in the previous proof. For the portion of the path after $\sigma(0,0)$, we use the bounds

$$c_3\varepsilon \le \mathbb{P}^z(\tau_{B(0,3)} < \sigma(-1,0)) \le \mathbb{P}^z(\tau_{B(0,3/2)} < \sigma(-1,0)) \le c_4\varepsilon$$

if $z = (0, \varepsilon)$. These are standard estimates and are almost identical to (5.1) and (5.2) with $\alpha = 0$.

For the portion of the path before $\sigma(0,0)$, we obtain the factor $c_5/\log(1/\varepsilon)$ in the upper bound from Proposition 5.10. A combination of Propositions 4.5 and 5.8 gives the factor $c_6/\log(1/\varepsilon)$ in the lower bound. □

6. The main estimate.

We define B, $C(j,\alpha)$ and D as in (5.30)-(5.32). Let

$$I(M,N)=\{\alpha : M/\log(1/\varepsilon)\leq \alpha \leq N/\log(1/\varepsilon), \alpha\log(1/\varepsilon)\in\mathbb{Z}\},$$

$$m(\alpha)=\varepsilon^{(3/\alpha\log(1/\varepsilon))-1}, \qquad n(\alpha)=\varepsilon^{(1/\alpha\log(1/\varepsilon))-1},$$

and

$$F(M,N)=\bigcap_{\alpha\in I(M,N)}\ \bigcap_{j\in[m(\alpha),n(\alpha)]} C(j,\alpha)^c.$$

The next theorem contains the conditional second moment argument. We define a new probability measure $\mathbb{Q}$ as a certain conditional probability. If R is the number of events $C(j,\alpha)$ that occur for j and α in appropriate ranges, the estimates of Section 5 will show that $\mathbb{E}_{\mathbb{Q}}R^2$ is bounded by a constant times $(\mathbb{E}_Q R)^2$; this then implies $\mathbb{Q}(R\geq 1)>0$.

Theorem 6.1. *There exist $\rho'\in(0,1)$, M_1, and N_1, all independent of ε, such that if $z=(x,y)$, $y<-1$, and $|z|>4$, then for small ε,*

$$\mathbb{P}^z(B\cap F(M_1,N_1)\mid D)<\rho'\mathbb{P}^z(B\mid D).$$

Proof. Define a probability measure $\mathbb{Q}$ by

$$\mathbb{Q}(A)=\frac{\mathbb{P}^z(A\cap B\mid D)}{\mathbb{P}^z(B\mid D)}.$$

We have

$$\mathbb{Q}(C(j,\alpha))=\frac{\mathbb{P}^z(B\cap C(j,\alpha)\mid D)}{\mathbb{P}^z(B\mid D)}.$$

We choose M_1 large enough so that $3/M_1<\delta_0$, where δ_0 is defined by Proposition 5.4, so that we can apply Proposition 5.9. That result and Proposition 5.12 imply

$$\mathbb{Q}(C(j,\alpha))\geq \frac{c_1\varepsilon}{j\log(1/\varepsilon)}\frac{\log(1/\varepsilon)}{c_2\varepsilon}=c_3/j. \tag{6.1}$$

Assume that $\alpha,\beta\in I(M_1,N_1)$. Then $|\beta-\alpha|<(\log(1/\varepsilon))^{-1/2}$ for small ε and so the assumption of Proposition 5.11(b) is satisfied. Propositions 5.11 and 5.12 yield

$$\mathbb{Q}(C(j,\alpha)\cap C(k,\beta))\leq \frac{c_4}{j(|k-j|+1)}, \tag{6.2}$$

if $\alpha=\beta$ and $j,k\in[m(\alpha),n(\alpha)]$;

$$\mathbb{Q}(C(j,\alpha)\cap C(k,\beta))\leq \frac{c_5}{j(|k-j|+1)}((|k-j|+1)\varepsilon)^{|\beta-\alpha|/4}, \tag{6.3}$$

provided $j \in [m(\alpha), n(\alpha)]$, $k \in [m(\beta), n(\beta)]$ and $k > j + 200j|\beta - \alpha|$; and

$$\mathbb{Q}(C(j,\alpha) \cap C(k,\beta)) \leq \frac{c_5}{j^2|\beta-\alpha|^2}, \tag{6.4}$$

when $j \in [m(\alpha), n(\alpha)]$, $k \in [m(\beta), n(\beta)]$ and $j \leq k \leq j + 200j|\beta - \alpha|$.

Let

$$R = \sum_{\alpha \in I(M_1,N_1)} \sum_{m(\alpha) \leq j \leq n(\alpha)} 1_{C(j,\alpha)}.$$

Since

$$\sum_{m(\alpha) \leq j \leq n(\alpha)} \frac{c_3}{j} \geq \frac{c_3}{2}(\log n(\alpha) - \log m(\alpha)) \geq \frac{-c_6 \log \varepsilon}{\alpha \log(1/\varepsilon)} = \frac{c_6}{\alpha}, \tag{6.5}$$

it follows from (6.1) that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}} R &= \sum_{\alpha \in I(M_1,N_1)} \sum_{m(\alpha) \leq j \leq n(\alpha)} \mathbb{Q}(C(j,\alpha)) \\
&\geq \sum_{\alpha \in I(M_1,N_1)} \sum_{m(\alpha) \leq j \leq n(\alpha)} \frac{c_3}{j} \\
&\geq \sum_{\alpha \in I(M_1,N_1)} \frac{c_6}{\alpha} \\
&\geq c_7 \log(1/\varepsilon)(\log N_1 - \log M_1).
\end{aligned}$$

We now compute the second moment of R under $\mathbb{E}_{\mathbb{Q}}$.

$$\mathbb{E}_{\mathbb{Q}} R^2 = \sum_{\alpha \in I(M_1,N_1)} \sum_{\beta \in I(M_1,N_1)} \sum_{m(\alpha) \leq j \leq n(\alpha)} \sum_{m(\beta) \leq k \leq n(\beta)} \mathbb{Q}(C(j,\alpha) \cap C(k,\beta)).$$

We first sum over the terms with $\alpha = \beta$. If $\alpha = \beta$, we use (6.2) to get the bound

$$\begin{aligned}
&\sum_{m(\alpha) \leq j \leq n(\alpha)} \sum_{m(\alpha) \leq k \leq n(\alpha)} \mathbb{Q}(C(j,\alpha) \cap C(k,\beta)) \\
&\qquad \leq \sum_{m(\alpha) \leq j \leq n(\alpha)} \sum_{m(\alpha) \leq k \leq n(\alpha)} \frac{c_4}{j(|k-j|+1)} \\
&\qquad \leq 2 \sum_{m(\alpha) \leq j \leq n(\alpha)} \frac{c_4}{j} \sum_{1 \leq k \leq n(\alpha)} \frac{1}{k} \\
&\qquad \leq c_8 \big(\log n(\alpha) - \log m(\alpha)\big) \log n(\alpha).
\end{aligned}$$

Since $\log n(\alpha) - \log m(\alpha) \leq c_9/\alpha$ and $\log n(\alpha) \leq c_{10} \log(1/\varepsilon)$, summing over $\alpha \in I(M_1, N_1)$ gives the bound

$$\begin{aligned}
&\sum_{\alpha \in I(M_1,N_1)} \sum_{m(\alpha) \leq j \leq n(\alpha)} \sum_{m(\alpha) \leq k \leq n(\alpha)} \mathbb{Q}(C(j,\alpha) \cap C(k,\beta)) \\
&\qquad \leq \sum_{\alpha \in I(M_1,N_1)} \frac{c_8 c_9 c_{10} \log(1/\varepsilon)}{\alpha} \\
&\qquad \leq c_{11} \log^2(1/\varepsilon)(\log N_1 - \log M_1)
\end{aligned} \tag{6.6}$$

for the $\alpha = \beta$ terms. This is less than or equal to $c_{12}(\mathbb{E}_{\mathbb{Q}} R)^2$ as long as N_1 is sufficiently large so that $\log N_1 - \log M_1 \geq 1$.

We next sum over the $\alpha < \beta$ terms. To start with we consider the sum over k greater than j and we look first at those k for which $k > j + 200j|\beta - \alpha|$. We have from (6.3)

$$\begin{aligned}\sum_{\substack{k=m(\beta)\\ k>j+200j|\beta-\alpha|}}^{n(\beta)} &\mathbb{Q}(C(j,\alpha)\cap C(k,\beta))\\ &\leq \sum_{\substack{k=m(\beta)\\ k>j+200j|\beta-\alpha|}}^{n(\beta)} \frac{c_5}{j(|k-j|+1)}((|k-j|+1)\varepsilon)^{(\beta-\alpha)/4}\\ &\leq \sum_{k=m(\beta)}^{n(\beta)} \frac{c_5}{j(|k-j|+1)}((|k-j|+1)\varepsilon)^{(\beta-\alpha)/4}\\ &\leq \frac{c_{13}}{j(\beta-\alpha)}\big(n(\beta)^{\beta-\alpha} - m(\beta)^{\beta-\alpha}\big)\varepsilon^{(\beta-\alpha)/4}\\ &= \frac{c_{13}}{j(\beta-\alpha)}\big[\varepsilon^{(\beta-\alpha)/4\beta\log(1/\varepsilon)} - \varepsilon^{3(\beta-\alpha)/4\beta\log(1/\varepsilon)}\big].\end{aligned}$$

Since $0 < (\beta-\alpha)/\beta \leq 1$, this yields

$$\sum_{\substack{k=m(\beta)\\ k>j+200j|\beta-\alpha|}}^{n(\beta)} \mathbb{Q}(C(j,\alpha)\cap C(k,\beta)) \leq \frac{c_{14}}{j(\beta-\alpha)}. \tag{6.7}$$

Using (6.4), the sum over those k between j and $j + 200j|\beta-\alpha|$ yields

$$\sum_{\substack{k=m(\beta)\\ j\leq k\leq j+200j|\beta-\alpha|}}^{n(\beta)} \mathbb{Q}(C(j,\alpha)\cap C(k,\beta)) \leq \sum_{\substack{k=m(\beta)\\ j\leq k\leq j+200j|\beta-\alpha|}}^{n(\beta)} \frac{c_5}{j^2|\beta-\alpha|^2} \leq \frac{200c_{15}}{j|\beta-\alpha|}.$$

This and (6.7) give

$$\sum_{\substack{k=m(\beta)\\ k\geq j}}^{n(\beta)} \mathbb{Q}(C(j,\alpha)\cap C(k,\beta)) \leq \frac{c_{16}}{j|\beta-\alpha|}.$$

Summing over j gives us

$$\sum_{j=m(\alpha)}^{n(\alpha)} \sum_{\substack{k=m(\beta)\\ k\geq j}}^{n(\beta)} \mathbb{Q}(C(j,\alpha)\cap C(k,\beta)) \leq \sum_{j=m(\alpha)}^{n(\alpha)} \frac{c_{16}}{j|\beta-\alpha|} \leq \frac{c_{17}}{\alpha|\beta-\alpha|}.$$

in a fashion similar to (6.5). Finally, summing over α and β,

$$\begin{aligned}\sum_{\alpha\in I(M_1,N_1)} &\sum_{\substack{\beta\in I(M_1,N_1)\\ \alpha<\beta}} \sum_{j=m(\alpha)}^{n(\alpha)} \sum_{\substack{k=m(\beta)\\ k\geq j}}^{n(\beta)} \mathbb{Q}(C(j,\alpha)\cap C(k,\beta))\\ &\leq \sum_{\alpha\in I(M_1,N_1)} \sum_{\substack{\beta\in I(M_1,N_1)\\ \alpha<\beta}} \frac{c_{17}}{\alpha|\beta-\alpha|} \leq c_{18}\log^2(1/\varepsilon)\log^2 N_1.\end{aligned}$$

Provided we take N_1 sufficiently large, large enough so that $\log N_1 \geq 2\log M_1$ (this can be done independently of ε), we see that the sum of those terms with $\alpha < \beta$ and $j \leq k$ is also bounded by a constant times $(\mathbb{E}_{\mathbb{Q}}R)^2$. We get a similar bound for $k < j$, using the remark following the proof of Proposition 5.7. The $\alpha > \beta$ terms are handled in an exactly similar fashion to the case we just did. This and (6.6) imply that there exist M_1, N_1 and c_{19}, independent of ε, and such that

$$\mathbb{E}_{\mathbb{Q}}R^2 \leq c_{19}(\mathbb{E}_{\mathbb{Q}}R)^2.$$

Moreover $E_{\mathbb{Q}}R > 2$ if ε is sufficiently small.

We now use a standard second moment argument:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}R &= \mathbb{E}_{Q}(R; R \geq 1) + \mathbb{E}_{\mathbb{Q}}(R; R < 1)\\ &\leq (\mathbb{E}_{\mathbb{Q}}R^2)^{1/2}\mathbb{Q}(R \geq 1)^{1/2} + 1\\ &\leq c_{19}^{1/2}(\mathbb{E}_{\mathbb{Q}}R)\mathbb{Q}(R \geq 1)^{1/2} + 1.\end{aligned}$$

Hence, if ε is so small that $E_{\mathbb{Q}}R > 2$,

$$\mathbb{Q}(R \geq 1)^{1/2} \geq \frac{\mathbb{E}_{\mathbb{Q}}R - 1}{c_{19}^{1/2}\mathbb{E}_{\mathbb{Q}}R} \geq c_{20} > 0,$$

with c_{20} independent of ε. Letting $\rho' = 1 - c_{20}^2/2$, we obtain $\mathbb{Q}(R = 0) < \rho'$. This implies

$$\begin{aligned}\frac{\mathbb{P}^z(F(M_1, N_1) \cap B \mid D)}{\mathbb{P}^z(B \mid D)} &= \frac{\mathbb{P}^z(\{R = 0\} \cap B \mid D)}{\mathbb{P}^z(B \mid D)}\\ &= \mathbb{Q}(R = 0) < \rho'.\end{aligned}$$

The proof is complete. □

Theorem 6.1 is perhaps the most important step in the proof. However, the factor of ρ is not good enough, and we need to iterate. We now look at approximate cut lines for much larger angles α than $N/\log(1/\varepsilon)$.

Starting at a point (w_1, w_2) on the line $\mathcal{L}(\varepsilon^{1/P-1}, P/\log(1/\varepsilon))$, a BM $\times$ Bes(3) is unlikely to return to the line $\mathcal{L}(\varepsilon^{1/2N-1}, N/\log(1/\varepsilon))$ if P is enough larger than N.

Lemma 6.2. *Let H be the upper half plane. Let Z_t be a BM$\times$ Bes(3) process, N an integer bigger than 2, and $\gamma > 0$. Let $\alpha = N/\log(1/\varepsilon)$ and $j = \varepsilon^{(1/2N)-1}$. There exist $P_0 \geq 4N$ with the property: if $P \geq P_0$, there exists $\varepsilon_0 = \varepsilon_0(P)$ so that if $\varepsilon \leq \varepsilon_0$, $\beta = P/\log(1/\varepsilon)$, and $k = \varepsilon^{(1/P)-1}$, then for all $w \in \mathcal{L}(k, \beta) \cap H$,*

$$\mathbb{P}^w_{BMBS}(\sigma(j, \alpha) < \tau_{B(0,2)}) < \gamma.$$

Proof. The lines $\mathcal{L}(j, \alpha)$ and $\mathcal{L}(k, \beta)$ intersect the x-axis at points $(R, 0)$ and $(R_1, 0)$, respectively. Let us first consider the case when $w_1 \leq R$, where $w = (w_1, w_2)$. Let $\lambda = \operatorname{dist}(w, \mathcal{L}(j, \alpha))$. Since $P \geq P_0 > 2N$, then $\beta > 2\alpha$, and

$$R_1 > (1/2)k\varepsilon/\beta > 8j\varepsilon/\alpha > 4R$$

for ε small. This easily implies that $w_2 \geq k\varepsilon/4$ for $w_1 \leq R$ and also that there exists $c_1 > 1$ such that

$$c_1^{-1} w_2 \leq \lambda \leq c_1 w_2 \tag{6.8}$$

whenever $w_1 \in (-\infty, R]$. Since a BM $\times$ Bes(3) process is a Brownian motion in the upper half plane h-path transformed by the harmonic function $h(x,y) = y$, (1.1) tells us that

$$\mathbb{P}^w_{BMBS}(\sigma(j,\alpha) < \tau_{B(0,2)}) = \frac{\mathbb{E}^w_{BM}(Y_{\sigma(j,\alpha)}; \sigma(j,\alpha) < \tau_{B(0,2)} \wedge \tau_H)}{h(w)}. \tag{6.9}$$

If $z = (x,y)$ lies on $\mathcal{L}(j,\alpha)$ a distance $\rho = \rho(z)$ from $(R,0)$, then $y = \rho \sin\alpha \leq \rho\alpha$. Let v be the projection of w onto $\mathcal{L}(j,\alpha)$ and let $r' = r'(z)$ be the distance from a point z on $\mathcal{L}(j,\alpha)$ to v. Let $r = r(z) = (\operatorname{sgn}(w_1 - z_1))r'(z)$. We relate $\rho(z)$ to $r(z)$. If u is the point of intersection of the lines $\mathcal{L}(j,\alpha)$ and $\{x = w_1\}$, then $|v - u| = \lambda \tan\alpha$ and $\rho(u) = (R - w_1)/\cos\alpha$. Therefore

$$\rho(v) = \lambda \tan\alpha + \frac{R - w_1}{\cos\alpha}. \tag{6.10}$$

For $z \in \mathcal{L}(j,\alpha) \cap H$,

$$y \leq \rho(z)\alpha = (\rho(v) + r(z))\alpha.$$

Recall that the distribution of where 2-dimensional Brownian motion hits $\mathcal{L}(j,\alpha)$ is given by the Poisson kernel (see Bass (1995), p. 91). We obtain

$$\mathbb{E}^w_{BM}(Y_{\sigma(j,\alpha)};\, \sigma(j,\alpha) < \tau_{B(0,2)} \wedge \tau_H) \leq \frac{1}{\pi} \int_{-\rho(v)}^{3} \frac{\lambda}{\lambda^2 + r^2} \alpha(\rho(v) + r)\, dr.$$

Since $\rho(v) + r \leq \rho(v)$ if $r \leq 0$ and $\pi^{-1} \int_{-\infty}^{\infty} \lambda/(\lambda^2 + r^2)\, dr = 1$, the right hand side is bounded by

$$c_2\alpha\Big[\rho(v) + \int_0^\lambda \frac{\lambda r}{\lambda^2 + r^2}\, dr + \int_\lambda^3 \frac{\lambda r}{\lambda^2 + r^2}\, dr\Big]. \tag{6.11}$$

The second term inside the brackets is bounded by

$$\lambda^{-1} \int_0^\lambda r\, dr = \lambda/2,$$

while the third term inside the brackets is bounded by

$$\lambda \int_\lambda^3 \frac{dr}{r} = \lambda(\log 3 - \log\lambda).$$

Since $\tan\alpha \leq 1$ and $\cos\alpha \geq 1/2$ if α is small, (6.10) shows that (6.11) is bounded by

$$c_3\alpha[\lambda + (R - w_1) - c_4\lambda\log\lambda].$$

Combining with (6.8) and (6.9),

$$\mathbb{P}^w_{BMBS}(\sigma(j,\alpha) < \tau_{B(0,2)}) \le \frac{c_5\alpha[w_2 + R - w_1 - c_6 w_2 \log(w_2/c_1)]}{w_2}.$$

Note that $R \le 2j\varepsilon/\alpha$ while $w_2 \ge k\varepsilon/4$. If $w_1 \le 0$, then $|w_1/w_2| \le 2/\beta$. Finally, since $w_2 \ge k\varepsilon/4$, then $-\log(w_2/c_1) \le c_7 + (1/P)\log(1/\varepsilon)$. Therefore

$$\begin{aligned}\mathbb{P}^w_{BMBS}(\sigma(j,\alpha) < \tau_{B(0,2)}) &\le c_8\Big[\alpha + \frac{j}{k} + \frac{\alpha}{\beta} + \frac{\alpha\log(1/\varepsilon)}{P}\Big] \qquad (6.12)\\ &= c_8\Big[\frac{N}{\log(1/\varepsilon)} + \varepsilon^{(1/2N)-(1/P)} + \frac{N}{P} + \frac{N}{P}\Big]\\ &\le c_8\Big[\frac{N}{\log(1/\varepsilon)} + \varepsilon^{1/4N} + 2N/P\Big]\end{aligned}$$

if P is large and ε is small.

Next we consider the case when $w_1 \ge R$. Let $h_1(x,y)$ denote the numerator of the right hand side of (6.9). This is a nonnegative function that is harmonic in $(B(0,2)\cap H) - \mathcal{L}(j,\alpha)$ with 0 boundary values on $\{(x,0) : x > R\}$. Let (R,S) be the intersection point of the lines $\mathcal{L}(k,\beta)$ and $\{x = R\}$ and let $M = \{(x,y) : y \in (0,S], y = -x + R + S\}$. By the boundary Harnack principle we see there exists $c_9 > 1$, independent of ε, such that

$$\frac{h_1(w_1,w_2)}{h(w_1,w_2)} \le c_9\frac{h_1(R,S)}{h(R,S)},$$

for all $(w_1,w_2) \in M$. Using (6.9) and (6.12), this implies for $w \in M$,

$$\begin{aligned}\mathbb{P}^w_{BMBS}(\sigma(j,\alpha) < \tau_{B(0,2)}) &\le c_9\mathbb{P}^{(R,S)}_{BMBS}(\sigma(j,\alpha) < \tau_{B(0,2)}) \qquad (6.13)\\ &\le c_9c_8\Big[\frac{N}{\log(1/\varepsilon)} + \varepsilon^{1/4N} + 2N/P\Big].\end{aligned}$$

This estimate also applies to $w \in \mathcal{L}(k,\beta)$ with $w_1 \ge R$ because the process starting from such a point would have to hit either M or a point of $\mathcal{L}(k,\beta)$ with $w_1 \le R$ before hitting $\mathcal{L}(j,\alpha)$. We could apply the strong Markov property, (6.12), and (6.13) at such a time. If we take ε_0 small enough so that $\varepsilon < \varepsilon_0$ implies $N/\log(1/\varepsilon) + \varepsilon^{1/4N} < \gamma/(2c_8c_9)$ and then take P_0 larger than $4c_8c_9N/\gamma$, the right hand sides of both (6.12) and (6.13) will be less than γ whenever $P \ge P_0$ and $\varepsilon < \varepsilon_0$. □

Let us let $\mathcal{L}_P$ be an abbreviation for $\mathcal{L}(\varepsilon^{1/P-1}, P/\log(1/\varepsilon))$ and σ_P the hitting time of $\mathcal{L}_P$. Let

$$\pi(j,\alpha) = \sup\{t < \sigma(0,0) : Z_t \in \mathcal{L}(j,\alpha)\}, \qquad (6.14)$$

the last time Z_t is in $\mathcal{L}(j,\alpha)$ before time $\sigma(0,0)$. Recall that $\mathcal{T}_K$ denotes the hitting time of a set K. Let

$$\begin{aligned}\widetilde{C}(j,\alpha,P) = \{&\sigma(j,\alpha) < \sigma(0,0) + \tau_{B(0,2)}\circ\theta_{\sigma(0,0)},\\ &(\tau_{B(0,2)}\wedge\sigma_P)\circ\theta_{\sigma(j,\alpha)} < \sigma(j-1,\alpha)\circ\theta_{\sigma(j,\alpha)}, |X_{\sigma(j,\alpha)}| \le j\varepsilon\}\\ &\cup\{\mathcal{T}_{B(0,2)} < \sigma(j,\alpha) < \pi(-\varepsilon^{1/P-1}, P/\log(1/\varepsilon))\}.\end{aligned}$$

The set $\widetilde{C}(j,\alpha,P)$ represents the event that there is an approximate cut line at $\mathcal{L}(j,\alpha)$ for the portion of the trajectory between $\pi(-\varepsilon^{1/P-1}, P/\log(1/\varepsilon))$ and σ_P. We then set

$$G(M,N,P) = \bigcap_{\alpha\in I(M,N)} \bigcap_{j\in[m(\alpha),n(\alpha)]} \widetilde{C}(j,\alpha,P)^c.$$

Note $C(j,\alpha) \subseteq \widetilde{C}(j,\alpha,P)$, hence $G(M,N,P) \subseteq F(M,N)$.

We show that if P is large enough, it does not make much difference to the estimates if we replace $F(M,N)$ by $G(M,N,P)$.

Proposition 6.3. *Let $\gamma > 0$. Suppose $z = (x,y)$ with $|z| \le 4$ and $y < -1$. There exist $P_0 \ge 4N$ (depending on γ, M, and N) with the following property: if $P \ge P_0$, there exists $\varepsilon_0 = \varepsilon_0(P)$ so that*

$$\mathbb{P}^z_{BM}(B \cap [F(M,N) - G(M,N,P)] \bigm| D) \le \gamma \mathbb{P}^z_{BM}(B \mid D)$$

provided $\varepsilon < \varepsilon_0$.

Proof. The largest angle α in $I(M,N)$ is $N/\log(1/\varepsilon)$. If $\mathcal{L}(j,\alpha)$ is a line with $\alpha \in I(M,N)$ and $j \in [m(\alpha), n(\alpha)]$, then $j\varepsilon \le \varepsilon^{1/\alpha\log(1/\varepsilon)} \le \varepsilon^{1/N}$. The x intercept of $\mathcal{L}(j,\alpha)$ is less than $2j\varepsilon/\sin\alpha \le 4j\varepsilon/\alpha$. So the largest x intercept of any such $\mathcal{L}(j,\alpha)$ is less than $4\varepsilon^{1/N}\log(1/\varepsilon)/N$. Let $\alpha' = N/\log(1/\varepsilon)$ and $j' = \varepsilon^{1/2N-1}$. Let W be the wedge with aperture angle α' lying in the upper half plane H bounded by $\mathcal{L}(0,0)$ and $\mathcal{L}(j',\alpha')$. Then if $\alpha \in I(M,N)$ and $j \in [m(\alpha), n(\alpha)]$, we see that for ε sufficiently small, $\mathcal{L}(j,\alpha) \cap H \subseteq W$. This implies that if $P \ge P_0 \ge 4N$, ε is sufficiently small, and $w \in \mathcal{L}_P$, then a BM $\times$ Bes(3) process started at w can hit one of the lines $\mathcal{L}(j,\alpha)$ with $\alpha \in I(M,N)$ and $j \in [m(\alpha), n(\alpha)]$ only if it first hits $\mathcal{L}(j',\alpha')$.

If ω is in $F(M,N)$ but not $G(M,N,P)$, then ω is in $\widetilde{C}(j,\alpha,P)$ but not in $C(j,\alpha)$ for some j and α. One way for this to happen is if after time $\sigma(j,\alpha)$ the path $Z_t(\omega)$ hits $\mathcal{L}_P$ before hitting $\mathcal{L}(j-1,\alpha)$ and hits $\mathcal{L}(j-1,\alpha)$ before exiting $B(0,2)$. By the discussion above, after time $\sigma(j,\alpha)$ the path must hit $\mathcal{L}_P$ before hitting $\mathcal{L}(j-1,\alpha)$ and then hit $\mathcal{L}(j',\alpha')$ before exiting $B(0,2)$.

