# Degenerate Stochastic Differential Equations with Hölder Continuous Coefficients and Super-Markov Chains

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**Abstract**

We consider the operator $\sum_{i,j=1}^d \sqrt{x_i x_j}\gamma_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}$ acting on functions in $C_b^2(\mathbb{R}_+^d)$. We prove uniqueness of the martingale problem for this degenerate operator under suitable nonnegativity and regularity conditions on $\gamma_{ij}$ and $b_i$. In contrast to previous work, the $b_i$ need only be nonnegative on the boundary rather than strictly positive, at the expense of the $\gamma_{ij}$ and $b_i$ being Hölder continuous. Applications to super-Markov chains are given. The proof follows Stroock and Varadhan's perturbation argument, but the underlying function space is now a weighted Hölder space and each component of the constant coefficient process being perturbed is the square of a Bessel process.

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## 1. Introduction.

Consider the operator

$$\mathcal{L}f(x) = \sum_{i,j=1}^{d} \sqrt{x_i x_j}\gamma_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x_i}(x) \tag{1.1}$$

on functions in $C_b^2(\mathbb{R}_+^d)$, the space of bounded $C^2$ functions on the non-negative orthant with bounded first and second order partial derivatives. We prove uniqueness of the martingale problem for $\mathcal{L}$ under suitable nonnegativity and regularity conditions on the $\gamma_{ij}$ and $b_i$. A precise statement is given below, but in essence we require that for each $x$ the matrix $\gamma_{ij}(x)$ is positive definite, the off-diagonal terms are small for $x$ in the boundary of $\mathbb{R}_+^d$, $b_i(x) \geq 0$ when $x_i = 0$, and the $\gamma_{ij}$ and $b_i$ are Hölder continuous of order $\alpha$ for some $\alpha$. We were motivated by some open uniqueness problems from the theory of superprocesses (see Example 1.4 below).

Let $\nu$ be a probability on $\mathbb{R}_+^d$. Let $\Omega' = C([0,\infty);\mathbb{R}_+^d)$ and furnish $\Omega'$ with the cylindrical Borel $\sigma$-field. Let $X_t(\omega) = \omega(t)$ for $\omega \in \Omega'$. Set $\mathcal{F}_t' = \cap_{u>t}\sigma(X_s : s \leq u)$. We say a probability measure $\mathbb{P}$ on $C(\mathbb{R}_+^d)$ solves the martingale problem $MP(\mathcal{L},\nu)$ if under $\mathbb{P}$ the law of $X_0$ is equal to $\nu$ and for all $f \in C_b^2(\mathbb{R}_+^d)$

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$$

is a local martingale under $\mathbb{P}$ with respect to the $\sigma$-fields $\mathcal{F}_t'$. In defining $C_b^2(\mathbb{R}_+^d)$ it is understood that for $x \in \partial\mathbb{R}_+^d$, the appropriate partial derivatives are interpreted as right-hand derivatives. Let $|x|$ denote the Euclidean norm of $x \in \mathbb{R}^d$ and $S_d^+$ denote the space of $d \times d$ symmetric strictly positive definite matrices. We assume that for some fixed $\alpha \in (0,1]$,

(H1) $(\gamma_{ij})_{i,j\leq d} : \mathbb{R}_+^d \to S_d^+$ is $\alpha$-Hölder continuous on compact sets.

(H2) $(b_i)_{i\leq d} : \mathbb{R}_+^d \to \mathbb{R}^d$ is $\alpha$-Hölder continuous on compact sets and for all $i \leq d$, $b_i(x) \geq 0$ whenever $x_i = 0$.

**Remark 1.1.** (a) Note that (H1) implies that if $a_{ij}(x) = \sqrt{x_i x_j}\gamma_{ij}(x)$, then the matrix $a(x)$ is symmetric non-negative definite for each $x \in \mathbb{R}_+^d$, is symmetric positive definite for each such $x$ outside of $\partial\mathbb{R}_+^d$ and is continuous in $x$. Standard results therefore show the existence of a solution to $MP(\mathcal{L},\nu)$ for any given law $\nu$ if we assume

$$|b(x)| \leq C(1 + |x|) \text{ for all } x \in \mathbb{R}_+^d \tag{1.2}$$

(see Theorem V.23.5 in [RW87] and (c) below, or the proof of Theorem 1.1 in [ABBP01]). Also [SV79] gives uniqueness of solutions up until the first hitting time of $\partial\mathbb{R}_+^d$ and so the main difficulty will be in resolving local uniqueness at points in $\partial\mathbb{R}_+^d$. (H1) also implies that for all $i \leq d$ and $x \in \mathbb{R}_+^d$, $\gamma_{ii}(x) > 0$.

(b) As is well known, uniqueness of the martingale problem $MP(\mathcal{L},\nu)$ is equivalent to the weak uniqueness of the corresponding stochastic differential equation. If $\sigma(x)\in S_d^+$ is the square root of the matrix $\gamma(x)$, then $\mathbb{P}$ is a solution of $MP(\mathcal{L},\nu)$ if and only if $\mathbb{P}$ is the law of a continuous process $(X_t, t\ge 0)$ defined on a probability space carrying a random vector $X(0)$ with law $\nu$ and an independent $d$-dimensional Brownian motion $(B^1,\dots,B^d)$ such that

$$X^i(t) = X^i(0) + \sum_{k=1}^d \int_0^t \sqrt{2X^i(s)}\sigma_{ik}(X(s))dB^k(s) + \int_0^t b_i(X(s))ds, \quad k=1,\dots,d. \tag{1.3}$$

(c) If we extend $\gamma$ and $b$ to $\mathbb{R}^d$, by setting $\gamma(x)=\gamma(x_1^+,\dots,x_d^+)$ and similarly for $b$, then (H2) and an easy comparison argument using the stochastic differential equation (1.3) (as in V.43 of [RW87]) will show that any solution to $MP(\mathcal{L},\nu)$ starting in $\mathbb{R}_+^d$ will remain there. Hence nothing is lost by restricting our attention to $\mathbb{R}_+^d$-valued solutions.

(d) Note that our formulation of the martingale problem considers $f\in C_b^2(\mathbb{R}_+^d)$, whereas in the usual formulation one considers $f\in C_b^2(\mathbb{R}^d)$. Suppose $f\in C_b^2(\mathbb{R}_+^d)$. For any solution of $MP(\mathcal{L},\nu)$, continuity of paths will imply that $T_R=\inf\{t: |X_t|\ge R\}$ tends to infinity a.s. as $R\to\infty$. To show that $f(X_t)-f(X_0)-\int_0^t \mathcal{L}f(X_s)ds$ is a local martingale, it suffices to show that for each $R$, $f(X_{t\wedge T_R})-f(X_0)-\int_0^{t\wedge T_R}\mathcal{L}f(X_s)ds$ is a local martingale, and we may thus suppose without loss of generality that $f$ has support in $[0,2R]^d$. By [St70], VI.4.6, a function in $C_b^2(\mathbb{R}_+^d)$ with compact support has an extension which is in $C_b^2(\mathbb{R}^d)$. In view of (c), this shows it is immaterial which formulation of martingale problem we use.

In view of the above our focus will be on the uniqueness of solutions to $MP(\mathcal{L},\nu)$. Our main result is

**Theorem 1.2.** *Suppose (H1) and (H2) hold. There is a positive constant $c_{1.1}=c_{1.1}(\alpha,d)$ such that if*

$$\sum_{i\ne j}|\gamma_{ij}(x)|\le c_{1.1}\min_i \gamma_{ii}(x) \quad \text{for all } x\in\partial\mathbb{R}_+^d, \tag{1.4}$$

*then for any probability $\nu$ on $\mathbb{R}_+^d$, there is at most one solution to $MP(\mathcal{L},\nu)$. If, in addition (1.2) holds, then there is exactly one solution to $MP(\mathcal{L},\nu)$.*

The above result is not as satisfactory as one would like because (1.4) requires the off-diagonal terms to be small on the boundary of the non-negative orthant. We have not been able to remove this condition but fortunately the main class of examples which motivated this work will trivially satisfy this condition as the off-diagonal terms will vanish. The following result, which follows immediately from Theorem 1.2, is therefore the result of major interest for us.

**Corollary 1.3.** *Suppose (H1), (H2) and (1.2) hold, and*

$$\gamma_{ij}(x) = 0 \text{ for all } x \in \partial\mathbb{R}^d_+ \text{ and all } i \neq j. \tag{1.5}$$

*Then there is exactly one solution to $MP(\mathcal{L}, \nu)$.*

Clearly in Corollary 1.3 one may weaken (H1), (H2) so that the coefficients are Hölder continuous of some index on each compact set as this suffices to prove uniqueness of solutions up to the exit time of any large ball. This is used implicitly in the examples discussed below.

**Example 1.4 (Super-Markov Chains).** Let $\gamma_i : \mathbb{R}^d_+ \to (0, \infty)$ be Hölder continuous on compacts and let $q_{ij}(x)$ be a $Q$-matrix for all $x \in \mathbb{R}^d_+$, that is, $q_{ij}(x) \geq 0$ for all $i \neq j$ and $\sum_j q_{ij}(x) = 0$ for all $i$. Suppose $q_{ij}(\cdot)$ is Hölder continuous on compacts, bounded, and let

$$\mathcal{L}_Q f(x) = \sum_{i=1}^{d} x_i \gamma_i(x) f_{ii}(x) + \sum_{i=1}^{d} \Big( \sum_{j=1}^{d} x_j q_{ji}(x) \Big) f_i(x).$$

If $q$ and $\gamma$ are constant, $MP(\mathcal{L}, \nu)$ is the martingale problem characterizing a superprocess with branching rate $\gamma_i$ in state $i$ and underlying spatial motion governed by a continuous time Markov chain with $Q$-matrix $(q_{ij})$. For $q$ and $\gamma$ as above $MP(\mathcal{L}, \nu)$ arises as the weak limit points of the large population, small mass and high branching rate limit of a branching particle system in which both the migration process and branching rate are now dependent on the empirical measure of the entire population. In other words the particles now interact with each other both through their $Q$-matrices $(q_{ij}(x))$ and branching rates $\gamma_i(x)$ (see the discussion in Section 1 of [ABBP01] for more details). A direct application of Corollary 1.3 with $b_i(x) = \sum_{j=1}^{d} x_j q_{ji}(x)$ and $\gamma_{ij}(x) = 1_{(i=j)}\gamma_i(x)$ shows there is exactly one solution to $MP(\mathcal{L}_Q, \nu)$ for any probability $\nu$ on $\mathbb{R}^d_+$. It therefore follows that the above interactive branching particle systems actually converge to the unique solution of $MP(\mathcal{L}, \nu)$.

It is now interesting to compare Theorem 1.2 to the following result which is essentially Theorem 1.1 from [ABBP01].

**Theorem A.** *Assume for $i \leq d$, $\gamma_{ii}$ and $b_i$ are continuous on $\mathbb{R}^d_+$, $\gamma_{ii}$ is strictly positive, and $b_i(x) > 0$ if $x_i = 0$. Assume also (1.2) holds and $\gamma_{ij} \equiv 0$ for $i \neq j$. Then for any probability $\nu$ on $\mathbb{R}^d_+$ there is exactly one solution to $MP(\mathcal{L}, \nu)$.*

The above theorem is slightly stronger than Theorem 1.1 in [ABBP01], which assumed $b_i(x) > 0$ on all of $\partial\mathbb{R}^d_+$. The change of measure argument used in the localization argument in the proof of Theorem 1.2 in Section 6 (see especially the argument in Case 2) can be easily modified to yield this improvement.

Theorem 1.2 imposes an additional Hölder continuity assumption on $b_i$ and $\gamma_i$ but allows (small) off-diagonal terms in the diffusion matrix and, more importantly, weakens the strict positivity condition on $b_i$ on $\{x_i = 0\}$ to non-negativity. This is precisely where the diffusion term degenerates and so it is perhaps not surprising that this improvement is rather delicate. This improvement turns out to be highly desirable from the perspective of applications such as Example 1.4. Example 1 in [ABBP01] gives the analogous uniqueness result to Example 1.4 above but instead assumes that $q_{ij}$ and $\gamma_i$ are only continuous and

$$q_{ij}(x) > 0 \text{ for all } i \neq j \text{ whenever } x_j = 0 \text{ and } x \neq 0. \tag{1.6}$$

The latter condition is needed there because of the strict positivity requirement on the boundary and rules out some of the most natural migration mechanisms such as nearest neighbor random walk. For ordinary super-Markov chains with $\gamma_i = 1$ and $q_{ij}(x)$ independent of $x$, it is known that just before extinction $X_t$ will hit a single axis (i.e. $X_t^i = 0$ for all but one value of $i$) infinitely often (see [T92] for much stronger results). This means that when (1.6) fails (as is allowed in Theorem 1.2), the effective drift $b_i(X_t)$ may indeed be 0 and the diffusion must know what to do when it hits corners, even if it starts in the interior of $\mathbb{R}_+^d$.

Similarly Example 2 of [ABBP01] deals with a family of mutually catalytic branching mechanisms in which there are $K$ types occupying $d$ sites, the branching rate of each type at site $i$ depends on the amount of mass of the other $K-1$ types at this site, and the migration of the each type depends on the configuration of the population of this type. Again in Example 2 of [ABBP01] one must assume (1.6) holds for the migration mechanism of each type, again ruling out the most natural local migration mechanisms. The present theorem allows us to obtain the uniqueness result given there without this condition (i.e. without assumption (1.13) in [ABBP01]) but assuming now the branching rates and $Q$-matrices are all locally Hölder continuous. The same comment applies to the stepping stone models treated in Example 3 in Section 1 of [ABBP01].

We feel the Hölder continuity condition is a small price to pay for the inclusion of the most natural migration mechanisms. It is, however, natural to ask if Corollary 1.3 is valid if we only assume continuity in (H1) and (H2) rather than Hölder continuity. In fact, the example given in Section 8 of [ABBP01] shows that even for $d = 1$ this is not the case. The same example showed that the strict positivity condition on $b_i|_{\partial\mathbb{R}_+^d}$ in Theorem 1.1 of [ABBP01] is also needed. We state it here for completeness.

**Counter-example 1.5.** *Let* $d = 1$, $b(x) = \left(c/\log^+\left(\frac{1}{x}\right)\right) \wedge 1$ *for* $x > 0$, *and let* $b(0) = b(0+) = 0$. *If* $c > 1$ *then the stochastic differential equation*

$$dX_t = (2X_t)^{1/2} dB_t + b(X_t)dt, \quad X_0 = 0, \tag{1.7}$$

*has a solution $X \geq 0$ which is not identically* 0*. Since 0 is also a solution, uniqueness in law fails for solutions of (1.7) and hence also for solutions of $MP(\mathcal{L}, \delta_0)$ where*

$$\mathcal{L}f(x) = x\frac{d^2 f}{dx^2}(x) + b(x)\frac{df}{dx}(x).$$

The reader is referred to Proposition 8.1 of [ABBP01] for a proof but this counterexample can also be viewed as an exercise in boundary classification for one-dimensional diffusions on $\mathbb{R}_+$.

Unlike the method of [ABBP01], which was a perturbation method in the space $L^2$, in this paper we use a perturbation method in a certain weighted Hölder space. Another advantage of this approach over that in [ABBP01] is that it appears to extend to infinite dimensions. One would like to obtain uniqueness in (1.1) when the underlying state space is countably infinite or even uncountably infinite, in which case one must deal with measure-valued martingale problems. This would allow one to establish analogues of Example 1.4 in infinite dimensions. The $L^2$ methods of [ABBP01] seem unlikely to succeed here, and even if they did, the infinite dimensional version of the required condition (1.6) is particularly unnatural. So far we have been able to extend the approach in this work to some infinite dimensional settings and we will pursue this further in a future work. Classical Hölder spaces have already been used in (different) infinite dimensional settings in [CD96].

We close with a brief outline of the paper. In Section 2 we introduce weighted Hölder spaces, which we denote by $\mathcal{C}_w^\alpha(\mathbb{R}_+^d)$. After some inequalities based on Gamma and Poisson distributions in Section 3, we begin developing our estimates in Section 4. We start by considering the one-dimensional process with generator

$$\gamma x f''(x) + bf'(x),$$

where $\gamma$ is a positive constant and $b$ is a nonnegative constant. Explicit formulas are known for the transition densities, and working with these and the inequalities of Section 3, we obtain estimates on the $L^\infty$ and $\mathcal{C}_w^\alpha(\mathbb{R}_+)$ norms of $(P_t f)'$ and $x(P_t f)''$ in terms of the $\mathcal{C}_w^\alpha(\mathbb{R}_+)$ norm of $f$; here $P_t$ is the corresponding semigroup.

In Section 5 we begin analyzing the operator

$$\sum_{i=1}^d \Big[x_i \gamma_i^0 \frac{\partial^2 f}{\partial x_i^2}(x) + b_i^0 \frac{\partial f}{\partial x_i}(x)\Big],$$

where the $\gamma_i^0$ and $b_i^0$ are constants. Since the semigroup corresponding to this case is given by a product of one-dimensional semigroups, we can derive estimates for the $L^\infty$ and $\mathcal{C}_w^\alpha(\mathbb{R}_+^d)$ norms of $\partial(P_t f)/\partial x_i$ and $x_i \partial^2(P_t f)/\partial x_i^2$ from the corresponding results in

Section 4. We then integrate these to obtain estimates for the resolvent $R_\lambda$ for the constant coefficient case.

The perturbation argument is carried out in Section 6, and Theorem 1.2 is proved there. Let $S_\lambda$ be the resolvent for the operator in (1.1). Under some additional assumptions on the coefficients we can write

$$S_\lambda f = R_\lambda f + R_\lambda \mathcal{B} R_\lambda + R_\lambda \mathcal{B} R_\lambda \mathcal{B} R_\lambda + \cdots$$

for a certain operator $\mathcal{B}$, and this leads to uniqueness of the martingale problem for $\mathcal{L}$. The estimates of Section 5 are exactly what are needed to show $\mathcal{B} R_\lambda$ is a bounded operator on $\mathcal{C}_w^\alpha(\mathbb{R}_+^d)$ with norm less than 1. Finally we remove the additional restrictions on the coefficients by a localization argument.

We use the letter $c$ with subscripts to indicate finite positive constants whose exact value does not matter and whose value may change from line to line.

**2. Preliminaries.**

Let $e_i$ be the unit vector in the $i$-th direction in $\mathbb{R}^d$. If $f \in C_b(\mathbb{R}_+^d)$, $\alpha \in (0,1]$ (eventually we will take $\alpha$ to be as in (H1) and (H2) but this is not important for now), and $1 \le i \le d$, let

$$|f|_{\alpha,i} = \sup\Big\{ \frac{|f(x+he_i) - f(x)|}{h^\alpha} x_i^{\alpha/2} : h > 0, x \in \mathbb{R}_+^d \Big\}.$$

If $\|f\|_\infty$ is the sup norm of $f$, set

$$|f|_\alpha = \sup_{i \le d} |f|_{\alpha,i}, \qquad \|f\|_\alpha = |f|_\alpha + \|f\|_\infty,$$

and

$$\mathcal{C}_w^\alpha(\mathbb{R}_+^d) = \{ f \in C_b(\mathbb{R}_+^d) : \|f\|_\alpha < \infty \}.$$

Here $C_b(\mathbb{R}_+^d)$ denotes the continuous bounded functions on $\mathbb{R}_+^d$. The same argument as for the usual Hölder spaces shows that $(\mathcal{C}_w^\alpha, \|\cdot\|_\alpha)$ is a Banach space. We will refer to it as the weighted $\alpha$-Hölder space.

Clearly, if $f$ is bounded, uniformly Hölder of index $\alpha$, and is constant outside of a bounded set, then $f$ is in $\mathcal{C}_w^\alpha$. On the other hand, as the following binary expansion argument of Lévy shows, $f \in \mathcal{C}_w^\alpha$ implies $f$ is uniformly Hölder of order $\alpha/2$. Here, the requirement that $f$ be continuous at $\partial \mathbb{R}_+^d$ is essential.

**Proposition 2.1.** *There exists $c_\alpha$ such that for all $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+^d)$*

$$|f(x) - f(y)| \le c_\alpha |f|_\alpha |x-y|^{\alpha/2} \qquad x, y \in \mathbb{R}_+^d. \tag{2.1}$$

**Proof.** Assume first $d=1$. Let $y>x\geq 0$. Let $x_n = 2^{-n}y+(1-2^{-n})x \downarrow x$. The continuity of $f$ shows that

$$\begin{aligned}|f(y)-f(x)| &\leq \sum_{n=0}^{\infty} |f(x_n)-f(x_{n+1})| \\ &\leq \sum_{n=0}^{\infty} |f|_\alpha (x_{n+1})^{-\alpha/2}|x_n - x_{n+1}|^\alpha \\ &= \sum_{n=0}^{\infty} |f|_\alpha \, |x+2^{-n-1}(y-x)|^{-\alpha/2}(2^{-n-1}|y-x|)^\alpha \\ &\leq |f|_\alpha \, |y-x|^{\alpha/2} \sum_{n=0}^{\infty} (2^{-n-1})^{\alpha/2} \\ &= c_\alpha |f|_\alpha |y-x|^{\alpha/2}.\end{aligned}$$

For general $d$ simply change one coordinate at a time to see that

$$|f(y)-f(x)| \leq c_\alpha \sum_{i=1}^{d} |f|_{\alpha,i}|y_i-x_i|^{\alpha/2} \leq c_\alpha |f|_\alpha \sum_{i=1}^{d} |y_i - x_i|^{\alpha/2} \leq c_\alpha |f|_\alpha |y-x|^{\alpha/2}.$$

□

We extend the definitions of $|f|_{\alpha,i}$ and $|f|_\alpha$ to $f:(0,\infty)^d \to \mathbb{R}$ by taking the supremum over $x\in(0,\infty)^d$ instead of over $\mathbb{R}_+^d$. If $f$ is continuous on $\mathbb{R}_+^d$ these definitions clearly coincide.

**Lemma 2.2.** *Let $f:(0,\infty)^d \to \mathbb{R}$ satisfy $|f|_\alpha<\infty$. Then $f$ is uniformly continuous and has a unique continuous extension to $\mathbb{R}_+^d$ which is uniformly Hölder $\alpha/2$ continuous.*

**Proof.** Note that $|f|_\alpha < \infty$ clearly implies that $f$ is continuous on $(0,\infty)^d$. Now take $y>x>0$ in the above 1-dimensional argument and then $y,x\in(0,\infty)^d$ in the above $d$-dimensional proof. (Note that the proof of Proposition 2.1 did not use the boundedness of $f$.) This gives the uniform $\alpha/2$ Hölder continuity on $(0,\infty)^d$ and the result follows. □

Let

$$\mathcal{L}^{\gamma,b}f(x) = \gamma x f''(x) + bf'(x), \qquad \gamma>0, b\geq 0$$

be the generator of $X_t^{\gamma,b}$. More precisely, $X^{\gamma,b}\geq 0$ is the process such that

$$M_t^f = f(X_t^{\gamma,b}) - f(X_0^{\gamma,b}) - \int_0^t \mathcal{L}^{\gamma,b} f(X_s^{\gamma,b})ds$$

is a $\sigma(X_s^{\gamma,b} : s \le t)$-martingale for all $f \in C_b^2(\mathbb{R}_+^d)$ and $X_0^{\gamma,b} = x$ is given. As is well known, the law of $X^{\gamma,b}$ is uniquely determined. We will need to work with the well known series expansions for the transition densities of $X^{\gamma,b}$; see, e.g., Gradinaru, Roynette, Vallois, and Yor [GRVY99].

If $b > 0$,

$$\mathbb{P}^x(X_t^{\gamma,b} \in dy) = q_t^{\gamma,b}(x,y)dy,$$

where

$$q_t^{\gamma,b}(x,y) = (\gamma t)^{-\frac{b}{\gamma}} y^{\frac{b}{\gamma}-1} \exp\Big\{-\Big(\frac{x+y}{\gamma t}\Big)\Big\}\Big[\sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\frac{b}{\gamma})} \frac{x^m y^m}{\gamma^{2m} t^{2m}}\Big]. \tag{2.2}$$

If $b = 0$,

$$\mathbb{P}^x(X_t^{\gamma,b} \in dy) = e^{-x/\gamma t}\delta_0(dy) + q_t^{\gamma,0}(x,y)dy = q_t^{\gamma,0}(x,dy), \tag{2.3}$$

where

$$q_t^{\gamma,0}(x,y) = (\gamma t)^{-1} \exp\Big\{-\Big(\frac{x+y}{\gamma t}\Big)\Big\} \sum_{m=0}^{\infty} \frac{1}{(m+1)!}\Big(\frac{x}{\gamma t}\Big)^{m+1} \frac{1}{m!}\Big(\frac{y}{\gamma t}\Big)^m \tag{2.4}$$

and $\delta_0$ is point mass at 0.

## 3. Some basic inequalities.

**Lemma 3.1.** *Let $S_k = \sum_{i=1}^k Z_i$ where the $Z_i$ are i.i.d. exponential random variables with parameter 1. Then for all $p > 0$ there exists $c_p \in (0,\infty)$ such that*

$$\mathbb{E}\Big(\Big|\frac{S_k - k}{\sqrt{k}}\Big|^p\Big) \le c_p, \qquad k \ge 1.$$

**Proof.** Let $a = \frac{1}{2}$. Since $\mathbb{E}\, e^{aZ_i} = 1/(1-a)$, then

$$\mathbb{E} \exp\Big(a((S_k - k)/\sqrt{k})\Big) = \Big(\frac{e^{-a/\sqrt{k}}}{(1-\frac{a}{\sqrt{k}})}\Big)^k,$$

which is bounded by a constant not depending on $k$. Performing a similar calculation with $a = -\frac{1}{2}$ and combining, we conclude

$$\sup_k \mathbb{E} \exp(|S_k - k|/(2\sqrt{k})) < \infty.$$

The conclusion is now immediate. $\square$

**Lemma 3.2.** *(a) For each $p > 0$ there exists a constant $c_p$ such that if $r > 0$, then*

$$\int |z-r|^p \frac{z^{r-1}}{\Gamma(r)} e^{-z} dz \le c_p (r^{p/2}+1).$$

*(b) For each $p > 0$ there exists a constant $c_p$ such that if $r > 0$ and $s \in (0, r\wedge 1)$, then*

$$\int |z-r|^p \frac{z^{r-1}}{\Gamma(r)} z^{-s} e^{-z} dz \le c_p \frac{\Gamma(r-s)}{\Gamma(r)} (r^{p/2}+1).$$

**Proof.** (a) Let $X_r$ be a gamma random variable with parameters $r$ and 1, let $k = [r]$, and let $S_k$ be a gamma random variable with parameters $k$ and 1 as in Lemma 3.1. Let $v = r - k$ and $Y_v$ a gamma random variable with parameters $v$ and 1 (with $Y_0 = 0$) and independent of $S_k$. Then $X_r$ is equal in law to $S_k + Y_v$. We have

$$\int |z-r|^p \frac{z^{r-1}}{\Gamma(r)} e^{-z} dz = \mathbb{E}\left(|S_k + Y_v - r|^p\right).$$

A simple application of Jensen's inequality allows us to reduce the result to the case where $p$ is a positive integer. The right hand side of the above equation is bounded by

$$c_p(\mathbb{E}\left(|S_k - k|^p\right) + \mathbb{E}\left((Y_v)^p\right) + |k-r|^p) \le c_p\Big(k^{p/2} + \frac{\Gamma(p+v)}{\Gamma(v)} + 1\Big) \le c_p(r^{p/2}+1).$$

In the first inequality we used Lemma 3.1 and in the last inequality we used $\Gamma(p+v) = (p+v-1)\cdots v\Gamma(v) \le p!\Gamma(v)$.

(b) This is immediate from (a) by replacing $r$ with $r-s$. □

Recall

$$\int_0^\infty e^{-z} \frac{z^k}{k!} dz = 1. \tag{3.1}$$

**Lemma 3.3.** *Let $w, r \ge 0$.*

*(a)*

$$\sum_{k=1}^\infty e^{-w} \frac{w^k}{k!} k^{-r} \le c_r (1 \wedge w^{-r}). \tag{3.2}$$

*(b)*

$$\sum_{k=1}^\infty e^{-w} \frac{w^k}{k!} k^{-r} |w-k| \le c_r (w \wedge w^{\frac{1}{2}-r}). \tag{3.3}$$

**Proof.** Let $N$ be a Poisson random variable with parameter $w$. The statement of the lemma is then equivalent to

$$\mathbb{E}\left[N^{-r} 1_{(N\ge 1)}\right] \le c_r (1 \wedge w^{-r}) \tag{3.4}$$

and

$$\mathbb{E}\,[N^{-r}1_{(N\geq 1)}|N-w|] \leq c_r(w^{\frac{1}{2}-r}\wedge w). \tag{3.5}$$

If $N \geq 1$, then $N^{-r} \leq 1$, so clearly the left hand side of (3.4) is bounded by 1. To get the bound of $c_r w^{-r}$, by Jensen's inequality it suffices to prove this bound with $r$ a non-negative integer. But

$$\mathbb{E}\,[N^{-r}1_{(N\geq 1)}] = \sum_{k=1}^{\infty} e^{-w}\frac{w^k}{k!}\frac{1}{k^r} \leq c_r e^{-w}w^{-r}\sum_{k=1}^{\infty}\frac{w^{k+r}}{(k+r)!} \leq c_r w^{-r}.$$

To prove (3.5), on the one hand we have from (3.4),

$$\begin{aligned}\mathbb{E}\,[N^{-r}1_{(N\geq 1)}|N-w|] &\leq \Big(\mathbb{E}\,|N-w|^2\Big)^{\frac{1}{2}}\Big(\mathbb{E}\,[N^{-2r}1_{(N\geq 1)}]\Big)^{\frac{1}{2}}\\ &\leq cw^{\frac{1}{2}}w^{-r}\end{aligned}$$

and on the other hand,

$$\begin{aligned}\mathbb{E}\,[N^{-r}1_{(N\geq 1)}|N-w|] &\leq \mathbb{E}\,[N^{1-r}1_{(N\geq 1)}] + w\mathbb{E}\,[N^{-r}1_{(N\geq 1)}]\\ &\leq \mathbb{E}\,N + w = 2w.\end{aligned}$$

□

**4. One-dimensional semigroups.**

In this section we take $d = 1$. Assume first $f \in C_b(R_+)$. We consider first $b = 0$ and write $P_t f(x) = \mathbb{E}^{\,x} f(X_t^{\gamma,b})$. When differentiating $P_t f$ at $x = 0$, it is understood we are always taking right hand derivatives.