The other way ω can be in $\widetilde{C}(j,\alpha,P)$ but not $C(j,\alpha)$ for some j and α is if $\sigma(j,\alpha) < \pi(-\varepsilon^{1/P-1}, P/\log(1/\varepsilon))$ for some j and α in the appropriate ranges. If we look at the time reversed process

$$\widetilde{Z}_t = (\widetilde{X}_t, \widetilde{Y}_t) = (-X_{\sigma(0,0)-t}, -Y_{\sigma(0,0)-t}), \tag{6.15}$$

then by the discussion following (5.29), $\widetilde{Z}_t$ is a BM $\times$ Bes(3) process conditioned to go to $(-x,-y)$. Let

$$\widetilde{\sigma}(j,\alpha) = \inf\{t : \widetilde{Z}_t \in \mathcal{L}(j,\alpha)\} \tag{6.16}$$

and define similarly $\widetilde{\sigma}_P$ and $\widetilde{\tau}_{B(0,2)}$. So we are asking that the paths of $\widetilde{Z}_t$ hit $\mathcal{L}(j,\alpha)$ after $\widetilde{\sigma}_P$ for some j,α. (Note that here the x coordinate of $\widetilde{Z}_t$ is the negative of the

x coordinate of the time reversed process in Proposition 5.8; this is to simplify the notation.) As we have seen above, this requires $\widetilde{Z}_t$ to hit $\mathcal{L}(j',\alpha')$ after $\widetilde{\sigma}_P$. If

$$E_1 = \{\sigma(j,\alpha) < \sigma(0,0) + \tau_{B(0,2)} \circ \theta_{\sigma(0,0)},\\ \sigma_P \circ \theta_{\sigma(j,\alpha)} < \sigma(j-1,\alpha) \circ \theta_{\sigma(j,\alpha)},\\ \sigma(j',\alpha') \circ \theta_{\sigma_P} < \tau_{B(0,2)} \circ \theta_{\sigma_P}\}$$

and

$$E_2 = \{\widetilde{\sigma}_P < \widetilde{\sigma}(j',\alpha') < \widetilde{\tau}_{B(0,2)}\},$$

then

$$F(M,N) - G(M,N,P) \subseteq E_1 \cup E_2.$$

We will now estimate $\mathbb{P}^z(B \cap E_1 \mid D)$. As in the proof of Proposition 5.9, we split the path at time $\sigma(0,0)$ and obtain our upper bound as the product of two factors. We use time reversal for the portion of the path prior to $\sigma(0,0)$ and apply Propositions 5.10 and 5.8 to obtain the factor $c_1/\log(1/\varepsilon)$.

We next use the strong Markov property at time $\sigma(0,0)$ and obtain a factor for the portion of the path after time $\sigma(0,0)$. Recall that we are conditioning $Z_{\sigma(0,0)}$ to be 0. In order for ω to be in $B \cap E_1$, the process Z_t started at z' must exit $B(0,2)$ before hitting $\mathcal{L}(-1,0)$. One possibility is that $\tau_{B(0,2)} < \sigma(L/\varepsilon,0) \wedge \sigma(-1,0)$, where L is to be chosen later.

Let $K = \{(x,y) : 0 < y < 1, |x| < R\}$, let $h(x,y) = y$, and let

$$h_1(x,y) = \mathbb{P}^{(x,y)}_{BM}(|X_{\tau_K}| = R).$$

By the boundary Harnack principle, there exists c_2 such that

$$h_1(0,\varepsilon) \le c_2 h_1(0,1/2) \frac{h(0,\varepsilon)}{h(0,1/2)} \le 2c_2\varepsilon h_1(0,1/2).$$

By (1.2), $h_1(0,1/2) \le c_3 e^{-c_4 R}$, so

$$\mathbb{P}^{(0,\varepsilon)}_{BM}(|X_{\tau_K}| = R) \le c_5 \varepsilon e^{-c_4 R}. \tag{6.17}$$

Now let $K' = \{(x,y) : -\varepsilon < y < L, |x| < 1\}$, where $L < 1/4$. By scaling and translation invariance applied to (6.17),

$$\mathbb{P}^{(0,0)}_{BM}(\tau_{B(0,2)} < \sigma(L/\varepsilon,0) \wedge \sigma(-1,0)) \le \mathbb{P}^{(0,0)}_{BM}(|X(\tau_{K'})| = 1) \le c_6 \varepsilon e^{-c_7/L}. \tag{6.18}$$

The other possibility is that $\sigma(L/\varepsilon,0) < \sigma(-1,0)$. The probability of this is given by

$$\mathbb{P}^{(0,0)}_{BM}(\sigma(L/\varepsilon,0) < \sigma(-1,0)) = \frac{1}{1+(L/\varepsilon)} \le \frac{\varepsilon}{L}. \tag{6.19}$$

Let κ be a constant, not depending on γ and ε, to be chosen later, and let $\gamma_0 = \kappa\gamma$. Choose γ' so that $\gamma' \log(1/\gamma') < \gamma_0$ and let $L = c_7/\log(1/\gamma')$. For the event E_1 to hold, Z_t must hit $\mathcal{L}(j',\alpha')$ after $\mathcal{L}_P$ and before $\tau_{B(0,2)}$. We consider two cases. The first one is when the process hits $\mathcal{L}(L/\varepsilon,0)$ after $\mathcal{L}_P$ and then returns to $\mathcal{L}(j',\alpha')$. Since the path is conditioned to hit $B(0,2)$ before hitting $\mathcal{L}(-1,0)$ and the line segment $\mathcal{L}(j',\alpha')\cap B(0,2)$ is within $c_8 N/\log(1/\varepsilon)$ of the real axis, the probability of hitting $\mathcal{L}(j',\alpha')$ after $\sigma(L/\varepsilon,0)$ is bounded by $c_9 N/L\log(1/\varepsilon)$. The second case is when the path after $\mathcal{L}_P$ first hits $\mathcal{L}(j',\alpha')$ and then $\mathcal{L}(L/\varepsilon,0)$. Conditional on the event $\{\sigma(L/\varepsilon,0)<\sigma(-1,0)\}$, the law of $(X_t,Y_t+\varepsilon)$ is that of a BM $\times$ Bes(3). By Lemma 6.2 with γ replaced by γ' and the remark following Lemma 6.2, the conditional probability of hitting $\mathcal{L}(j',\alpha')$ before $\mathcal{L}(L/\varepsilon,0)$ is less than γ' if $P\geq P_0$, P_0 is sufficiently large, and ε_0 is sufficiently small. Thus, taking into account (6.18) and (6.19), the factor for the portion of the path after $\sigma(0,0)$ is bounded by

$$\begin{aligned} c_6\varepsilon e^{-c_7/L}+\frac{\varepsilon}{L}\Big(\gamma'+\frac{c_9N}{L\log(1/\varepsilon)}\Big) &\leq c_6\varepsilon e^{-\log(1/\gamma')}+\frac{\varepsilon\gamma'\log(1/\gamma')}{c_7}+\frac{c_9N\varepsilon}{L^2\log(1/\varepsilon)}\\ &\leq c_6\varepsilon\gamma'+\frac{\varepsilon\gamma_0}{c_7}+\frac{c_9N\log^2(1/\gamma')\varepsilon}{c_7^2\log(1/\varepsilon)}\leq c_{10}\varepsilon\gamma_0, \end{aligned}$$

for small ε. Therefore combining with the factor obtained for the portion of the path before $\sigma(0,0)$, we conclude

$$\mathbb{P}^z_{BM}(B\cap E_1\mid D)\leq \frac{c_1}{\log(1/\varepsilon)}c_{10}\varepsilon\gamma_0. \tag{6.20}$$

We now turn to $\mathbb{P}^z_{BM}(B\cap E_2\mid D)$. We again split the path at $\sigma(0,0)$. To be in B, the portion of the path after $\sigma(0,0)$ must exit $B(0,2)$ before returning to $\sigma(-1,0)$, and by (1.3) the probability of this is bounded by $c_{11}\varepsilon$. So by the strong Markov property at time $\sigma(0,0)$ we obtain the factor $c_{11}\varepsilon$ for the portion of the path after $\sigma(0,0)$.

Let us use time reversal at time $\sigma(0,0)$. The process $\widetilde{Z}_t$ is a BM $\times$ Bes(3) conditioned to go to $(-x,-y)$, but let us temporarily suppose that $\widetilde{Z}_t$ is a BM $\times$ Bes(3) (we will remedy this further on by applying Proposition 5.8). To be in B, one possibility is for

$$\sup_{s\leq\widetilde{\sigma}(\varepsilon^{-1/2},0)}|\widetilde{X}_s-\widetilde{X}_0|>\varepsilon^{1/4},$$

where we write $\widetilde{Z}_t=(\widetilde{X}_t,\widetilde{Y}_t)$. By (1.2) and scaling the probability of this is bounded by

$$c_{12}\exp(-c_{13}\varepsilon^{1/4}/\varepsilon^{1/2})=c_{12}\exp(-c_{13}\varepsilon^{-1/4}). \tag{6.21}$$

We note that if $P>4$ and ε is small, then the process $\widetilde{Z}_t$ starting from $(0,0)$ cannot hit $\mathcal{L}_P$ before hitting either $\mathcal{L}(\varepsilon^{-1/2},0)$ or one of the lines $\{|x|=2\varepsilon^{1/4}\}$. Also, after hitting $\mathcal{L}(\varepsilon^{-1/2},0)$ and then hitting $\mathcal{L}_P$, $\widetilde{Z}_t$ cannot hit $\mathcal{L}(-j',\alpha')$ without first hitting $\mathcal{L}(0,\alpha')$. So the other possible way to be in B is for $\widetilde{Z}_t$ to hit $\mathcal{L}(\varepsilon^{-1/2},0)$ without any approximate points of increase in the $\alpha=0$ direction; then to hit $\mathcal{L}_P$; and then to return to $\mathcal{L}(0,\alpha')$ before exiting $B(0,2)$.

The probability of no approximate points of increase up to level $\varepsilon^{1/2}$ is, by Proposition 4.7, bounded by $c_{14}/\log(1/\varepsilon)$. Starting at a point (x'', y'') in $\mathcal{L}(\varepsilon^{-1/2}, 0)$ and using the strong Markov property at the hitting time of $\mathcal{L}_P$, Lemma 6.2 tells us that the probability of hitting $\mathcal{L}_P$ and then $\mathcal{L}(0, \alpha')$ before exiting $B(0,2)$ is bounded by γ_0 if P_0 is sufficiently large. So by the strong Markov property at $\widetilde{\sigma}(\varepsilon^{-1/2}, 0)$, we obtain the bound

$$c_{14}\gamma_0/\log(1/\varepsilon)$$

for the second possibility. So, still assuming that $\widetilde{Z}_t$ is a BM $\times$ Bes(3) process and using (6.21), we obtain the estimate

$$c_{12}\exp(-c_{13}\varepsilon^{-1/4}) + c_{14}\gamma_0/\log(1/\varepsilon)$$

for the portion of the path before $\sigma(0,0)$. Now $\widetilde{Z}_t$ is not a BM $\times$ Bes(3) but rather a BM $\times$ Bes(3) conditioned to go to $(-x, -y)$. By Proposition 5.8 it will suffice to change the estimate by a constant factor. Hence the factor for the portion of the path before $\sigma(0,0)$ is majorized by

$$c_{15}[c_{12}\exp(-c_{13}\varepsilon^{-1/4}) + c_{14}\gamma_0/\log(1/\varepsilon)].$$

Combining with the factor $c_{11}\varepsilon$ for the portion of the path after $\sigma(0,0)$, we obtain

$$\mathbb{P}^z_{BM}(B \cap E_2 \mid D) \le c_{11}c_{15}\varepsilon\big[c_{12}\exp(-c_{13}\varepsilon^{-1/4}) + c_{14}\gamma_0/\log(1/\varepsilon)\big].$$

If ε is sufficiently small,

$$\mathbb{P}^z_{BM}(B \cap E_2 \mid D) \le c_{16}\gamma_0\varepsilon/\log(1/\varepsilon). \tag{6.22}$$

From (6.21) and (6.22),

$$\mathbb{P}^z_{BM}(B \cap [F(M,N) - G(M,N,P)] \mid D) \le c_{17}\gamma_0\varepsilon/\log(1/\varepsilon)$$

if P_0 is sufficiently large and ε is sufficiently small. Comparing with the bound

$$\mathbb{P}^z_{BM}(B \mid D) \ge c_{18}\varepsilon/\log(1/\varepsilon)$$

derived in Proposition 5.12 yields

$$\mathbb{P}^z_{BM}(B \cap [F(M,N) - G(M,N,P)] \mid D) \le c_{19}\gamma_0\mathbb{P}^z_{BM}(B \mid D).$$

If we now let $\kappa = (c_{19} \vee 1)^{-1}$, then $c_{19}\gamma_0 \le \gamma$ and we obtain the formula in the statement of the proposition. □

We recall some definitions from the proof of Proposition 6.3. Let

$$\widetilde{Z}_t = (\widetilde{X}_t, \widetilde{Y}_t) = (-X_{\sigma(0,0)-t}, -Y_{\sigma(0,0)-t}) \tag{6.23}$$

and $\widetilde{\sigma}(j,\alpha) = \inf\{t : \widetilde{Z}_t \in \mathcal{L}(j,\alpha)\}$. We define similarly $\widetilde{\sigma}_P$, $\widetilde{\tau}_{B(0,2)}$, etc. Let

$$\begin{aligned}\widehat{A}_n = \widehat{A}_n(P) = \{&(\widetilde{\tau}_{B(0,2)} \wedge \widetilde{\sigma}_P) \circ \theta_{\widetilde{\sigma}(n,0)} < \widetilde{\sigma}(n-1,0) \circ \theta_{\widetilde{\sigma}(n,0)},\\ &\widetilde{\sigma}(n,0) < \widetilde{\sigma}(\varepsilon^{1/P-1}/2, P/\log(1/\varepsilon)) \wedge \widetilde{\tau}_{B(0,2)}\},\\ \widetilde{B}(P) = A_0 \cap &\widehat{A}^c_1 \cap \cdots \cap \widehat{A}^c_{\varepsilon^{2/P-1}},\end{aligned}$$

where A_0 is defined by (5.29).

The next two propositions say that it also does not make much difference to the estimates if we replace B by $\widetilde{B}(P)$.

Proposition 6.4. *Let $\gamma > 0$. Suppose $z = (x, y)$ with $|z| \geq 4$ and $y < -1$. There exists $P_0 \geq 4N$ (depending on γ) such that*

$$\mathbb{P}^z(B - \widetilde{B}(P) \mid D) \leq \gamma \mathbb{P}^z(B \mid D)$$

provided $P \geq P_0$ and $\varepsilon < \varepsilon_0(P)$.

Proof. To estimate $\mathbb{P}^z(B - \widetilde{B}(P) \mid D)$ we split the path at $\sigma(0,0)$ as usual. To be in $B \subseteq A_0$, the portion of the path after $\sigma(0,0)$ must exit $B(0,2)$ before returning to $\mathcal{L}(-1,0)$, and by (1.3) the probability of this is bounded by $c_1\varepsilon$.

We now examine the portion of the path before $\sigma(0,0)$ by using time reversal at $\sigma(0,0)$. Let $E = \{\widetilde{Y}_{\widetilde{\sigma}_P} < \varepsilon^{3/2P}\}$. Let (R_1, R_2) be the point where $\mathcal{L}(\varepsilon^{1/P-1}/2, P/\log(1/\varepsilon))$ intersects $\mathcal{L}(\varepsilon^{(3/2P)-1}, 0)$. Then $R_1 \geq \varepsilon^{5/4P}$ if ε is small, so

$$\mathbb{P}^0_{BMBS}(|X_{\sigma(\varepsilon^{(3/2P)-1},0)}| > R_1) \leq c_2 \exp(-c_3\varepsilon^{-1/4P})$$

by (1.2) and scaling. So by Proposition 5.8,

$$\mathbb{P}^z(E) \leq c_4 c_2 \exp(-c_3 \varepsilon^{-1/4P}).$$

Next we analyze the case when E^c holds. For ω to be in B but not in $\widetilde{B}(P)$, ω cannot be in any $\widetilde{A}_n$ for $n \leq 1/\varepsilon$ but ω must be in $\widehat{A}_n$ for some $n \leq \varepsilon^{2/P-1}$. Since $B \subseteq A_0$ and $y < -1$, then $\sigma(-n, 0) < \infty$ with $\mathbb{P}^z$ probability one for all $n \leq 1/\varepsilon$. Consequently we have that for some $n \leq \varepsilon^{2/P-1}$, after the process $\widetilde{Z}_t$ hits $\mathcal{L}(n, 0)$, it must hit $\mathcal{L}_P$ before hitting $\mathcal{L}(n-1, 0)$ and then hit $\mathcal{L}(n-1,0)$ before exiting $B(0,2)$. So after hitting $\mathcal{L}_P$ the process must hit $\mathcal{L}(\varepsilon^{2/P-1}, 0)$ before exiting $B(0,2)$. If $w = (w_1, w_2) \in \mathcal{L}_P$ with $w_2 \geq \varepsilon^{3/2P}$, then

$$\begin{aligned}\mathbb{P}^w_{BMBS}(\sigma(\varepsilon^{2/P-1}, 0) < \tau_{B(0,2)}) &\leq \mathbb{P}^w_{BMBS}(\sigma(\varepsilon^{2/P-1}, 0) < \infty) \\ &\leq \varepsilon^{2/P}/\varepsilon^{3/(2P)} = \varepsilon^{1/2P}\end{aligned}$$

by (2.1). By Proposition 5.8, we can apply this estimate to $\widetilde{Z}_t$ provided we multiply the right hand side by a constant. It follows that the factor we get for the portion of the path before $\sigma(0,0)$ is at most

$$c_4 c_2 \exp(-c_3\varepsilon^{-1/4P}) + c_5 \varepsilon^{1/2P}.$$

We combine this with the estimate for the portion of the path after $\sigma(0,0)$ to obtain

$$\mathbb{P}^z(B - \widetilde{B}(P) \mid D) \leq c_5 \varepsilon\Big(c_4 c_2 \exp(-c_3 \varepsilon^{-1/4P}) + c_5 \varepsilon^{1/2P}\Big).$$

By Proposition 5.12, $\mathbb{P}^z(B \mid D) \geq c_6 \varepsilon/\log(1/\varepsilon)$. So for small ε

$$\mathbb{P}^z(B - \widetilde{B}(P) \mid D) < \gamma \mathbb{P}^z(B \mid D).$$

This completes the proof. □

Lemma 6.5. *Let $\gamma > 0$. Suppose $z = (x, y)$ with $|z| \geq 4$ and $y < -1$. There exists $P_0 \geq 4N$ (depending on γ) such that*

$$\mathbb{P}^z(\widetilde{B}(P) - B \mid D) \leq \gamma \mathbb{P}^z(B \mid D)$$

provided $P \geq P_0$ and $\varepsilon < \varepsilon_0(P)$.

Proof. In order to estimate the left hand side we split the path at $\sigma(0,0)$. The probability that the process after $\sigma(0,0)$ does not return to $\mathcal{L}(-1,0)$ before exiting $B(0,2)$ is bounded by $c_1\varepsilon$ by (1.3).

Next we consider the part of the trajectory before $\sigma(0,0)$. We will examine the time-reversed process $\widetilde{Z}$ which has been defined in (6.23). By Proposition 5.8 we may assume that $\widetilde{Z}$ is a BM $\times$ Bes(3) process; doing so will introduce at most a multiplicative constant in our estimates. We now split the path of $\widetilde{Z}$ at $\sigma(\varepsilon^{2/P-1}, 0)$ into two pieces, the initial piece $W_1(t)$, which is the portion of the path of $\widetilde{Z}$ before time $\sigma(\varepsilon^{2/P-1}, 0)$, and the second piece $W_2(t)$, which is the portion of the path after that time. By the strong Markov property, $W_2(t)$ is a BM $\times$ Bes(3) process.

For ω to be in $\widetilde{B}(P)$ but not B, ω cannot be in any $\widehat{A}_n$ for any $n \leq \varepsilon^{4/P-1}$ but ω must be in $\widetilde{A}_n$ for some $n \in [\varepsilon^{2/P-1}, \varepsilon^{-1}]$.

If R_1 is the x coordinate of the point where $\mathcal{L}(\varepsilon^{1/P-1}/2, P/\log(1/\varepsilon))$ intersects $\mathcal{L}(\varepsilon^{4/P-1}, 0)$, then $R_1 \geq \varepsilon^{2/P}$ if ε is small, and by (1.2) and scaling,

$$\mathbb{P}^0_{BMBS}(|X_{\sigma(\varepsilon^{4/P-1},0)}| > R_1) \leq c_2 \exp(-c_3\varepsilon^{-2/P}).$$

Hence the probability that $W_1(t)$ hits $\mathcal{L}(\varepsilon^{1/P-1}/2, P/\log(1/\varepsilon))$ or exits $\tau_{B(0,2)}$ before hitting $\mathcal{L}(\varepsilon^{4/P-1}, 0)$ is bounded by $c_2 \exp(-c_3\varepsilon^{-2/P})$.

If $W_1(t)$ does not hit $\mathcal{L}(\varepsilon^{1/P-1}/2, P/\log(1/\varepsilon))$ or exit $\tau_{B(0,2)}$ before hitting the line $\mathcal{L}(\varepsilon^{4/P-1}, 0)$ but $\omega \in \widetilde{B}(P)$, then $\widetilde{Z}$ can have no approximate lines of increase for $n \leq \varepsilon^{4/P-1}$. This implies that either (i) $W_1(t)$ has no approximate cut lines for $n \leq \varepsilon^{4/P-1}$ or (ii) $W_2(t)$ hits $\mathcal{L}(\varepsilon^{4/P-1}, 0)$. By Proposition 4.7, the first probability is bounded by $c_4|4/P - 1|/\log(1/\varepsilon)$, and by (2.1) the probability of (ii) is majorized by $\varepsilon^{4/P}/\varepsilon^{2/P} = \varepsilon^{2/P}$.

Take $\delta > 0$ small. We bound the probability that ω is in $\widetilde{A}_n$ for some $n \in [\varepsilon^{2/P-1}, \varepsilon^{-1}]$ by applying Proposition 4.6 to $W_2(t)$. By choosing $P_1(\delta)$ sufficiently large and taking $P \geq P_1(\delta)$, we can make this probability less than δ. If (i) holds, we combine with the probability that $W_1(t)$ has no approximate cut lines to obtain the bound $c_4\delta|4/P - 1|/\log(1/\varepsilon)$. If (ii) holds, we use the strong Markov property at time $\sigma(\varepsilon^{4/P-1}, 0)$ and obtain the estimate $\delta\varepsilon^{2/P}$. Summing and combining with the portion of the path of Z_t after $\sigma(0,0)$, we conclude

$$\mathbb{P}^z(\widetilde{B}(P) - B \mid D) \leq c_1\varepsilon\big[\delta(c_4|4/P - 1|/\log(1/\varepsilon) + \varepsilon^{2/P}) + c_2\exp(-c_3\varepsilon^{-2/P})\big].$$

Proposition 5.12 gives $\gamma c_5\varepsilon/\log(1/\varepsilon)$ as a lower bound for $\gamma\mathbb{P}^z(B \mid D)$. We choose $\delta < \gamma c_5/(2c_1c_4)$ and $P_0 = P_1(\delta)$. The lemma follows if we take $\varepsilon_0(P)$ sufficiently small. □

Recall that σ_P is the hitting time of the line $\mathcal{L}(\varepsilon^{1/P-1}, P/\log(1/\varepsilon))$. The next lemma says that the probability of hitting $\mathcal{L}_P$ far to the right is small.

Lemma 6.6. *Let Z_t be a BM× Bes(3) process. If $\varepsilon < \varepsilon_0(P)$, then*

$$\mathbb{P}^{(0,0)}_{BMBS}(|Z_{\sigma_P}| \geq \varepsilon^{1/2P}) \leq \varepsilon^3.$$

Proof. Let S be the sector $\{re^{i\theta} : 0 < r < R, 0 < \theta < \alpha\}$ and suppose that $R < 1$. First we will show that there exists c_1 such that if $0 < x < R/2$ and $\alpha \leq \pi/16$, then

$$\mathbb{P}^{(x,0)}_{BMBS}(|Z_{\tau_S}| = R) \leq \frac{c_1}{\alpha} x^{(\pi/\alpha)-1} R^{-\pi/\alpha}. \tag{6.24}$$

Let $h(x', y') = y'$ and $h_1(x', y') = \mathbb{P}^{(x',y')}_{BM}(|Z_{\tau_S}| = R\}$. Both h and h_1 are nonnegative harmonic functions in S with boundary value 0 on the x axis. By the boundary Harnack principle in $(x-\alpha x/2, x+\alpha x/2)\times(0, \alpha x/2)$, there exists c_2 such that

$$\frac{h_1(x, y')}{h(x, y')} \leq c_2 \frac{h_1(x, \alpha x/8)}{h(x, \alpha x/8)}, \qquad x < R/2, \quad y' < \alpha x/4.$$

By (1.1),

$$\begin{aligned}\mathbb{P}^{(x,y')}_{BMBS}(|Z_{\tau_S}| = R) &= \frac{h_1(x, y')}{h(x, y')} \leq c_2 \frac{h_1(x, \alpha x/8)}{h(x, \alpha x/8)} \\ &= \frac{8c_2}{\alpha x} h_1(x, \alpha x/8).\end{aligned}$$

By continuity,

$$\mathbb{P}^{(x,0)}_{BMBS}(|Z_{\tau_S}| = R) \leq \frac{8c_2}{\alpha x} h_1(x, \alpha x/8).$$

Using the conformal mapping $z \to z^{\pi/\alpha}$ taking S onto the upper half disk, the conformal invariance of Brownian motion implies that

$$h_1(x, \alpha x/8) = \mathbb{P}^{(x,\alpha x/8)}_{BM}(|Z_{\tau_S}| = R) = \mathbb{P}^{z_1}_{BM}(|Z_{\tau(H\cap B(0,1))}| = R^{\pi/\alpha}), \tag{6.25}$$

where H is the upper half plane and z_1 is the image of $(x, \alpha x/8)$. There exists c_3 such that $|z_1| \leq c_3 x^{\pi/\alpha}$. So by (1.3) and scaling,

$$\mathbb{P}^{(x,0)}_{BMBS}(|Z_{\tau_S}| = R) \leq \frac{c_4}{\alpha x} x^{\pi/\alpha}/R^{\pi/\alpha} = \frac{c_4}{\alpha} x^{(\pi/\alpha)-1} R^{-\pi/\alpha}.$$

This proves (6.24).

Next we apply (6.24) to prove the lemma. Let $\alpha = P/\log(1/\varepsilon)$. The x-coordinate x_0 of the point of intersection of $\mathcal{L}(\varepsilon^{1/P-1}, P/\log(1/\varepsilon))$ with the x-axis is bounded above by $c_5\varepsilon^{1/P}\log(1/\varepsilon)/P$. Let $R = \varepsilon^{1/2P}/2$. Suppose that $|x| \leq \varepsilon\log(1/\varepsilon)$. Translation invariance and (6.24) show that for small ε,

$$\begin{aligned}\mathbb{P}^{(0,0)}_{BMBS}(|Z_{\sigma_P}| \geq \varepsilon^{1/2P}) &\leq \frac{c_1}{\alpha} x_0^{(\pi/\alpha)-1} R^{-\pi/\alpha} \\ &\leq \frac{c_1}{P/\log(1/\varepsilon)} (c_5\varepsilon^{1/P}\log(1/\varepsilon)/P)^{(\pi/\alpha)-1} (\varepsilon^{1/2P}/2)^{-\pi/\alpha} \\ &= (c_1/c_5)(2c_5)^{\pi/\alpha} P^{-\pi/\alpha} (\log(1/\varepsilon))^{\pi/\alpha} \varepsilon^{(1/2P)(\pi/\alpha)-(1/P)} \\ &\leq c_6 \varepsilon^{(1/4P)(\pi/\alpha)} \\ &= c_6 \varepsilon^{\pi\log(1/\varepsilon)/4P^2}.\end{aligned} \tag{6.26}$$

The exponent of ε in the last line will be larger than 3 if ε is small enough, so the right hand side of (6.26) is less than ε^3 for $\varepsilon < \varepsilon_0(P)$. □

We are now ready for the iteration step.