**Lemma 4.1.** *Let $f \in C_b(\mathbb{R}^d_+)$. (a) We have*

$$\begin{aligned}(P_t f)'(x) &= e^{-x/\gamma t}\int_0^{\infty}[f(z\gamma t)-f(0)]e^{-z}\frac{dz}{\gamma t}\\ &\quad + e^{-x/\gamma t}\sum_{k=1}^{\infty}\Big(\frac{x}{\gamma t}\Big)^k\frac{1}{k!}\int_0^{\infty}f(z\gamma t)e^{-z}\Big[\frac{z^k}{k!}-\frac{z^{k-1}}{(k-1)!}\Big]\frac{dz}{\gamma t}.\end{aligned}$$

*The series converges uniformly in $x$ on compacts in $[0,\infty)$ for all $t > 0$.*

*(b) We have*

$$\begin{aligned}(P_t f)''(x) &= \frac{1}{\gamma t}e^{-x/\gamma t}\int_0^{\infty}[f(z\gamma t)-f(0)]e^{-z}(z-2)\frac{dz}{\gamma t}\\ &\quad + \frac{1}{\gamma t}\sum_{k=1}^{\infty}e^{-x/\gamma t}\Big(\frac{x}{\gamma t}\Big)^k\frac{1}{k!}\int_0^{\infty}f(z\gamma t)e^{-z}\Big[\frac{z^{k+1}}{(k+1)!}-2\frac{z^k}{k!}+\frac{z^{k-1}}{(k-1)!}\Big]\frac{dz}{\gamma t}.\end{aligned}$$

*The series converges uniformly in $x$ on compacts in $[0,\infty)$ for all $t>0$. In particular, $(P_tf)'$ and $(P_tf)''$ exist for all $x \ge 0$ and are continuous.*

**Proof.** If $\widehat{f}(x)=f(x)-f(0)$, then $P_t\widehat{f}(x)=P_tf(x)-f(0)$ and so it follows that the left hand sides in (a) and (b) are the same for $f$ and $\widehat{f}$. From (3.1) we see that the right hand sides are also the same for $f$ and $\widehat{f}$. Hence we may assume without loss of generality that $f(0)=0$. By Fubini and the substitution $z=y/(\gamma t)$,

$$P_tf(x)=e^{-x/\gamma t}\sum_{m=0}^{\infty}\frac{1}{(m+1)!}\Big(\frac{x}{\gamma t}\Big)^{m+1}\frac{1}{m!}\int_0^\infty f(z\gamma t)z^m e^{-z}dz.$$

It is easy to differentiate through the summation (e.g., the resulting series is uniformly and absolutely convergent on compacts as $f$ is bounded) and so this gives (a). A second differentiation gives (b). □

We use the following notation. If $f\in C_b(\mathbb{R}_+)$ let

$$F_k(t)=\int_0^\infty f(z\gamma t)e^{-z}\frac{z^k}{k!}\frac{dz}{\gamma t} \tag{4.1}$$

and

$$G_k(w)=e^{-w}w^{k+1}/k!, \qquad H_k(w)=e^{-w}w^k/k!. \tag{4.2}$$

For $k\ge 1$ set

$$J_k(t)=\int_0^\infty [f(z\gamma t)-f(k\gamma t)]e^{-z}\frac{z^{k-1}}{k!}[z-k]\frac{dz}{\gamma t} \tag{4.3}$$

and set

$$J_0(t)=\int_0^\infty [f(z\gamma t)-f(0)]e^{-z}\frac{dz}{\gamma t}. \tag{4.4}$$

**Lemma 4.2.** *(a) If $f\in\mathcal{C}_w^\alpha(\mathbb{R}_+)$, then*

$$|J_0(t)|\le c_\alpha|f|_\alpha(\gamma t)^{\frac{\alpha}{2}-1}.$$

*(b) For all $k\ge 1$*

$$|J_k(t)|\le c_\alpha|f|_\alpha(\gamma t)^{\frac{\alpha}{2}-1}k^{-\frac{1}{2}}.$$

**Proof.** (a) Using Proposition 2.1

$$|J_0(t)|\le c_\alpha|f|_\alpha(\gamma t)^{\frac{\alpha}{2}-1}\int_0^\infty z^{\frac{\alpha}{2}}e^{-z}dz,$$

and (a) follows immediately.

(b) We have

$$|J_k(t)| \le \int_0^\infty \Big|f(z\gamma t) - f(k\gamma t)\Big| e^{-z}\Big|\frac{z^k}{k!} - \frac{z^{k-1}}{(k-1)!}\Big|\frac{dz}{\gamma t}.$$

Since $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+)$, this is less than or equal to

$$c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}\int_0^\infty |z-k|^{\alpha+1}\frac{z^{k-1}e^{-z}}{k!}[z^{-\frac{\alpha}{2}} + k^{-\frac{\alpha}{2}}]dz.$$

Stirling's formula shows that $\Gamma(k-\frac{\alpha}{2})/\Gamma(k) \le c_\alpha k^{-\frac{\alpha}{2}}$. This and Lemma 3.2 gives (b). □

**Lemma 4.3.** *(a) There exists $c_\alpha > 0$ such that for all $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+)$*

*(i)*

$$|(P_tf)'(x)| \le \frac{c_\alpha}{2}|f|_\alpha(\gamma t)^{\frac{\alpha}{2}-1}\Big(\Big(\frac{x}{\gamma t}\Big)^{-\frac{1}{2}}\wedge 1\Big) \le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-\frac{1}{2}}(x+\gamma t)^{-\frac{1}{2}}.$$

*(ii)*

$$|x(P_tf)''(x)| \le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}\Big(\frac{x}{\gamma t}\wedge 1\Big).$$

*(b) Suppose $f \in C_b(\mathbb{R}_+)$. There exists $c_0 > 0$ such that*

*(i)*

$$|(P_tf)'(x)| \le \frac{c_0}{2}\frac{\|f\|_\infty}{\gamma t}\Big(\Big(\frac{x}{\gamma t}\Big)^{-\frac{1}{2}}\wedge 1\Big) \le c_0\|f\|_\infty(\gamma t)^{-\frac{1}{2}}\Big(x+\gamma t\Big)^{-\frac{1}{2}}.$$

*(ii)*

$$|x(P_tf)''(x)| \le c_0\frac{\|f\|_\infty}{\gamma t}\Big(\frac{x}{\gamma t}\wedge 1\Big).$$

**Proof.** (a)(i) From Lemma 4.1 and (3.1),

$$(P_tf)'(x) = e^{-x/\gamma t}J_0(t) + e^{-x/\gamma t}\sum_{k=1}^\infty \Big(\frac{x}{\gamma t}\Big)^k\frac{1}{k!}J_k(t).$$

By Lemma 4.2,

$$|(P_tf)'(x)| \le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}e^{-x/\gamma t}\Big[1 + \sum_{k=1}^\infty\Big(\frac{x}{\gamma t}\Big)^k\frac{1}{k!}k^{-\frac{1}{2}}\Big]. \tag{4.5}$$

Using Lemma 3.3(a) and the elementary inequality $e^{-w} \le (1\wedge w^{-\frac{1}{2}})$ for $w \ge 0$, we obtain (i).

(ii) We may assume without loss of generality that $f(0)=0$. Then from Lemma 4.1(b), setting $w=x/\gamma t$, we have

$$x(P_tf)''(x)=\sum_{k=1}^\infty G_k(w)[F_{k-1}(t)-2F_k(t)+F_{k+1}(t)]+G_0(w)[F_1(t)-2F_0(t)].$$

Summing by parts, we see that

$$\begin{aligned} x(P_tf)''(x) &\\ &=\sum_{k=1}^\infty [G_k(w)-G_{k-1}(w)][F_{k-1}(t)-F_k(t)]-G_0(w)F_0(t) \qquad (4.6)\\ &=\sum_{k=1}^\infty \frac{e^{-w}}{k!}w^k[w-k]\int_0^\infty f(z\gamma t)e^{-z}\frac{z^{k-1}}{k!}(k-z)\frac{dz}{\gamma t}-e^{-w}w\int_0^\infty f(z\gamma t)e^{-z}\frac{dz}{\gamma t}. \end{aligned}$$

By (3.1) and the fact that $f(0)=0$, this in turn is equal to

$$-\sum_{k=1}^\infty e^{-w}\frac{w^k}{k!}[w-k]J_k(t)-we^{-w}J_0(t).$$

To prove (ii) we bound $|J_k(t)|$ and $|J_0(t)|$ by using Lemma 4.2, and then use Lemma 3.3(b) and the elementary bound $we^{-w}\le w\wedge 1$ for $w\ge 0$.

(b) (i) By Lemma 4.1(a)

$$|(P_tf)'(x)|\le \frac{e^{-x/\gamma t}}{\gamma t}2\|f\|_\infty\int_0^\infty e^{-z}dz+\frac{e^{-x/\gamma t}\|f\|_\infty}{\gamma t}\sum_{k=1}^\infty\Big(\frac{x}{\gamma t}\Big)^k\frac{1}{k!}\int_0^\infty e^{-z}\frac{z^{k-1}}{k!}|z-k|dz.$$

Using Lemma 3.2 this is bounded by

$$c_0\frac{e^{-x/\gamma t}\|f\|_\infty}{\gamma t}\Big[1+\sum_{k=1}^\infty\Big(\frac{x}{\gamma t}\Big)^k\frac{1}{k!}k^{-\frac12}\Big],$$

and (i) follows from an application of Lemma 3.3(a) and the fact that $e^{-w}\le w^{-1/2}\wedge 1$ for $w\ge 0$.

(ii) We may assume without loss of generality that $f(0)=0$ (or else consider $f-f(0)$). By (4.6) and Lemma 3.2, if $w=x/\gamma t$,

$$\begin{aligned} |x(P_tf)''(x)| &\le \frac{\|f\|_\infty}{\gamma t}\sum_{k=1}^\infty e^{-w}w^k\frac{|w-k|}{k!}\int_0^\infty e^{-z}\frac{z^{k-1}}{k!}|z-k|dz+\frac{\|f\|_\infty}{\gamma t}e^{-w}w\\ &\le c_0\frac{\|f\|_\infty}{\gamma t}\Big[\sum_{k=1}^\infty e^{-w}w^k\frac{|w-k|}{k!}k^{-\frac12}+we^{-w}\Big]. \end{aligned}$$

We now apply Lemma 3.3(b). $\square$

**Lemma 4.4.** *(a) There exists $c_\alpha > 0$ such that for all $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+)$ and for all $x, \Delta \geq 0$ we have*

$$\Big|(P_t f)'(x+\Delta) - (P_t f)'(x)\Big| \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-\frac{3}{2}} (x+\gamma t)^{-\frac{1}{2}} \Delta.$$

*(b) There exists $c_\alpha > 0$ such that for $f, x, \Delta$ as above*

$$|(x+\Delta)(P_t f)''(x+\Delta) - x(P_t f)''(x)| \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-\frac{3}{2}} (x+\gamma t)^{-\frac{1}{2}} \Delta.$$

**Proof.** (a) From Lemma 4.1(a) (recall the definition of $H_k$ in (4.2)) and (3.1),

$$|(P_t f)'(x+\Delta) - (P_t f)'(x)| \leq \sum_{k=0}^{\infty} \Big| H_k\Big(\frac{x+\Delta}{\gamma t}\Big) - H_k\Big(\frac{x}{\gamma t}\Big)\Big| \, |J_k(t)|.$$

Since $H_k'(w) = e^{-w} w^{k-1}(k-w)/k!$ for $k \geq 1$ and $H_0'(w) = -e^{-w}$, we get from Lemma 4.2 that

$$\begin{aligned}
&|(P_t f)'(x+\Delta) - (P_t f)'(x)| \\
&\qquad \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1} e^{-x/\gamma t} \Big(\frac{\Delta}{\gamma t}\Big) \\
&\qquad\quad + c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1} \sum_{k=1}^{\infty} \Big| \int_{x/\gamma t}^{(x+\Delta)/\gamma t} y^{-1} e^{-y} y^k \frac{k-y}{k!} dy \Big| k^{-\frac{1}{2}}.
\end{aligned}$$

By Lemma 3.3(b)

$$\begin{aligned}
&|(P_t f)'(x+\Delta) - (P_t f)'(x)| \\
&\qquad \leq c_\alpha |f|_\alpha \Delta (\gamma t)^{\frac{\alpha}{2}-2} e^{-x/\gamma t} + c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1} \int_{x/\gamma t}^{(x+\Delta)/\gamma t} \frac{(y \wedge 1)}{y} dy.
\end{aligned}$$

Note $(y \wedge 1)/y \leq c(y+1)^{-\frac{1}{2}}$ for $y \geq 0$ and so

$$\int_{x/\gamma t}^{(x+\Delta)/\gamma t} \frac{(y \wedge 1)}{y} dy \leq c \int_{x/\gamma t}^{(x+\Delta)/\gamma t} (y+1)^{-\frac{1}{2}} dy \leq c \frac{\Delta}{\gamma t} \Big(\frac{x}{\gamma t} + 1\Big)^{-\frac{1}{2}}.$$

Use this and the elementary inequality $e^{-x/\gamma t} \leq c(\gamma t/(x+\gamma t))^{\frac{1}{2}}$ to derive (a).

(b) Lemma 4.1(b) and (3.1) imply

$$\begin{aligned}
x(P_t f)''(x) &= H_1\Big(\frac{x}{\gamma t}\Big) \int_0^\infty (f(z\gamma t) - f(0)) e^{-z} (z-2) \frac{dz}{\gamma t} \\
&+ \sum_{k=1}^{\infty} H_{k+1}\Big(\frac{x}{\gamma t}\Big)(k+1) \int_0^\infty (f(z\gamma t) - f(k\gamma t)) e^{-z} \Big[\frac{z^{k+1}}{(k+1)!} - \frac{2z^k}{k!} + \frac{z^{k-1}}{(k-1)!}\Big] \frac{dz}{\gamma t}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&|(x+\Delta)(P_tf)''(x+\Delta)-x(P_tf)''(x)| \qquad (4.7)\\
&\qquad \le \Big|H_1\Big(\frac{x+\Delta}{\gamma t}\Big)-H_1\Big(\frac{x}{\gamma t}\Big)\Big|\Big|\int_0^\infty (f(z\gamma t)-f(0))e^{-z}(z-2)\frac{dz}{\gamma t}\Big|\\
&\qquad\quad +\sum_{k=1}^\infty\Big|H_{k+1}\Big(\frac{x+\Delta}{\gamma t}\Big)-H_{k+1}\Big(\frac{x}{\gamma t}\Big)\Big|\,|K_k(t)|,
\end{aligned}
$$

where

$$
K_k(t)=\int_0^\infty (f(z\gamma t)-f(k\gamma t))e^{-z}\frac{z^{k-1}}{(k-1)!}\frac{[(z-k)^2+k-2z]}{k}\frac{dz}{\gamma t}.
$$

Note that $H_1'(w)=e^{-w}(1-w)$, so by (2.1) we have

$$
\begin{aligned}
&\Big|H_1\Big(\frac{x+\Delta}{\gamma t}\Big)-H_1\Big(\frac{x}{\gamma t}\Big)\Big|\Big|\int_0^\infty (f(z\gamma t)-f(0))e^{-z}(z-2)\frac{dz}{\gamma t}\Big| \qquad (4.8)\\
&\qquad \le c_\alpha|f|_\alpha(\gamma t)^{\frac{\alpha}{2}-1}\int_0^\infty z^{\alpha/2}e^{-z}|z-2|dz\Big|\int_{x/\gamma t}^{(x+\Delta)/\gamma t} e^{-w}(1-w)dw\Big|\\
&\qquad \le c_\alpha|f|_\alpha(\gamma t)^{\frac{\alpha}{2}-1}\Big|\int_{x/\gamma t}^{(x+\Delta)/\gamma t} e^{-w}(1-w)dw\Big|.
\end{aligned}
$$

We also have for $k\ge 1$,

$$
|K_k(t)|\le c_\alpha|f|_\alpha(\gamma t)^{\frac{\alpha}{2}-1}\int_0^\infty [z^{-\alpha/2}+k^{-\alpha/2}]|z-k|^\alpha\frac{(z-k)^2+k+2z}{k}\frac{z^{k-1}}{(k-1)!}e^{-z}dz.
$$

Lemma 3.2 and a short calculation gives

$$
|K_k(t)|\le c_\alpha|f|_\alpha(\gamma t)^{\frac{\alpha}{2}-1}. \qquad (4.9)
$$

Use (4.8), (4.9), and $H_k'(w)=e^{-w}w^{k-1}(k-w)/k!$ in (4.7) to get

$$
\begin{aligned}
&|(x+\Delta)(P_tf)''(x+\Delta)-x(P_tf)''(x)| \qquad (4.10)\\
&\qquad \le c_\alpha|f|_\alpha(\gamma t)^{\frac{\alpha}{2}-1}\sum_{k=1}^\infty\Big|\int_{x/\gamma t}^{(x+\Delta)/\gamma t} w^{-1}e^{-w}\frac{w^k}{k!}(k-w)dw\Big|.\\
&\qquad \le c_\alpha|f|_\alpha(\gamma t)^{\frac{\alpha}{2}-1}\int_{x/\gamma t}^{(x+\Delta)/\gamma t}(w+1)^{-\frac12}dw\le c_\alpha|f|_\alpha(\gamma t)^{\frac{\alpha}{2}-\frac32}(x+\gamma t)^{-\frac12}\Delta,
\end{aligned}
$$

where we also use Lemma 3.3(b) in the last line as well as the fact that $1\wedge w^{-1/2}\le\frac{c}{\sqrt{1+w}}$.
□

We now turn to the case where $b>0$. We write $P_t^bf(x)=\mathbb{E}^xf(X_t^{\gamma,b})$.

The proof of the following lemma is nearly identical to that of Lemma 4.1 and is therefore omitted.

**Lemma 4.5.** *(a) If $f \in C_b(\mathbb{R}_+)$, then*

$$(P_t^b f)'(x) = \sum_{k=0}^{\infty} e^{-x/\gamma t}\Big(\frac{x}{\gamma t}\Big)^k \frac{1}{k!}\int_0^\infty f(z\gamma t)e^{-z}\Big(\frac{z^{k+\frac{b}{\gamma}}}{\Gamma(k+\frac{b}{\gamma}+1)} - \frac{z^{k+\frac{b}{\gamma}-1}}{\Gamma(k+\frac{b}{\gamma})}\Big)\frac{dz}{\gamma t},$$

*and the above series converges uniformly in $x$ on compacts in $[0,\infty)$ for all $t>0$.*

*(b) We have*

$$(P_t^b f)''(x) = \frac{1}{\gamma t}\sum_{k=1}^{\infty} e^{-x/\gamma t}\Big(\frac{x}{\gamma t}\Big)^{k-1}\frac{1}{k!}$$
$$\times k\int_0^\infty f(z\gamma t)e^{-z}\Big[\frac{z^{k+\frac{b}{\gamma}}}{\Gamma(k+\frac{b}{\gamma}+1)} - 2\frac{z^{k+\frac{b}{\gamma}-1}}{\Gamma(k+\frac{b}{\gamma})} + \frac{z^{k+\frac{b}{\gamma}-2}}{\Gamma(k+\frac{b}{\gamma}-1)}\Big]\frac{dz}{\gamma t},$$

*and the above series converges uniformly in $x$ on compacts in $[0,\infty)$ for all $t>0$.*

Note that this implies that $(P_t^b f)''$ exists and is continuous for $x \geq 0$, with the understanding that we take right hand derivatives at $x=0$.

All constants are independent of $\gamma$ and $b$ unless noted otherwise.

**Lemma 4.6.** *(a) There exists $c_\alpha > 0$ such that for all $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+)$*

*(i)*

$$|(P_t^b f)'(x)| \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}\Big(\Big(\frac{x}{\gamma t}\Big)^{-\frac{1}{2}} \wedge 1\Big) \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-\frac{1}{2}}(x+\gamma t)^{-\frac{1}{2}}.$$

*(ii)*

$$|x(P_t^b f)''(x)| \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}\Big(\Big(\frac{x}{\gamma t}\Big) \wedge 1\Big).$$

*(b) There exists $c_0 > 0$ such that for all $f \in C_b(\mathbb{R}_+)$,*

*(i)*

$$|(P_t^b f)'(x)| \leq \frac{c_0}{2}\|f\|_\infty \frac{1}{\gamma t}\Big(\Big(\frac{x}{\gamma t}\Big)^{-\frac{1}{2}} \wedge 1\Big) \leq c_0\|f\|_\infty (\gamma t)^{-\frac{1}{2}}(x+\gamma t)^{-\frac{1}{2}}.$$

*(ii)*

$$|x(P_t^b f)''(x)| \leq c_0 \frac{\|f\|_\infty}{\gamma t}\Big(\Big(\frac{x}{\gamma t}\Big) \wedge 1\Big).$$

**Proof.** (a)(i) For $k \geq 0$ let

$$J_k^b(t) = \int_0^\infty (f(z\gamma t) - f((k+\tfrac{b}{\gamma})\gamma t))e^{-z}\Big(\frac{z^{k+\frac{b}{\gamma}}}{\Gamma(k+\frac{b}{\gamma}+1)} - \frac{z^{k+\frac{b}{\gamma}-1}}{\Gamma(k+\frac{b}{\gamma})}\Big)\frac{dz}{\gamma t}. \tag{4.11}$$

By making minor changes in the proof of Lemma 4.2(b) we have for $k \geq 1$

$$|J_k^b(t)| \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}[k+\tfrac{b}{\gamma}]^{-\frac{1}{2}} \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}k^{-\frac{1}{2}}. \tag{4.12}$$

For the case $k=0$, by (2.1)

$$|J_0^b(t)| \le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1} \int_0^\infty |z-\tfrac{b}{\gamma}|^{1+\frac{\alpha}{2}} e^{-z} \frac{z^{\frac{b}{\gamma}-1}}{\Gamma(\frac{b}{\gamma}+1)} dz.$$

For $b/\gamma$ large we use Lemma 3.2(a) to bound the integral in the above by

$$c_\alpha \frac{1}{b/\gamma} \big( (\tfrac{b}{\gamma})^{(1+\frac{\alpha}{2})/2} + 1 \big) \le c_\alpha.$$

For $b/\gamma$ small we bound the integral by

$$\begin{aligned} c_\alpha \int_0^\infty e^{-z} \frac{z^{\frac{b}{\gamma}+\frac{\alpha}{2}}}{\Gamma(\frac{b}{\gamma}+1)} dz + c_\alpha (\tfrac{b}{\gamma})^{1+\frac{\alpha}{2}} \int_0^\infty e^{-z} \frac{z^{\frac{b}{\gamma}-1}}{\Gamma(\frac{b}{\gamma}+1)} dz \\ \le c_\alpha \frac{\Gamma(\frac{b}{\gamma}+\frac{\alpha}{2}+1)}{\Gamma(\frac{b}{\gamma}+1)} + c_\alpha (\tfrac{b}{\gamma})^{1+\frac{\alpha}{2}} \frac{\Gamma(\frac{b}{\gamma})}{\Gamma(\frac{b}{\gamma}+1)} \le c_\alpha. \end{aligned}$$

Thus

$$|J_0^b(t)| \le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}. \tag{4.13}$$

Lemma 4.5(a) implies

$$(P_t^b f)'(x) = \sum_{k=0}^\infty e^{-x/\gamma t} \frac{(x/\gamma t)^k}{k!} J_k^b(t).$$

Use (4.12) and (4.13) to bound $|(P_t^b f)'(x)|$ by the right side of (4.5) and the result follows as in the proof of Lemma 4.3(a).
(ii) Using Lemma 4.5(b) and summing by parts as in the proof of Lemma 4.3(a)(ii), with $w = x/\gamma t$, we derive

$$\begin{aligned} |x(P_t^b f)''(x)| &\le \sum_{k=1}^\infty e^{-w} \Big| \frac{w^k}{(k-1)!} - \frac{w^{k+1}}{k!} \Big| |J_k^b(t)| + e^{-w} w |J_0^b(t)| \\ &\le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1} \Big[ \sum_{k=1}^\infty e^{-w} \frac{w^k}{k!} |w-k| k^{-\frac{1}{2}} + e^{-w} w \Big]. \end{aligned} \tag{4.14}$$

Here (4.12) and (4.13) have been used in the last line. (a)(ii) now follows from an application of Lemma 3.3(b).

(b)(i) If $k \ge 1$, then by Lemma 3.2

$$\begin{aligned} |J_k^b(t)| &\le \frac{\|f\|_\infty}{\gamma t} \int_0^\infty e^{-z} \frac{z^{k-1+\frac{b}{\gamma}}}{\Gamma(k+\frac{b}{\gamma}+1)} |z-(k+\tfrac{b}{\gamma})| dz \\ &\le c \frac{\|f\|_\infty}{\gamma t} \frac{(k+\frac{b}{\gamma})^{\frac{1}{2}}+1}{k+\frac{b}{\gamma}} \le c \frac{\|f\|_\infty}{\gamma t} k^{-\frac{1}{2}}. \end{aligned} \tag{4.15}$$

Also

$$|J_0^b(t)| \le \frac{\|f\|_\infty}{\gamma t}\int_0^\infty e^{-z}\Big|\frac{z^{\frac{b}{\gamma}}}{\Gamma(\frac{b}{\gamma}+1)} - \frac{z^{\frac{b}{\gamma}-1}}{\Gamma(\frac{b}{\gamma})}\Big|dz \le 2\frac{\|f\|_\infty}{\gamma t}. \tag{4.16}$$

(b)(i) now follows from Lemma 4.5(a) and Lemma 3.3(a).
(ii) Use (4.14), (4.15), (4.16) and Lemma 3.3(b) just as in the proof of Lemma 4.3(b)(ii).
□

**Lemma 4.7.** *(a) There exists $c_\alpha > 0$ such that for all $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+)$ and all $x, \Delta \ge 0$*

$$|(P_t^b f)'(x+\Delta) - (P_t^b f)'(x)| \le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-\frac{3}{2}}(x+\gamma t)^{-\frac{1}{2}}\Delta.$$

*(b) There exists $c_\alpha > 0$ such that for all $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+)$ and all $x, \Delta \ge 0$*

$$|(x+\Delta)(P_t^b f)''(x+\Delta) - x(P_t^b f)''(x)| \le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-\frac{3}{2}}(x+\gamma t)^{-\frac{1}{2}}\Delta.$$

**Proof.** (a) This argument is very similar to that given for $b = 0$ in Lemma 4.4(a).
(b) Lemma 4.5(b) shows that if $D_k^b(t) = k(J_k^b(t) - J_{k-1}^b(t))$ for $k \ge 1$, then

$$|(x+\Delta)(P_t^b f)''(x+\Delta) - x(P_t^b f)''(x)| \le \sum_{k=1}^\infty \Big| H_k\Big(\frac{x+\Delta}{\gamma t}\Big) - H_k\Big(\frac{x}{\gamma t}\Big)\Big| |D_k^b(t)|.$$

It suffices to prove the analogue of (4.9) for $b > 0$, that is,

$$|D_k^b(t)| \le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}. \tag{4.17}$$

The result then would follow just as in the proof of Lemma 4.5(b). For $k \ge 2$, (4.17) is easily proved by making the obvious modifications in the proof of (4.9) in Lemma 4.4(b). We give a derivation of the $k = 1$ case of (4.17) because some additional care is needed to ensure that the constant $c_\alpha$ remains bounded as $b$ approaches 0 (recall our constants do not depend on $b$ and $\gamma$). Use (3.1) and Proposition 2.1 to see that

$$\begin{aligned}
|D_1^b(t)| &= \Big|\int_0^\infty (f(z\gamma t) - f(0))e^{-z}\Big[\frac{z^{1+\frac{b}{\gamma}}}{\Gamma(2+\frac{b}{\gamma})} - \frac{2z^{\frac{b}{\gamma}}}{\Gamma(1+\frac{b}{\gamma})} + \frac{z^{\frac{b}{\gamma}-1}}{\Gamma(\frac{b}{\gamma})}\Big]\frac{dz}{\gamma t}\Big| \\
&\le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}\int_0^\infty \frac{e^{-z}z^{\frac{\alpha}{2}+\frac{b}{\gamma}-1}}{\Gamma(\frac{b}{\gamma}+2)}|(z-(1+\tfrac{b}{\gamma}))^2 - (1+\tfrac{b}{\gamma})|dz \\
&\le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}\Big[\frac{\Gamma(1+\frac{b}{\gamma}+\frac{\alpha}{2}-1)}{\Gamma(\frac{b}{\gamma}+2)}((1+\tfrac{b}{\gamma})+1) \\
&\qquad + (1+\tfrac{b}{\gamma})\tfrac{\Gamma(\frac{\alpha}{2}+\frac{b}{\gamma})}{\Gamma(\frac{b}{\gamma}+2)}\Big] \\
&\le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}\Big[\Big(\Big(\frac{2+\frac{b}{\gamma}}{1+\frac{b}{\gamma}}\Big)+1\Big)\frac{\Gamma(\frac{b}{\gamma}+\frac{\alpha}{2})}{\Gamma(\frac{b}{\gamma}+1)}\Big].
\end{aligned}$$

Lemma 3.2(b) is used in the next to last inequality. (4.17) follows for the $k = 1$ case because the expression in square brackets in the last line is uniformly bounded as a function of $b/\gamma \geq 0$ for each fixed $\alpha \in (0, 1]$. $\square$

**Lemma 4.8.** *Let $b \geq 0$. There exists $c_\alpha > 0$ such that if $t \geq 0$, $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+)$, and $x, \Delta \geq 0$, then*

$$|P_t^b f(x+\Delta) - P_t^b f(x)| \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-\frac{1}{2}} \Delta (x+\Delta+\gamma t)^{-1/2}.$$

**Proof.** From Lemma 4.6(a) if $b > 0$ and Lemma 4.3(a)(i) if $b = 0$, and the fundamental theorem of calculus,

$$\begin{aligned} &|P_t^b f(x+\Delta) - P_t^b f(x)| \\ &\qquad \leq \int_x^{x+\Delta} c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-\frac{1}{2}} (y+\gamma t)^{-1/2} dy \\ &\qquad \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-\frac{1}{2}} (x+\Delta+\gamma t)^{-1/2} \Delta. \end{aligned}$$

$\square$

**Lemma 4.9.** *For all $p \in (0, 1]$ there exists $c_{\alpha,p} > 0$ such that if $b \geq 0$, $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+)$, $t > 0$, and $x, \Delta \geq 0$, then*

$$|(x+\Delta)^p (P_t^b f)'(x+\Delta) - x^p (P_t^b f)'(x)| \leq c_{\alpha,p} |f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1} (x+\Delta)^{p-1} \Delta.$$

**Proof.** First let $b > 0$. For $k \geq 0$ let

$$L_k^b(t) = \int_0^\infty f(z\gamma t) e^{-z} \Big[\frac{z^{k+\frac{b}{\gamma}}}{\Gamma(k+\frac{b}{\gamma}+1)} - \frac{z^{k+\frac{b}{\gamma}-1}}{\Gamma(k+\frac{b}{\gamma})}\Big] (\gamma t)^{p-1} dz.$$

Lemma 4.5(a) implies

$$x^p (P_t^b f)'(x) = \sum_{k=0}^\infty e^{-x/\gamma t} \frac{(x/\gamma t)^{k+p}}{k!} L_k^b(t). \tag{4.18}$$

Let $M_k(w) = e^{-w} w^{k+p}/k!$ so that

$$M_k'(w) = e^{-w} w^{k+p-1} [k+p-w]/k!.$$

Use (4.12) and (4.13) to see that

$$|L_k^b(t)| = (\gamma t)^p |J_k^b(t)| \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}+p-1} (k+1)^{-\frac{1}{2}} \tag{4.19}$$

for all $k \geq 0$. Therefore

$$|(x+\Delta)^p (P_t f)'(x+\Delta) - x^p (P_t f)'(x)| \le c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}+p-1} \int_{x/\gamma t}^{(x+\Delta)/\gamma t} \sum_{k=0}^{\infty} e^{-w} \frac{w^{k+p-1}}{k!} [|k-w|+p](k+1)^{-\frac{1}{2}} dw. \tag{4.20}$$

Lemma 3.3 shows the sum over $k$ is at most $cw^{p-1}$. Now use the elementary inequality $(x+\Delta)^p - x^p \leq (x+\Delta)^{p-1}\Delta$, which holds for $0 < p \leq 1$, to see that the left hand side of (4.20) is less than or equal to

$$c_{\alpha,p}|f|_\alpha (\gamma t)^{\frac{\alpha}{2}-1}(x+\Delta)^{p-1}\Delta.$$

Next set $b = 0$. By replacing $f$ with $f - f(0)$, which does not affect $(P_t f)'$ or $|f|_\alpha$, we may assume $f(0) = 0$. Let

$$L_k(t) = \int_0^t f(z\gamma t) e^{-z} \Big[\frac{z^k}{k!} - \frac{z^{k-1}}{(k-1)!}\Big] (\gamma t)^{p-1} dz.$$

Lemma 4.1(a) shows

$$x^p (P_t f)'(x) = \Big(\frac{x}{\gamma t}\Big)^p e^{-x/\gamma t} \int_0^\infty f(z\gamma t) e^{-z} (\gamma t)^{p-1} dz + \sum_{k=1}^{\infty} M_k(x/\gamma t) L_k(t). \tag{4.21}$$

By (4.3) and Lemma 4.2, for $k \geq 1$,

$$|L_k(t)| = |J_k(t)|(\gamma t)^p \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}+p-1} k^{-\frac{1}{2}}.$$

Therefore by (4.21)

$$\begin{aligned}
&|(x+\Delta)^p (P_t f)'(x+\Delta) - x^p (P_t f)'(x)| \\
&\quad \leq \Big|\int_{x/\gamma t}^{(x+\Delta)/\gamma t} e^{-w} w^{p-1}(p-w) dw\Big| \Big|\int_0^\infty f(z\gamma t) e^{-z} (\gamma t)^{p-1} dz\Big| \\
&\qquad + c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}+p-1} \sum_{k=1}^{\infty} \Big|\int_{x/\gamma t}^{(x+\Delta)/\gamma t} e^{-w} \frac{w^{k+p-1}}{k!} (k+p-w) dw\Big| k^{-\frac{1}{2}} \\
&\quad \leq c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}+p-1} \int_{x/\gamma t}^{(x+\Delta)/\gamma t} e^{-w} w^{p-1} |p-w| dw \int_0^\infty z^{\alpha/2} e^{-z} dz \\
&\qquad + c_\alpha |f|_\alpha (\gamma t)^{\frac{\alpha}{2}+p-1} \sum_{k=1}^{\infty} \int_{x/\gamma t}^{(x+\Delta)/\gamma t} e^{-w} \frac{w^{k+p-1}}{k!} |k+p-w| dw k^{-\frac{1}{2}} \\
&\quad \leq c_\alpha |f|_\alpha (\gamma t)^{\alpha/2+p-1} \int_{x/\gamma t}^{(x+\Delta)/\gamma t} \Big(\sum_{k=0}^{\infty} e^{-w} \frac{w^{k+p-1}}{k!} \frac{|k-w|+p}{(k+1)^{1/2}}\Big) dw.
\end{aligned}$$