Theorem 6.7. *There exist $\rho \in (0,1)$ and a sequence $M_1, N_1, M_2, N_2, \ldots$ not depending on ε satisfying*

(a)

$$4 < M_1 < N_1 < 16N_1 \leq M_2 < N_2 < 16N_2 \leq M_3 < \cdots$$

and

(b) if $z = (x, y)$, $|z| > 4$, and $y < -1$, and $i \geq 1$, then

$$\limsup_{\varepsilon \to 0} \frac{\mathbb{P}^z(B \cap F(M_1, N_i) \mid D)}{\mathbb{P}^z(B \mid D)} < \rho^i. \tag{6.27}$$

Proof. Since $\mathbb{P}^z(B \mid D) \geq c_1\varepsilon/\log(1/\varepsilon)$ by Proposition 5.12, it will suffice to prove

$$\limsup_{\varepsilon \to 0} \frac{\mathbb{P}^z(B \cap F(M_1, N_i) \mid D) - i\varepsilon^2}{\mathbb{P}^z(B \mid D)} < \rho^i. \tag{6.28}$$

The proof is by induction. If we set ρ equal to the ρ' of Theorem 6.1, the case $i = 1$ follows from that theorem. Let us suppose that (6.28) holds for i and we will show it holds for $i + 1$.

Fix z. If (6.28) holds for i, there exists $\delta > 0$ such that

$$\mathbb{P}^z(B \cap F(M_1, N_i) \mid D) < (\rho^i - \delta)\mathbb{P}^z(B \mid D) + i\varepsilon^2 \tag{6.29}$$

if ε is sufficiently small. Propositions 6.3, 6.4, and 6.5 tell us that

$$\mathbb{P}^z(B \cap [F(M_1, N_i) - G(M_1, N_i, P_i)] \mid D) \leq \gamma\mathbb{P}^z(B \mid D),$$

$$\mathbb{P}^z(B - \widetilde{B}(P_i) \mid D) \leq \gamma\mathbb{P}^z(B \mid D),$$

and

$$\mathbb{P}^z(\widetilde{B}(P_i) - B \mid D) \leq \gamma\mathbb{P}^z(B \mid D)$$

if $P_i \geq 4N_i$ is sufficiently large and ε is sufficiently small. Taking $\gamma = \rho\delta/4$,

$$\begin{aligned}\mathbb{P}^z([B \cap F(M_1,N_i)] - [\widetilde{B}(P_i) \cap G(M_1, N_i, P_i)] \mid D) \\ \leq \mathbb{P}^z(B \cap [F(M_1, N_i) - G(M_1, N_i, P_i)] \mid D) + \mathbb{P}^z(B - \widetilde{B}(P_i) \mid D) \\ < \frac{\rho\delta}{2}\mathbb{P}^z(B \mid D).\end{aligned} \tag{6.30}$$

Suppose we show there exist $M_{i+1} > 16P_i$ and $N_{i+1} > M_{i+1}$ such that

$$\begin{aligned}\mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \cap F(M_{i+1}, N_{i+1}) \mid D) \\ < \rho\mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \mid D) + \varepsilon^2\end{aligned} \tag{6.31}$$

for ε sufficiently small. Then using (6.29) and (6.30), if ε is sufficiently small,

$$\begin{aligned}
\mathbb{P}^z(B \cap F(M_1, N_{i+1}) \mid D) &\le \mathbb{P}^z(B \cap F(M_1, N_i) \cap F(M_{i+1}, N_{i+1}) \mid D) \\
&\le \mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \cap F(M_{i+1}, N_{i+1}) \mid D) \\
&\qquad + \mathbb{P}^z([B \cap F(M_1, N_i)] - [\widetilde{B}(P_i) \cap G(M_1, N_i, P_i)] \mid D) \\
&\le \rho\mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \mid D) + \varepsilon^2 + \frac{\rho\delta}{2}\mathbb{P}^z(B \mid D) \\
&\le \rho\mathbb{P}^z(B \cap G(M_1, N_i, P_i) \mid D) + \rho\gamma\mathbb{P}^z(B \mid D) + \varepsilon^2 + \frac{\rho\delta}{2}\mathbb{P}^z(B \mid D) \\
&\le \rho\mathbb{P}^z(B \cap F(M_1, N_i) \mid D) + \varepsilon^2 + \frac{3\rho\delta}{4}\mathbb{P}^z(B \mid D) \\
&< \rho(\rho^i - \delta)\mathbb{P}^z(B \mid D) + (i+1)\varepsilon^2 + \frac{3\rho\delta}{4}\mathbb{P}^z(B \mid D) \\
&= (\rho^{i+1} - \frac{\rho\delta}{4})\mathbb{P}^z(B \mid D) + (i+1)\varepsilon^2,
\end{aligned}$$

which implies (6.28) for $i+1$.

So it suffices to show (6.31). Recall (6.14), (6.23) and the definitions directly preceding and following them. Let

$$\begin{aligned}
E_1 &= \{\sigma(j,\alpha) < \sigma(-1,0) \wedge \tau_{B(0,3/2)}, |X_{\sigma(j,\alpha)}| \le j\varepsilon\}, \\
E_2 &= \{\tau_{B(0,3)} \circ \theta_{\sigma(j,\alpha)} < (\sigma(j-1,\alpha) \wedge \sigma(0,0)) \circ \theta_{\sigma(j,\alpha)}\}, \\
E_3 &= \{\tau_{B(0,2)} < \sigma(-1,0)\}, \\
E_4 &= \{\sigma(j,\alpha) \circ \theta_{\sigma(0,0)} > \sigma_{P_i} \circ \theta_{\sigma(0,0)}, |X(\sigma_{P_i})| \le \varepsilon^{1/2P_i}\}, \\
E_5 &= \{\widetilde{\sigma}(j,\alpha) > \widetilde{\sigma}_{P_i}, |\widetilde{X}(\widetilde{\sigma}_{P_i})| \le \varepsilon^{1/2P_i}\}, \\
E_6 &= \{\sigma(j,\alpha) \circ \theta_{\sigma(0,0)} < (\sigma(-1,0) \wedge \tau_{B(0,3/2)}) \circ \theta_{\sigma(0,0)}, |X_{\sigma(j,\alpha)}| \le j\varepsilon\}.
\end{aligned}$$

We will first prove the following modification of (6.31),

$$\begin{aligned}
\mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \cap F(M_{i+1}, N_{i+1}) \cap E_4 \cap E_5 \mid D) \\
\le \rho''\mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \cap E_4 \cap E_5 \mid D).
\end{aligned} \tag{6.32}$$

We take $M_{i+1} = 16P_i + 1$ and we consider a lower bound for

$$\mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \cap E_4 \cap E_5 \cap C(j,\alpha) \mid D), \tag{6.33}$$

for $\alpha \in I(M_{i+1}, N_{i+1})$ and $j \in [m(\alpha), n(\alpha)]$, where we will choose N_{i+1} in a moment. Instead of splitting the path once at $\sigma(0,0)$, we now split the path into three portions. The first portion is the piece of the path up until time $\pi(-\varepsilon^{1/P_i - 1}, P_i/\log(1/\varepsilon))$. The second piece is the portion of the path of Z_t from $\pi(-\varepsilon^{1/P_i - 1}, P_i/\log(1/\varepsilon))$ to $\sigma(\varepsilon^{1/P_i - 1}, P_i/\log(1/\varepsilon))$, and the third portion is the piece after $\sigma(\varepsilon^{1/P_i - 1}, P_i/\log(1/\varepsilon))$. A lower bound for (6.33) will be obtained by multiplying three factors corresponding to the three portions of the path. The contribution from the second, i.e., middle part, is

$$\mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \cap E_4 \cap E_5 \mid D). \tag{6.34}$$

For the event in (6.33) to hold, the line $\mathcal{L}(j,\alpha)$ must be an approximate cut line for the third portion of the path. By the strong Markov property at the stopping time $\sigma(\varepsilon^{1/P_i-1}, P_i/\log(1/\varepsilon))$, the third portion is a Brownian motion conditioned not to return to $\mathcal{L}(-1,0)$ before exiting $B(0,2)$. Let $w=(w_1,w_2)$ represent the starting point for the third portion. If E_4 holds, then $w\in\mathcal{L}_{P_i}$ with $|w_1|\le\varepsilon^{1/2P_i}$. By Proposition 5.5(b) we have

$$\mathbb{P}^w(E_1\cap E_2)\ge c_1(w_2+\varepsilon)/j.$$

By (1.3) and the fact that

$$\mathbb{P}^w_{BMBS}(E_3)\ge\mathbb{P}^w_{BMBS}(\sigma(3/\varepsilon,0)<\sigma(-1,0))=\frac{w_2+\varepsilon}{3+\varepsilon},$$

we have

$$c_2(w_2+\varepsilon)\le\mathbb{P}^w(E_3)\le c_3(w_2+\varepsilon). \tag{6.35}$$

Hence

$$\mathbb{P}^w(E_1\cap E_2)\ge\frac{c_4}{j}\mathbb{P}^w(E_3).$$

It follows that c_4/j is the factor corresponding to the third portion of the path.

Now let us examine the first portion of the path. If $\widetilde{Z}_t$ is defined by (6.23), then $\widetilde{Z}_t$ is a BM$\times$ Bes(3) process conditioned to go to $(-x,-y)$. The time $\pi(-\varepsilon^{1/P_i-1}, P_i/\log(1/\varepsilon))$ for Z_t becomes the first time $\widetilde{Z}_t$ hits $\mathcal{L}(\varepsilon^{1/P_i-1}, P_i/\log(1/\varepsilon))$. By the strong Markov property for $\widetilde{Z}_t$, the first portion of the path of Z_t, if run in reversed order, is also a BM$\times$ Bes(3) process conditioned to go to $(-x,-y)$. Let w now represent the starting point for this BM$\times$ Bes(3) process. Assuming that E_5 holds we have $w\in\mathcal{L}_{P_i}$ with $|w_1|\le\varepsilon^{1/2P_i}$. Suppose that $\alpha=\widetilde{M}/\log(1/\varepsilon)$ with $\widetilde{M}\in[M_{i+1},N_{i+1}]$ and $j\ge m(\alpha)$. Since the distance d from w to the intersection point of $\mathcal{L}(-j,\alpha)$ and the x-axis is greater than $\varepsilon^{3/\widetilde{M}}$, Corollary 5.3 and Proposition 5.8 show that there exists c_5 such that the probability that a BM$\times$ Bes(3) process started at w does not hit $\mathcal{L}(-j,\alpha)$ before exiting $B(0,2)$ is greater than

$$c_5 d^\alpha\ge c_5(\varepsilon^{3/\widetilde{M}})^{\widetilde{M}/\log(1/\varepsilon)}=c_5e^{-3}.$$

By multiplying this estimate by those corresponding to the other two portions of the path we obtain

$$\begin{aligned}\mathbb{P}^z(\widetilde{B}(P_i)\cap & G(M_1,N_i,P_i)\cap E_4\cap E_5\cap C(j,\alpha)\mid D)\\ &\ge\frac{c_4}{j}c_5e^{-3}\mathbb{P}^z(\widetilde{B}(P_i)\cap G(M_1,N_i,P_i)\cap E_4\cap E_5\mid D).\end{aligned} \tag{6.36}$$

In a similar fashion, if $\beta\in I(M_{i+1},N_{i+1})$ and $k\in[m(\beta),n(\beta)]$, we obtain an upper bound for

$$\mathbb{P}^z(\widetilde{B}(P)\cap G(M_1,N_i,P_i)\cap E_4\cap E_5\cap C(j,\alpha)\cap C(k,\beta)\mid D). \tag{6.37}$$

We split the path into two pieces at the stopping time σ_P. An upper bound for (6.37) will be obtained by multiplying the estimates for both portions of the path. An estimate for the first portion of the path is

$$\mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \cap E_4 \cap E_5 \mid D). \tag{6.38}$$

Let $w = (w_1, w_2)$ be the starting point for the second portion. We have $w \in \mathcal{L}_{P_i}$ with $|w_1| \le \varepsilon^{1/2P_i}$, assuming that E_4 holds. The second portion is a Brownian motion conditioned to exit $B(0,2)$ before returning to $\mathcal{L}(-1,0)$. We need an estimate for the probability that both lines $\mathcal{L}(j,\alpha)$ and $\mathcal{L}(k,\beta)$ are approximate cut lines. Propositions 5.6(b) and 5.7(b) and (6.35) show that the probability is bounded by

$$\frac{c_6}{j(|k-j|+1)}, \tag{6.39}$$

if $\alpha = \beta$ and $j, k \in [m(\alpha), n(\alpha)]$; by

$$\frac{c_7}{j(|k-j|+1)}((|k-j|+1)\varepsilon)^{|\beta-\alpha|}, \tag{6.40}$$

provided $j \in [m(\alpha), n(\alpha)]$, $k \in [m(\beta), n(\beta)]$ and $k > j + 200j|\beta - \alpha|$; and by

$$\frac{c_8}{j^2|\beta-\alpha|^2}, \tag{6.41}$$

when $j \in [m(\alpha), n(\alpha)]$, $k \in [m(\beta), n(\beta)]$ and $j \le k \le j + 200j|\beta - \alpha|$. An upper bound for (6.37) is obtained by multiplying each of the above estimates (6.39)-(6.41) by (6.38). Note that we ignore the possibility that the first portion of the path may hit one or both of the lines $\mathcal{L}(j,\alpha)$ and $\mathcal{L}(k,\beta)$ because we need only an upper bound for the probability in (6.37).

Setting

$$\mathbb{Q}(F) = \frac{\mathbb{P}^z(F \cap B \cap G(M_1, N_i, P_i) \cap E_4 \cap E_5 \mid D)}{\mathbb{P}^z(B \cap G(M_1, N_i, P_i) \cap E_4 \cap E_5 \mid D)}$$

we obtain from (6.36), (6.38) and (6.39)-(6.41) the following inequalities. For $\alpha, \beta \in I(M_{i+1}, N_{i+1})$, $j \in [m(\alpha), n(\alpha)]$ and $k \in [m(\beta), n(\beta)]$,

$$\mathbb{Q}(C(j,\alpha)) \ge c_9/j; \tag{6.42}$$

$$\mathbb{Q}(C(j,\alpha) \cap C(k,\beta)) \le \frac{c_{10}}{j(|k-j|+1)}, \tag{6.43}$$

if $\alpha = \beta$ and $j, k \in [m(\alpha), n(\alpha)]$;

$$\mathbb{Q}(C(j,\alpha) \cap C(k,\beta)) \le \frac{c_{11}}{j(|k-j|+1)}((|k-j|+1)\varepsilon)^{|\beta-\alpha|}, \tag{6.44}$$

provided $j \in [m(\alpha), n(\alpha)]$, $k \in [m(\beta), n(\beta)]$ and $k > j + 200j|\beta - \alpha|$; and

$$\mathbb{Q}(C(j,\alpha) \cap C(k,\beta)) \le \frac{c_{12}}{j^2|\beta-\alpha|^2}, \tag{6.45}$$

when $j \in [m(\alpha), n(\alpha)]$, $k \in [m(\beta), n(\beta)]$ and $j \le k \le j + 200j|\beta - \alpha|$.

Estimates (6.42)-(6.45) are completely analogous to (6.1)-(6.4). The constant ρ' in Theorem 6.1 depends only on the constants in (6.1)-(6.4) and does not depend on M_1 and N_1 as long as $\log N_1 - \log M_1 \ge 1$ and $\log N_1 \ge 2 \log M_1$. The proof of Theorem 6.1 does not use any other specific properties of the measure $\mathbb{Q}$ besides (6.1)-(6.4), so by following the same argument we can conclude from (6.42)-(6.45) that there exist N_{i+1} and ρ'' (independent of the M_n and N_n) such that for sufficiently small ε we have

$$\begin{aligned}
\mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \cap F(M_{i+1}, N_{i+1}) \cap E_4 \cap E_5 \mid D) \\
\le \rho'' \mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \cap E_4 \cap E_5 \mid D).
\end{aligned}$$

This finishes the proof of (6.32).

Recall event A_0 defined in (5.29) and the fact that $\widetilde{B}(P_i) \subset A_0$, by definition. By Lemma 6.6, time reversal and Proposition 5.8,

$$\mathbb{P}^z(E_5^c \mid D) \le c_8 \varepsilon^3. \tag{6.46}$$

Similarly,

$$\mathbb{P}^z(A_0 \cap E_4^c \mid D) \le c_8 \varepsilon^3. \tag{6.47}$$

Recall that (6.28) holds for $i = 1$ with $\rho = \rho'$. Choose $\rho \in ((\rho' \vee \rho''), 1)$ so that both (6.28) and (6.32) hold with this choice of ρ. Combining (6.32) and (6.46)-(6.47), we obtain for sufficiently small ε,

$$\begin{aligned}
&\mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \cap F(M_{i+1}, N_{i+1}) \mid D) \\
&\quad \le \mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \cap F(M_{i+1}, N_{i+1}) \cap E_4 \cap E_5 \mid D) \\
&\qquad + \mathbb{P}^z(A_0 \cap E_4^c \mid D) + \mathbb{P}^z(E_5^c \mid D) \\
&\quad \le \rho \mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \cap E_4 \cap E_5 \mid D) + 2c_8 \varepsilon^3 \\
&\quad \le \rho \mathbb{P}^z(\widetilde{B}(P_i) \cap G(M_1, N_i, P_i) \mid D) + \varepsilon^2.
\end{aligned}$$

This completes the proof of (6.31) and of the theorem. □

Recall the definitions of $\widetilde{Z}_t$ and $\widetilde{\sigma}(j, \alpha)$ from (6.23). Let

$$\begin{aligned}
J[a, b) &= \{\alpha \in [a, b) : \alpha \log(1/\varepsilon) \in \mathbb{Z}\}, \qquad (6.48)\\
A_n(r) &= \{\tau_{B(0,2)} \circ \theta_{\sigma(n,0)} < \sigma(n-1, 0) \circ \theta_{\sigma(n,0)}, \sigma(n, 0) < \infty, |Z_{\sigma(n,0)}| \le r\}, \\
\widetilde{A}_n(r) &= \{\widetilde{\tau}_{B(0,2)} \circ \theta_{\widetilde{\sigma}(n,0)} < \widetilde{\sigma}(n-1, 0) \circ \theta_{\widetilde{\sigma}(n,0)}, \widetilde{\sigma}(n, 0) < \infty, |\widetilde{Z}_{\widetilde{\sigma}(n,0)}| \le r\}, \\
B(r) &= A_0 \cap \widetilde{A}_2^c(r) \cap \cdots \cap \widetilde{A}_{1/\varepsilon}^c(r), \\
C(j, \alpha, r) &= C(j, \alpha) \cap \{|Z_{\sigma(j,\alpha)}| \le r\}, \\
F(r) &= \bigcap_{0 \le j \le \varepsilon^{-1}} \ \bigcap_{\alpha \in J[0, \pi/16)} C(j, \alpha, r)^c.
\end{aligned}$$

We want to be able to restrict where our approximate cut lines take place.

Lemma 6.8. *Suppose that $z=(x,y)$, $|z|>4$ and $y<-1$.*

(a) There exists c_1 such that for sufficiently small ε,

$$\mathbb{P}^z(B(\varepsilon^{1/2}) \mid D) \le c_1\varepsilon/\log(1/\varepsilon).$$

(b) Let $\gamma,\kappa\in(0,1/2)$. For sufficiently small ε,

$$\mathbb{P}^z(B(\kappa) - B \mid D) \le \gamma\varepsilon/\log(1/\varepsilon).$$

(c) Let $\gamma,\kappa\in(0,1/2)$. Then

$$\mathbb{P}^z(B(\kappa)\cap F(\kappa) \mid D) \le \gamma\varepsilon/\log(1/\varepsilon),$$

for sufficiently small ε.

Proof. (a) By applying the same argument as in the proof of Proposition 5.12 we reduce the problem to showing that

$$\mathbb{P}^{(0,0)}_{BMBS}(A_2^c(\varepsilon^{1/2})\cap\cdots\cap A^c_{1/\varepsilon}(\varepsilon^{1/2})) \le c_2/\log(1/\varepsilon). \tag{6.49}$$

Let $\widehat{A}$ be the event that the process escapes from $B(0,\varepsilon^{1/2})$ before hitting $\mathcal{L}(\varepsilon^{-1/4},0)$. Let A^* denote the event that the process returns to $\mathcal{L}(\varepsilon^{-1/8},0)$ after hitting $\mathcal{L}(\varepsilon^{-1/4},0)$, and let

$$A'_n = \{\sigma(\varepsilon^{-1/4},0)\circ\theta_{\sigma(n,0)} < \sigma(n-1,0)\circ\theta_{\sigma(n,0)}\}.$$

Then

$$\begin{aligned} &\mathbb{P}^{(0,0)}_{BMBS}(A_2^c(\varepsilon^{1/2})\cap\cdots\cap A^c_{1/\varepsilon}(\varepsilon^{1/2})) \\ &\qquad \le \mathbb{P}^{(0,0)}_{BMBS}(\widehat{A}) + \mathbb{P}^{(0,0)}_{BMBS}(A^*) + \mathbb{P}^{(0,0)}_{BMBS}((A'_2)^c\cap\cdots\cap(A'_{\varepsilon^{-1/8}})^c). \end{aligned} \tag{6.50}$$

The probability of $\widehat{A}$ is bounded by $c_3\exp(-c_4\varepsilon^{-1/4})$ by the remark following (1.2). Equation (2.1) shows that the probability of A^* is $\varepsilon^{7/8}/\varepsilon^{3/4}=\varepsilon^{1/8}$. Proposition 4.7 and scaling show that the last term in (6.50) is bounded by $c_5/\log(1/\varepsilon)$. Hence the probability in (6.49) is bounded by

$$c_3\exp(-c_4\varepsilon^{-1/4}) + \varepsilon^{1/8} + c_5/\log(1/\varepsilon).$$

This completes the proof of (6.49) and part (a) of the lemma.

(b) The proof is similar to that of part (a). Fix some $\delta>0$. First we will show that there exists c_6 such that

$$\mathbb{P}^{(0,0)}_{BMBS}([A_2^c(\kappa)\cap\cdots\cap A^c_{1/\varepsilon}(\kappa)] - [A_2^c\cap\cdots\cap A^c_{1/\varepsilon}]) \le \delta c_6/\log(1/\varepsilon). \tag{6.51}$$

Take $\zeta>0$, where we will choose the value of ζ in a moment. Suppose ω is in the event in (6.51). Then for some $n\le 1/\varepsilon$ there is an approximate cut line at $\sigma(n,0)$

but $|Z_{\sigma(n,0)}| > \kappa$. Since $\varepsilon^{\zeta/2} < \kappa$ for ε small, one of the following three possibilities must hold. The first possibility is that there may be an approximate cut line at time $\sigma(n,0)$ with $Z_{\sigma(n,0)} \in U$ where $U = \{(x,y) : 0 \leq y \leq \varepsilon^{\zeta/2}, |x| \geq \varepsilon^{\zeta/8}\}$. The second possibility is that the process returns to $\mathcal{L}(\varepsilon^{2\zeta-1}, 0)$ after hitting $\mathcal{L}(\varepsilon^{\zeta-1}, 0)$. Finally, the third possibility is that the portion of the trajectory $\{Z_t, 0 \leq t \leq \sigma(\varepsilon^{\zeta-1}, 0)\}$ may have no approximate cut line at any time $\sigma(n,0)$ with $n \leq \varepsilon^{2\zeta-1}$ and at the same time there is an approximate cut line for some $n \geq \varepsilon^{\zeta-1}$.

Choose $\zeta > 0$ as in Proposition 4.6 so that

$$\mathbb{P}^{(0,\varepsilon^\zeta)}_{BMBS}(A_{\varepsilon^{\zeta-1}} \cup \cdots \cup A_{1/\varepsilon}) \leq \delta.$$

By Proposition 4.7 and scaling, the probability that $\{Z_t, 0 \leq t \leq \sigma(\varepsilon^{\zeta-1}, 0)\}$ has no approximate cut line at any time $\sigma(n,0)$ with $n \leq \varepsilon^{2\zeta-1}$ is bounded by $c_7/\log(1/\varepsilon)$. This and the strong Markov property applied at $\sigma(\varepsilon^{\zeta-1}, 0)$ imply that the probability for the first option is bounded by $\delta c_7/\log(1/\varepsilon)$.

The chance of returning to $\mathcal{L}(\varepsilon^{2\zeta-1}, 0)$ after hitting $\mathcal{L}(\varepsilon^{\zeta-1}, 0)$ is ε^ζ by (2.1).

Let $\widehat{A}$ be the event that the process ever hits the set U. The event $\widehat{A}$ can be realized in two ways. First, the process may exit $B(0, \varepsilon^{\zeta/8}/2)$ before hitting $\mathcal{L}(\varepsilon^{\zeta/4-1}, 0)$; the probability of this happening is bounded by $c_8 \exp(-c_9 \varepsilon^{-\zeta/8})$ by (1.2) and the remark following it. The second way $\widehat{A}$ might happen is if the process returns to $\mathcal{L}(\varepsilon^{\zeta/2-1}, 0)$ after hitting $\mathcal{L}(\varepsilon^{\zeta/4-1}, 0)$. The probability of this event is $\varepsilon^{\zeta/4}$. It follows that for small ε,

$$\mathbb{P}^{(0,0)}_{BMBS}(\widehat{A}) \leq c_8 \exp(-c_9 \varepsilon^{-\zeta/8}) + \varepsilon^{\zeta/4} \leq 2\varepsilon^{\zeta/4}.$$

By adding the probabilities for the three possible ways the event in (6.51) may be realized, we see that its probability is bounded by

$$\delta c_7/\log(1/\varepsilon) + \varepsilon^\zeta + 2\varepsilon^{\zeta/4} \leq \delta c_{10}/\log(1/\varepsilon),$$

for small ε. By adjusting the value of δ we obtain (6.51).

Given (6.51), part (b) of the lemma can be obtained by appealing to path decomposition at $\sigma(0,0)$ and Proposition 5.8, as in the proof of Proposition 5.12. In this way we obtain the bound $\delta c_{11}\varepsilon/\log(1/\varepsilon)$. Since $\delta > 0$ is arbitrarily small, the proof of part (b) is complete.