This is precisely the expression that appears in (4.20), so the proof now proceeds as in the case $b > 0$. □

**5. Multidimensional semigroups and resolvents.**

Fix $b^0 = (b_1^0, \dots, b_d^0) \in [0,\infty)^d$, $\gamma^0 = (\gamma_1^0, \dots, \gamma_0^d) \in (0,\infty)^d$. Let $X_t^{\gamma^0,b^0} \in \mathbb{R}_+^d$ be the diffusion with generator

$$\mathcal{L}^{\gamma^0,b^0} f(x) = \sum_{i=1}^d [\gamma_i^0 x_i f_{ii}(x) + b_i^0 f_i(x)].$$

Hence $X_t = (X_t^1, \dots, X_t^d)$, where the $X^i$ are independent and $X_t^i$ has the semigroup $P_t^{\gamma_i^0, b_i^0}$ which was studied in Section 4. We write $P_t$ for the semigroup of $X_t$ and

$$P_t^i f(x) = \int_0^\infty q_t^i(x, dy) f(y)$$

for the semigroup of $X^i$, where $q_t^i = q_t^{\gamma_i^0, b_i^0}$ is defined by (2.2) or (2.3). We will keep track of the dependence on $\gamma^0$ and $b^0$, so all constants $c_\alpha$ will be independent of $\gamma^0, b^0$. Let $(e_1, \dots, e_d)$ be the standard basis for $\mathbb{R}^d$. Partial derivatives with respect to $x_i$ when $x_i = 0$ are understood to be right-hand derivatives. If $x = (x_1, \dots, x_d) \in \mathbb{R}_+^d$, let $\widehat{x}_i = (x_k)_{k \neq i} \in \mathbb{R}_+^{d-1}$, $\widehat{x}_{ij} = (x_k)_{k \neq i,j} \in \mathbb{R}_+^{d-2}$, etc. If $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+^d)$ and $1 \le i,j \le d$ we set

$$F_{fi}(y_i; t, \widehat{x}_i) = F(y_i; t, \widehat{x}_i) = \int \prod_{i \neq j} q_t^j(x_j, dy_j) f(y), \tag{5.1}$$

and

$$G_{fij}(y_i, y_j; t, \widehat{x}_{ij}) = G(y_i, y_j; t, \widehat{x}_{ij}) = \int \prod_{k \neq i,j} q_t^k(x_k, dy_k) f(y). \tag{5.2}$$

We frequently write $g_i$ for $\partial g / \partial x_i$ and similarly for second partials.

**Proposition 5.1.** *If $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+^d)$ and $t > 0$, then $(P_t f)_i$ and $(P_t f)_{ii}$ exist on $\mathbb{R}_+^d$ and satisfy*

*(a)*

$$\|(P_t f)_i\|_\infty + \|x_i (P_t f)_{ii}\|_\infty \le c_\alpha |f|_{\alpha,i} (\gamma_i^0 t)^{\frac{\alpha}{2}-1}.$$

*(b)*

$$|(P_t f)_i(x)| \le c_\alpha |f|_{\alpha,i} (\gamma_i^0 t)^{\alpha/2-1} \Big( \Big( \frac{x_i}{\gamma_i^0 t} \Big)^{-\frac{1}{2}} \wedge 1 \Big). \tag{5.3}$$

*(c)*

$$|x_i (P_t f)_{ii}(x)| \le c_\alpha |f|_{\alpha,i} (\gamma_i^0 t)^{\alpha/2-1} \Big( \frac{x_i}{\gamma_i^0 t} \wedge 1 \Big). \tag{5.4}$$

**Proof.** We have

$$P_t f(x) = \int q_t^i(x_i, dy_i) F(y_i; t, \widehat{x}_i), \tag{5.5}$$

Note that if $\Delta > 0$

$$\begin{aligned}|F(y_i + \Delta; t, \widehat{x}_i) - &F(y_i; t, \widehat{x}_i)| \\ &\leq \int |f(y + \Delta e_i) - f(y)| \prod_{j \neq i} q_t^j(x_j, dy_j) \\ &\leq |f|_{\alpha,i} y_i^{\alpha/2} \Delta^\alpha\end{aligned}$$

and so $F(\cdot; t, \widehat{x}_i) \in \mathcal{C}_w^\alpha(\mathbb{R}_+)$ with

$$|F(\cdot; t, \widehat{x}_i)|_\alpha \leq |f|_{\alpha,i}. \tag{5.6}$$

It follows from Lemmas 4.1 and 4.5 that

$$(P_t f)_i(x) = (P_t^i F(\cdot; t, \widehat{x}_i))'(x_i) \tag{5.7}$$

and

$$(P_t f)_{ii}(x) = (P_t^i F(\cdot; t, \widehat{x}_i))''(x_i) \tag{5.8}$$

both exist. The assertions of the Proposition now follow from Lemmas 4.3 and 4.6, and (5.6). □

**Proposition 5.2.** *Let $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+^d)$, $t > 0$. For $i = 1, \ldots, d$, $(P_t f)_i$, $(P_t f)_{ii}$, and $x_i(P_t f)_{ii}$ are bounded and continuous on $\mathbb{R}_+^d$ and for all $\Delta > 0$, $x \in \mathbb{R}_+^d$, and $1 \leq i, j \leq d$:*

*(a)* $|(P_t f)_i(x + \Delta e_j) - (P_t f)_i(x)| \leq c_\alpha |f|_{\alpha,j} \frac{\gamma_j^0}{\gamma_i^0} (\gamma_j^0 t)^{\frac{\alpha}{2} - \frac{3}{2}} (x_j + \gamma_j^0 t)^{-\frac{1}{2}} \Delta.$

*(b)* $|(x + \Delta e_j)_i (P_t f)_{ii}(x + \Delta e_j) - x_i (P_t f)_{ii}(x)| \leq c_\alpha |f|_{\alpha,j} \frac{\gamma_j^0}{\gamma_i^0} (\gamma_j^0 t)^{\frac{\alpha}{2} - \frac{3}{2}} (x_j + \gamma_j^0 t)^{-\frac{1}{2}} \Delta.$

**Proof.** First consider $i = j$.

(a) Use (5.6), (5.7), Lemma 4.7(a), and Lemma 4.4(a) to see that if $F$ is as in (5.1), then

$$\begin{aligned}|(P_t f)_i(x + \Delta e_i) &- (P_t f)_i(x)| \\ &= |(P_t^i F(\cdot; t, \widehat{x}_i))'(x_i + \Delta) - (P_t^i F(\cdot; t, \widehat{x}_i))')(x_i)| \\ &\leq c_\alpha |F(\cdot; t, \widehat{x}_i)|_\alpha (\gamma_i^0 t)^{\frac{\alpha}{2} - \frac{3}{2}} (x_i + \gamma_i^0 t)^{-\frac{1}{2}} \Delta \\ &\leq c_\alpha |f|_{\alpha,i} (\gamma_i^0 t)^{\frac{\alpha}{2} - \frac{3}{2}} (x_i + \gamma_i^0 t)^{-\frac{1}{2}} \Delta.\end{aligned}$$

For (b) (with $i = j$) we proceed as in (a), but now use (5.8), Lemma 4.7(b) and Lemma 4.4(b).

Next consider (a) and (b) in the slightly more involved case when $i \neq j$. Recall the definition of $G$ from (5.2). Then

$$P_t f(x) = \int q_t^j(x_j, dy_j) \int q_t^i(x_i, dy_i) G(y_i, y_j; t, \widehat{x}_{ij}). \tag{5.9}$$

As for (5.6) we have

$$|G(\cdot, y_j; t, \widehat{x}_{ij})|_\alpha \le |f|_{\alpha,i}, \tag{5.10}$$

and so Lemmas 4.3 and 4.6(a) imply

$$\Big|\frac{\partial}{\partial x_i} \int q_t(x_i, dy_i) G(y_i, y_j; t, \widehat{x}_{ij})\Big| \le c_\alpha |f|_{\alpha,i} (\gamma_i^0 t)^{\frac{\alpha}{2}-1} \tag{5.11}$$

and

$$\Big|\frac{\partial^2}{\partial x_i^2} \int q_t(x_i, dy_i) G(y_i, y_j; t, \widehat{x}_{ij})\Big| \le c_\alpha |f|_{\alpha,i} (\gamma_i^0 t)^{\frac{\alpha}{2}-2}. \tag{5.12}$$

These bounds allow us to differentiate through the $y_j$ integral in (5.9) and conclude that if $K(y_j, \widehat{x}_j)(x_i) = \frac{\partial}{\partial x_i} P_t^i(G(\cdot, y_j; t, \widehat{x}_{ij}))(x_i)$ and $L(y_j, \widehat{x}_j)(x_i) = x_i \frac{\partial^2}{\partial x_i^2} P_t^i(G(\cdot, y_j; t, \widehat{x}_{ij}))(x_i)$, then

$$(P_t f)_i(x) = \int q_t^j(x_j, dy_j) K(y_j, \widehat{x}_j) \tag{5.13}$$

and

$$(P_t f)_{ii}(x) = \int q_t^j(x_j, dy_j) L(y_j, \widehat{x}_j). \tag{5.14}$$

If $\Delta > 0$, Lemma 4.3(b)(i) or Lemma 4.6(b)(i) and (2.1) and (5.10) imply that

$$\begin{aligned}|K(y_j + \Delta, \widehat{x}_j) - K(y_j, \widehat{x}_j)| &\le c_0 \|G(\cdot, y_j + \Delta; t, \widehat{x}_j) - G(\cdot, y_j; t, \widehat{x}_j)\|_\infty (\gamma_i^0 t)^{-1} \\ &\le c_\alpha |f|_{\alpha,j} [y_j^{-\frac{\alpha}{2}} \Delta^\alpha \wedge \Delta^{\frac{\alpha}{2}}] (\gamma_i^0 t)^{-1}.\end{aligned}$$

This proves $K(\cdot, \widehat{x}_j) \in C_w^\alpha(\mathbb{R}_+)$ and

$$|K(\cdot, \widehat{x}_j)|_\alpha \le c_\alpha |f|_{\alpha,j} (\gamma_i^0 t)^{-1}. \tag{5.15}$$

A similar argument using Lemma 4.3(b)(ii) (if $b_i^0 = 0$) or Lemma 4.6(b)(ii) (if $b_i^0 > 0$) gives $L(\cdot, \widehat{x}_j) \in C_w^\alpha(\mathbb{R}_+)$ and

$$|L(\cdot, \widehat{x}_j)|_\alpha \le c_\alpha |f|_{\alpha,j} (\gamma_i^0 t)^{-1} \Big(\frac{x_i}{\gamma_i^0 t} \wedge 1\Big). \tag{5.16}$$

By (5.13), (5.15) and Lemma 4.8,

$$\begin{aligned}|(P_t f)_i(x + \Delta e_j) - (P_t f)_i(x)| &\le c_\alpha |K(\cdot, \widehat{x}_j)|_\alpha (\gamma_j^0 t)^{\frac{\alpha}{2}-\frac{1}{2}} \Delta (\Delta + x_j + \gamma_j^0 t)^{-\frac{1}{2}} \\ &\le c_\alpha |f|_{\alpha,j} \frac{\gamma_j^0}{\gamma_i^0} (\gamma_j^0 t)^{\frac{\alpha}{2}-\frac{3}{2}} (\Delta + x_j + \gamma_j^0 t)^{-\frac{1}{2}} \Delta.\end{aligned}$$

Similarly by (5.14), Lemma 4.8 and (5.16),

$$
\begin{aligned}
&|x_i(P_tf)_{ii}(x+\Delta e_j)-x_i(P_tf)_{ii}(x)| &(5.17)\\
&\qquad\le c_\alpha |f|_{\alpha,j}\frac{\gamma_j^0}{\gamma_i^0}(\gamma_j^0 t)^{\frac{\alpha}{2}-\frac{3}{2}}(\Delta+x_j+\gamma_j^0 t)^{-\frac{1}{2}}\Delta\Big(\frac{x_i}{\gamma_i^0 t}\wedge 1\Big).
\end{aligned}
$$

This completes the derivation of (a) and (b) for all $i, j$.

The continuity of $(P_tf)_i$ and $x_i(P_tf)_{ii}$ are immediate from (a) and (b), and their boundedness is given by Proposition 5.1. It remains to show $(P_tf)_{ii} \in C_b(\mathbb{R}^d_+)$. Boundedness is implied by (5.4). (5.17) shows that

$$
|(P_tf)_{ii}(\widehat{x}_i', x_i)-(P_tf)_{ii}(\widehat{x}_i, x_i)| \le c(\gamma^0,\alpha)t^{\frac{\alpha}{2}-3}|f|_{\alpha,j}|\widehat{x}_i'-\widehat{x}_i|. \tag{5.18}
$$

Lemmas 4.1 and 4.5 (which give the continuity of the second derivative of the one-dimensional semigroup) together with (5.8) imply that for all $\widehat{x}_i \in \mathbb{R}^{d-1}_+$ the function $x_i \mapsto (P_tf)_{ii}(\widehat{x}_i, x_i)$ is continuous on $\mathbb{R}_+$. The continuity of $(P_tf)_{ii}$ on $\mathbb{R}^d_+$ is implied by this and (5.18). □

Let

$$
R_\lambda f(x)=\int_0^\infty e^{-\lambda t}P_tf(x)dt \tag{5.19}
$$

be the resolvent operator for $X^{\gamma_0,b_0}$.

**Proposition 5.3.** *Let $f \in \mathcal{C}^\alpha_w(\mathbb{R}^d_+)$. Then $(R_\lambda f)_i$ exists and is continuous on $\mathbb{R}^d_+$, $(R_\lambda f)_{ii}$ exists and is continuous on $\{x \in \mathbb{R}^d_+ : x_i > 0\}$, and $x_i(R_\lambda f)_{ii}(x)$ has a continuous extension to $\mathbb{R}^d_+$ (which we also denote by $x_i(R_\lambda f)_{ii}(x)$). Moreover, $(R_\lambda f)_i, x_i(R_\lambda f)_{ii}(x) \in \mathcal{C}^\alpha_w(\mathbb{R}^d_+)$,*

$$
(R_\lambda f)_i(x)=\int_0^\infty e^{-\lambda t}(P_tf)_i(x)dt, \qquad x\in\mathbb{R}^d_+, \tag{5.20}
$$

$$
(R_\lambda f)_{ii}(x)=\int_0^\infty e^{-\lambda t}(P_tf)_{ii}(x)dt \quad \text{on } \{x\in\mathbb{R}^d_+ : x_i>0\}, \tag{5.21}
$$

*and there exists $c_\alpha$ such that*

*(a)*

$$
\|(R_\lambda f)_i\|_\infty+\|x_i(R_\lambda f)_{ii}\|_\infty \le c_\alpha|f|_{\alpha,i}(\gamma_i^0)^{\frac{\alpha}{2}-1}\lambda^{-\frac{\alpha}{2}}
$$

*(b)*

$$
|(R_\lambda f)_i|_{\alpha,j} \le c_\alpha|f|_{\alpha,i}^{1-\alpha}|f|_{\alpha,j}^{\alpha}(\gamma_i^0)^{-1}(\gamma_i^0/\gamma_j^0)^{(1-\alpha)\alpha/2}
$$

*(c)*

$$
|x_i(R_\lambda f)_{ii}|_{\alpha,j} \le c_\alpha|f|_{\alpha,i}^{1-\alpha}|f|_{\alpha,j}^{\alpha}(\gamma_i^0)^{-1}(\gamma_i^0/\gamma_j^0)^{(1-\alpha)\alpha/2}.
$$

**Proof.** The bound in Proposition 5.1(a) allows us to differentiate through the integral and conclude (5.20). It follows from Proposition 5.2, Proposition 5.1(a), and dominated convergence that $(R_\lambda f)_i$ is continuous on $\mathbb{R}^d_+$. If $x_i > 0$, then the bound on $\|x_i(P_t f)_{ii}\|_\infty$ in Proposition 5.1(a) allows us to differentiate again through the integral on the right hand side of (5.20) and obtain (5.21). Again the continuity of $(P_t f)_{ii}$ from Proposition 5.2, the bound on $\|x_i(P_t f)_{ii}\|_\infty$ from Proposition 5.1, and dominated convergence shows that $(R_\lambda f)_{ii}$ is continuous on $\{x \in \mathbb{R}^d_+ : x_i > 0\}$.