(c) For fixed M, N and κ, for j and α as in the definition of $F(M,N)$, and for small ε, we have $C(j,\alpha,\kappa) = C(j,\alpha)$ by the definition of $C(j,\alpha)$ (see (5.31)). Hence Theorem 6.7 says that for some $\rho < 1$ and arbitrary fixed i,

$$\mathbb{P}^z(B \cap F(\kappa) \mid D) \leq \rho^i \mathbb{P}^z(B \mid D),$$

if ε is small. By Proposition 5.12, $\mathbb{P}^z(B \mid D) \leq c_{12}\varepsilon/\log(1/\varepsilon)$. Choose i so that $\rho^i < \gamma/2c_{12}$. Then

$$\mathbb{P}^z(B \cap F(\kappa) \mid D) \leq (\gamma/2)\varepsilon/\log(1/\varepsilon).$$

For small ε we have

$$\mathbb{P}^z(B(\kappa) - B \mid D) \le (\gamma/2)\varepsilon/\log(1/\varepsilon),$$

by part (b) of the lemma. This and the previous estimate immediately imply (c). □

Let Λ be the square $\{(x,y) : |x| + |y| < 1/4\}$ and let $D(j,\alpha,v)$ denote $\{Z_{\sigma(j,\alpha)} = v\}$. Suppose that Z hits Λ and let $T_0 = \sup\{t < \mathcal{T}_\Lambda : Z_t \in \partial B(0,2)\}$, the last exit time from $\partial B(0,2)$ before hitting Λ. Let $T_1 = \inf\{t > \mathcal{T}_\Lambda : Z_t \in \partial B(0,2)\}$, the first hitting time of $\partial B(0,2)$ after hitting Λ. Let $\widehat{Z}_t = Z_{(T_0+t)\wedge T_1}$. We make the following definitions relative to $\widehat{Z}$. We say that $\mathcal{L}(j,\alpha)$ is an approximate cut line if $\tau_{B(0,2)} \circ \theta_{\sigma(j,\alpha)} < \sigma(j-1,\alpha) \circ \theta_{\sigma(j,\alpha)}$. Let $\widehat{B}(j,\alpha,\kappa)$ be the event that $\mathcal{L}(j,\alpha)$ is an approximate cut line but there is no approximate cut line $\mathcal{L}(n,\alpha)$ with $n < j$ and $|\widehat{Z}_{\sigma(n,\alpha)} - \widehat{Z}_{\sigma(j,\alpha)}| \le \kappa$. Let $\widehat{F}(j,\alpha,\kappa)$ be the event that there is no line of increase $\mathcal{L}(n,\beta)$ with $\beta \in J(\alpha,\alpha+\kappa)$ and $|\widehat{Z}_{\sigma(n,\beta)} - \widehat{Z}_{\sigma(j,\alpha)}| \le \kappa$.

We modify our previous results to take care of the case where $Z_{\sigma(0,0)} \neq 0$ and to replace $\mathcal{L}(0,0)$ by other lines.

Lemma 6.9. *Suppose that $z = (x,y)$, $|z| > 4$, $y < -1$, $|j| \le 1/\varepsilon$, $\alpha \in J[0,\pi/16)$, $\kappa \le \pi/16$, and $v \in \Lambda$.*

(a) For sufficiently small ε,

$$\mathbb{P}^z(\widehat{B}(j,\alpha,\varepsilon^{1/2}) \mid D(j,\alpha,v)) \le c_1\varepsilon/\log(1/\varepsilon).$$

(b) Let $\gamma > 0$. Then

$$\mathbb{P}^z(\widehat{B}(j,\alpha,\kappa) \cap \widehat{F}(j,\alpha,\kappa) \mid D(j,\alpha,v)) \le \gamma\varepsilon/\log(1/\varepsilon),$$

for $\varepsilon < \varepsilon_0$, where ε_0 depends only on κ and γ.

Proof. Let $\overline{Z}_n(t) = Z_{\sigma(n,0)-t}$, the process Z_t time-reversed at time $\sigma(n,0)$. By rotation and translation invariance, $\overline{Z}_n$ has the same law as that obtained by taking a BM $\times$ Bes(3), rotating it an angle π, translating it by $Z_{\sigma(n,0)}$, and finally taking the resulting process and conditioning it to go to z. By (1.2) and Proposition 5.8,

$$\mathbb{P}^z(|\overline{Z}_n(\sigma(n-1,0)) - Z_{\sigma(n,0)}| > \varepsilon^{1/2}/2) \le c_2 \exp(-c_3\varepsilon^{1/2}).$$

So if G is the event that for some $n \in [-1/\varepsilon, 1/\varepsilon]$ the line $\mathcal{L}(n,0)$ is an approximate cut line and $|\overline{Z}_n(\sigma(n-1,0)) - Z_{\sigma(n,0)}| > \varepsilon^{1/2}/2$, then

$$\mathbb{P}^z(G) \le \varepsilon^2$$

for ε sufficiently small.

Suppose the event $\widehat{B}(0,0,\varepsilon^{1/2}) \cap G^c$ holds. Then $\mathcal{L}(0,0)$ is an approximate cut line and if $n < 0$, either $\mathcal{L}(n,0)$ is not an approximate cut line or else $\mathcal{L}(n,0)$

is an approximate cut line but $|Z_{\sigma(n,0)} - Z_{\sigma(0,0)}| > \varepsilon^{1/2}$. In the second case, since $\mathcal{L}(n,0)$ is an approximate cut line and $\omega \notin G$,

$$|\overline{Z}_n(\sigma(n-1,0)) - Z_{\sigma(0,0)}| > \varepsilon^{1/2}/2.$$

Hence if $Z_{\sigma(0,0)} = 0$, then $\widehat{B}(0,0,\varepsilon^{1/2}) \subseteq B(\varepsilon^{1/2}/2) \cup G$. So by Lemma 6.8(a) with $\varepsilon^{1/2}$ replaced by $\varepsilon^{1/2}/2$,

$$\begin{aligned}\mathbb{P}^z(\widehat{B}(0,0,\varepsilon^{1/2}) \mid D(0,0,0)) &\le \mathbb{P}^z(G \mid D) + \mathbb{P}^z(B(\varepsilon^{1/2}/2) \mid D) \\ &\le \varepsilon^2 + c_4\varepsilon/\log(1/\varepsilon)\end{aligned}$$

if ε is sufficiently small. This implies (a) with $v = (0,0)$, $j = 0$, and $\alpha = 0$. A similar argument implies (b) with $v = (0,0), j = 0, \alpha = 0$, and $\kappa = \pi/16$.

The assertions (a) and (b) may be proved for any fixed $\kappa \le \pi/16$, $v = (0,0)$, and $j = \alpha = 0$ by minor modifications to the above and to the proofs of Sections 5 and 6. These proofs apply as well to other values of j, α and v by translation and rotation invariance of Brownian motion. Here we have to use the remarks preceding (5.3) since the problem is not quite translation invariant. For example, there may be no line of the form $\mathcal{L}(k,\beta)$ passing through v. Note that for $v \in \Lambda$, the distance from v to $\partial B(0,2)$ is at least 1 and this is the only assumption that plays an important role in our proofs. □

If $U \subseteq B(0,2)$, let

$$H(j,\alpha,U) = \{\tau_{B(0,2)} \circ \theta_{\sigma(j,\alpha)} < \sigma(j-1,\alpha) \circ \theta_{\sigma(j,\alpha)}, Z_{\sigma(j,\alpha)} \in U\}.$$

This is the event that $\mathcal{L}(j,\alpha)$ is an approximate cut line and Z hits the line $\mathcal{L}(j,\alpha)$ inside the set U. Let

$$\widetilde{H} = \bigcup_{-1/\varepsilon \le j \le 1/\varepsilon} \; \bigcup_{\alpha \in J(0,\pi/16)} H(j,\alpha,\Lambda).$$

The following is the main goal of this section.

Theorem 6.10. *If $\gamma > 0$ and $z = (x,y)$ with $|z| > 4$ and $y < -1$, then $\mathbb{P}^z(\widetilde{H}) < \gamma$ for ε sufficiently small.*

Proof. Fix $\delta > 0$, $\kappa \in (0, 1/32)$, and let

$$\begin{aligned}\Lambda(b) &= \{z \in \Lambda : \mathrm{dist}\,(z, \partial\Lambda) \le b\}, \\ \Lambda(a,b) &= \{z \in \Lambda : a \le \mathrm{dist}\,(z, \partial\Lambda) \le b\}.\end{aligned}$$

If $\widetilde{H}$ holds, then there must be a largest α such that $H(j,\alpha,\Lambda)$ holds for some value of j. For this value of α there is a smallest value of j for which $H(j,\alpha,\Lambda)$ holds. Hence if $\widetilde{H}$ holds then one of the following must occur:

(i) there exist $j \in [-1/\varepsilon, 1/\varepsilon]$ and $\alpha \in J[0, \pi/16)$ such that $H(j, \alpha, \Lambda(2\varepsilon^{1/2}))$ holds; or

(ii) there exist $j \in [-1/\varepsilon, 1/\varepsilon]$ and $\alpha \in J[0, \pi/16 - \kappa)$ such that $H(j, \alpha, \Lambda(\varepsilon^{1/2}, 2\kappa))$ holds but no event $H(n, \alpha, \Lambda)$ holds with $n \in [-1/\varepsilon, j)$; or

(iii) there exist $j \in [-1/\varepsilon, 1/\varepsilon]$ and $\alpha \in J[0, \pi/16 - \kappa)$ such that $H(j, \alpha, \Lambda - \Lambda(\kappa))$ holds but no event $H(n, \beta, \Lambda)$ holds with $\beta \in J(\alpha, \alpha + \kappa)$ and $n \in [-1/\varepsilon, 1/\varepsilon]$ and also no event $H(n, \alpha, \Lambda)$ holds with $n \in [-1/\varepsilon, j)$; or

(iv) there exist $j \in [-1/\varepsilon, 1/\varepsilon]$ and $\alpha \in J[\pi/16 - \kappa, \pi/16)$ such that $H(j, \alpha, \Lambda - \Lambda(\varepsilon^{1/2}))$ holds.

First we will estimate the probability of $H(j, \alpha, \Lambda(2\varepsilon^{1/2}))$. By the formula for the Poisson kernel in the half plane, there exists c_1 such that the density with respect to length measure on $\mathcal{L}(j, \alpha) \cap \Lambda$ of the $\mathbb{P}^z$-distribution of $Z_{\sigma(j,\alpha)}$ is bounded by c_1 for all $|j| \le 1/\varepsilon$, $\alpha \in J(0, \pi/16)$, and $|z| > 4$. The total length of $\mathcal{L}(j, \alpha) \cap \Lambda(2\varepsilon^{1/2})$ does not exceed $4\sqrt{2}\varepsilon^{1/2}$. It follows that

$$\mathbb{P}^z(Z_{\sigma(j,\alpha)} \in \Lambda(2\varepsilon^{1/2})) \le c_2 \varepsilon^{1/2}. \tag{6.52}$$

The probability that Brownian motion starting from a point of $\mathcal{L}(j, \alpha) \cap \Lambda$ will hit $\partial B(0, 2)$ before hitting $\mathcal{L}(j - 1, \alpha)$ is bounded by $c_3 \varepsilon$ by the same argument that proves (1.3). Therefore

$$\mathbb{P}^z\big(H(j, \alpha, \Lambda(2\varepsilon^{1/2}))\big) \le c_3 \varepsilon \cdot c_2 \varepsilon^{1/2} = c_4 \varepsilon^{3/2}. \tag{6.53}$$

Since there are at most $2/\varepsilon + 1$ possible values of j and at most $(\pi/16)\log(1/\varepsilon)$ possible values of α,

$$\mathbb{P}^z\Big(\bigcup_{-1/\varepsilon \le j \le 1/\varepsilon}\ \bigcup_{\alpha \in J(0,\pi/16)} H(j, \alpha, \Lambda(2\varepsilon^{1/2}))\Big) \le c_5 \varepsilon^{1/2} \log(1/\varepsilon), \tag{6.54}$$

which gives a bound for (i).

Let us consider the event (ii). If $H(j, \alpha, \Lambda(\varepsilon^{1/2}, 2\kappa))$ occurs but $H(n, \alpha, \Lambda)$ does not occur for $n \in [-1/\varepsilon, j)$, then either $Z_{\sigma(j,\alpha)} \in \Lambda(2\varepsilon^{1/2})$ or else $H(j, \alpha, \Lambda(\varepsilon^{1/2}, 2\kappa))$ occurs and $H(n, \alpha, \Lambda)$ does not occur for $n \in [-1/\varepsilon, j)$. Therefore the event described by (ii) is contained in the union of the event described by (i) and the event described by

(ii′) there exist $j \in [-1/\varepsilon, 1/\varepsilon]$ and $\alpha \in J[0, \pi/16 - \kappa)$ such that $H(j, \alpha, \Lambda(\varepsilon^{1/2}, 2\kappa))$ holds but no event $H(n, \alpha, \Lambda)$ occurs with $n \in [-1/\varepsilon, j)$.

We now deal with the event described in (ii′). We have

$$\mathbb{P}^z(Z_{\sigma(j,\alpha)} \in \Lambda(\varepsilon^{1/2}, 2\kappa)) \le \kappa c_2,$$

by the same argument we used to prove (6.52). If we integrate the inequality in Lemma 6.9(a) over $v \in \Lambda(\varepsilon^{1/2}, 2\kappa)$, we have that the probability that

$H(j,\alpha,\Lambda(\varepsilon^{1/2},\kappa))$ holds but $H(n,\alpha,\Lambda)$ does not for $n \in [-1/\varepsilon, j)$ if $\alpha \le \pi/16 - \kappa$ and $|j| \le 1/\varepsilon$ is bounded by $\kappa c_6 \varepsilon/\log(1/\varepsilon)$. Now, similarly to (6.54), summing over all such values of j and α, we obtain a bound κc_7 for the probability of the event (ii′).

Just as with (ii), the event (iii) is contained in the union of the event described in (ii) and the event described by

(iii′) there exist $j \in [-1/\varepsilon, 1/\varepsilon]$ and $\alpha \in J[0, \pi/16 - \kappa)$ such that $H(j,\alpha,\Lambda - \Lambda(\kappa))$ holds but no event $H(n,\beta,\Lambda)$ holds with $\beta \in J(\alpha, \alpha+\kappa)$ and $n \in [-1/\varepsilon, 1/\varepsilon]$ and also no event $H(n,\alpha,\Lambda)$ holds with $n \in [-1/\varepsilon, j)$.

We look at (iii′). We have restricted our event so that we can apply Lemma 6.9 (b). If we take the inequality in Lemma 6.9(b) with γ replaced by δ and integrate over $v \in \Lambda - \Lambda(\kappa)$, we obtain the bound $c_8 \delta\varepsilon/\log(1/\varepsilon)$ for the probability of the event in (iii′) for a fixed j and α. As usual, we sum over j and α to see that the probability of the event in (iii′) is bounded by δc_9.

Finally, we deal with (iv). Fix $\alpha \in J[\pi/16 - \kappa, \pi/16)$ and find the smallest j for which $H(j,\alpha,\Lambda)$ holds. The argument given in cases (ii) and (iii) applies here also and gives us a bound $c_{10}/\log(1/\varepsilon)$ for the probability of

$$\bigcup_{-1/\varepsilon \le j \le 1/\varepsilon} H(j,\alpha,\Lambda - \Lambda(\varepsilon^{1/2})).$$

There are at most $\kappa \log(1/\varepsilon)$ values of α in $J[\pi/16 - \kappa, \pi/16)$. Summing over all such α, we get a bound κc_{10} for the event in (iv).

The sum of the probabilities of the events in (i), (ii′), (iii′), and (iv) is thus no more than

$$c_5 \varepsilon^{1/2} \log(1/\varepsilon) + \kappa c_7 + \delta c_9 + \kappa c_{10},$$

for small ε. If we choose δ and κ sufficiently small, the theorem follows. □

7. Estimates for wedges.

Recall that the definition of $\mathcal{L}(j,\alpha)$ is relative to a fixed point $\widetilde{w}$. The distance from $\mathcal{L}(j,\alpha)$ to $\widetilde{w}$ is an integer multiple of ε. We set $\widetilde{w}=(0,0)$ until further notice. In this way, $\mathcal{L}(0,\alpha)$ passes through $(0,0)$ for every α.

We begin by deriving upper bounds for the hitting distribution of Brownian motion in a wedge.

Proposition 7.1. *Let $F=\{re^{i\theta}: r>0, 0<\theta<\pi-\alpha\}$ and $z_0=(x_0,y_0)$, where $0<y_0\le 2$, $x_0\ge 0$, and $0<\alpha<\pi/8$. Let $E(s,t)=\{re^{i(\pi-\alpha)}: s\le r<t\}$. There exists c_1 such that the following hold.*

(a) For $0\le s<t$,

$$\mathbb{P}^{z_0}_{BM}(Z_{\tau_F}\in E(s,t))\le c_1\int_s^t \frac{y_0}{y_0^2+(x_0+r)^2}\,dr.$$

(b) If $1\le s<t\le 2$ then

$$\mathbb{P}^{z_0}_{BM}(Z_{\tau_F}\in E(s,t),\tau_F<\tau_{B(0,2)})\le c_1\int_s^t y_0(2-r)dr.$$

Proof. (a) Since $z\mapsto \mathbb{P}^z(Z_{\tau_F}\in E(s,t))$ is a harmonic function of z, by Harnack's inequality we may assume without loss of generality that $x_0\ge 8y_0$. We map F into the upper half plane H by the conformal map $z\longrightarrow z^{\pi/(\pi-\alpha)}$. Let $z_1=(x_1,y_1)$ be the image of z_0. The image of $E(s,t)$ is $(-t^{\pi/(\pi-\alpha)},-s^{\pi/(\pi-\alpha)}]$. We recall that Brownian motion is conformally invariant and we use the explicit formula for the exit distribution of Brownian motion in a half plane (Bass (1995), p. 91) to see that

$$\mathbb{P}^{z_0}_{BM}(Z_{\tau_F}\in E(s,t))=\frac{1}{\pi}\int_{s^{\pi/(\pi-\alpha)}}^{t^{\pi/(\pi-\alpha)}}\frac{y_1}{y_1^2+(x_1+r)^2}\,dr.$$

Define

$$h(t)=\int_0^{t^{\pi/(\pi-\alpha)}}\frac{y_1}{y_1^2+(x_1+r)^2}\,dr,$$

and so

$$h'(t)=\frac{\pi}{\pi-\alpha}t^{\alpha/(\pi-\alpha)}\frac{y_1}{y_1^2+(x_1+t^{\pi/(\pi-\alpha)})^2}.$$

As in the proof of Proposition 5.2, there exist c_3 and c_4 such that

$$c_3x_0^{\alpha/(\pi-\alpha)}y_0\le y_1\le c_4x_0^{\alpha/(\pi-\alpha)}y_0$$

and

$$c_3x_0^{\pi/(\pi-\alpha)}\le x_1\le c_4x_0^{\pi/(\pi-\alpha)}.$$

By our assumptions that $x_0\ge 8y_0$ and $\alpha\le\pi/8$, we have

$$x_0+t\ge x_0\ge y_0 \qquad\text{and}\qquad x_1+t\ge x_1\ge y_1.$$

If $t \le x_0$, so that $t^{\pi/(\pi-\alpha)} \le c_5 x_1$, then

$$h'(y) \le c_6 \frac{t^{\alpha/(\pi-\alpha)} y_1}{x_1^2} \le c_7 \frac{y_0 t^{\alpha/(\pi-\alpha)}}{x_0^2 x_0^{\alpha/(\pi-\alpha)}} \le c_7 \frac{y_0}{x_0^2} \le c_8 \frac{y_0}{(x_0+t)^2} \le c_9 \frac{y_0}{y_0^2 + (x_0+t)^2}.$$

If $t > x_0$, so that $t^{\pi/(\pi-\alpha)} \ge c_{10} x_1$, then

$$\begin{aligned} h'(t) &\le c_{11} \frac{t^{\alpha/(\pi-\alpha)} y_1}{t^2 t^{2\alpha/(\pi-\alpha)}} \le c_{12} \frac{y_0 x_0^{\alpha/(\pi-\alpha)}}{t^2 t^{\alpha/(\pi-\alpha)}} \le c_{12} \frac{y_0}{t^2} \\ &\le c_{13} \frac{y_0}{(x_0+t)^2} \le c_{14} \frac{y_0}{y_0^2 + (x_0+t)^2}. \end{aligned}$$

In either case, $h'(t) \le c_{15} y_0 / (y_0^2 + (x_0+t)^2)$. Since

$$\mathbb{P}^{z_0}(Z_{\tau_F} \in E(s,t)) = \pi^{-1}[h(t) - h(s)],$$

part (a) follows.

(b) Let F' be the wedge $\{(x,y) : x > 0, y > 0\}$. An argument using conformal mappings, similar to that in part (a) of the proof, shows that the hitting density on $\partial F'$ for Brownian motion starting from $(0.1, 0.1)$ is bounded by $c_{16} v_1$ at a point $v = (v_1, 0) \in \partial F'$ provided $v_1 \le 1$. Let F'' be the wedge with aperture angle $\pi/2$ which contains $B(0,2) \cap F$, whose vertex is the point of intersection of $\mathcal{L}(0,\alpha)$ with $\partial B(0,2)$ and which has one edge lying on $\mathcal{L}(0,\alpha)$. Let z_1 be the point inside F'' which is equidistant from the two edges of F'' and which is a distance $(0.1)\sqrt{2}$ from the vertex of F''. By translation and rotation invariance, for $1 \le s < t \le 2$,

$$\mathbb{P}^{z_1}_{BM}(Z_{\tau_F} \in E(s,t), \tau_F < \tau_{B(0,2)}) \le \mathbb{P}^{z_1}_{BM}(Z_{\tau_{F''}} \in E(s,t)) \le c_{16} \int_s^t (2-r)dr.$$

The function $z \to \mathbb{P}^{z}_{BM}(Z_{\tau_F} \in E(s,t), \tau_F < \tau_{B(0,2)})$ is harmonic in $F \cap B(0,2)$. Let $h_1(x,y)$ be the harmonic function in $F \cap B(0,2)$ which has the same boundary values as $\mathbb{P}^{(x,y)}_{BM}(Z_{\tau_F} \in E(1/4,1/2))$ for $(x,y) \in \partial(F \cap B(0,2))$ with $x < -1/8$. We let h_1 have 0 boundary values elsewhere on the boundary. It is easy to see that $h_1(z_1) \ge c_{17}$ where c_{17} does not depend on α. Using part (a),

$$h_1(x,y) \le \mathbb{P}^{(x,y)}_{BM}(Z_{\tau_F} \in E(1/4,1/2)) \le c_{18} y,$$

for $(x,y) \in F \cap B(0,2)$ with $x \ge 0$. An application of the boundary Harnack principle shows that for z_0 as in the statement of the proposition, and $1 \le s < t \le 2$,

$$\begin{aligned} \mathbb{P}^{z_0}_{BM}(Z_{\tau_F} \in E(s,t), \tau_F < \tau_{B(0,2)}) &\le c_{19} \mathbb{P}^{z_1}_{BM}(Z_{\tau_F} \in E(s,t), \tau_F < \tau_{B(0,2)}) \frac{h_1(z_0)}{h_1(z_1)} \\ &\le c_{19} \frac{c_{18} y_0}{c_{17}} c_{16} \int_s^t (2-r)dr. \end{aligned}$$

□

Recall that $\mathbb{P}_h$ denotes the distribution of Brownian motion conditioned by h. Let H be the upper half-plane.

Proposition 7.2. *Let F be as in Proposition 7.1. Let h be a positive harmonic function in F with zero boundary values except for a pole at a point $z_0 \in \partial F - \partial H$.*

(a) There exists c_1 such that if $|(x_1, y_1)| < |z_0| < \lambda$, $x_1, y_1 \geq 0$, then

$$\mathbb{P}_h^{(x_1,y_1)}(\sup_{t \leq \tau_F} Y_t > \lambda) \leq c_1(|z_0|/\lambda)^{2\pi/(\pi-\alpha)}.$$

(b) There exist c_2 and c_3 such that if $|(x_1, y_1)| < |z_0|$, $x_1 \geq 0$ and $y_1 < \lambda < |z_0|$, then

$$\mathbb{P}_h^{(x_1,y_1)}(\sup_{t \leq \tau_F} Y_t < \lambda) \leq c_2 \exp(-c_3|z_0|/\lambda).$$

Proof. (a) The function $h_1(x, y) = y/(y^2 + (x + a)^2)$ is positive and harmonic in H. It has a pole at $(-a, 0)$ and zero boundary values everywhere else on ∂H. Brownian motion conditioned by h_1 exits H at $(-a, 0)$.