Now let

$$\bar{t} = \frac{\Delta^2}{\gamma_j^0 x_j} \Big(\frac{|f|_{\alpha,j}}{|f|_{\alpha,i}}\Big)^2 \Big(\frac{\gamma_j^0}{\gamma_i^0}\Big)^\alpha.$$

Use Proposition 5.2(b) for $t \geq \bar{t}$ and Proposition 5.1(a) for $t < \bar{t}$ in (5.21) to obtain for $x_i > 0$

$$\begin{aligned} &|(x+\Delta e_j)_i(R_\lambda f)_{ii}(x+\Delta e_j) - x_i(R_\lambda f)_{ii}(x)| \qquad (5.22)\\ &\leq c_\alpha |f|_{\alpha,i} \int_0^{\bar{t}} e^{-\lambda t}(\gamma_i^0 t)^{\frac{\alpha}{2}-1} dt + c_\alpha |f|_{\alpha,j} \frac{\gamma_j^0}{\gamma_i^0} \int_{\bar{t}}^\infty e^{-\lambda t}(\gamma_j^0 t)^{\frac{\alpha}{2}-\frac{3}{2}}(x_j + \gamma_j^0 t)^{-\frac{1}{2}} \Delta\, dt\\ &\leq c_\alpha |f|_{\alpha,i} (\gamma_i^0)^{-1} (\gamma_i^0 \bar{t})^{\frac{\alpha}{2}} + c_\alpha |f|_{\alpha,j} (\gamma_j^0 \bar{t})^{\frac{\alpha}{2}-\frac{1}{2}} (\gamma_i^0)^{-1} x_j^{-\frac{1}{2}} \Delta\\ &\leq c_\alpha |f|_{\alpha,i}^{1-\alpha} |f|_{\alpha,j}^\alpha (\gamma_i^0)^{-1} \Big(\frac{\gamma_i^0}{\gamma_j^0}\Big)^{(1-\alpha)\frac{\alpha}{2}2} x_j^{-\frac{\alpha}{2}} \Delta^\alpha. \end{aligned}$$

Lemma 2.2 now shows that $x_i(R_\lambda f)_{ii}$ has a Hölder $\alpha/2$ continuous extension to $\mathbb{R}^d_+$ which by (5.22) satisfies (c). Exactly the same calculation using (5.20), Proposition 5.1(a), and Proposition 5.2(a) shows (b). Recall here that we have already shown that $(R_\lambda f)_i$ is continuous on all of $\mathbb{R}^d_+$.

Use (5.20), (5.21), and Proposition 5.1(a) to see that for $x_i > 0$

$$\begin{aligned} &|(R_\lambda f)_i(x)| + |x_i(R_\lambda f)_{ii}(x)|\\ &\leq c_\alpha |f|_{\alpha,i} \int_0^\infty e^{-\lambda t}(\gamma_i^0 t)^{\frac{\alpha}{2}-1} dt\\ &\leq c_\alpha |f|_{\alpha,i} (\gamma_i^0)^{\frac{\alpha}{2}-1} \lambda^{-\frac{\alpha}{2}}. \end{aligned}$$

This estimate then also follows for $x_i = 0$ by continuity, and (a) is proved. Finally, (b), (c), and this bound imply $(R_\lambda f)_i, x_i(R_\lambda f)_{ii} \in \mathcal{C}_w^\alpha(\mathbb{R}^d_+)$. □

**Remark 5.4.** If we use (5.4), instead of Proposition 5.1(a), to bound $x_i(P_t f)_{ii}$ for $t > \frac{x_i}{\gamma_i^0}$, then by (5.21) we get for $x_i > 0$,

$$\begin{aligned} &|x_i(R_\lambda f)_{ii}(x)| \qquad (5.23)\\ &\leq c_\alpha |f|_{\alpha,i} \int_0^{x_i/\gamma_i^0} e^{-\lambda t}(\gamma_i^0 t)^{\frac{\alpha}{2}-1} dt + c_\alpha |f|_{\alpha,i} \int_{x_i/\gamma_i^0}^\infty e^{-\lambda t}(\gamma_i^0 t)^{\frac{\alpha}{2}-2} x_i dt\\ &\leq c_\alpha |f|_{\alpha,i} (\gamma_i^0)^{-1} x_i^{\frac{\alpha}{2}}. \end{aligned}$$

This shows the continuous extension of $x_i(R_\lambda f)_{ii}$ to $\mathbb{R}^d_+$ is obtained by setting $x_i(R_\lambda f)_{ii}(x) = 0$ on $\{x_i = 0\}$.

**Remark 5.5.** If ordinary Hölder $\alpha$ norms (i.e., the usual space $C^\alpha$) had been used instead of weighted Hölder norms, one obtains for a bound on the left hand side of (5.22) an expression like

$$c_\alpha \|f\|_{C^\alpha}\Big(\frac{x_i^{\alpha/2}}{x_j^{\alpha/2}}\Big)\Delta^\alpha,$$

which is not bounded as $x_j \to 0$ for $i \neq j$. Thus we were forced to use weighted Hölder norms.

We now turn to the mixed partial terms. For the next lemma fix $i \neq j$ and recall that $G(y_i, y_j; t, \widehat{x}_{ij}) = \int \prod_{k\neq i,j} q_t^k(x_k, dy_k) f(y)$.

**Lemma 5.6.** *$(P_t f)_{ji}(x)$ is a bounded uniformly Lipschitz function on $\mathbb{R}^d_+$,*

$$(P_t f)_{ji}(x) = \frac{\partial}{\partial x_i}\int q_t^i(x_i, dy_i)\Big[\frac{\partial}{\partial x_j}\int q_t^j(x_j, dy_j) G(y_i, y_j; t, \widehat{x}_{ij})\Big], \tag{5.24}$$

*and there exists $c_\alpha > 0$ such that*

$$|(P_t f)_{ji}(x)| \le c_\alpha |f|_{\alpha,i} (\gamma_i^0)^{\alpha/2} (\gamma_i^0\gamma_j^0)^{-\frac12} t^{\frac{\alpha}{2}-1} (x_j + \gamma_j^0 t)^{-\frac12} (x_i + \gamma_i^0 t)^{-\frac12}, \tag{5.25}$$

$$\begin{aligned} &|\sqrt{x_i+\Delta}\sqrt{x_j}(P_t f)_{ji}(x+\Delta e_i) - \sqrt{x_i x_j}(P_t f)_{ji}(x)| \\ &\qquad\qquad \le c_\alpha |f|_{\alpha,i} (\gamma_i^0)^{\frac{\alpha}{2}-1} (\gamma_j^0)^{-\frac12} t^{\frac{\alpha}{2}-\frac32} \Delta (x_i + \Delta)^{-\frac12}, \end{aligned} \tag{5.26}$$

*and for $k \neq i, j$*

$$\begin{aligned} &|\sqrt{x_i x_j}(P_t f)_{ji}(x+\Delta e_k) - \sqrt{x_i x_j}(P_t f)_{ji}(x)| \\ &\qquad\qquad \le c_\alpha |f|_{\alpha,k} (\gamma_k^0)^{\alpha/2} (\gamma_i^0\gamma_j^0\gamma_k^0)^{-\frac12} t^{\frac{\alpha}{2}-\frac32} \Delta (x_k + \gamma_k^0 t)^{-\frac12}. \end{aligned} \tag{5.27}$$

**Proof.** Clearly $G(\cdot,\cdot; t, \widehat{x}_{ij}) \in \mathcal{C}^\alpha_w(\mathbb{R}^2_+)$ and

$$|G(\cdot,\cdot; t, \widehat{x}_{ij})|_{\alpha,l} \le |f|_{\alpha,l} \tag{5.28}$$

for $l = i, j$. Lemma 4.1(a) or Lemma 4.5(a) shows that

$$Q(y_i, x_j, \widehat{x}_{ij}) = \frac{\partial}{\partial x_j}\int q_t^j(x_j, dy_j) G(y_i, y_j; t, \widehat{x}_{ij}) \tag{5.29}$$

exists and Lemma 4.3(a)(i) or Lemma 4.6(a)(i), and (5.28) give the bound

$$\|Q\|_\infty \le c_\alpha |f|_{\alpha,j} (\gamma_j^0 t)^{\frac{\alpha}{2}-\frac12} (x_j + \gamma_j^0 t)^{-\frac12} \le c_\alpha |f|_{\alpha,j} (\gamma_j^0 t)^{\frac{\alpha}{2}-1}. \tag{5.30}$$

This bound allows us to differentiate through the integral and conclude

$$\frac{\partial}{\partial x_j}(P_t f)(x) = \int q_t^i(x_i, dy_i) Q(y_i, x_j, \widehat{x}_{ij}). \tag{5.31}$$

Since

$$|G(y_i + \Delta, y_j; t, \widehat{x}_{ij}) - G(y_i, y_j; t, \widehat{x}_{ij})| \le c_\alpha |f|_{\alpha,i} (\Delta^\alpha y_i^{-\frac{\alpha}{2}} \wedge \Delta^{\frac{\alpha}{2}})$$

by (2.1) and (5.28), then Lemma 4.3(b)(i) or Lemma 4.6(b)(i) tells us

$$\begin{aligned} &|Q(y_i + \Delta, x_j, \widehat{x}_{ij}) - Q(y_i, x_j, \widehat{x}_{ij})| \\ &\qquad \le c_\alpha (\gamma_j^0 t)^{-\frac{1}{2}} (x_j + \gamma_j^0 t)^{-\frac{1}{2}} |f|_{\alpha,i} (\Delta^\alpha y_i^{-\frac{\alpha}{2}} \wedge \Delta^{\frac{\alpha}{2}}). \end{aligned} \tag{5.32}$$

This and (5.30) imply $Q(\cdot, x_j, \widehat{x}_{ij}) \in \mathcal{C}_w^\alpha(\mathbb{R}_+)$ and

$$|Q(\cdot, x_j, \widehat{x}_{ij})|_\alpha \le c_\alpha (\gamma_j^0 t)^{-\frac{1}{2}} (x_j + \gamma_j^0 t)^{-\frac{1}{2}} |f|_{\alpha,i}. \tag{5.33}$$

Lemmas 4.1(a), 4.3(a)(i), and 4.4(a) (for $b_j^0 = 0$) or Lemmas 4.5(a), 4.6(a)(i), and 4.7(a) (for $b_j^0 > 0$) imply $(P_t f)_{ji}(x) = \frac{\partial}{\partial x_i} \int q_t^i(x_i, dy_i) Q(y_i, x_j, \widehat{x}_{ij})$ exists on $\mathbb{R}_+^d$ (this also gives (5.24)) and satisfies

$$|(P_t f)_{ji}(x)| \le c_\alpha |f|_{\alpha,i} (\gamma_i^0)^{\frac{\alpha}{2}} (\gamma_i^0 \gamma_j^0)^{-\frac{1}{2}} t^{\frac{\alpha}{2}-1} (x_j + \gamma_j^0 t)^{-\frac{1}{2}} (x_i + \gamma_i^0 t)^{-\frac{1}{2}} \tag{5.34}$$

and

$$\begin{aligned} &|(P_t f)_{ji}(x + \Delta e_i) - (P_t f)_{ji}(x)| \\ &\qquad \le c_\alpha |f|_{\alpha,i} (\gamma_i^0)^{\frac{\alpha}{2}-\frac{3}{2}} (\gamma_j^0)^{-\frac{1}{2}} t^{\frac{\alpha}{2}-2} (x_j + \gamma_j^0 t)^{-\frac{1}{2}} (x_i + \gamma_i^0 t)^{-\frac{1}{2}} \Delta. \end{aligned} \tag{5.35}$$

To check continuity in $x_j$ (at this point we do not yet know $(P_t f)_{ij} = (P_t f)_{ji}$ and so cannot appeal to symmetry) first use Lemma 4.4(a) or Lemma 4.8(a), and (5.28) to see that

$$|Q(y_i, x_j + \Delta, \widehat{x}_{ij}) - Q(y_i, x_j, \widehat{x}_{ij})| \le c_\alpha |f|_{\alpha,j} (\gamma_j^0 t)^{\frac{\alpha}{2}-\frac{3}{2}} (x_j + \gamma_j^0 t)^{-\frac{1}{2}} \Delta, \tag{5.36}$$

and then Lemma 4.3(b)(i) or Lemma 4.6(b)(i) to see that

$$\begin{aligned} &|(P_t f)_{ji}(x + \Delta e_j) - (P_t f)_{ji}(x)| \\ &\qquad = \Big| \frac{\partial}{\partial x_i} \int q_t^i(x_i, dy_i) (Q(y_i, x_j + \Delta, \widehat{x}_{ij}) - Q(y_i, x_j, \widehat{x}_{ij})) \Big| \\ &\qquad \le c_\alpha |f|_{\alpha,j} (\gamma_0 t)^{\frac{\alpha}{2}-\frac{3}{2}} (x_j + \gamma_j^0 t)^{-\frac{1}{2}} \Delta (\gamma_i^0 t)^{-\frac{1}{2}} (x_i + \gamma_i^0 t)^{-\frac{1}{2}}. \end{aligned} \tag{5.37}$$

If $k \neq i,j$, then similarly, using Lemma 4.3(b)(i) or 4.6(b)(i) twice, we get

$$\begin{aligned} &|(P_tf)_{ji}(x+\Delta e_k)-(P_tf)_{ji}(x)| \\ &\quad \le c(\gamma_i^0 t)^{-\frac12}(x_i+\gamma_i^0 t)^{-\frac12}(\gamma_j^0 t)^{-\frac12}(x_j+\gamma_j^0 t)^{-\frac12} \\ &\qquad \times \sup_{y_i,y_j}|G(y_i,y_j;t,\widehat{x}_{ij}+\Delta e_k)-G(y_i,y_j;t,\widehat{x}_{ij})|. \end{aligned} \tag{5.38}$$

Let

$$H(y_i,y_j,y_k;t,\widehat{x}_{ijk})=\int\prod_{\ell\neq i,j,k} q_t^\ell(x_\ell,dy_\ell)f(dy),$$

where $\widehat{x}_{ijk}$ is defined immediately preceding (5.1). By Lemma 4.8

$$\begin{aligned} &|G(y_i,y_j;t,\widehat{x}_{ij}+\Delta e_k)-G(y_i,y_j;t,\widehat{x}_{ij}| \\ &\qquad \le c_\alpha|H(y_i,y_j,\cdot;t,\widehat{x}_{ijk})|_\alpha(\gamma_k^0 t)^{\frac{\alpha-1}{2}}\Delta(\Delta+x_k+\gamma_k^0 t)^{-\frac12} \\ &\qquad \le c_\alpha|f|_{\alpha,k}(\gamma_k^0 t)^{\frac{\alpha-1}{2}}\Delta(\Delta+x_k+\gamma_k^0 t)^{-\frac12}. \end{aligned}$$

Put this into (5.38) to obtain

$$\begin{aligned} &|(P_tf)_{ji}(x+\Delta e_k)-(P_tf)_{ji}(x)| \\ &\quad \le c_\alpha|f|_{\alpha,k}(\gamma_i^0 t)^{-\frac12}(\gamma_j^0 t)^{-\frac12}(\gamma_k^0 t)^{\frac{\alpha-1}{2}}(x_i+\gamma_i^0 t)^{-\frac12}(x_j+\gamma_j^0 t)^{-\frac12}(x_k+\gamma_k^0 t)^{-\frac12}\Delta \end{aligned} \tag{5.39}$$

for $k\neq i,j$. (5.35), (5.37), and (5.39) give the uniform Lipschitz continuity on $\mathbb{R}_+^d$. The boundedness of $(P_tf)_{ij}$ and (5.25) are given by (5.34). (5.39) implies (5.27), and (5.24) was already derived. It remains only to prove (5.26).

We first use (5.24) with $i$ and $j$ interchanged on the right hand side, and use Lemmas 4.3(b)(i) or 4.6(b)(i) to handle the first ($x_j$) differentiation to see that

$$\begin{aligned} &|\sqrt{x_i+\Delta}\sqrt{x_j}(P_tf)_{ji}(x+\Delta e_i)-\sqrt{x_i}\sqrt{x_j}(P_tf)_{ji}(x)| \\ &\quad \le c_\alpha(\gamma_j^0 t)^{-\frac12}(x_j/(x_j+\gamma_j^0 t))^{\frac12} \\ &\qquad \times \sup_{y_j}\Big|\sqrt{x_i+\Delta}\frac{\partial}{\partial x_i}\int q_t^i(x_i+\Delta,dy_i)G(y_i,y_j;t,\widehat{x}_{ij}) \\ &\qquad\qquad -\sqrt{x_i}\frac{\partial}{\partial x_i}\int q_t^i(x_i,dy_i)G(y_i,y_j;t,\widehat{x}_{ij})\Big| \\ &\quad \le c_\alpha(\gamma_j^0 t)^{-\frac12}(\gamma_i^0 t)^{\frac{\alpha}{2}-1}(x_i+\Delta)^{-\frac12}\Delta\sup_{y_j}|G(\cdot,y_j;t,\widehat{x}_{ij})|_\alpha \\ &\quad \le c_\alpha|f|_{\alpha,i}(\gamma_i^0)^{\frac{\alpha}{2}-1}(\gamma_j^0)^{-\frac12}t^{\frac{\alpha}{2}-\frac32}\Delta(x_i+\Delta)^{-\frac12}. \end{aligned}$$

In the next to last inequality we used Lemma 4.9 with $p=\frac12$, and in the last inequality we used (5.28) with $l=i$. □

**Proposition 5.7.** *Let $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+^d)$ and let $i \neq j$. Then $(R_\lambda f)_{ji}$ exists and is continuous on $\{x \in \mathbb{R}_+^d : x_i, x_j > 0\}$. The function $\sqrt{x_i x_j}(R_\lambda f)_{ji}$ has a continuous extension to $\mathbb{R}_+^d$ (which we denote by the same expression), and this function is in $\mathcal{C}_w^\alpha(\mathbb{R}_+^d)$. Moreover there exists $c_\alpha$ such that*

*(a)*

$$\|\sqrt{x_i x_j}(R_\lambda f)_{ji}\|_\infty \leq c_\alpha(\gamma_i^0)^{\frac{\alpha}{2}}(\gamma_i^0\gamma_j^0)^{-\frac{1}{2}}|f|_{\alpha,i}\lambda^{-\frac{\alpha}{2}}.$$

*(b)*

$$|\sqrt{x_i x_j}(R_\lambda f)_{ji}|_{\alpha,k} \leq c_\alpha(\gamma_i^0\gamma_j^0)^{-\frac{1}{2}}(\gamma_i^0/\gamma_k^0)^{(1-\alpha)\alpha/2}|f|_{\alpha,i}^{1-\alpha}|f|_{\alpha,k}^\alpha,\ k = 1, \ldots, d.$$

**Proof.** If $x_i, x_j > 0$, (5.25) and dominated convergence allows us to differentiate through the integral to conclude

$$(R_\lambda f)_{ji}(x) = \int_0^\infty e^{-\lambda t}(P_t f)_{ji}(x)dt. \tag{5.40}$$

The continuity of $(P_t f)_{ji}$ from Lemma 5.6, (5.25), and dominated convergence imply the continuity of $(R_\lambda f)_{ji}$ if $x_i, x_j > 0$.

To examine the uniform continuity of $\sqrt{x_i x_j}(R_\lambda f)_{ji}$ on $x_i, x_j > 0$, proceed as in Proposition 5.3. Fix $k$ and $x_i, x_j, x_k, \Delta > 0$ and set

$$\widetilde{t} = \frac{\Delta^2}{x_k}|f|_{\alpha,k}^2|f|_{\alpha,i}^{-2}(\gamma_k^0)^{1-\alpha}(\gamma_i^0)^\alpha.$$

Use (5.40) to write

$$(x + \Delta e_k)_i^{1/2}(x + \Delta e_k)_j^{1/2}(R_\lambda f)_{ji}(x + \Delta e_k) - (x_i x_j)^{1/2}(R_\lambda f)_{ji}(x)$$

as an integral over $t \geq 0$. Then split this integral at $\widetilde{t}$, use (5.25) for $t < \widetilde{t}$, and (5.26) or (5.27) for $t \geq \widetilde{t}$ to prove (b). Lemma 2.2 shows that $\sqrt{x_i x_j}(R_\lambda f)_{ji}$ has a Hölder continuous extension to $\mathbb{R}_+^d$. (a) follows easily from (5.25) first for $x_i, x_j > 0$ and then on all of $\mathbb{R}_+^d$. That $\sqrt{x_i x_j}(R_\lambda f)_{ji} \in \mathcal{C}_w^\alpha(\mathbb{R}_+^d)$ now follows. □

**Remark 5.8.** By being a little more careful, we can show that the continuous extension of $\sqrt{x_i x_j}(R_\lambda f)_{ij}$ is 0 if $x_i = 0$ or $x_j = 0$, as the notation suggests. To see this first note that by symmetry and (5.25), if $c_{5.9} = c_\alpha \min((\gamma_i^0)^{\frac{\alpha}{2}}|f|_{\alpha,i}, (\gamma_j^0)^{\frac{\alpha}{2}}|f|_{\alpha,j})(\gamma_i^0\gamma_j^0)^{-1}$ and $z_i = x_i/\gamma_i^0$, then

$$|(P_t f)_{ji}(x)| \leq c_{5.9}t^{\frac{\alpha}{2}-1}(z_j \vee t)^{-\frac{1}{2}}(z_i \vee t)^{-\frac{1}{2}}.$$

Assume $x_i, x_j > 0$ and, without loss of generality, that $z_j \geq z_i$. Decompose the integral in (5.40) according to $t \leq z_i$, $z_i < t < z_j$ or $t \geq z_j$ and use the above bound to conclude that

$$\begin{aligned}|\sqrt{x_i x_j}(R_\lambda f)_{ji}(x)| &\leq c_\alpha c_{5.9}(x_i x_j)^{\frac{1}{2}}[z_i^{\frac{\alpha}{2}}(z_j z_i)^{-\frac{1}{2}} + z_i^{\frac{\alpha-1}{2}} z_j^{-\frac{1}{2}} + z_j^{\frac{\alpha}{2}-1}] \\ &\leq c_\alpha c_{5.9}(x_i x_j)^{\frac{1}{2}}(z_i z_j)^{-\frac{1}{2}}[2 z_i^{\frac{\alpha}{2}} + z_j^{\frac{\alpha-1}{2}} z_i^{\frac{1}{2}}] \\ &\leq c_\alpha c_{5.9}(\gamma_i^0 \gamma_j^0)^{\frac{1}{2}} \min(z_i^{\frac{\alpha}{2}}, z_j^{\frac{\alpha}{2}})\end{aligned} .$$

We have used $z_j^{\frac{\alpha-1}{2}} \leq z_i^{\frac{\alpha-1}{2}}$ in the last line, and the result clearly follows.

**6. Uniqueness.**

We now consider the operator $\mathcal{L}f(x)$ defined in (1.1). Let $\alpha \in (0,1]$, and we suppose Hypotheses (H1) and (H2) are in force throughout this section.

Let $\gamma_i^0 > 0$ and $b_i^0 \geq 0$ be constants and let

$$\mathcal{L}^{\gamma^0, b^0} f(x) \equiv \mathcal{L}^0 f(x) = \sum_{i=1}^d \gamma_i^0 x_i f_{ii}(x) + \sum_{i=1}^d b_i^0 f_i(x)$$

denote the generator studied in the previous section. We continue to let $P_t$ denote its semigroup and $R_\lambda$ the corresponding resolvent. Let $\mathcal{B} = \mathcal{L} - \mathcal{L}^0$. By Propositions 5.3 and 5.7 we see that $\mathcal{B}R_\lambda f$ is continuous on $\mathbb{R}_+^d$ if $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+^d)$ and that if the $\gamma_{ij}$ and $b_i$ are bounded, then $\mathcal{B}R_\lambda f$ is also bounded.

Let $\delta_x$ be point mass at $x$. Assume for now that the $\gamma_{ij}$ and $b_i$ are uniformly bounded. Assume also that we have a family of probability measures $\mathbb{P}^x$ such that for each $x$ the probability $\mathbb{P}^x$ is a solution to $MP(\mathcal{L}, \delta_x)$ and that the family $(X_t, \mathbb{P}^x)$ is a strong Markov process. We will remove these assumptions later on. Let

$$S_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t) dt$$

be the resolvent of $X$.

**Lemma 6.1.** *If $\lambda_b = \max(\|b_i\|_\infty, 1 \leq i \leq d)$, then for all $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+^d)$,*

$$S_\lambda f = R_\lambda f + S_\lambda \mathcal{B} R_\lambda f \qquad \textit{for all } \lambda > \lambda_b.$$

**Proof.** Let $g \in C_b^2(\mathbb{R}_+^d)$ and let $\mathbb{P}^x$ be a solution to $MP(\mathcal{L}, \delta_x)$. Since $\gamma$ and $b$ are bounded, standard estimates for solutions to (1.3) using Gronwall's lemma on $\mathbb{E}^x X_t^i$, imply $\mathbb{E}^x |X_t| \leq \sqrt{d}|x| e^{\lambda_b t}$ for all $t$. Hence $M_g(t) = g(X_t) - g(X_0) - \int_0^t \mathcal{L}g(X_s) ds$ satisfies

$\mathbb{E}^x \sup_{s\le t} |M_g(s)| < \infty$ and so $M_g$ is actually a martingale and not just a local martingale. Therefore

$$\mathbb{E}^x g(X_t) = g(x) + \int_0^t \mathbb{E}^x \mathcal{L}g(X_s)ds.$$

Fix $\lambda > \lambda_b$. If we multiply this equation by $\lambda e^{-\lambda t}$, integrate over $t$ from 0 to $\infty$, and use Fubini, we obtain

$$\lambda S_\lambda g(x) = g(x) + S_\lambda(\mathcal{L}g)(x) = g(x) + S_\lambda \mathcal{B}g(x) + S_\lambda(\mathcal{L}^0 g)(x). \tag{6.1}$$

Here we have used the bound $\mathbb{E}^x|\mathcal{L}g(X_s)| \le c\mathbb{E}^x|X_s| \le c|x|e^{\lambda_b s}$ to apply Fubini's theorem. Let

$$g_\delta(x) = \int_\delta^\infty e^{-\lambda t} P_t f(x)dt$$

for $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+^d)$. We see that $g_\delta$ is in $C_b^2(\mathbb{R}_+^d)$ by Propositions 5.2 and 5.1, (5.4), Lemma 5.6, (5.25), and dominated convergence.

Dominated convergence implies that $g_\delta$ converges boundedly and pointwise to $R_\lambda f$ and $\lambda S_\lambda g_\delta$ converges boundedly and pointwise to $\lambda S_\lambda R_\lambda f$. Proposition 5.1(c), (5.21), and dominated convergence if $x_i > 0$, and Remark 5.4 if $x_i = 0$, show that $x_i(g_\delta)_{ii}(x) = \int_\delta^\infty e^{-\lambda t} x_i (P_t f)_{ii}(x)dt$ converges boundedly and pointwise to $x_i(R_\lambda f)_{ii}$. Similarly, by (5.25), (5.40), dominated convergence, and Remark 5.8,

$$\sqrt{x_i x_j}(g_\delta)_{ij}(x) = \int_\delta^\infty e^{-\lambda t} \sqrt{x_i x_j}(P_t f)_{ij}(x)dt$$

converges boundedly and pointwise to $\sqrt{x_i x_j}(R_\lambda f)_{ij}$ on $\mathbb{R}_+^d$. Proposition 5.1(a), (5.20), and dominated convergence show that $(g_\delta)_i(x) = \int_\delta^\infty e^{-\lambda t}(P_t f)_i(x)dt$ converges boundedly and pointwise to $(R_\lambda f)_i$. We therefore have

$$\mathcal{B}g_\delta \to \mathcal{B}R_\lambda f$$

boundedly and pointwise on $\mathbb{R}_+^d$.

Since $\mathcal{L}^0$ is the infinitesimal generator of $P_t$, a straightforward calculation, using the fact that the law of $X^{\gamma^0, b^0}$ is a solution to $MP(\mathcal{L}^0, \delta_x)$ and that the local martingales in this martingale problem are in fact martingales, shows that

$$\mathcal{L}^0 g_\delta = \lambda g_\delta - e^{-\lambda\delta} P_\delta f,$$

which converges boundedly and pointwise to $\lambda R_\lambda f - f$.

Replacing $g$ in (6.1) by $g_\delta$, letting $\delta \downarrow 0$, and using the above results, we obtain

$$\lambda S_\lambda R_\lambda f(x) = R_\lambda f(x) + S_\lambda(\mathcal{B}R_\lambda f)(x) + \lambda S_\lambda R_\lambda f(x) - S_\lambda f(x),$$

and the result follows. $\square$

For any bounded linear operator $S : C^\alpha_w(\mathbb{R}^d_+) \to C^\alpha_w(\mathbb{R}^d_+)$, we denote its norm by $\|S\|_\alpha$. Let $\mu^0 = \min_{1\le i\le d} \gamma^0_i$,

$$\varepsilon_0 = \sum_{i=1}^d \|\gamma^0_i - \gamma_{ii}\|_\infty + \sum_{i=1}^d \|b^0_i - b_i\|_\infty + \sum_{i\ne j} \|\gamma_{ij}\|_\infty,$$

and

$$A_0 = \max_k \Big[ \sum_{i,j} |\gamma_{ij}|_{\alpha,k} + \sum_i |b_i|_{\alpha,k} + 1 \Big].$$

**Proposition 6.2.** *Suppose $A_0 < \infty$. There exists $c_{6.2} = c_{6.2}(\alpha)$ such that if*

$$c_{6.2}\varepsilon_0 \le \mu^0 \tag{6.2}$$

*and*

$$\lambda \ge c_{6.2}[\mu^0 + (\mu^0)^{1-\frac{2}{\alpha}}]A_0^{\frac{2}{\alpha}}, \tag{6.3}$$

*then $\mathcal{B}R_\lambda$ is a bounded operator on $\mathcal{C}^\alpha_w(\mathbb{R}^d_+)$ with*

$$\|\mathcal{B}R_\lambda\|_\alpha \le \tfrac{1}{2}. \tag{6.4}$$

**Remark.** Note we are assuming $A_0 < \infty$ in this result. This will be the case, for example, if $\gamma$ and $b$ are $\alpha$-Hölder continuous and are constant outside of a bounded set.