Suppose $0 < |z| \leq a < b$, $z = (x, y) \in H$. Since $\mathbb{P}^z(\mathcal{T}_{\partial B(0,b)} < \mathcal{T}_{\partial H}) \leq c_4 y/b$, by (1.3) and scaling, we obtain

$$\begin{aligned} \mathbb{P}_{h_1}^z(\mathcal{T}_{\partial B(0,b)} < \infty) &= \frac{\mathbb{E}^z[h_1(Z_{\mathcal{T}(\partial B(0,b))}); \mathcal{T}_{\partial B(0,b)} < \infty]}{h_1(z)} \qquad (7.1) \\ &\leq \frac{\Big(\sup_{v \in \partial B(0,b)} h_1(v)\Big)\mathbb{P}^z(\mathcal{T}_{\partial B(0,b)} < \tau_F)}{h_1(z)} \\ &\leq c_5 \frac{(1/b)(y/b)}{y/a^2} = c_6(a/b)^2. \end{aligned}$$

The function $f(z) = z^{\pi/(\pi-\alpha)}$ maps F onto H. We have $f(z_0) = (-|z_0|^{\pi/(\pi-\alpha)}, 0)$ and $|f(x_1, y_1)| < |f(z_0)|$, if $|(x_1, y_1)| < |z_0|$. Let $H_\lambda = \{(x, y) : y \geq \lambda\}$. Note that if $z \in H_\lambda$, then $|f(z)| \geq \lambda^{\pi/(\pi-\alpha)}$. By the conformal invariance of conditioned Brownian motion, we can apply inequality (7.1) with $a = |z_0|^{\pi/(\pi-\alpha)}$ to see that

$$\begin{aligned} \mathbb{P}_h^{(x_1,y_1)}(\sup_{s \leq \tau_F} Y_t > \lambda) &= \mathbb{P}_{h_1}^{f(x_1,y_1)}(\mathcal{T}(f(H_\lambda)) < \infty) \\ &\leq \mathbb{P}_{h_1}^{f(x_1,y_1)}(\mathcal{T}(\partial B(0, \lambda^{\pi/(\pi-\alpha)})) < \infty) \\ &\leq c_6(|z_0|^{\pi/(\pi-\alpha)}/\lambda^{\pi/(\pi-\alpha)})^2. \end{aligned}$$

By the strong Markov property applied at the hitting time of the y-axis, it will suffice to prove (b) for $(x_1, y_1) = (0, y)$ with $y < \lambda < |z_0|$. Suppose $0 < b < a/4$ and let $D = \{(x, y) : 0 < y < b, -a/2 < x < a/2\}$ and $U = \{(x, b) : |x| < a/2\}$. Let u be the positive harmonic function in D that has boundary value 0 on ∂D if $y = 0$ or $y = b$ and boundary value 1 on ∂D if $|x| = a/2$. For z with $|z| < b$,

$$\mathbb{P}_{h_1}^z(\mathcal{T}_U > \mathcal{T}_{\partial D}) = \mathbb{P}_{h_1}^z(|X_{\tau_D}| = a/2) = \mathbb{E}_{h_1}^z u(Z_{\tau_D}).$$

Since h_1 is bounded above on ∂D by $b/(0^2 + (a/2)^2) = 4b/a^2$,

$$\mathbb{E}_{h_1}^z u(Z_{\tau_D}) = \mathbb{E}_{h_1}^z \Big(\frac{u}{h_1}\Big)(Z_{\tau_D}) h_1(Z_{\tau_D}) \leq 4(b/a^2)\mathbb{E}_{h_1}^z \Big(\frac{u}{h_1}\Big)(Z_{\tau_D}).$$

The function u/h_1 is harmonic with respect to the process $(\mathbb{P}^z_{h_1}, Z_{t\wedge\tau_D})$, so the right hand side is equal to $4(b/a^2)(u/h_1)(z)$. It is easy to see that for $|x| < b$, $h_1(x,y)$ is an increasing function of y for $y \le a - b$. Since u is symmetric about the line $y = b/2$, then $(u/h_1)(x,y) \le (u/h_1)(x,y')$, where $y' = y \wedge (b-y)$. Thus without loss of generality let us assume $y \le b/2$. By the boundary Harnack principle in D (Theorem 1.1), there exists c_7 such that

$$\frac{u(x,y)}{h_1(x,y)} \le c_7 \frac{u(x,b/2)}{h_1(x,b/2)}.$$

The constant c_7 can be chosen independently of b by scaling. By (1.2), for $|x| < b$, $u(x,b/2) \le e^{-c_8(a/b)}$, while $h(x,b/2) \ge c_9 b/a^2$. It follows that for $z = (x,y)$ with $|z| < b$,

$$\begin{aligned}\mathbb{P}^z_{h_1}(\mathcal{T}_U > \mathcal{T}_{\partial D}) &= \mathbb{E}^z_{h_1} u(Z_{\tau_D}) \le 4(b/a^2)(\frac{u}{h_1})(z) \\ &\le 4(b/a^2)c_7\frac{u(x,b/2)}{h(x,b/2)} \le 4(b/a^2)c_7\frac{e^{-c_8(a/b)}}{c_9 b/a^2} = c_{10}e^{-c_8(a/b)}.\end{aligned} \tag{7.2}$$

Recall the mapping $f(z) = z^{\pi/(\pi-\alpha)}$. Note that $f(\partial H_\lambda)$ lies below the line $\{(x,y) : y = 2\lambda|z_0|^{\alpha/(\pi-\alpha)}\}$ between the lines $\{(x,y) : |x| = |z_0|\}$. Now let $a = |z_0|^{\pi/(\pi-\alpha)}$ and $b = 2\lambda|z_0|^{\alpha/(\pi-\alpha)}$. Provided $|z_0|/\lambda > 8$, we apply conformal invariance just as in part (a) and use (7.2) to see that

$$\begin{aligned}\mathbb{P}^{(0,y)}_h(\sup_{s\le\tau_F} Y_t < \lambda) &\le \mathbb{P}^z_{h_1}(\mathcal{T}_U > \mathcal{T}(\partial D)) \le c_{10}e^{-c_8(a/b)} \\ &\le c_{10}\exp(-c_8(|z_0|^{\pi/(\pi-\alpha)}/(2\lambda|z_0|^{\alpha/(\pi-\alpha)})) \\ &\le c_{10}\exp(-c_{11}|z_0|/\lambda).\end{aligned} \tag{7.3}$$

Taking c_{10} larger if necessary, we again have (7.3) if $|z_0|/\lambda \le 8$ since probabilities are bounded by 1. □

Proposition 7.3. *There exists c_1 independent of ε such that if $\alpha < \pi/8$,*

$$\mathbb{P}^{(0,\varepsilon)}_{BM}(\sigma(0,\alpha) < \tau_{B(0,2)} < \sigma(0,0)) \le c_1\alpha\varepsilon\log(1/\varepsilon).$$

Proof. Let $\psi(s)$ be the point in the upper half plane on the line $\mathcal{L}(0,\alpha)$ whose distance from $(0,0)$ is s. Using (1.3),

$$\mathbb{P}^{\psi(s)}(\tau_{B(0,2)} < \sigma(0,0)) \le c_2 s\sin\alpha \le c_2\alpha s, \tag{7.4}$$

for $s \le 1$. When $1 \le s < 2$, we have from (1.3) and scaling

$$\mathbb{P}^{\psi(s)}(\tau_{B(0,2)} < \sigma(0,0)) \le \mathbb{P}^{\psi(s)}(\tau_{B(\psi(s),(2-s)/2)} < \sigma(0,0)) \le c_3\alpha/(2-s). \tag{7.5}$$

Using the strong Markov property at $\sigma(0,\alpha)$, conditioning on the value of $|Z_{\sigma(0,\alpha)}| = \psi(s)$, and using Proposition 7.1, we have

$$\begin{aligned}\mathbb{P}^{(0,\varepsilon)}(\sigma(0,\alpha) < \tau_{B(0,2)} < \sigma(0,0)) &\le c_4 \int_0^1 \frac{\varepsilon}{\varepsilon^2+s^2}\alpha s\, ds + c_5 \int_1^2 \varepsilon(2-s)\frac{\alpha}{2-s}\, ds \\ &\le c_6\alpha\varepsilon\log(1/\varepsilon) + c_7\alpha\varepsilon \le c_8\alpha\varepsilon\log(1/\varepsilon). \qquad \square\end{aligned}$$

Remark. We will need the following elementary property of conditioned Brownian motion. Suppose that $D_1 \subseteq D_2 \subseteq D$ are open connected sets with $\partial D_1 \cap \partial D_2 \subseteq \partial D$. Assume that these sets have Lipschitz boundaries and h is a positive harmonic function in D which vanishes on an open superset of $\partial D \cap \partial D_2$. Then, by the boundary Harnack principle, for $z_1, z_2 \in D_1$, $A \subseteq \partial D_2$,

$$\begin{aligned}\mathbb{P}_h^{z_1}(Z(\tau_{D_2}) \in A) &= \frac{1}{h(z_1)}\mathbb{E}^{z_1}(h(Z(\tau_{D_2}); Z(\tau_{D_2}) \in A) \\ &\le c_1 \frac{1}{h(z_2)}\mathbb{E}^{z_2}(h(Z(\tau_{D_2}); Z(\tau_{D_2}) \in A) = c_1\mathbb{P}_h^{z_2}(Z(\tau_{D_2}) \in A).\end{aligned}$$

It follows that the Radon-Nikodym derivative for the hitting distribution of ∂D_2 for the h-process started at z_1 with respect to the one for the h-process started at z_2 is bounded away from 0 and ∞. This and the strong Markov property applied at the stopping time τ_{D_2} imply that for any event B defined in terms of the post-τ_{D_2} process, we have $\mathbb{P}_h^{z_1}(B) \le c_2\mathbb{P}_h^{z_2}(B)$ for $z_1, z_2 \in D_1$.

We next need an estimate on the event given in Proposition 7.3 together with there being no approximate points of increase in the y direction after hitting 0. The estimate will be given in Proposition 7.6 below; it will be preceded by a few preliminary results.

Let

$$\begin{aligned}A &= \{\sigma(0,\alpha) < \tau_{B(0,2)} < \sigma(0,0)\}, \qquad (7.6)\\ C_j &= \{\sigma(j-1,0)\circ\theta_{\sigma(j,0)} < \tau_{B(0,2)}\circ\theta_{\sigma(j,0)}, \sigma(j,0) < \infty\}.\end{aligned}$$

Recall that we are interested only in small angles α. Let F be the wedge in the upper half plane with obtuse angle, bounded by $\mathcal{L}(0,0)$ and $\mathcal{L}(0,\alpha)$. Let $\psi(s)$ be the point on $\mathcal{L}(0,\alpha)$ whose y coordinate is positive and whose distance from the origin is s. We will write κ for $\log(1/\varepsilon)$. Suppose $\alpha \le 1/\kappa$. Let G be the subdomain of F bounded by $\mathcal{L}(0,0)$, $\mathcal{L}(0,\alpha)$, and $\mathcal{L}(\alpha s/\varepsilon\kappa^3, 0)$. Consider Brownian motion in F conditioned to exit F at $\psi(s)$. Next consider this conditioned process further conditioned to exit G through $\mathcal{L}(\alpha s/\varepsilon\kappa^3, 0)$. This final process is an h-path transform of Brownian motion in G, say by the harmonic function h_1.

Proposition 7.4. *There exists c_1 independent of ε such that if $0 \le x_0 \le s/\kappa^6$, $0 < y_0 < 2\alpha s/\kappa^6$, $\varepsilon\kappa^8 > \alpha s$, and $1 > s > \alpha s > \varepsilon\kappa^7$, then*

$$\mathbb{P}_{h_1}^{(x_0,y_0)}(C_1 \cap \cdots \cap C_{\alpha s/\varepsilon\kappa^7}) \le c_1 \frac{1 \vee \log(y_0/\varepsilon)}{\log(\alpha s/\varepsilon\kappa^8)}$$

for ε sufficiently small.

Proof. Let

$$M = \{(x,y) \in G : \text{dist}\,((x,y),\partial F) \geq \varepsilon/2, 0 \leq x \leq s/\kappa^6, 0 < y < 2\alpha s/\kappa^6\}.$$

We will first prove the proposition for $(x_0, y_0) \in M$. Let h_2 be the harmonic function in G that has boundary values 0 on $\mathcal{L}(0,\alpha)$ and $\mathcal{L}(0,0)$ and boundary value 1 on $\mathcal{L}(\alpha s/\varepsilon\kappa^3, 0)$. Let

$$\begin{aligned} D &= \{(x,y) \in G : |x - x_0| \leq \alpha s/\kappa^3, y < \alpha s/\kappa^5\}, \\ S &= \{(x,y) \in \partial D : |x - x_0| = \alpha s/\kappa^3\}, \\ B &= \bigcap_{1 \leq j \leq \alpha s/2\varepsilon\kappa^8} \{\sigma(j-1,0) \circ \theta_{\sigma(j,0)} < \tau_D \circ \theta_{\sigma(j,0)}, \sigma(j,0) < \tau_D\}. \end{aligned}$$

Note that $(x_0, y_0) \in D$. We have

$$\mathbb{P}_{h_1}^{(x_0,y_0)}(C_1 \cap \cdots \cap C_{\alpha s/\varepsilon\kappa^7}) \leq \mathbb{P}_{h_1}^{(x_0,y_0)}(Z_{\tau_D} \in S) + \mathbb{P}_{h_1}^{(x_0,y_0)}(B). \tag{7.7}$$

By the boundary Harnack principle in $((x_0 - 2\alpha s/\kappa^3, x_0 + 2\alpha s/\kappa^3) \times (0, \alpha s/2\kappa^3)) \cap G$, there exists c_2 (not depending on ε) such that

$$\frac{h_1(z)}{h_1(x_0,y_0)} \leq c_2 \frac{h_2(z)}{h_2(x_0,y_0)}, \qquad z \in \overline{D} \cap G.$$

For the first term on the right of (7.7) we write

$$\begin{aligned} \mathbb{P}_{h_1}^{(x_0,y_0)}(Z_{\tau_D} \in S) &= \int_S \frac{h_1(z)}{h_1(x_0,y_0)} \mathbb{P}^{(x_0,y_0)}(Z_{\tau_D} \in dz) \\ &\leq c_2 \int_S \frac{h_2(z)}{h_2(x_0,y_0)} \mathbb{P}^{(x_0,y_0)}(Z_{\tau_D} \in dz) \\ &= c_2 \frac{\mathbb{E}^{(x_0,y_0)}[h_2(Z_{\tau_D}); Z_{\tau_D} \in S]}{h_2(x_0,y_0)}. \end{aligned} \tag{7.8}$$

Since h_2 is bounded by 1, this expression is bounded by

$$\mathbb{P}^{(x_0,y_0)}(Z_{\tau_D} \in S)/h_2(x_0,y_0).$$

We have

$$\mathbb{P}^{(x_0,y_0)}(Z_{\tau_D} \in S) \leq c_3 e^{-c_4\kappa^2} \tag{7.9}$$

by (1.2) and scaling. We will next estimate $h_2(x_0,y_0)$.

Let h_3 be the harmonic function in $F \cap B(0, \alpha s/\kappa^3)$ with boundary values 1 on $F \cap \partial B(0, \alpha s/\kappa^3)$ and 0 on ∂F. Let $K = F \cap \partial B(0, \alpha s/\kappa^3)$ and $v_0 = (0, \alpha s/2\kappa^3)$. Consider a point $(x,y) \in M$ with $x \geq 2y$. The boundary Harnack principle shows that

$$h_2(x,y) \geq c_5 h_3(x,y) \frac{h_2(v_0)}{h_3(v_0)} = c_5 h_3(x,y) \frac{\mathbb{P}^{v_0}(Z_{\tau_G} \in \mathcal{L}(\alpha s/\varepsilon\kappa^3, 0))}{\mathbb{P}^{v_0}(\mathcal{T}_K < \mathcal{T}_{\partial F})}.$$

By the support theorem for Brownian motion and scaling, both probabilities on the right hand side of the last formula are bounded away from 0 and 1 and so $h_2(x,y) \ge c_6 h_3(x,y)$. By Proposition 5.2 and scaling,

$$h_2(x,y) \ge c_6 h_3(x,y) = c_6 \mathbb{P}^{(x,y)}(\mathcal{T}_K < \mathcal{T}_{\partial F}) \ge c_7 x^\alpha y (\kappa^3/\alpha s)^{1+\alpha} \ge c_8 (\varepsilon\kappa^3/\alpha s)^{1+\alpha}.$$

It follows from our assumptions that $\varepsilon\kappa^3/\alpha s \ge \varepsilon\kappa^3 \ge \varepsilon$. Since we have assumed that $\alpha \le 1/\kappa$, we see that $\alpha \log \varepsilon$ is bounded below by -1 and so $\varepsilon^\alpha \ge e^{-1}$. Hence we obtain the following bound,

$$h_2(x,y) \ge c_8(\varepsilon\kappa^3/\alpha s)\cdot(\varepsilon\kappa^3/\alpha s)^\alpha \ge c_8(\varepsilon\kappa^3/\alpha s)\varepsilon^\alpha \ge c_9 \varepsilon\kappa^3/\alpha s.$$

By the Harnack inequality,

$$h_2(x,y) \ge c_{10}\varepsilon\kappa^3/\alpha s, \tag{7.10}$$

for all $(x,y) \in M$, not necessarily just those points satisfying $x \ge 2y$. Recall that $\varepsilon\kappa^3/\alpha s \ge \varepsilon$. This, (7.8), (7.9) and (7.10) show that the first term on the right hand side of (7.7) is bounded by $c_{11}\varepsilon^{-1}e^{-c_4 \log^2(1/\varepsilon)}$. Since $\alpha s/\varepsilon\kappa^6 \le 1/\varepsilon\kappa^6$, we have

$$1/\log(\alpha s/\varepsilon\kappa^6) \ge c_{12}/\log(1/\varepsilon) \ge c_{11}\varepsilon^{-3}e^{-c_4\log^2(1/\varepsilon)}$$

for ε sufficiently small. It follows that the first term on the right hand side of (7.7) is bounded by $1/\log(\alpha s/\varepsilon\kappa^6)$.

We turn to the second term on the right hand side of (7.7). With a similar argument to that used in (7.8),

$$\mathbb{P}_{h_1}^{(x_0,y_0)}(B) \le c_2 \frac{\mathbb{E}^{(x_0,y_0)}[h_2(Z_{\tau_D}); B]}{h_2(x_0,y_0)}. \tag{7.11}$$

By the strong Markov property, this is the same as

$$c_2 \frac{\mathbb{E}^{(x_0,y_0)}[h_2(Z_{\tau_G}); B]}{h_2(x_0,y_0)} = c_2 \frac{\mathbb{P}^{(x_0,y_0)}(Z_{\tau_G} \notin \partial F \cup \mathcal{L}(0,0); B)}{\mathbb{P}^{(x_0,y_0)}(Z_{\tau_G} \notin \partial F \cup \mathcal{L}(0,0))}. \tag{7.12}$$

An easy modification of Proposition 5.11 gives

$$\mathbb{P}_{BMBS}^{(x_0,y_0)}(B) \le c_{13}\frac{1 \vee \log(y_0/\varepsilon)}{\log(\alpha s/\varepsilon\kappa^8)}.$$

Brownian motion conditioned to hit $\mathcal{L}(\alpha s/\varepsilon\kappa^3, 0)$ before hitting $\mathcal{L}(0,0)$ is a BM $\times$ Bes(3) process. Suppose for a moment that $(x,y) \in M$ with $x \ge 2y$. By Proposition 5.3 and scaling, the chance that a BM $\times$ Bes(3) process starting from (x,y) will not hit ∂F before hitting $\mathcal{L}(\alpha s/\varepsilon\kappa^3, 0)$ is bounded below by

$$c_{14}(x\varepsilon\kappa^3/\alpha s)^\alpha \ge c_{15}(\varepsilon^2\kappa^3/\alpha s)^\alpha \ge c_{16}(\varepsilon^2\kappa^3)^{1/\kappa} \ge c_{17}.$$

This bound can be extended to all $(x,y)\in M$ by the Harnack inequality. So

$$\mathbb{P}^{(x_0,y_0)}_{BM}(Z_{\tau_G}\notin\partial F\cup\mathcal{L}(0,0))\geq c_{17}\mathbb{P}^{(x_0,y_0)}_{BM}(\sigma(\alpha s/\varepsilon\kappa^3,0)<\sigma(0,0)).$$

It follows that

$$\begin{aligned}\mathbb{P}^{(x_0,y_0)}_{h_1}(B)&\leq c_2\frac{\mathbb{P}^{(x_0,y_0)}_{BM}(Z_{\tau_G}\notin\partial F\cup\mathcal{L}(0,0);B)}{\mathbb{P}^{(x_0,y_0)}_{BM}(Z_{\tau_G}\notin\partial F\cup\mathcal{L}(0,0))}\\&\leq c_2\frac{\mathbb{P}^{(x_0,y_0)}_{BM}(\sigma(\alpha s/\varepsilon\kappa^3,0)<\sigma(0,0);B)}{c_{17}\mathbb{P}^{(x_0,y_0)}_{BM}(\sigma(\alpha s/\varepsilon\kappa^3,0)<\sigma(0,0))}\\&=c_{18}\mathbb{P}^{(x_0,y_0)}_{BMBS}(B)\\&\leq c_{19}\frac{1\vee\log(y_0/\varepsilon)}{\log(\alpha s/\varepsilon\kappa^8)}.\end{aligned}$$

This provides a sufficiently sharp bound for the second term in (7.7) and so completes the proof in the case $(x_0,y_0)\in M$. The remaining case is when $\operatorname{dist}((x_0,y_0),\partial F)\leq\varepsilon/2$. This follows easily from the proposition applied to $(x_0,y_0)=(0,\varepsilon)$ and the boundary Harnack principle applied as indicated in the remark preceding the proposition. □

Proposition 7.5. *Let F, $\psi(s)$, and α be as in Proposition 7.4. Let h_1 be the harmonic function whose boundary values are 0 everywhere on ∂F except at $\psi(s)$, where it has a pole. There exists c_1 independent of ε such that if $z_0=(x_0,y_0)\in F$, $x_0<2$ and $y_0\geq\alpha s/\kappa^3$, then*

$$\mathbb{P}^{z_0}_{h_1}(\inf_t Y_t<\alpha s/\kappa^7)\leq c_1/\kappa^2$$

for ε sufficiently small.

Proof. First, let $v\in\mathbb{R}$ and consider Brownian motion in the upper half plane H started at $z_0=(x_0,y_0)$ and conditioned by the harmonic function $h_2(x,y)=y/(y^2+(x-v)^2)$. Let $R=|z_0|$ and suppose $4r<R$ and $2R<|v|$. We first show that

$$\mathbb{P}^{z_0}_{h_2}(\mathcal{T}_{B(0,r)}<\infty)\leq c_2\frac{r^2}{R^2}. \tag{7.13}$$

If $z\in\partial B(0,r)\cap H$, the support theorem for Brownian motion and scaling show there exists c_3 not depending on r such that $\mathbb{P}^z_{BM}(|X_{\tau_H}|<2r)\geq c_3$. On the other hand, by the expression for the Poisson kernel in the half plane and the fact that $|z_0|=R>4r$,

$$\mathbb{P}^{z_0}_{BM}(|X_{\tau_H}|\leq 2r)=\frac{1}{\pi}\int_{-2r}^{2r}\frac{y_0}{y_0^2+(x_0-u)^2}\,du\leq\frac{c_4y_0r}{R^2}.$$

By the strong Markov property,

$$\begin{aligned} c_4 y_0 r R^{-2} &\geq \mathbb{P}^{z_0}_{BM}(|X_{\tau_H}| \leq 2r) \qquad (7.14)\\ &\geq \mathbb{P}^{z_0}_{BM}(|X_{\tau_H}| \leq 2r; \mathcal{T}_{B(0,r)} < \tau_H)\\ &= \mathbb{E}^{z_0}_{BM}\big(\mathbb{P}^{Z_{\mathcal{T}(B(0,r))}}_{BM}(|X_{\tau_H}| \leq 2r); \mathcal{T}_{B(0,r)} < \tau_H\big)\\ &\geq c_3 \mathbb{P}^{z_0}_{BM}(\mathcal{T}_{B(0,r)} < \tau_H). \end{aligned}$$

Note h_2 is bounded above by $c_5 r/v^2$ on $\partial B(0,r) \cap H$ and $h_2(z_0) \geq y_0/v^2$. This, (1.1), and (7.14) imply

$$\begin{aligned} \mathbb{P}^{z_0}_{h_2}(\mathcal{T}_{B(0,r)} < \infty) &= \frac{\mathbb{E}^{z_0}_{BM}[h_2(Z_{\mathcal{T}(B(0,r))}); \mathcal{T}_{B(0,r)} < \tau_H]}{h_2(z_0)}\\ &\leq \frac{(c_5 r/v^2)\mathbb{P}^{z_0}_{BM}(\mathcal{T}_{B(0,r)} < \tau_H)}{h_2(z_0)}\\ &\leq \frac{(c_5 r/v^2)(c_3^{-1} c_4 y_0 r/R^2)}{y_0/v^2}\\ &= \frac{c_6 r^2}{R^2}. \end{aligned}$$

This completes the proof of (7.13).

Next let us return to the wedge F and Brownian motion conditioned to go to $\psi(s)$. Recall h_1 is the positive harmonic function in F with pole at $\psi(s)$ and zero boundary values elsewhere on ∂F. The conformal invariance of conditioned Brownian motion under the conformal map $f(z) = z^{\pi/(\pi-\alpha)}$ shows that if $v = -s^{\pi/(\pi-\alpha)}$, $h_2(x,y) = y/(y^2+(x-v)^2)$, $|z| < s/8$, and $\rho < |z|/4$, then

$$\mathbb{P}^z_{h_1}(\mathcal{T}_{B(0,\rho)} < \infty) = \mathbb{P}^{f(z)}_{h_2}(\mathcal{T}_{B(0,\rho^{\pi/(\pi-\alpha)})} < \infty).$$

By (7.13) we have that

$$\mathbb{P}^z_{h_1}(\mathcal{T}_{B(0,\rho)} < \infty) \leq c_2 \frac{\rho^{2\pi/(\pi-\alpha)}}{|z|^{2\pi/(\pi-\alpha)}} \leq c_2 \frac{\rho^2}{|z|^2}. \qquad (7.15)$$

We now estimate $\mathbb{P}^z_{h_1}(\inf_t Y_t < \alpha s/\kappa^7)$. By the strong Markov property at time $\sigma(\alpha s/\varepsilon\kappa^3, 0)$, it suffices to prove the proposition for $z = (x,y)$ with $y = \alpha s/\kappa^3$. If we let $U = (F \cap \{0 < y < \alpha s/\kappa^7\}) - B(0, \alpha s/\kappa^4)$ and $B = \{\mathcal{T}_U < \infty\}$, then by (7.15)

$$\begin{aligned} \mathbb{P}^z_{h_1}(\inf_t Y_t < \alpha s/\kappa^7) &\leq \mathbb{P}^z_{h_1}(\mathcal{T}_{B(0,\alpha s/\kappa^4)} < \infty) \qquad (7.16)\\ &\qquad + \mathbb{P}^z_{h_1}(\inf_t Y_t < \alpha s/\kappa^7, \mathcal{T}_{B(0,\alpha s/\kappa^4)} = \infty)\\ &\leq c_2 \kappa^{-2} + \mathbb{P}^z_{h_1}(B; \mathcal{T}_{B(0,\alpha s/\kappa^4)} = \infty). \end{aligned}$$

Suppose $w = (w_1, w_2)$ with $w_2 = \alpha s/\kappa^5$ and $w \notin B(0, \alpha s/\kappa^4)$. We claim

$$\mathbb{P}^w_{h_1}(B) \leq c_7 \exp(-c_8 \kappa^2) + c_9 \mathbb{P}^w_{BMBS}(B). \qquad (7.17)$$

The proof of (7.17) is very similar to the argument in the proof of Proposition 7.4, so we only give a sketch and refer the reader to Proposition 7.4 for details. Let $h_3(x,y) = y$. We construct a rectangle D centered at w whose bottom side is on the x-axis and whose width is κ^2 times its height. By the boundary Harnack principle, the $\mathbb{P}^w_{h_1}$ probability of exiting D through the left or right sides is comparable to the $\mathbb{P}^w_{h_3}$ probability. Brownian motion conditioned by the harmonic function h_3 is a BM $\times$ Bes(3); using (1.2) yields the first term on the right of (7.17). The other possibility is that Brownian motion conditioned by h_1 exits D through the top and B holds. By the boundary Harnack principle, the $\mathbb{P}^w_{h_1}$ probability of this is bounded by a constant times the $\mathbb{P}^w_{h_3}$ probability. This gives the second term on the right hand side of (7.17).

By (2.1)

$$\mathbb{P}^w_{BMBS}(B) \le \frac{\alpha s/\kappa^7}{\alpha s/\kappa^5} = \kappa^{-2}.$$

So for ε sufficiently small, $\mathbb{P}^w_{h_1}(B) \le c_{10}\kappa^{-2}$. We apply the strong Markov property at $\sigma(\alpha s/\varepsilon\kappa^5, 0)$ and obtain

$$\mathbb{P}^z_{h_1}(B; \mathcal{T}_{B(0,\alpha s/\kappa^4)} = \infty) \le c_{10}\kappa^{-2}.$$

Substituting in (7.16) gives the desired estimate. $\square$

Remark. Consider Brownian motion on $F \cap B(0,2)$ conditioned to exit on $\partial B(0,2)$. Then condition this process again to exit $F \cap B(0,2)$ at a point $z_0 = (x_0, y_0)$ with $y_0 \ge \alpha s/\kappa^3$. Proposition 7.5 applies to this process. This can be seen either by repeating the arguments of the last proof with minor adjustments or by deriving this new result from Proposition 7.7 using the boundary Harnack principle.

Proposition 7.6. *Suppose that $\alpha < 1/\log(1/\varepsilon)$, let A and C_j be defined by (7.6), and let $A_1 = A \cap C_1 \cap \cdots \cap C_{\varepsilon^{-1/16}}$. There exists c_1 independent of ε such that*

$$\mathbb{P}^{(0,\varepsilon)}_{BM}(A_1) \le c_1 \alpha^{15/16} \varepsilon/\kappa^{1/16}.$$

Proof. Recall from Proposition 7.4 the definition of the wedge F bounded by $\mathcal{L}(0,0)$ and $\mathcal{L}(0,\alpha)$. For the event A_1 to happen, A must hold, and so we must have $\tau_F = \sigma(0,\alpha) < \tau_{B(0,2)}$. We break up our desired probability according to various ranges of $|Z_{\sigma(0,\alpha)}|$; we consider them in the order: $(0,\varepsilon]$, $[\varepsilon, \varepsilon\kappa^{100}]$, $[\varepsilon^{1/2}, 2]$, and $[\varepsilon\kappa^{100}, \varepsilon^{1/2}]$.

Case 1: $|Z_{\sigma(0,\alpha)}| \le \varepsilon$.

Recall from (7.4) that Brownian motion starting at $\psi(s)$ exits $B(0,2)$ before hitting $\mathcal{L}(0,0)$ with probability at most $c_2\alpha s$; that is, for $s \le 1$,

$$\mathbb{P}^{\psi(s)}(\tau_{B(0,2)} < \sigma(0,0)) \le c_2\alpha s. \tag{7.18}$$

Using Proposition 7.1(a) and conditioning on the value of $|Z_{\sigma(0,\alpha)}| = s$, we have

$$\begin{aligned}\mathbb{P}^{(0,\varepsilon)}(|Z_{\sigma(0,\alpha)}| \le \varepsilon; A) &\le c_3 \int_0^\varepsilon \frac{\varepsilon}{\varepsilon^2+s^2}\alpha s\,ds \\ &\le c_3\alpha\varepsilon^{-1}\int_0^\varepsilon s\,ds = c_3\alpha\varepsilon/2.\end{aligned} \tag{7.19}$$

Since $\alpha \le 1/\log(1/\varepsilon)$, then $\alpha \le \alpha^{15/16}/\kappa^{1/16}$, and so

$$\alpha\varepsilon \le \alpha^{15/16}\varepsilon/\kappa^{1/16}. \tag{7.20}$$

This and (7.19) give

$$\mathbb{P}^{(0,\varepsilon)}(A_1; |Z_{\sigma(0,\alpha)}| \le \varepsilon) \le \mathbb{P}^{(0,\varepsilon)}(|Z_{\sigma(0,\alpha)}| \le \varepsilon; A) \le c_3\alpha^{15/16}\varepsilon/2\kappa^{1/16}. \tag{7.21}$$

Case 2: $|Z_{\sigma(0,\alpha)}| \in [\varepsilon, \varepsilon\kappa^{100}]$.

Let $V = \sup\{Y_t : t \le \tau_F\}$. We will look at the Brownian motion started at $(0,\varepsilon)$ and conditioned to exit F at $\psi(s)$ for $s \in [\varepsilon, \varepsilon\kappa^{100}]$. By Proposition 7.2 (a),

$$\mathbb{P}^{(0,\varepsilon)}\big(V \ge \varepsilon\kappa^{110} \;\big|\; Z_{\tau_F} = \psi(s)\big) \le c_4\kappa^{-20}. \tag{7.22}$$

Using the strong Markov property at $\sigma(0,\alpha)$, (7.18), and (7.22), we calculate as in (7.19),

$$\begin{aligned}\mathbb{P}^{(0,\varepsilon)}(\varepsilon \le |Z_{\tau_F}| \le \varepsilon\kappa^{100}, V \ge \varepsilon\kappa^{110}; A) & \\ \le c_5 \int_\varepsilon^{\varepsilon\kappa^{100}} \frac{\varepsilon}{\varepsilon^2+s^2}\frac{1}{\kappa^{20}}\alpha s\,ds & \\ \le c_5\alpha\varepsilon\kappa^{-20}\int_\varepsilon^{\varepsilon\kappa^{100}} \frac{ds}{s} & \\ \le c_6\alpha\varepsilon\kappa^{-20}\log\log(1/\varepsilon) \le c_7\alpha\varepsilon. &\end{aligned} \tag{7.23}$$

Next consider the event where $V < \varepsilon\kappa^{110}$, $|Z_{\tau_F}| \in [\varepsilon, \varepsilon\kappa^{100}]$, $|Z_{\sigma(\kappa^{110},0)}| > 1/2$, and A_1 holds. For this to happen, the Brownian motion must hit $\mathcal{L}(\kappa^{110},0)$ after $\sigma(0,\alpha) = \tau_F$ and before $\sigma(0,0)$ and $|Z_{\sigma(\kappa^{110},0)}| > 1/2$. We use the strong Markov property at τ_F. Since $|Z_{\tau_F}| \le \varepsilon\kappa^{100}$, then $Y_{\tau_F} \le \alpha\varepsilon\kappa^{100}$. Starting at Z_{τ_F}, then, the probability that a Brownian motion hits $\mathcal{L}(\kappa^{110},0)$ before returning to the x-axis is bounded by α/κ^{10}. Conditional on hitting $\mathcal{L}(\kappa^{110},0)$ before hitting $\mathcal{L}(0,0)$, Brownian motion is a BM $\times$ Bes(3) up to time $\sigma(\kappa^{110},0)$, and by (1.2) and scaling, the probability that $|Z_{\sigma(\kappa^{110},0)}|$ exceeds 1/2 is bounded by $c_8 \exp(-c_9\kappa^{110})$. So

$$\mathbb{P}^{(0,\varepsilon)}(\varepsilon \le |Z_{\tau_F}| \le \varepsilon\kappa^{100}, V < \varepsilon\kappa^{110}, |Z_{\sigma(\kappa^{110},0)}| > 1/2; A_1) \le c_8\alpha\kappa^{-10}\exp(-c_9\kappa^{110}). \tag{7.24}$$

We now consider the event where $V < \varepsilon\kappa^{110}$, $|Z_{\tau_F}| \in [\varepsilon, \varepsilon\kappa^{100}]$, $|Z_{\sigma(\kappa^{110},0)}| \le 1/2$, and A_1 holds. We will bound the probability of this event by the product of several factors, using the strong Markov property several times. Let us consider the portion of the path after $\sigma(0,\alpha)$. Since the y-coordinate of $\psi(s)$ does not exceed αs, the probability that the process started at $\psi(s)$ reaches $\mathcal{L}(\kappa^{110}, 0)$ before hitting $\mathcal{L}(0,0)$ is at most $\alpha s/\varepsilon\kappa^{110}$. Using the strong Markov property at time $\sigma(\kappa^{110},0)$ and (1.3), there is probability at most $c_{10}\varepsilon\kappa^{110}$ of exiting $B(0,2)$ before hitting $\mathcal{L}(0,0)$. For the portion of the path starting at $Z_{\sigma(\kappa^{110},0)}$, we will apply Proposition 5.10. We may do this since we have that $|Z_{\sigma(\kappa^{110},0)}| \le 1/2$. Thus, given that the process exits $B(0,2)$ before hitting $\mathcal{L}(0,0)$, the probability starting at $Z_{\sigma(\kappa^{110},0)}$ that there are no approximate points of increase, i.e., that $C_{\kappa^{110}} \cap \cdots \cap C_{\varepsilon^{-1/16}}$ holds, is bounded by $c_{11}\log(\kappa^{110})/\log(1/\varepsilon)$. Therefore, by conditioning on the value of $|Z_{\sigma(0,\alpha)}| = s$,

$$\begin{aligned} \mathbb{P}^{(0,\varepsilon)}(|Z_{\tau_F}| \in [\varepsilon, \varepsilon\kappa^{100}], |Z_{\sigma(\kappa^{110},0)}| < 1/2, V < \varepsilon\kappa^{110}; A_1) \\ \le c_{12}\int_\varepsilon^{\varepsilon\kappa^{100}} \frac{\varepsilon}{\varepsilon^2+s^2}\frac{\alpha s}{\varepsilon\kappa^{110}}\varepsilon\kappa^{110}\frac{\log(\kappa^{110})}{\log(1/\varepsilon)}\,ds \\ \le 110c_{12}\alpha\varepsilon\frac{\log\log(1/\varepsilon)}{\log(1/\varepsilon)}\int_\varepsilon^{\varepsilon\kappa^{100}}\frac{ds}{s} \le c_{13}\alpha\varepsilon. \end{aligned} \tag{7.25}$$

Combining this with (7.20), (7.23), and (7.24) yields

$$\begin{aligned} \mathbb{P}^{(0,\varepsilon)}(A_1; |Z_{\tau_F}| \in [\varepsilon, \varepsilon\kappa^{100}]) &\le c_7\alpha\varepsilon + c_8\alpha\kappa^{-10}\exp(-c_9\kappa^{110}) + c_{13}\alpha\varepsilon \\ &\le c_{14}\alpha^{15/16}\varepsilon/\kappa^{1/16}. \end{aligned} \tag{7.26}$$

Case 3: $|Z_{\sigma(0,\alpha)}| \in [\varepsilon^{1/2}, 2]$.

With V defined as above, using Propositions 7.1 and 7.2(b), estimates (7.4), (7.5) and following the argument of (7.19),

$$\begin{aligned} &\mathbb{P}^{(0,\varepsilon)}(\varepsilon^{1/2} \le |Z_{\tau_F}| \le 2, V \le |Z_{\tau_F}|/\kappa^2; A) \\ &\le c_{15}\Big(\int_{\varepsilon^{1/2}}^1 \frac{\varepsilon}{\varepsilon^2+s^2}\alpha s e^{-c_{16}\kappa^2}\,ds + \int_1^2 \varepsilon(2-s)\frac{\alpha}{2-s}e^{-c_{16}\kappa^2}\,ds\Big) \\ &\le c_{17}\alpha\varepsilon\log(1/\varepsilon)e^{-c_{16}\kappa^2} \le c_{18}\alpha\varepsilon. \end{aligned} \tag{7.27}$$

We continue to discuss bounds for the probability of the event A. We use two steps to break the process into three pieces. Let W_1 be the portion of the path up to time $\sigma(0,\alpha)$ and W_2 the portion of the path after $\sigma(0,\alpha)$ up to time $\tau_{B(0,2)}$.

Let $s = |Z_{\sigma(0,\alpha)}|$ and $\psi(s) = Z_{\sigma(0,\alpha)}$. The process W_1 is a Brownian motion started at $(0, \varepsilon)$ and conditioned to exit F at $\psi(s)$ before hitting $\partial B(0, 2)$. The process W_2 is Brownian motion started at $\psi(s)$. We now split W_1 at time $\sigma(\alpha s/\varepsilon\kappa^2, 0)$. We let W_{11} be the portion of the path up to time $\sigma(\alpha s/\varepsilon\kappa^2, 0)$ and W_{12} the portion from $\sigma(\alpha s/\varepsilon\kappa^2, 0)$ to $\sigma(0, \alpha)$. Then W_{11} is a Brownian motion started at $(0, \varepsilon)$ conditioned to hit $\mathcal{L}(\alpha s/\varepsilon\kappa^2, 0)$ before exiting F in $\mathcal{L}(0, \alpha)$ and W_{12} is a Brownian motion conditioned to exit F at $\psi(s)$ before hitting $\partial B(0, 2)$. We thus have three pieces: W_{11}, W_{12}, and W_2. Note that if the event A holds, then W_2 cannot touch $\mathcal{L}(0, 0)$.

By (2.1) and a minor modification of Proposition 5.8, the probability that W_2 gets below $\alpha s/\kappa^7$ is less than c_{19}/κ^7. By Proposition 7.5 and the remark preceding Proposition 7.6, the probability that W_{12} gets below $\alpha s/\kappa^7$ is less than c_{20}/κ^2. At this point we include the event A_1 into our considerations. If neither W_{12} nor W_2 gets below $\alpha s/\kappa^7$ and the process Z_t does not have any approximate points of increase up to level $\alpha s/\kappa^7$, then W_{11} must not have any approximate points of increase up to level $\alpha s/\kappa^7$. By Proposition 7.4, the probability that the first piece W_{11} has no approximate points of increase of size ε up to level $\alpha s/\kappa^7$ is bounded by

$$\frac{c_{21}}{\log(\alpha s/\varepsilon\kappa^8)} \wedge 1 \leq \frac{c_{21}}{\log(\alpha\varepsilon^{1/2}/\varepsilon\kappa^8)} \wedge 1. \tag{7.28}$$

Recall $\alpha\kappa \leq 1$. If $\alpha > \varepsilon^{1/16}$, then $\alpha\varepsilon^{1/2}/\varepsilon\kappa^8 \geq c_{22}\varepsilon^{3/4}$, and

$$\frac{1}{\log(\alpha\varepsilon^{1/2}/\varepsilon\kappa^8)} \leq \frac{c_{23}}{\log(1/\varepsilon)} \leq \frac{c_{23}}{\alpha^{1/16}\kappa^{1/16}\log(1/\varepsilon)}.$$

If $\alpha \leq \varepsilon^{1/16}$, we use the fact that (7.28) is bounded by 1; in this case

$$\alpha^{1/16}\kappa^{1/16} \leq c_{24}\varepsilon^{1/64} \leq c_{25}/\log(1/\varepsilon),$$

and

$$1 \leq \frac{c_{25}}{\alpha^{1/16}\kappa^{1/16}\log(1/\varepsilon)}.$$

So in either case we obtain a contribution from (7.28) of a factor

$$\frac{c_{26}}{\alpha^{1/16}\kappa^{1/16}\log(1/\varepsilon)}.$$

Recall that the contributions from W_{12} and W_2 are c_{19}/κ^7 and c_{20}/κ^2. Since $\alpha < 1/\kappa$, each of these quantities is bounded by $c_{27}/\alpha^{1/16}\kappa^{1/16}\log(1/\varepsilon)$. Therefore,

$$\begin{aligned} &\mathbb{P}^{(0,\varepsilon)}(\varepsilon^{1/2} \leq |Z_{\tau_F}| \leq 2, V > \varepsilon^{1/2}/\kappa^2; A_1) \qquad (7.29)\\ &\leq c_{28}\Big(\int_{\varepsilon^{1/2}}^1 \frac{\varepsilon}{\varepsilon^2+s^2}\alpha s \frac{1}{\alpha^{1/16}\kappa^{1/16}\log(1/\varepsilon)}\,ds \\ &\qquad + \int_1^2 \varepsilon(2-s)\frac{\alpha}{2-s}\frac{1}{\alpha^{1/16}\kappa^{1/16}\log(1/\varepsilon)}\,ds\Big) \\ &\leq c_{28}\Big(\frac{\alpha^{15/16}\varepsilon}{\kappa^{1/16}}\Big)\frac{1}{\log(1/\varepsilon)}\Big(\int_{\varepsilon^{1/2}}^1 \frac{ds}{s} + \int_1^2 1\,ds\Big) \\ &\leq \frac{c_{29}\alpha^{15/16}\varepsilon}{\kappa^{1/16}}. \end{aligned}$$

Combining this with (7.20) and (7.27) gives

$$\mathbb{P}^{(0,\varepsilon)}(A_1; |Z_{\tau_F}| \in [\varepsilon^{1/2}, 2]) \leq c_{30}\alpha^{15/16}\varepsilon/\kappa^{1/16}. \tag{7.30}$$

Case 4: $|Z_{\sigma(0,\alpha)}| \in [\varepsilon\kappa^{100}, \varepsilon^{1/2}]$.

We bound

$$\mathbb{P}^{(0,\varepsilon)}(\varepsilon\kappa^{100} \leq |Z_{\tau_F}| \leq \varepsilon^{1/2}, V < |Z_{\tau_F}|/\kappa^2; A) \tag{7.31}$$

by $c_{31}\alpha\varepsilon$ as in (7.27). We bound

$$\mathbb{P}^{(0,\varepsilon)}(\varepsilon\kappa^{100} \leq |Z_{\tau_F}| \leq \varepsilon^{1/2}, V > |Z_{\tau_F}|\kappa^5; A) \tag{7.32}$$

by $c_{32}\alpha\varepsilon$ as in (7.23). We must thus consider

$$\mathbb{P}^{(0,\varepsilon)}(\varepsilon\kappa^{100} \leq |Z_{\tau_F}| \leq \varepsilon^{1/2}, |Z_{\tau_F}|/\kappa^2 \leq V \leq |Z_{\tau_F}|\kappa^5; A_1). \tag{7.33}$$

We condition on $|Z_{\tau_F}| = s$. If the event in (7.33) holds then we cannot have any approximate points of increase up to level $\alpha s/\kappa^7$, and we obtain an estimate similar to (7.28), namely, $c_{33}/\log(\alpha s/\varepsilon\kappa^8) \wedge 1$. If $\alpha > \kappa^{-20}$, then $\alpha s/\varepsilon\kappa^8 \geq s/\varepsilon\kappa^{28}$, so

$$\frac{1}{\log(\alpha s/\varepsilon\kappa^8)} \leq \frac{\alpha^{-1/16}\kappa^{-1/16}}{\log(s/\varepsilon\kappa^{28})}.$$

If $\alpha \leq \kappa^{-20}$, then

$$\alpha^{1/16}\kappa^{1/16} \leq \kappa^{-1} = \frac{1}{\log(1/\varepsilon)} \leq \frac{c_{34}}{\log(\varepsilon^{1/2}/\varepsilon\kappa^{28})} \leq \frac{c_{35}}{\log(s/\varepsilon\kappa^{28})},$$

and

$$1 \leq \frac{c_{35}\alpha^{-1/16}\kappa^{-1/16}}{\log(s/\varepsilon\kappa^{28})}.$$

In either case,

$$\frac{c_{33}}{\log(\alpha s/\varepsilon\kappa^8)} \wedge 1 \leq \frac{c_{36}\alpha^{-1/16}\kappa^{-1/16}}{\log(s/\varepsilon\kappa^{28})}. \tag{7.34}$$

If the event in (7.33) holds, we cannot have any approximate points of increase from the level $\mathcal{L}(s\kappa^5/\varepsilon, 0)$ up to level $\mathcal{L}(1/\varepsilon^{1/16}, 0)$. By the same argument as in (7.24) and (7.25), the probability of this is bounded by $c_{37}\log(s\kappa^5/\varepsilon)/\log(1/\varepsilon)$. We therefore bound (7.33) by

$$c_{38}\int_{\varepsilon\kappa^{100}}^{\varepsilon^{1/2}} \frac{\varepsilon}{\varepsilon^2+s^2}\alpha s \frac{\alpha^{-1/16}\kappa^{-1/16}}{\log(s/\varepsilon\kappa^{28})}\frac{\log(s\kappa^5/\varepsilon)}{\log(1/\varepsilon)}\,ds. \tag{7.35}$$

For s in the range $[\varepsilon\kappa^{100}, \varepsilon^{1/2}]$,

$$\log(s\kappa^5/\varepsilon) = \log(s/\varepsilon) + 5\log\kappa \leq 2\log(s/\varepsilon)$$

and

$$\log(s/\varepsilon\kappa^{28}) = \log(s/\varepsilon) - 28\log\kappa \geq (72/100)\log(s/\varepsilon).$$

The ratio of these two terms is bounded by 3, and thus (7.35) is less than

$$\frac{3c_{38}}{\log(1/\varepsilon)}\alpha^{15/16}\kappa^{-1/16}\varepsilon\int_{\varepsilon\kappa^{100}}^{\varepsilon^{1/2}}\frac{ds}{s} \leq c_{39}\alpha^{15/16}\varepsilon/\kappa^{1/16}.$$

This, (7.20), (7.31), (7.32), and (7.33) give

$$\mathbb{P}^{(0,\varepsilon)}(A_1; |Z_{\tau_F}| \in [\varepsilon\kappa^{100}, \varepsilon^{1/2}]) \leq c_{40}\alpha^{15/16}\varepsilon/\kappa^{1/16}.$$

The proposition follows from this, (7.21), (7.26), and (7.30). □

We now turn to examining the paths that return to $\mathcal{L}(0,\alpha)$ and $\mathcal{L}(0,-\alpha)$ and then exit $B(0,2)$ before hitting $\mathcal{L}(0,0)$.

Proposition 7.7. *There exists c_1 independent of ε such that if $\alpha < 1/\log(1/\varepsilon)$, then*

$$\mathbb{P}_{BM}^{(0,\varepsilon)}(\sigma(0,-\alpha) \vee \sigma(0,\alpha) < \tau_{B(0,2)} < \sigma(0,0)) \leq c_1\alpha^2\varepsilon\log^2(1/\varepsilon).$$

Proof. For the event to occur, the Brownian motion must hit both $\mathcal{L}(0,\alpha)$ and $\mathcal{L}(0,-\alpha)$ before exiting $B(0,2)$. By symmetry we may suppose that $\mathcal{L}(0,\alpha)$ is hit first and we thus need a bound on

$$\mathbb{P}^{(0,\varepsilon)}(\sigma(0,\alpha) < \sigma(0,-\alpha) < \tau_{B(0,2)} < \sigma(0,0)).$$

Let $\psi_\alpha(s)$ be the point on $\mathcal{L}(0,\alpha)$ that has positive y-coordinate and is a distance s from 0, and let $\psi_{-\alpha}(t)$ be the point on $\mathcal{L}(0,-\alpha)$ with positive y-coordinate that is a distance t from 0.

To hit $\mathcal{L}(0,\alpha)$ before $\mathcal{L}(0,-\alpha)$ or $\mathcal{L}(0,0)$, Z_t must exit the obtuse wedge in the upper half plane formed by $\mathcal{L}(0,0)$ and $\mathcal{L}(0,\alpha)$ at some point $\psi_\alpha(s)$, and we have by Proposition 7.1

$$\mathbb{P}^{(0,\varepsilon)}(|Z_{\sigma(0,\alpha)}| \in ds) \leq c_2\frac{\varepsilon}{\varepsilon^2+s^2}\,ds. \tag{7.36}$$

Starting at $\psi_\alpha(s) \in \mathcal{L}(0,\alpha)$, the process Z_t must then exit the obtuse wedge in the upper half plane formed by $\mathcal{L}(0,-\alpha)$ and $\mathcal{L}(0,0)$ at some point $\psi_{-\alpha}(t)$. Since $\psi_\alpha(s) = (-s\cos\alpha, s\sin\alpha)$, by Proposition 7.1 again we have

$$\begin{aligned}\mathbb{P}^{\psi_\alpha(s)}(|Z_{\sigma(-\alpha,0)}| \in dt) &\leq c_3\frac{s\sin\alpha}{(s\sin\alpha)^2+(t+s\cos\alpha)^2}\,dt \\ &\leq c_3\frac{\alpha s}{(t+s\cos\alpha)^2}\,dt \\ &\leq 8c_3\frac{\alpha s}{(s+t)^2}\,dt.\end{aligned} \tag{7.37}$$

By Proposition 7.1(b), for $1 \le t < 2$,

$$\mathbb{P}^{\psi_\alpha(s)}(|Z_{\sigma(-\alpha,0)}| \in dt, \sigma(-\alpha,0) < \tau_{B(0,2)}) \le c_4(2-t)s\sin\alpha \le c_4(2-t)s\alpha. \quad (7.38)$$

Recall the following estimates from (7.4)-(7.5): for $t \le 1$,

$$\mathbb{P}^{\psi_{-\alpha}(t)}(\tau_{B(0,2)} < \sigma(0,0)) \le c_5\alpha t; \quad (7.39)$$

when $1 \le t \le 2$,

$$\mathbb{P}^{\psi_{-\alpha}(t)}(\tau_{B(0,2)} < \sigma(0,0)) \le c_6\alpha/(2-t). \quad (7.40)$$

Using the strong Markov property first at $\sigma(0,\alpha)$ and then at $\sigma(0,-\alpha)$ and conditioning on $|Z_{\sigma(0,\alpha)}| = s$ and $|Z_{\sigma(0,-\alpha)}| = t$, we obtain from (7.37)-(7.40),

$$\begin{aligned}&\mathbb{P}^{(0,\varepsilon)}(\sigma(0,\alpha) < \sigma(0,-\alpha) < \tau_{B(0,2)} < \sigma(0,0)) \quad (7.41)\\ &\qquad \le c_7 \int_0^2 \frac{\varepsilon}{\varepsilon^2+s^2}\Big(\int_0^1 \frac{\alpha s}{(s+t)^2}\alpha t\,dt + \int_1^2 (2-t)s\alpha\frac{\alpha}{2-t}dt\Big)ds,\end{aligned}$$

and it remains to estimate the integrals.

Observe that for $s \in (0,2]$,

$$\int_0^1 \frac{t}{(s+t)^2}\,dt = \log((1+s)/s) - 1/(1+s) \le c_8(1+|\log s|).$$

We can thus bound the first double integral in (7.41) by

$$\begin{aligned}c_8\alpha^2\varepsilon\int_0^2 \frac{s}{\varepsilon^2+s^2}\Big[\int_0^1\frac{t}{(s+t)^2}\,dt\Big]\,ds &\le c_9\alpha^2\varepsilon\int_0^2\frac{s(1+|\log s|)}{\varepsilon^2+s^2}\,ds \quad (7.42)\\ &\le c_{10}\alpha^2\varepsilon\Big[\int_0^\varepsilon\frac{s|\log s|}{\varepsilon^2}\,ds + \int_\varepsilon^1\frac{|\log s|}{s}\,ds + \int_0^2\frac{s}{\varepsilon^2+s^2}\,ds\Big]\\ &\le c_{11}\alpha^2\varepsilon\log^2(1/\varepsilon).\end{aligned}$$

We have

$$\int_1^2 (2-t)s\alpha\frac{\alpha}{2-t}dt = s\alpha^2,$$

and so

$$\begin{aligned}c_7\int_0^2\frac{\varepsilon}{\varepsilon^2+s^2}\Big[\int_1^2(2-t)s\alpha\frac{\alpha}{2-t}dt\Big]\,ds &\le c_{12}\alpha^2\varepsilon\int_0^2\frac{s}{\varepsilon^2+s^2}\,ds \quad (7.43)\\ &\le c_{13}\alpha^2\varepsilon\log(1/\varepsilon).\end{aligned}$$

It follows from (7.41)-(7.43) that

$$\mathbb{P}^{(0,\varepsilon)}(\sigma(0,\alpha) < \sigma(0,-\alpha) < \tau_{B(0,2)} < \sigma(0,0)) \le c_{14}\alpha^2\varepsilon\log^2(1/\varepsilon).$$

The proof is complete. □

Corollary 7.8. *There exists c_1 such that if $\alpha \le 1/\log(1/\varepsilon)$, then*

$$\mathbb{P}^{(0,0)}_{BMBS}(\sigma(1,\alpha) < \tau_{B(0,2)}, \sigma(0,\alpha)\circ\theta_{\sigma(1,\alpha)} < \tau_{B(0,2)}\circ\theta_{\sigma(1,\alpha)}) \le c_1\alpha\log(1/\varepsilon), \quad (7.44)$$

and

$$\mathbb{P}^{(0,0)}_{BMBS}(\sigma(1,\alpha) < \tau_{B(0,2)}, \sigma(0,\alpha)\circ\theta_{\sigma(1,\alpha)} < \tau_{B(0,2)}\circ\theta_{\sigma(1,\alpha)}, \quad (7.45)$$
$$\sigma(1,-\alpha) < \tau_{B(0,2)}, \sigma(0,-\alpha)\circ\theta_{\sigma(1,-\alpha)} < \tau_{B(0,2)}\circ\theta_{\sigma(1,-\alpha)}) \le c_1\alpha^2\log^2(1/\varepsilon).$$

Proof. Let $D = \{(x,y) : |x| < 3\varepsilon/4, 0 < y < 3\varepsilon/4\}$ and let A denote the event in (7.44). Note that A is defined in terms of the post-τ_D process. Recall that BM $\times$ Bes(3) is the same as 2-dimensional Brownian motion conditioned by the function $h(x,y) = y$. Hence, by the remark following the proof of Proposition 7.3, we have

$$\mathbb{P}^{(0,0)}_{BMBS}(A) \le c_2\mathbb{P}^{(0,\varepsilon/2)}_{BMBS}(A). \quad (7.46)$$

Note that

$$\mathbb{P}^{(0,\varepsilon/2)}_{BMBS}(A) \le \mathbb{P}^{(0,\varepsilon/2)}_{BMBS}(\sigma(0,\alpha) < \tau_{B(0,2)}). \quad (7.47)$$

Let h_1 be the harmonic function in the upper half of $B(0,3)$ which is equal to 0 on the x-axis and 1 elsewhere on the boundary. An argument analogous to Proposition 5.8 shows that

$$\begin{aligned}\mathbb{P}^{(0,\varepsilon/2)}_{BMBS}(\sigma(0,\alpha) < \tau_{B(0,2)}) &= \mathbb{P}^{(0,\varepsilon/2)}_{h}(\sigma(0,\alpha) < \tau_{B(0,2)}) \\ &\le c_3\mathbb{P}^{(0,\varepsilon/2)}_{h_1}(\sigma(0,\alpha) < \tau_{B(0,2)}).\end{aligned} \quad (7.48)$$

We have

$$\begin{aligned}\mathbb{P}^{(0,\varepsilon/2)}_{h_1}(\sigma(0,\alpha) < \tau_{B(0,2)}) &= \frac{\mathbb{P}^{(0,\varepsilon/2)}_{BM}(\sigma(0,\alpha) < \tau_{B(0,2)}, \tau_{B(0,3)} < \sigma(0,0))}{\mathbb{P}^{(0,\varepsilon/2)}_{BM}(\tau_{B(0,3)} < \sigma(0,0))} \\ &\le \frac{\mathbb{P}^{(0,\varepsilon/2)}_{BM}(\sigma(0,\alpha) < \tau_{B(0,3)} < \sigma(0,0))}{\mathbb{P}^{(0,\varepsilon/2)}_{BM}(\tau_{B(0,3)} < \sigma(0,0))}.\end{aligned}$$

The numerator on the right hand side is bounded by $c_4\alpha\varepsilon\log(1/\varepsilon)$, by Proposition 7.3, scaling, and the Harnack principle. The denominator is greater than $c_5\varepsilon$. It follows that

$$\mathbb{P}^{(0,\varepsilon/2)}_{h_1}(\sigma(0,\alpha) < \tau_{B(0,2)}) \le c_6\alpha\log(1/\varepsilon).$$

This and (7.46)-(7.48) show that

$$\mathbb{P}^{(0,0)}_{BMBS}(A) \le c_7\alpha\log(1/\varepsilon).$$

The proof of (7.45) is analogous except that it uses Proposition 7.7 instead of Proposition 7.3. □

Lemma 7.9. *For* $\varepsilon < 1/2$,

(*a*) $$\int_0^\varepsilon \int_0^2 \frac{1}{(\varepsilon^2+s^2)} \frac{st}{(s+t)^2}\, ds\, dt \le 3/2.$$

(*b*) $$\int_\varepsilon^2 \int_0^\varepsilon \frac{1}{(\varepsilon^2+s^2)} \frac{st}{(s+t)^2}\, ds\, dt \le \log(1/\varepsilon).$$

(*c*) $$\int_\varepsilon^2 \int_t^2 \frac{1}{(\varepsilon^2+s^2)} \frac{st}{(s+t)^2}\, ds\, dt \le 2\log(1/\varepsilon).$$

Proof. We bound the expression in (a) by

$$\int_0^\varepsilon \int_0^\varepsilon \frac{1}{\varepsilon^2} \frac{st}{(s+t)^2}\, ds\, dt + \int_0^\varepsilon \int_\varepsilon^2 \frac{1}{s^2} \frac{st}{(s+t)^2}\, ds\, dt.$$

The first integral is bounded by

$$\varepsilon^{-1} \int_0^\varepsilon \int_0^\varepsilon \frac{t}{(s+t)^2}\, ds\, dt = \varepsilon^{-1} \int_0^\varepsilon \Big(\frac{1}{t} - \frac{1}{t+\varepsilon}\Big) t\, dt \le \varepsilon^{-1} \int_0^\varepsilon 1\, dt \le 1,$$

while the second is bounded by

$$\int_0^\varepsilon \int_\varepsilon^2 \frac{1}{s^2} \frac{s\varepsilon}{s^2}\, ds\, dt \le \frac{1}{2\varepsilon} \int_0^\varepsilon dt \le 1/2.$$

Turning to (b), the integral does not exceed

$$\int_\varepsilon^2 \int_0^\varepsilon \frac{1}{\varepsilon^2} \frac{st}{t^2}\, ds\, dt \le \log(1/\varepsilon),$$

for $\varepsilon < 1/2$. Finally, considering (c), the integral is less than or equal to

$$\int_\varepsilon^2 \int_t^2 \frac{1}{s^2} \frac{st}{s^2}\, ds\, dt \le \int_\varepsilon^2 \frac{t}{t^2}\, dt \le 2\log(1/\varepsilon).$$

□

Let

$$F = \{\sigma(0,-\alpha) \vee \sigma(0,\alpha) < \tau_{B(0,2)} < \sigma(0,0)\},$$

let C_j be as in (7.6), and

$$F_1 = F \cap C_1 \cap \cdots \cap C_{\varepsilon^{-1/16}}.$$

The most complicated estimate we have in this section is the following.

Proposition 7.10. *There exists c_1 independent of ε such that if $\alpha < 1/\log(1/\varepsilon)$, then*

$$\mathbb{P}^{(0,\varepsilon)}(F_1) \le c_1 \alpha^{15/8} \varepsilon \log(1/\varepsilon)/\kappa^{1/8}.$$

Proof. By symmetry it will suffice to consider only the case when $\sigma(0,\alpha) < \sigma(0,-\alpha)$. As in Proposition 7.7 we condition on $Z_{\sigma(0,\alpha)} = \psi_\alpha(s)$ and $Z_{\sigma(0,-\alpha)} = \psi_{-\alpha}(t)$.

We will show that it suffices to suppose $t \ge \varepsilon$, $s \ge \varepsilon$ and $t \ge s$ because the probability of $\{|Z_{\sigma(0,-\alpha)}| < \varepsilon\} \cup \{|Z_{\sigma(0,\alpha)}| < \varepsilon\} \cup \{|Z_{\sigma(0,-\alpha)}| < |Z_{\sigma(0,\alpha)}|\}$ is small. The same argument which gives (7.41) shows that

$$\begin{aligned} &\mathbb{P}^{(0,\varepsilon)}(\sigma(0,\alpha) < \sigma(0,-\alpha) < \tau_{B(0,2)} < \sigma(0,0), |Z_{\sigma(0,-\alpha)}| < \varepsilon) \\ &\quad \le c_2 \int_0^2 \frac{\varepsilon}{\varepsilon^2 + s^2} \int_0^\varepsilon \frac{\alpha s}{(s+t)^2} \alpha t \, dt \, ds. \end{aligned}$$

This is bounded by $(3/2)c_2 \varepsilon \alpha^2 \le c_3 \alpha^{15/8} \varepsilon \log(1/\varepsilon)/\kappa^{1/8}$ by Lemma 7.9 (a). Similarly, we have

$$\begin{aligned} &\mathbb{P}^{(0,\varepsilon)}(\sigma(0,\alpha) < \sigma(0,-\alpha) < \tau_{B(0,2)} < \sigma(0,0), |Z_{\sigma(0,-\alpha)}| \ge \varepsilon, |Z_{\sigma(0,\alpha)}| < \varepsilon) \\ &\quad \le c_2 \int_0^\varepsilon \frac{\varepsilon}{\varepsilon^2 + s^2} \int_\varepsilon^1 \frac{\alpha s}{(s+t)^2} \alpha t \, dt \, ds + c_2 \int_0^\varepsilon \frac{\varepsilon}{\varepsilon^2 + s^2} \int_1^2 (2-t) s \alpha \frac{\alpha}{2-t} dt \, ds. \end{aligned}$$

The first integral is bounded by $c_2 \log(1/\varepsilon) \varepsilon \alpha^2$ by Lemma 7.9 (b), and the second one is bounded by

$$c_2 \alpha^2 \varepsilon \int_0^\varepsilon \frac{s}{\varepsilon^2 + s^2} \, ds \le c_2 \alpha^2 \varepsilon / 2. \tag{7.49}$$

Hence, the sum of the two terms is bounded by $c_4 \alpha^{15/8} \varepsilon \log(1/\varepsilon)/\kappa^{1/8}$. Finally,

$$\begin{aligned} &\mathbb{P}^{(0,\varepsilon)}(\sigma(0,\alpha) < \sigma(0,-\alpha) < \tau_{B(0,2)} < \sigma(0,0), |Z_{\sigma(0,-\alpha)}| \ge \varepsilon, |Z_{\sigma(0,\alpha)}| > |Z_{\sigma(0,-\alpha)}|) \\ &\quad \le c_2 \int_\varepsilon^2 \int_t^2 \frac{\varepsilon}{\varepsilon^2 + s^2} \frac{\alpha s}{(s+t)^2} \alpha t \, ds \, dt + c_2 \int_0^2 \frac{\varepsilon}{\varepsilon^2 + s^2} \int_1^2 (2-t) s \alpha \frac{\alpha}{2-t} dt \, ds. \end{aligned}$$

Now we use Lemma 7.9(c) and (7.49) to obtain $c_5 \alpha^2 \varepsilon \log(1/\varepsilon)$ as the upper bound for the sum of these terms. The bound is less than $c_6 \alpha^{15/8} \varepsilon \log(1/\varepsilon)/\kappa^{1/8}$. This completes our estimates. From now on we will assume that $t = |Z_{\sigma(0,-\alpha)}| \ge \varepsilon$, $s = |Z_{\sigma(0,\alpha)}| \ge \varepsilon$ and $t = |Z_{\sigma(0,-\alpha)}| \ge s = |Z_{\sigma(0,\alpha)}|$.

The rest of the proof follows along the same lines as the proof of Proposition 7.6 except that we have a larger number of cases to cover. There are 6 cases:

(i) $\varepsilon \le |Z_{\sigma(0,\alpha)}| \le |Z_{\sigma(0,-\alpha)}| \le \varepsilon \kappa^{100}$,
(ii) $\varepsilon \le |Z_{\sigma(0,\alpha)}| \le \varepsilon \kappa^{100} \le |Z_{\sigma(0,-\alpha)}| \le \varepsilon^{1/2}$,
(iii) $\varepsilon \le |Z_{\sigma(0,\alpha)}| \le \varepsilon \kappa^{100}, \varepsilon^{1/2} \le |Z_{\sigma(0,-\alpha)}| \le 2$,
(iv) $\varepsilon \kappa^{100} \le |Z_{\sigma(0,\alpha)}| \le |Z_{\sigma(0,-\alpha)}| \le \varepsilon^{1/2}$,
(v) $\varepsilon \kappa^{100} \le |Z_{\sigma(0,\alpha)}| \le \varepsilon^{1/2} \le |Z_{\sigma(0,-\alpha)}| \le 2$,
(vi) $\varepsilon^{1/2} \le |Z_{\sigma(0,\alpha)}| \le |Z_{\sigma(0,-\alpha)}| \le 2$.

We will discuss only case (iv), the most complicated one. The other cases can be dealt with in a way similar to (iv), using arguments and estimates which appeared in the proofs of Propositions 7.6 and 7.8.

We proceed to discuss the case $\varepsilon\kappa^{100} \le |Z_{\sigma(0,\alpha)}| \le |Z_{\sigma(0,-\alpha)}| \le \varepsilon^{1/2}$. Let

$$V_s = \sup_{r \le \sigma(0,\alpha)} Y_r, \qquad V_t = \sup_{r \le \sigma(0,-\alpha)} Y_r.$$

We use Propositions 7.1 and 7.2 (a)-(b) in a manner similar to that in proofs of Propositions 7.6 and 7.7 to see that,

$$\begin{aligned}
&\mathbb{P}^{(0,\varepsilon)}\Big(F; \varepsilon\kappa^{100} \le |Z_{\sigma(0,\alpha)}| \le |Z_{\sigma(0,-\alpha)}| \le \varepsilon^{1/2}, \sigma(0,\alpha) < \sigma(0,-\alpha),\\
&\qquad \{|Z_{\sigma(0,\alpha)}|/\kappa^2 \ge V_{Z(\sigma(0,\alpha))} \text{ or } V_{Z(\sigma(0,\alpha))} \ge |Z_{\sigma(0,\alpha)}|\kappa^5\},\\
&\qquad \{|Z_{\sigma(0,-\alpha)}|/\kappa^2 \ge V_{Z(\sigma(0,-\alpha))} \text{ or } V_{Z(\sigma(0,-\alpha))} \ge |Z_{\sigma(0,-\alpha)}|\kappa^5)\}\Big)\\
&\quad \le c_7 \int_{\varepsilon\kappa^{100}}^{\varepsilon^{1/2}} \frac{\varepsilon}{\varepsilon^2 + s^2}(c_8\kappa^{-10} + c_9 e^{-c_{10}\kappa^2})\\
&\qquad \times \int_s^{\varepsilon^{1/2}} \frac{\alpha s}{(s+t)^2}\alpha t(c_8\kappa^{-10} + c_9 e^{-c_{10}\kappa^2})\, dt\, ds\\
&\quad \le c_{11}\varepsilon\alpha^2\kappa^{-20} \int_{\varepsilon\kappa^{100}}^{\varepsilon^{1/2}} \frac{s}{\varepsilon^2+s^2} \int_s^{\varepsilon^{1/2}} \frac{t}{(s+t)^2}\, dt\, ds\\
&\quad \le c_{11}\varepsilon\alpha^2\kappa^{-20} \int_{\varepsilon\kappa^{100}}^{\varepsilon^{1/2}} \frac{s}{\varepsilon^2+s^2} ds \int_s^{\varepsilon^{1/2}} \frac{1}{t}\, dt\\
&\quad \le c_{11}\varepsilon\alpha^2\kappa^{-20}\log(1/\varepsilon)\log(1/\varepsilon) \le c_{12}\alpha^{15/8}\varepsilon\log(\varepsilon)/\kappa^{1/8}.
\end{aligned}$$

We see that we need an estimate for $\mathbb{P}^{(0,\varepsilon)}(F_2)$, where

$$\begin{aligned}
F_2 = \{&F \cap C_1 \cap \cdots \cap C_{\varepsilon^{-1/16}}, \varepsilon\kappa^{100} \le |Z_{\sigma(0,\alpha)}| \le |Z_{\sigma(0,-\alpha)}| \le \varepsilon^{1/2}, \sigma(0,\alpha) < \sigma(0,-\alpha),\\
&|Z_{\sigma(0,\alpha)}|/\kappa^2 \le V_{Z(\sigma(0,\alpha))} \le |Z_{\sigma(0,\alpha)}|\kappa^5, |Z_{\sigma(0,-\alpha)}|/\kappa^2 \le V_{Z(\sigma(0,-\alpha))} \le |Z_{\sigma(0,-\alpha)}|\kappa^5\}.
\end{aligned}$$

If we omit for a moment the term $C_1 \cap \cdots \cap C_{\varepsilon^{-1/16}}$ and condition on $Z_{\sigma(0,\alpha)} = \psi_\alpha(s)$ and $Z_{\sigma(0,-\alpha)} = \psi_{-\alpha}(t)$, we obtain the following upper bound analogous to the first double integral in (7.41):

$$c_{13}\alpha^2\varepsilon \int_{\varepsilon\kappa^{100}}^{\varepsilon^{1/2}} \int_{\varepsilon\kappa^{100}}^{t} \frac{1}{s^2}\frac{s}{(s+t)^2} t\, ds\, dt. \tag{7.50}$$

We now see how (7.50) is modified when we reintroduce $C_1 \cap \cdots \cap C_{\varepsilon^{-1/16}}$.

Suppose first that $t > \kappa^{40}s$. As in Case 4 of Proposition 7.6 (see especially (7.34)), we obtain a factor $c_{14}\alpha^{-1/16}\kappa^{-1/16}/\log(s/\varepsilon\kappa^{28})$ because there can be no approximate points of increase at levels between 1 and $\alpha s/\kappa^7$. Next we need a factor corresponding to there being no approximate points of increase at levels between $\alpha s\kappa^5$ and $\alpha t/\kappa^2$. An estimate analogous to Proposition 7.4 gives an upper bound of

$1 \wedge (c_{15} \log(\alpha s\kappa^5/\varepsilon)/\log(\alpha t/\kappa^3\varepsilon))$. If $\kappa \geq \alpha > \kappa^{-20}$ then $\log(\alpha s\kappa^5/\varepsilon) \leq \log(s\kappa^6/\varepsilon)$ and

$$c_{15}\frac{\log(\alpha s\kappa^5/\varepsilon)}{\log(\alpha t/\kappa^3\varepsilon)} \leq c_{15}\alpha^{-1/16}\kappa^{-1/16}\frac{\log(s\kappa^6/\varepsilon)}{\log(t/\kappa^{23}\varepsilon)}.$$

In the case $\alpha \leq \kappa^{-20}$ we use an estimate from the proof of (7.34) to see that

$$c_{16}\alpha^{-1/16}\kappa^{-1/16}\frac{\log(s\kappa^5/\varepsilon)}{\log(t/\kappa^{23}\varepsilon)} \geq c_{16}\alpha^{-1/16}\kappa^{-1/16}\frac{1}{\log(t/\kappa^{23}\varepsilon)} \geq 1.$$

In either case,

$$1 \wedge c_{15}\frac{\log(\alpha s\kappa^5/\varepsilon)}{\log(\alpha t/\kappa^3\varepsilon)} \leq c_{17}\alpha^{-1/16}\kappa^{-1/16}\frac{\log(s\kappa^6/\varepsilon)}{\log(t/\kappa^{23}\varepsilon)}.$$

Because there can be no approximate points of increase at levels between $\alpha t\kappa^5$ and $\varepsilon^{1/4}$ we get in a similar way a factor $c_{18}\log(t\kappa^6/\varepsilon)/\log(\varepsilon^{1/4}/\varepsilon)$. In order to obtain an upper bound for $\mathbb{P}^{(0,\varepsilon)}(F_2 \cap \{|Z_{\sigma(0,-\alpha)}| > \kappa^{40}|Z_{\sigma(0,\alpha)}|\})$, we use the integral (7.50) except that we replace the integrand $t/(s(s+t)^2)$ by

$$c_{14}c_{17}c_{18}\alpha^{-1/8}\kappa^{-1/8}\frac{t}{s(s+t)^2}\frac{1}{\log(s/\varepsilon\kappa^{28})}\frac{\log(s\kappa^6/\varepsilon)}{\log(t/\varepsilon\kappa^{23})}\frac{\log(t\kappa^6/\varepsilon)}{\log(\varepsilon^{-3/4})}. \tag{7.51}$$

Since $s/\varepsilon \geq \kappa^{100}$,

$$\frac{\log(s\kappa^6/\varepsilon)}{\log(s/\varepsilon\kappa^{28})} = \frac{\log(s/\varepsilon)+\log\kappa^6}{\log(s/\varepsilon)-\log\kappa^{28}} \leq \frac{2\log(s/\varepsilon)}{(72/100)\log(s/\varepsilon)} \leq 4,$$

and similarly,

$$\frac{\log(t\kappa^6/\varepsilon)}{\log(t/\varepsilon\kappa^{23})} = \frac{\log(t/\varepsilon)+\log\kappa^6}{\log(t/\varepsilon)-\log\kappa^{23}} \leq \frac{2\log(t/\varepsilon)}{(77/100)\log(t/\varepsilon)} \leq 4.$$

Hence the expression in (7.51) is bounded by

$$c_{19}\alpha^{-1/8}\kappa^{-1/8}\frac{t}{s(s+t)^2}\frac{1}{\log(1/\varepsilon)}. \tag{7.52}$$

We use (7.42) to see that

$$\begin{aligned}
&\mathbb{P}^{(0,\varepsilon)}(F_2 \cap \{|Z_{\sigma(0,-\alpha)}| > \kappa^{40}|Z_{\sigma(0,\alpha)}|\})\\
&\quad\leq c_{20}\alpha^{-1/8}\kappa^{-1/8}\frac{1}{\log(1/\varepsilon)}\alpha^2\varepsilon\int_{\varepsilon\kappa^{100}}^{\varepsilon^{1/2}}\int_{\varepsilon\kappa^{100}}^{t}\frac{1}{s^2}\frac{s}{(s+t)^2}t\,ds\,dt\\
&\quad\leq c_{20}\alpha^{-1/8}\kappa^{-1/8}\frac{1}{\log(1/\varepsilon)}\alpha^2\varepsilon c_{21}\log^2(1/\varepsilon)\\
&\quad= c_{22}\alpha^{15/8}\kappa^{-1/8}\varepsilon\log(1/\varepsilon).
\end{aligned} \tag{7.53}$$

If $s \leq t \leq s\kappa^{40}$, in the derivation of (7.51) we must omit the middle factor and we obtain instead

$$c_{14}c_{17}c_{18}\alpha^{-1/8}\kappa^{-1/8}\frac{t}{s(s+t)^2}\frac{1}{\log(s/\varepsilon\kappa^{28})}\frac{\log(t\kappa^6/\varepsilon)}{\log(\varepsilon^{-3/4})}.$$

Since $t \leq \kappa^{40}s$ and $s/\varepsilon \geq \kappa^{100}$,

$$\frac{\log(t\kappa^6/\varepsilon)}{\log(s/\varepsilon\kappa^{28})} \leq \frac{\log(s/\varepsilon)+\log\kappa^{46}}{\log(s/\varepsilon)-\log\kappa^{28}} \leq \frac{2\log(s/\varepsilon)}{(72/100)\log(s/\varepsilon)} \leq 4,$$

and proceeding as above, we obtain the same bound for the probability of $F_2 \cap \{|Z_{\sigma(0,-\alpha)}| \leq \kappa^{40}|Z_{\sigma(0,\alpha)|}\}$ as in (7.53). □

8. Filling in the gaps.

Our first lemma shows that if there is no approximate point of increase of size ε at level $i\varepsilon/r$ for $i=0,1,2,\ldots,r-1$, then there can be no approximate point of increase of size $(1-r^{-1})\varepsilon$ at level y for any $y \in [0,\varepsilon]$.

The original definition (5.3) of $\sigma(j,\alpha)$ was stated for integer j only. It extends to non-integer j in an obvious way.

Lemma 8.1. *Let $\varepsilon > 0$, $r \in \mathbb{Z}$, $s \le 1-1/r$. Suppose Z_t is a Brownian motion starting in the lower half plane, $\sigma(2,0) < \tau_{B(0,2)}$, and for $y_i = i/r$, $i=0,1,2,\ldots,r-1$,*

$$\sigma(y_i,0)\circ\theta_{\sigma(y_i+1,0)} < \tau_{B(0,2)}\circ\theta_{\sigma(y_i+1,0)}.$$

Then if $y \in [0,\varepsilon]$,

$$\sigma(y/\varepsilon,0)\circ\theta_{\sigma(y/\varepsilon+s,0)} < \tau_{B(0,2)}\circ\theta_{\sigma(y/\varepsilon+s,0)}.$$

Proof. Suppose $y/\varepsilon \in [y_i, y_{i+1}]$, $i \le r$. Since $\sigma(2,0) < \tau_{B(0,2)}$, the path after time $\sigma(y/\varepsilon+s,0)$ must hit $\mathcal{L}(y_i+1,0)$ before exiting $B(0,2)$. Because $\sigma(y_i,0)\circ\theta_{\sigma(y_i+1,0)} < \tau_{B(0,2)}\circ\theta_{\sigma(y_i+1,0)}$, the post-$\sigma(y/\varepsilon+s,0)$ process returns to $\mathcal{L}(y_i,0)$ before exiting $B(0,2)$, which means it also has returned to $\mathcal{L}(y/\varepsilon,0)$ before escaping $B(0,2)$. □

Remark. The last result extends easily to approximate cut lines in other directions and at levels y between v and $v+\varepsilon$, for arbitrary $|v| \le 1$.

Let

$$\begin{aligned}
\Lambda &= \{Z_{\sigma(0,0)} = (0,0)\},\\
A &= \{\sigma(-1,0)\circ\theta_{\sigma(0,0)} > \tau_{B(0,2)}\circ\theta_{\sigma(0,0)}\},\\
B_\alpha &= \{\sigma(-1,\alpha)\circ\theta_{\sigma(0,\alpha)} < \tau_{B(0,2)}\circ\theta_{\sigma(0,\alpha)}\}\\
&\quad \cap \Big\{\sup\{t<\sigma(0,0) : Z_t \in \mathcal{L}(-1,\alpha)\} > \sup\{t < \sigma(0,0) : Z_t \in B(0,2)\}\Big\}.
\end{aligned}$$

Proposition 8.2. *There exists c_1 independent of ε such that if $\alpha \le 1/\log(1/\varepsilon)$, $z=(x,y)$ with $|z|>4$ and $y<-1$, then*

$$\mathbb{P}^z(A \cap B_\alpha \cap B_{-\alpha} \mid \Lambda) \le c_1 \alpha^2 \varepsilon \log^2(1/\varepsilon).$$

Proof. If $\omega \in A\cap\Lambda$, then after time $\sigma(0,0)$, the process Z_t will exit $B(0,2)$ before returning to $\mathcal{L}(-1,0)$. If $\omega \in B_\alpha \cap B_{-\alpha} \cap \Lambda$, then by time $\sigma(0,0)$ the process will have already hit $\mathcal{L}(0,\alpha)$ and $\mathcal{L}(0,-\alpha)$ (at time $\sigma(0,0)$, if not before). There are three possibilities. The first is that before $\sigma(0,0)$ the process will have hit $\mathcal{L}(0,\alpha)$ and then $\mathcal{L}(-1,\alpha)$ and also will have hit $\mathcal{L}(0,-\alpha)$ and then $\mathcal{L}(-1,-\alpha)$. The second is that by time $\sigma(0,0)$ the process will have hit $\mathcal{L}(0,\alpha)$ and then $\mathcal{L}(-1,\alpha)$, but will not have hit $\mathcal{L}(-1,-\alpha)$; the process will then have to hit $\mathcal{L}(-1,-\alpha)$ after time $\sigma(0,0)$. (We include in this possibility the same situation with the roles of α and $-\alpha$ reversed.) The third possibility is that the process will hit both $\mathcal{L}(-1,\alpha)$ and $\mathcal{L}(-1,-\alpha)$ after time $\sigma(0,0)$.

Let us consider these three possibilities for an $\omega \in A\cap B_\alpha \cap B_{-\alpha}\cap\Lambda$. For the first, let us split the path at $\sigma(0,0)$. If we examine the portion of the path before $\sigma(0,0)$, by time reversal and the argument of Proposition 5.8, the probability of these paths is the same as the probability for a BM $\times$ Bes(3) started at 0 to hit $\mathcal{L}(1,\alpha)$ and then hit $\mathcal{L}(0,\alpha)$ and also to hit $\mathcal{L}(1,-\alpha)$ and then hit $\mathcal{L}(0,-\alpha)$. By Corollary 7.8, the probability of such ω's is bounded by $c_2\alpha^2\log^2(1/\varepsilon)$. For the portion of the path after $\sigma(0,0)$ we use the strong Markov property. The process must escape $B(0,2)$ before returning to $\mathcal{L}(-1,0)$. Using (1.3) we obtain a factor $c_3\varepsilon$. So altogether we have the bound $c_2c_3\alpha^2\varepsilon\log^2(1/\varepsilon)$.

For the second possibility we again split the path at time $\sigma(0,0)$. By the argument we just gave in the preceding paragraph, using the first formula of Corollary 7.8, we obtain a factor $c_4\alpha\log(1/\varepsilon)$ for the portion of the path before $\sigma(0,0)$. By the strong Markov property, Proposition 7.3, and the remark preceding Proposition 5.1, we get a factor $c_5\alpha\varepsilon\log(1/\varepsilon)$ for the portion of the path after $\sigma(0,0)$. So the second possibility is bounded by $c_4c_5\alpha^2\varepsilon\log^2(1/\varepsilon)$. (By symmetry, the probability of the second possibility with the roles of α and $-\alpha$ reversed has the same bound.)

For the third possibility we use the strong Markov property at $\sigma(0,0)$, examine only the portion of the path after $\sigma(0,0)$, and use Proposition 7.7 to get the desired bound. □

Let C_j be as in (7.6).

Proposition 8.3. *If z is as in Proposition 8.2 and $\alpha \le 1/\log(1/\varepsilon)$, then*

$$\mathbb{P}^z(A\cap B_\alpha\cap B_{-\alpha}\cap C_1\cap\cdots\cap C_{\varepsilon^{-1/16}} \mid \Lambda) \le c_1\varepsilon\alpha^{15/8}\log^{7/8}(1/\varepsilon).$$

Proof. The proof is very similar to Proposition 8.2, where we look at the same three possibilities, but this time we require $C_1\cap\cdots\cap C_{\varepsilon^{-1/16}}$ to hold.

For the first possibility, we proceed exactly as in Proposition 8.2 for the portion of the path before $\sigma(0,0)$. For the portion of the path after $\sigma(0,0)$ we again obtain a factor $c_2\varepsilon$. However, we also require $C_1\cap\cdots\cap C_{\varepsilon^{-1/16}}$ to hold, so by Proposition 4.7 and the same argument as in Proposition 5.8, we also get an additional factor $c_3/\log(1/\varepsilon)$. Thus the probability is bounded by

$$c_4\varepsilon\alpha^2\log^2(1/\varepsilon)/\log(1/\varepsilon) \le c_4\varepsilon\alpha^{15/8}\log^{7/8}(1/\varepsilon).$$

For the second possibility we proceed as in Proposition 8.2 but we use Proposition 7.6 in place of Proposition 7.3 for the portion of the path after $\sigma(0,0)$. We thus obtain the estimate

$$c_5\alpha\log(1/\varepsilon)\alpha^{15/16}\varepsilon/\log^{1/16}(1/\varepsilon) \le c_5\varepsilon\alpha^{15/8}\log^{7/8}(1/\varepsilon).$$

For the third possibility we use Proposition 7.10 instead of Proposition 7.6 and again deduce the desired bound. $\square$

Remark. It is easy to see that Propositions 8.1 and 8.2 hold if we replace the conditioning on $\Lambda = \{Z_{\sigma(0,0)} = (0,0)\}$ by conditioning on $\{Z_{\sigma(0,0)} = (x,0)\}$ for any fixed $(x,0) \in B(0,1/2)$. We have to accordingly modify the definition of B_α so that it involves hitting times of straight lines passing through $Z_{\sigma(0,0)}$. We have to change the value of the constants, but we can do so independently of x. The estimates similarly generalize to hitting times of lines inclined at different angles lying at different distances from the origin.

Let $\overline{\mathcal{L}}(x,\alpha)$ be the line with slope $-\tan\alpha$, with y intercept $x/\cos\alpha$, so that its distance from the origin is equal to $|x|$. Define

$$\rho(x,\alpha) = \inf\{t : Z_t \in \overline{\mathcal{L}}(x,\alpha)\}.$$

Note that for integer j, we have $\overline{\mathcal{L}}(j\varepsilon,\alpha) = \mathcal{L}(j,\alpha)$ and $\rho(j\varepsilon,\alpha) = \sigma(j,\alpha)$. Suppose $\zeta \in (0,\pi/16)$ and set

$$\begin{aligned} D(x,\alpha,\varepsilon) &= \{\rho(x,\alpha)\circ\theta_{\rho(x+\varepsilon,\alpha)} > \tau_{B(0,2)}\circ\theta_{\rho(x+\varepsilon,\alpha)}\}, \\ E(\alpha,\varepsilon) &= \bigcup_{|x|<1/64} D(x,\alpha,\varepsilon) \\ G_\zeta &= \{Z_{\rho(x,\alpha)} \in B(0,1/8), |x| < 1/16, |\alpha| < \zeta\}. \end{aligned}$$

Proposition 8.4. *Let $\gamma > 0$, $\zeta \in (0,\pi/16)$. There exists $\varepsilon_0 > 0$ and a dense subset A_{ε_0} of $(-\zeta,\zeta)$ such that if $z = (x,y)$ with $|z| > 4$ and $y < -1$, then*

$$\mathbb{P}^z\Big(G_\zeta \cap \bigcup_{\alpha\in A_{\varepsilon_0}} E(\alpha,\varepsilon_0)\Big) < \gamma.$$

Proof. Let K_1 be chosen so that

$$\prod_{j=K_1}^{\infty}(1-j^{-2}) \geq 1/2 \quad \text{and} \quad \sum_{j=K_1}^{\infty} j^{-2} \leq \frac{1}{64}.$$

Suppose $K > K_1$ and $M = M(K)$ are large integers whose values will be chosen later. By Theorem 6.10 and rotation invariance, there exists $\varepsilon \in (0,1)$ such that

$$\mathbb{P}^z\Big(G_\zeta \cap \bigcup_{\substack{\alpha\log(1/\varepsilon)\in\mathbb{Z}\\ |\alpha|<\zeta}} \bigcup_{\substack{x\varepsilon\in\mathbb{Z}\\ |x|<1/16}} D(x,\alpha,\varepsilon)\Big) < \gamma/M. \tag{8.1}$$

We will assume that ε is small enough so that $\varepsilon^{1/4} < 1/64$. Let

$$\begin{aligned} \varepsilon_k &= \varepsilon\prod_{j=K}^{k}(1-j^{-2}), \\ d_k &= \varepsilon\sum_{j=K}^{k} j^{-2}, \\ b_k &= \sum_{j=K}^{k} 2^{-j}/\log(1/\varepsilon), \\ I_k &= \{\alpha : |\alpha| < \zeta - b_k, 2^k\alpha\log(1/\varepsilon) \in \mathbb{Z}\}. \end{aligned}$$

Note that by our choice of K_1, K, and ε, we have $(1/16) - d_k > 1/32$ for all $k \geq K$. Let

$$E_k(\alpha) = \bigcup_{|x|<(1/16)-d_k} D(x, \alpha, \varepsilon_k).$$

By translation and rotation invariance we obtain from (8.1) for any fixed j and m,

$$\mathbb{P}^z\Big(G_\zeta \cap \bigcup_{\substack{j2^{-K}+\alpha\log(1/\varepsilon)\in\mathbb{Z}\\ |\alpha|<\zeta}} \bigcup_{\substack{(m/K^2)+x\varepsilon\in\mathbb{Z}\\ |x|<1/16}} D(x,\alpha,\varepsilon)\Big) < \gamma/M.$$

Summing over 2^K values of j and K^2 values of m yields

$$\mathbb{P}^z\Big(G_\zeta \cap \bigcup_{\alpha\in I_K} \bigcup_{\substack{\varepsilon x K^2\in\mathbb{Z}\\ |x|<1/16}} D(x,\alpha,\varepsilon)\Big) < K^2 2^K \gamma/M. \tag{8.2}$$

By Lemma 8.1 and the remark following it,

$$\mathbb{P}^z\Big(G_\zeta \cap \bigcup_{\alpha\in I_K} E_K(\alpha)\Big) < K^2 2^K \gamma/M.$$

Note that the range of values of x has to shrink by ε/K^2 due to the form of the result proved in Lemma 8.1. For similar reasons we have introduced E_k and I_k; the range of values of x and α shrinks each time we make an inductive step in our bisecting procedure later in the proof.

Let

$$F_k = \bigcup_{\alpha\in I_k} E_k(\alpha).$$

We now begin bisecting angles. Since $\varepsilon_k > \varepsilon/2$ by our choice of K_1, then $\log(1/\varepsilon)/\log(1/\varepsilon_k)$ is bounded above and below by constants not depending on k. If β is an angle of the form $i/(2^{k+1}\log(1/\varepsilon))$ with i odd and $|\beta| < \pi/16$, then by Proposition 8.2 (see also the remark following Proposition 8.3),

$$\mathbb{P}^z(G_\zeta \cap D(x,\beta,\varepsilon_k) \cap F_k^c) \leq c_1\Big(\frac{1}{2^{k+1}\log(1/\varepsilon)}\Big)^2 \varepsilon_k \log^2(1/\varepsilon_k) \leq c_2\varepsilon_k 2^{-2(k+1)}, \tag{8.3}$$

for $x \in (1/4 - \varepsilon_k^{1/4}, 1/4)$ for which x/ε_k is an integer. There are at most $2\varepsilon_k^{-3/4}$ such values. Proposition 8.3 similarly yields

$$\begin{aligned}
&\mathbb{P}^z(G_\zeta \cap D(x,\beta,\varepsilon_k) \cap F_k^c \cap D(x+\varepsilon_k,\beta,\varepsilon_k)^c \cap \cdots \cap D(x+\varepsilon_k^{1/4},\beta,\varepsilon_k)^c)\\
&\quad\leq c_3\Big(\frac{1}{2^{k+1}\log(1/\varepsilon)}\Big)^{15/8} \varepsilon_k \log^{7/8}(1/\varepsilon_k)\\
&\quad\leq c_4\varepsilon_k 2^{-15(k+1)/8}/\log(1/\varepsilon_k),
\end{aligned} \tag{8.4}$$

for $x \in (-1/4, 1/4 - \varepsilon_k^{1/4})$ for which x/ε_k is an integer. There are at most $2/\varepsilon_k$ such values. Summing over x in (8.3) and (8.4), and using translations by $i\varepsilon_k/k^2$,

$i = 0, 1, \ldots, k^2 - 1$, we have

$$\begin{aligned}\mathbb{P}^z\Big(G_\zeta \cap \bigcup_{\substack{k^2x/\varepsilon_k \in \mathbb{Z}\\ |x|<1/4}} D(x,\beta,\varepsilon_k) \cap F_k^c\Big) \\ \le c_5 k^2 \Big(2\varepsilon_k^{-3/4} c_2 \varepsilon_k 2^{-2(k+1)} + 2\varepsilon_k^{-1} c_4 \varepsilon_k 2^{-15(k+1)/8}/\log(1/\varepsilon_k)\Big) \\ \le c_6 k^2 2^{-15(k+1)/8}/\log(1/\varepsilon_k).\end{aligned}$$

By Lemma 8.1,

$$\mathbb{P}^z(G_\zeta \cap E_{k+1}(\beta) \cap F_k^c) \le c_6 k^2 2^{-15(k+1)/8}/\log(1/\varepsilon_k). \tag{8.5}$$

The number of angles β for which we have the estimate (8.5) is at most $2^k \log(1/\varepsilon)$. Therefore

$$\mathbb{P}^z(G_\zeta \cap F_{k+1} \cap F_k^c) \le c_7 k^2 2^k 2^{-15(k+1)/8} \le c_8 2^{-k/2}.$$

We obtain

$$\begin{aligned}\mathbb{P}^z\Big(G_\zeta \cap \bigcup_{k=K}^\infty F_k\Big) &\le \mathbb{P}^z(G_\zeta \cap F_K) + \sum_{k=K}^\infty \mathbb{P}^z(G_\zeta \cap F_{k+1} \cap F_k^c) \\ &\le K^2 2^K \gamma/M + \sum_{k=K}^\infty c_8 2^{-k/2}.\end{aligned} \tag{8.6}$$

We can make the left hand side of (8.6) less than $\gamma/2$ if we first take K sufficiently large and then M sufficiently large. We have proved that with probability greater than $1-\gamma$, there are no approximate points of increase of size ε_k at any level $x \in (-1/32, 1/32)$ at any direction $\alpha \in I_k$, for any $k \ge K$.

We set $\varepsilon_0 = \varepsilon/4$ and $A_{\varepsilon_0} = (\cup_{k=K}^\infty I_k) \cap (-\zeta, \zeta)$. By Lemma 8.1, if $\alpha \in A_\varepsilon$ and there are no approximate points of increase of size ε_k at any level $x \in (-1/32, 1/32)$, then there are no approximate points of increase of size ε for any level $x \in (-1/64, 1/64)$. □

In the following proof, $\mathcal{L}$ will denote a straight line. We let $\theta(\mathcal{L})$ denote the angle that $\mathcal{L}$ makes with the x-axis. Let $H(\mathcal{L}, z)$ denote the half plane determined by $\mathcal{L}$ that contains the point z.

Proof of Theorem 0.7. Recall the definition of $\mathcal{S}$ from (0.1). If $\omega \in \mathcal{S}$, then there exists a straight line $\mathcal{L}$, an integer $M > 16$ and rationals x_1, y_1, r_1, p, ζ, and r_2 with the following properties. First of all, $\theta(\mathcal{L}) \in (p-\zeta/4, p+\zeta/4)$. If $z_1 = (x_1, y_1)$, then $Z_{\mathcal{T}_\mathcal{L}} \in B(z_1, r_1/4)$, the portion of the path from $\mathcal{T}_{B(z_1,r_1)}$ to $\mathcal{T}_\mathcal{L}$ lies on one side of $\mathcal{L}$, and the portion of the path from $\mathcal{T}_\mathcal{L}$ to $\tau_{B(z_1,r_1)}$ lies on the other side of $\mathcal{L}$. We have $r_2 < r_1/M$ and $Z_{\mathcal{T}_\mathcal{L}} \in B(z_1, r_2/4)$, and $Z(\mathcal{T}_{B(z_1,r_1)})$ is a distance at least $16 r_2$ from $\mathcal{L}$. We have $\zeta \in (-\pi/16, \pi/16)$. Finally, if $\mathcal{L}'$ is any line intersecting $B(z_1, r_2/4)$ with $\theta(\mathcal{L}') \in (p-\zeta, p+\zeta)$, then the location where Z_t first hits $\mathcal{L}'$ after time $\mathcal{T}_{B(z_1,r_1)}$ will lie in $B(z_1, r_2/2)$.

By using the strong Markov property at $\mathcal{T}_{B(z_1,4r_2)}$, translation invariance, rotational invariance, and scaling, it thus suffices to show that if $|z| > 4$, $z = (x, y)$, and $y < -4$, then $\mathbb{P}^z(\mathcal{S}_1) = 0$, where $\mathcal{S}_1$ is the event

$$\begin{aligned}\{\exists \mathcal{L} : & Z_{\mathcal{T}_\mathcal{L}} \in B(0, 1/2), Z_t \notin H(\mathcal{L}, z) \text{ for } \mathcal{T}_\mathcal{L} < t < \tau_{B(0,2)}, \theta(\mathcal{L}) \in (-\zeta/4, \zeta/4), \\ & Z_{\mathcal{T}(\mathcal{L}')} \in B(0, 1/2) \text{ for } \mathcal{L}' \text{ such that } \mathcal{L}' \cap B(0, 1/4) \neq \emptyset \text{ and } \theta(\mathcal{L}') < \zeta\}.\end{aligned}$$

Suppose $\mathbb{P}^z(\mathcal{S}_1) > 0$. Choose large $N < \infty$ and small $\gamma > 0$ such that

$$\mathbb{P}^z\Big(\mathcal{S}_1 \cap \{\sup_{t \le \tau_{B(0,2)} \circ \theta_{\mathcal{T}(B(0,1))}} |Z_t| < N\}\Big) > \gamma. \tag{8.7}$$

By Proposition 8.4 and scaling we can find ε and a dense subset A_ε of $(-\pi/16, \pi/16)$ such that the probability that there exists an approximate point of increase of size ε at any level $x \in (-1, 1)$ for any $\alpha \in A_\varepsilon$ is less than $\gamma/2$. We see that there must exist an ω which belongs to the event in (8.7) but such that there is no approximate point of increase of size ε at any level $x \in (-1, 1)$ for any $\alpha \in A_\varepsilon$. Consider such an ω and let $\mathcal{L}$ be a cut line as in the definition of $\mathcal{S}_1$. Since A_ε is dense in $(-\pi/16, \pi/16)$, there exist two parallel lines $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\theta(\mathcal{L}_1) \in A_\varepsilon$, the process hits $\mathcal{L}_1$ before hitting $\mathcal{L}_2$, $Z_{\mathcal{T}(\mathcal{L}_1)} \in B(0, 3/4)$, the distance between $\mathcal{L}_1$ and $\mathcal{L}_2$ is ε, and $\mathcal{L} \cap B(0, N)$ lies strictly between $\mathcal{L}_1$ and $\mathcal{L}_2$. After hitting $\mathcal{L}_2$ the process cannot return to $\mathcal{L}_1$ before exiting $B(0, 2)$ because it cannot cross the cut line $\mathcal{L}$. Hence there exists an approximate point of increase of size ε at some level $x \in (-1, 1)$ for the angle $\theta(\mathcal{L}_1)$. This is a contradiction. The proof of Theorem 0.7 is complete. □

9. Further results and problems.

Cut planes.

In this section we will sketch a proof of Theorem 0.6. Recall the result.

Theorem 0.6. *Let X_t denote d-dimensional Brownian motion, where $d \geq 3$. With positive probability, there exists $t \in (0,1)$ such that $X([0,t))$ and $X((t,1])$ lie on the opposite sides of some $(d-1)$-dimensional hyperplane.*

We first consider 2 dimensional Brownian motion. Recall the definition of $\mathcal{L}(j,\alpha)$ and $\sigma(j,\alpha)$ from (5.3) and the event $D = \{X_{\sigma(0,0)} = 0\}$ from (5.32). Let $C(j,\alpha)$ be the event that $\mathcal{L}(j,\alpha)$ is an approximate cut line of size ε, i.e.,

$$C(j,\alpha) = \{\tau_{B(0,2)} \circ \theta_{\sigma(j,\alpha)} < \sigma(j-1,\alpha) \circ \theta_{\sigma(j,\alpha)}\}.$$

Lemma 9.1. *Consider $\alpha < \pi/16$ and suppose $z = (x,y)$ with $|z| \geq 4$ and $y < -1$. There exists c_1 not depending on ε such that for $j \leq \alpha/\varepsilon$,*

$$\mathbb{P}^z(C(0,0) \cap C(j,\alpha) \mid D) \leq \frac{c_1 \varepsilon}{j}(j\varepsilon)^\alpha.$$

For $j \geq \alpha/\varepsilon$,

$$\mathbb{P}^z(C(0,0) \cap C(j,\alpha) \mid D) \leq \frac{c_1 \varepsilon}{j}.$$

Proof. (Sketch) Let v be the point where $\mathcal{L}(j,\alpha)$ and $\mathcal{L}(0,0)$ intersect. Given D, the process hits $\mathcal{L}(0,0)$ at $(0,0)$. It must then hit $\mathcal{L}(j,\alpha)$ before hitting $\mathcal{L}(-1,0)$. So we require the y coordinate of the Brownian motion to hit $j\varepsilon/2$ before hitting $-\varepsilon$, and the probability of this is $2/j$. There is a chance that the process X_t will hit $\mathcal{L}(j,\alpha)$ at a point closer to v then one would expect, but it is easy to show that this probability is relatively small.

After hitting $\mathcal{L}(j,\alpha)$, the process X_t must exit $B(0,2)$ before hitting either $\mathcal{L}(j-1,\alpha)$ or $\mathcal{L}(1,0)$. For $j \geq \alpha/\varepsilon$, we use a crude bound $c_2\varepsilon$. Suppose that $j \leq \alpha/\varepsilon$. By Proposition 5.2, the probability of the same event is bounded by a constant times $\varepsilon(j\varepsilon)^\alpha$. Again there is a difficulty if X_t hits $\mathcal{L}(j,\alpha)$ at a point much further from v than one would expect; a conformal mapping gives the exact hitting distribution of $\mathcal{L}(j,\alpha) \cup \mathcal{L}(-1,0)$ starting from $(0,0)$, and this can be used to estimate this latter probability. The first estimate in the lemma follows by multiplying the bounds $2/j$ and $\varepsilon(j\varepsilon)^\alpha$, and the second one can be obtained in a similar way. □

Proof of Theorem 0.6. (Sketch) By projection, if a 3 dimensional Brownian motion can be cut by a plane, then d-dimensional Brownian motion can be cut by a $(d-1)$-dimensional hyperplane, $d \geq 3$. So it suffices to consider the case $d = 3$.

Let $\varepsilon > 0$ and let $v(k,m)$ be the unit vector starting at the origin whose terminal point has spherical coordinates $(1, k/\log(1/\varepsilon), m/\log(1/\varepsilon))$. Let $P(j,k,m)$ be the plane which has $v(k,m)$ as its normal vector and which passes through the point whose spherical coordinates are $(j\varepsilon, k/\log(1/\varepsilon), m/\log(1/\varepsilon))$. Let $A(j,k,m)$ be the event that 3 dimensional Brownian motion hits $P(j,k,m)$ before exiting $B(0,1)$ and then exits $B(0,2)$ before returning to $P(j-1,k,m)$.

Let $z \in \partial B(0,2) \cap P(-1/2\varepsilon, 0, 0)$. Note there exists c_2 such that $\mathbb{P}^z(A(j,k,m)) \geq c_2\varepsilon$. If we sum over j from 1 to $1/2\varepsilon$ and k and m from 1 to $\pi\log(1/\varepsilon)/16$, we see

$$\sum_{j,k,m} \mathbb{P}^z(A(j,k,m)) \geq c_3 \log^2(1/\varepsilon).$$

Let α be the angle between $P(0,0,0)$ and $P(j',k',m')$. By projecting onto the plane containing $v(0,0)$ and $v(k',m')$ and using Lemma 9.1,

$$\mathbb{P}^z(A(0,0,0) \cap A(j',k',m') \mid D) \leq \frac{c_4\varepsilon}{j'}(j'\varepsilon)^\alpha, \qquad \text{if } j' \leq \alpha/\varepsilon,$$

$$\mathbb{P}^z(A(0,0,0) \cap A(j',k',m') \mid D) \leq \frac{c_4\varepsilon}{j'}, \qquad \text{if } j' \geq \alpha/\varepsilon.$$

We obtain

$$\sum_{j'=1}^{1/(2\varepsilon)} \mathbb{P}^z(A(0,0,0) \cap A(j',k',m') \mid D) \leq \sum_{j'=1}^{\alpha/\varepsilon} \frac{c_4\varepsilon}{j'}(j'\varepsilon)^\alpha + \sum_{j'=\alpha/\varepsilon}^{1/(2\varepsilon)} \frac{c_4\varepsilon}{j'} \leq \frac{c_6\varepsilon}{\alpha}.$$

It is easy to remove the conditioning on the event D, i.e., to see that

$$\sum_{j'=1}^{1/(2\varepsilon)} \mathbb{P}^z(A(0,0,0) \cap A(j',k',m')) \leq \frac{c_6\varepsilon}{\alpha}.$$

Now summing over k' and m' from 1 to $\pi\log(1/\varepsilon)/16$, routine calculations lead to

$$\sum_{j',k',m'} \mathbb{P}^z(A(0,0,0) \cap A(j',k',m')) \leq c_7\varepsilon \log^2(1/\varepsilon).$$

Using the rotation and translation invariance of Brownian motion and making minor modifications, we have

$$\sum_{j',k',m'} \mathbb{P}^z(A(j,k,m) \cap A(j',k',m')) \leq c_8\varepsilon \log^2(1/\varepsilon)$$

with c_8 not depending on j, k, or m. Finally summing over j, k, and m, we obtain

$$\sum_{j,k,m,j',k',m'} \mathbb{P}^z(A(j,k,m) \cap A(j',k',m')) \leq c_9 \log^4(1/\varepsilon).$$

If

$$R = \sum_{j,k,m} 1_{A(j,k,m)},$$

we thus have shown that

$$\mathbb{E}^z R^2 \le c_{10}(\mathbb{E}^z R)^2$$

with c_{10} independent of ε. A standard second moment argument (see the last paragraph of the proof of Theorem 6.1) shows there exists c_{11} independent of ε such that

$$\mathbb{P}^z(R \ge 1) \ge c_{11}.$$

This implies that, with probability greater than $c_{12} > 0$, for every $\varepsilon_1 > 0$, there is a cut plane of size $\varepsilon \in (0, \varepsilon_1)$. A compactness argument readily yields the existence of cut planes with positive probability. □

Open problems.

1. Logarithmic dimension. Cut planes exist for all dimensions 3 and larger. Evans' (1985) result implies that the set of times

$$C_d = \{t \in (0,1) : X([0,t)) \text{ lies on one side of a hyperplane,} \\ X((t,1]) \text{ lies on the other side of the hyperplane}\}$$

has Hausdorff dimension zero. Yet perhaps a more sensitive measure of C_d will show dependence on the dimension d. Recall the Hausdorff dimension of a set A is the infimum of those α for which the x^α-Hausdorff measure of A is zero. The logarithmic dimension of a set A is defined to be the infimum of those α for which the $1/\log^\alpha(1/x)$–Hausdorff measure is zero. What is the logarithmic dimension of C_d?

2. Curved lines. By the result of Burdzy (1989) on the existence of cut points, the trace of two dimensional Brownian motion can be cut by Lipschitz curves. We have shown that it cannot be cut by straight lines. How smooth may a curve be and still cut the trace of Brownian motion? We conjecture that the curve may be C^1 but cannot have a derivative that is too smooth.

3. Local times. For one dimensional Brownian motion, the local time at level s at time 1 is never 0 if s is in the interval $(\inf_{r\le 1} X_r, \sup_{r\le t} X_r)$. Now consider two dimensional Brownian motion and let $L_t(s,v)$ be the local times on straight lines introduced by Bass (1984). This is a jointly continuous version of the local time at s of $X_t \cdot v$. Although cut lines of X_t do not exist, it is still conceivable that $L_t(s,v)$ can be 0 for some v and some $s \in (\inf_{r\le t} X_r \cdot v, \sup_{r\le t} X_r \cdot v)$. Does this happen or not?

4. Excursions from the convex hull. Let D_t denote the convex hull of the trace $X([0,t])$ of 2 dimensional Brownian motion X. Do there exist Brownian excursions from the convex hull, i.e., do there exist times $t \in (0,1)$ such that D_t and $X((t,1])$

are disjoint? If t is such a time, then $X((t,1])$ is a part of an excursion of X from the set D_t. The shape of an excursion and its "likelihood" depend on the shape of ∂D_t. The boundary of D_t contains line segments and it would not be hard to argue that there are no excursions from D_t starting in the middle of one of those line segments. The curvature of ∂D_t near its minimum point has been described by Burdzy and San Martin (1989) and Mountford (1992). The same results hold simultaneously for the extremal points of D_t in almost all directions, by Fubini's theorem. The results of Burdzy (1987) on the path behavior of Brownian excursions in Lipshitz domains show that a typical Brownian excursion starting from such a point on ∂D_t would have the same local properties as an excursion from a straight line. This suggests that there are no excursions from the convex hull starting at such points. One cannot rule out, however, existence of a sufficiently large set of points on ∂D_t with exceptionally large "curvature" and excursions $X((t,1])$ from D_t starting from such points.

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Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195-4350

bass@math.washington.edu burdzy@math.washington.edu