**Proof.** We have

$$\begin{aligned}\mathcal{B}R_\lambda f(x) = \sum_{i=1}^d &[(\gamma_i(x) - \gamma^0_i)(x_i(R_\lambda f)_{ii}(x)) + (b_i(x) - b^0_i)(R_\lambda f)_i(x)] \\ &+ \sum_{i\ne j} \gamma_{ij}(x)\sqrt{x_i x_j}(R_\lambda f)_{ij}(x).\end{aligned}$$

Propositions 5.3 and 5.7 and (6.2) and (6.3) imply that

$$\begin{aligned}\|\mathcal{B}R_\lambda f\|_\infty &\le c_\alpha \varepsilon_0 (\mu^0)^{-1}(\mu^0/\lambda)^{\alpha/2}|f|_\alpha \\ &\le c_\alpha c_{6.2}^{-\frac{\alpha}{2}-1}|f|_\alpha \\ &\le \tfrac{1}{4}|f|_\alpha, \end{aligned} \tag{6.5}$$

providing $c_{6.2}$ is sufficiently large. Let $\gamma^0_{ii} = \gamma^0_i$ and $\gamma^0_{ij} = 0$. If $\Delta, x_k > 0$, then we may use Propositions 5.3 and 5.7 to obtain

$$\begin{aligned}
&|\mathcal{B}R_\lambda f(x+\Delta e_k) - \mathcal{B}R_\lambda f(x)| \\
&\quad \le \sum_{i=1}^d \sum_{j=1}^d |\gamma_{ij}(x+\Delta e_k) - \gamma^0_{ij}| \\
&\qquad\qquad \times |(x+\Delta e_k)_i^{1/2}(x+\Delta e_k)_j^{1/2}(R_\lambda f)_{ij}(x+\Delta e_k) - \sqrt{x_i x_j}(R_\lambda f)_{ij}(x)| \\
&\qquad + \sum_{i=1}^d |b_i(x+\Delta e_k) - b^0_i|\,|(R_\lambda f)_i(x+\Delta e_k) - (R_\lambda f)_i(x)| \\
&\qquad + \sum_{i=1}^d \sum_{j=1}^d |\gamma_{ij}(x+\Delta e_k) - \gamma_{ij}(x)|\,|\sqrt{x_i x_j}(R_\lambda f)_{ij}(x)| \\
&\qquad + \sum_{i=1}^d |b_i(x+\Delta e_k) - b_i(x)|\,|(R_\lambda f)_i(x)| \\
&\quad \le c_\alpha |f|_\alpha [\varepsilon_0(\mu^0)^{-1} + A_0(\mu^0)^{\frac{\alpha}{2}-1}\lambda^{-\frac{\alpha}{2}}]\Delta^\alpha x_k^{-\frac{\alpha}{2}},
\end{aligned}$$

where in the last line we used the elementary inequality

$$(\gamma^0_i\gamma^0_j)^{-\frac{1}{2}}(\gamma^0_i/\gamma^0_k)^{(1-\alpha)\alpha/2} \le (\mu^0)^{-1} \quad \text{for all } i,j,k \le d.$$

This shows that

$$\begin{aligned}
|\mathcal{B}R_\lambda f|_{\alpha,k} &\le c_\alpha(\mu^0)^{-1}[\varepsilon_0 + (\mu^0/\lambda)^{\frac{\alpha}{2}}A_0]|f|_\alpha \\
&\le c_\alpha[c_{6.2}^{-1} + c_{6.2}^{-\frac{\alpha}{2}}]|f|_\alpha \qquad \text{(by (6.2) and (6.3))},
\end{aligned}$$

and so for $c_{6.2}$ large enough we have $|\mathcal{B}R_\lambda f|_\alpha \le \frac{1}{4}|f|_\alpha$. This together with (6.5) shows that $\mathcal{B}R_\lambda f \in \mathcal{C}^\alpha_w(\mathbb{R}^d_+)$ (recall that this function is continuous) and

$$\|\mathcal{B}R_\lambda f\|_\alpha \le \tfrac{1}{2}|f|_\alpha.$$

□

**Remark.** It is much simpler to show that for any $\lambda > 0$, $R_\lambda$ is a bounded operator on $\mathcal{C}^\alpha_w(\mathbb{R}^d_+)$ with

$$\|R_\lambda\|_\alpha \le c_\alpha(\mu^0 \wedge 1)^{(\alpha-1)/2}(\lambda^{-1} \vee \lambda^{-(1+\alpha)/2}),$$

but we will not need this here. This follows easily from Proposition 5.2 and the fundamental theorem of calculus.

**Proposition 6.3.** *Suppose $\gamma$, $b$ satisfy (H1), (H2), are $\alpha$-Hölder continuous on $\mathbb{R}_+^d$ and constant outside of a bounded set, and for some $\gamma_i^0 > 0$, $b_i^0 \geq 0$, (6.2) holds. Then for any probability $\nu$ on $\mathbb{R}_+^d$, there is one and only one solution to the martingale problem $MP(\mathcal{L}, \nu)$.*

**Proof.** Note $b$ and $\gamma$ are bounded (by (6.2)) and continuous, and the fact that $b_i(x) \geq 0$ if $x_i = 0$ ensures solutions remain in $\mathbb{R}_+^d$ (see, e.g., the proof of Theorem 1.1 in [ABBP01]). Existence is then standard, keeping in mind Remark 1.1.

We therefore focus on uniqueness. A standard conditioning argument (e.g., [B97], p. 136) allows us to assume $\nu = \delta_x$. By using Krylov's Markov selection theorem (see Th. 12.2.4 of [SV79] and the proof of Proposition 2.1 in [AABP01]), we need only consider uniqueness for families of strong Markov solutions.

Let $\lambda_0$ denote the right-hand side of (6.3) and note that it is finite by our assumptions on $\gamma$ and $b$. Let $\lambda_b$ be as in Lemma 6.1 and suppose $\lambda > \lambda_0 \vee \lambda_b$. Using Propositions 6.1 and 6.2, if $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+^d)$, we have for any solution to the martingale problem $MP(\mathcal{L}, \delta_x)$ that

$$S_\lambda f = R_\lambda f + S_\lambda \mathcal{B} R_\lambda f,$$

and $\mathcal{B} R_\lambda f$ is again in $\mathcal{C}_w^\alpha(\mathbb{R}_+^d)$. By iterating,

$$S_\lambda f = \sum_{i=0}^{n} R_\lambda (\mathcal{B} R_\lambda)^i f + S_\lambda (\mathcal{B} R_\lambda)^{n+1} f.$$

Since the $\mathcal{C}_w^\alpha(\mathbb{R}_+^d)$ norm of $\mathcal{B} R_\lambda$ is less than or equal to $\frac{1}{2}$, and the $L^\infty$ norm is bounded by the $\mathcal{C}_w^\alpha(\mathbb{R}_+^d)$ norm, we can let $n \to \infty$ and obtain

$$S_\lambda f(x) = \sum_{i=0}^{\infty} R_\lambda (\mathcal{B} R_\lambda)^i f(x), \tag{6.6}$$

where the series converges in $\|\cdot\|_\infty$.

The above equation is valid for any strong Markov family of solutions to the martingale problem, and therefore $S_\lambda f(x)$ is uniquely determined for $f \in \mathcal{C}_w^\alpha(\mathbb{R}_+^d)$ and $\lambda > \lambda_0 \vee \lambda_b$. Since $\mathcal{C}_w^\alpha(\mathbb{R}_+^d)$ is dense in the set of bounded measurable functions with respect to bounded pointwise convergence, $S_\lambda f(x)$ is uniquely determined for all bounded, measurable $f$, provided $\lambda > \lambda_0 \vee \lambda_b$. Standard techniques (see [B97], Theorem VI.3.2) then yield our Proposition. □

**Proof of Theorem 1.2.** In view of Remark 1.1(a) we only need consider the uniqueness part of Theorem 1.2. We will set $c_{1.1} = (2c_{6.2})^{-1}$, where $c_{6.2}$ is as in Proposition 6.2.

A standard conditioning argument allows us to assume $\nu = \delta_z$, where $z \in \mathbb{R}^d_+$ (see [B97], p.136). If $T_R$ is the exit time from $[0,R]^d$, it suffices to show that $P(X(\cdot \wedge T_R) \in \cdot)$ is unique for any $R > 0$ and any solution $P$ of $MP(\mathcal{L}, \delta_z)$. Therefore, by changing $b$ and $\gamma$ outside $[0,R]^d$ we may assume that $b$ and $\gamma$ are bounded. Let $MP_0(\mathcal{L}, \nu)$ denote the martingale problem in which the local martingales in $MP$ are now assumed to be martingales; that is, if $f \in C^2_b(\mathbb{R}^d_+)$, then $f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$ is a martingale and not just a local martingale. It is easy to see that for our bounded $\gamma$ and $b$ these two martingale problems are equivalent. By a localization argument of Stroock and Varadhan (Theorem 6.6.1 of [SV79] – see also Theorem VI.3.4 in [B97] and the proof of Theorem 1.1 in [ABBP01]), it suffices to show that for each $x^0 \in \mathbb{R}^d_+$ there is an $r = r(x^0) > 0$ and continuous maps $(\widetilde{a}_{ij}) : \mathbb{R}^d_+ \to S^+_d$, $(\widetilde{b}_i) : \mathbb{R}^d_+ \to \mathbb{R}^d$ which agree with $(a_{ij}) = (\sqrt{x_i x_j}\gamma_{ij})$ and $(b_i)$, respectively, on $B(x^0, r) \cap \mathbb{R}^d_+$ and such that there is exactly one solution to $MP_0(\widetilde{\mathcal{L}}, \delta_z)$ for any $z \in \mathbb{R}^d_+$, where

$$\widetilde{\mathcal{L}}f(x) = \sum_{i,j} \widetilde{a}_{ij}(x) f_{ij}(x) + \sum_i \widetilde{b}_i(x) f_i(x). \tag{6.7}$$

A few comments concerning the application of this localization argument are in order. Firstly, as we are only using it to prove uniqueness, the boundedness of $a$, assumed in [SV79], is not needed. Local boundedness suffices for the proof of uniqueness up to $T_R$ and $T_R \to \infty$ as $R \to \infty$ as we are dealing with globally defined solutions. Secondly, the Borel measurability in $z$ of the solutions to $MP_0(\widetilde{\mathcal{L}}, \delta_z)$ assumed in [SV79] follows from the uniqueness of the solution to this martingale problem just as in Exercise 6.7.4 of [SV79]; here it is more convenient to work with $MP_0(\widetilde{\mathcal{L}}, \delta_z)$ instead of $MP(\widetilde{\mathcal{L}}, \delta_z)$.

If $x^0_i \neq 0$ for some $i$, set $r_0 = \frac{1}{4}\min\{x^0_i : 1 \le i \le d, x^0_i \neq 0\}$; otherwise set $r_0 = 1$. For $r > 0$, let $\rho_r : [0,\infty) \to [0,1]$ be the function that is 1 on $[0,r]$, 0 on $[2r,\infty)$, and linear in between.

For the sake of clarity, we first consider a special case.

**Case 1.** $b^0_i = b_i(x^0) \ge 0$ for $i \le d$.
Let $\gamma^0_{ij} = \gamma_{ij}(x^0)$, $\mu^0 = \min_{1\le i\le d}\gamma^0_{ii} > 0$ and

$$\overline{\varepsilon}(r) = \sup_{x \in B(x^0,r)\cap\mathbb{R}^d_+} \Big[\sum_{i=1}^d |\gamma^0_{ii} - \gamma_{ii}(x)| + \sum_{i=1}^d |b^0_i - b_i(x)| + \sum_{i\neq j} |\gamma_{ij}(x)|\Big].$$

Choose $r \in (0, r_0)$ such that $c_{6.2}\overline{\varepsilon}(2r) \le \mu^0$. This is possible by (1.4), our choice of $c_{1.1}$, and the continuity of our coefficients. For $x \in \mathbb{R}^d_+$, let

$$\widetilde{\gamma}_{ij}(x) = \rho_r(|x - x^0|)\gamma_{ij}(x) + [1 - \rho_r(|x - x^0|)]\gamma^0_{ij},$$

and

$$\widetilde{b}_i(x) = \rho_r(|x - x^0|)b_i(x) + [1 - \rho_r(|x - x^0|)]b_i^0.$$

Set $\widetilde{a}_{ij}(x) = \sqrt{x_i x_j}\widetilde{\gamma}_{ij}(x)$. Clearly $\widetilde{a}$ and $\widetilde{b}$ agree with $a$ and $b$, respectively, on $B(x^0, r)$. Let us check that these coefficients satisfy the hypotheses of Proposition 6.3. It is easy to check that $\widetilde{\gamma}$ and $\widetilde{b}$ satisfy (H1) and (H2) since $\gamma$ and $b$ do, and $S_d^+$ is convex. These coefficients are clearly constant outside of $B(x^0, 2r)$ and are uniformly $\alpha$-Hölder continuous. Finally our choice of $r$ and a short calculation shows that (6.2) is valid. Therefore Proposition 6.3 implies the existence of a unique solution to $MP(\widetilde{\mathcal{L}}, \delta_z)$ for each $z$ in the positive orthant. In view of the boundedness of $\widetilde{\gamma}$ and $\widetilde{b}$ this martingale problem is equivalent to $MP_0(\widetilde{\mathcal{L}}, \delta_z)$.

We now consider the general case.

**Case 2.** By relabeling the axes if necessary, let us choose $0 \leq m \leq d$ and assume $x_i^0 = 0$ for $i \leq m$. By (H2)

$$b_i(x^0) \geq 0 \text{ for all } i \leq m. \tag{6.8}$$

If $m = d$, we are in Case 1, so we assume $m < d$. Let $a_{ij}(x) = \sqrt{x_i x_j}\gamma_{ij}(x)$ and choose $r \in (0, r_0]$. By this choice of $r$ and (H1) there is an $\varepsilon_0(r) > 0$ such that if $x \in \overline{B(x^0, 2r)} \cap \mathbb{R}_+^d$, then for any $(v_i)_{i>m} \in \mathbb{R}^{d-m}$,

$$\sum_{i>m}\sum_{j>m} v_i a_{ij}(x) v_j \geq \varepsilon_0(r) \sum_{i>m} x_i v_i^2 \geq \varepsilon_0(r) r \sum_{i>m} v_i^2.$$

This proves that the continuous $(d-m)\times(d-m)$-matrix valued map $(a_{ij})_{i,j>m}$ is uniformly positive definite on $\overline{B(x^0, 2r)} \cap \mathbb{R}_+^d$. Therefore it has a bounded continuous inverse $(g_{ij}(x))$ for $x$ as above. Define

$$\delta_i(x) = 1_{(i>m)}\rho_r(|x - x^0|) \sum_{j>m} g_{ij}(x) b_j(x) \quad \text{for } x \in \mathbb{R}_+^d.$$

Then $\delta$ is bounded and continuous on the positive orthant. If $\bar{b}_i(x) = 1_{(i\leq m)} b_i(x)$, then $\bar{b}_i(x^0) \geq 0$ for all $i \leq d$ and so $\bar{b}$ satisfies the hypotheses of Case 1. That argument shows by taking $r$ smaller, if necessary, we may assume there are $\widehat{b} \in \mathbb{R}^d$ and $(\widetilde{\gamma}_{ij}) \in S_d^+$ which satisfy the hypotheses of Proposition 6.3, agree with $\bar{b}$ and $\gamma$, respectively, on $B(x^0, r)$, and for which $MP_0(\widehat{\mathcal{L}}, \delta_z)$ has exactly one solution for each $z \in \mathbb{R}_+^d$. Here $\widehat{\mathcal{L}}$ is defined as in (6.7) but with $\widehat{b}$ and $\widetilde{\gamma}$ instead of $\widetilde{b}$ and $\widetilde{\gamma}$. Now set

$$\widetilde{b}_i(x) = \widehat{b}_i(x) + \sum_j \sqrt{x_i x_j}\widetilde{\gamma}_{ij}(x)\delta_j(x),$$

and note that

$$\text{if } x_i = 0 \text{ then } \widetilde{b}_i(x) = \widehat{b}_i(x) \geq 0, \tag{6.9}$$

where we have used the fact that $\widehat{b}$ satisfies (H2) in the last inequality. Let $\widetilde{\mathcal{L}}$ denote the usual operator with coefficients $\widetilde{b}$ and $\widetilde{\gamma}$. The existence of a solution to $MP_0(\widetilde{\mathcal{L}}, \delta_z)$ is again standard as $\widetilde{\gamma}$ is bounded and continuous, $\widetilde{b}$ has linear growth, and (6.9) ensures solutions will remain in the first orthant. Girsanov's theorem (see V.27 in [RW87]) and the fact that there is a unique solution to $MP_0(\widehat{\mathcal{L}}, \delta_z)$ shows that the solution to $MP_0(\widetilde{\mathcal{L}}, \delta_z)$ is unique. Here note that although $\widetilde{a}_{ij}(x) = \sqrt{x_i x_j}\widetilde{\gamma}_{ij}(x)$ is unbounded, one can still apply Girsanov's theorem to show that the law of $P(X(\cdot \wedge T_R) \in \cdot)$ is unique where $T_R$ is the exit time from $[0, R]^d$ and this gives the result. Note also that Girsanov's theorem applies without change in our $\mathbb{R}^d_+$-valued setting. Finally, if $x \in B(x^0, r)$, then

$$\begin{aligned}
\widetilde{b}_i(x) &= 1_{(i\le m)} b_i(x) + \sum_{j=1}^{d} \sqrt{x_i x_j}\gamma_{ij}(x)\delta_j(x) \\
&= 1_{(i\le m)} b_i(x) + \sum_{k>m}\sum_{j>m} a_{ij}(x) g_{jk}(x) b_k(x) \\
&= 1_{(i\le m)} b_i(x) + \sum_{k>m} \delta_{ik} b_k(x) \\
&= b_i(x).
\end{aligned}$$

This completes the construction of the required coefficients which agree with $b$ and $\gamma$ locally and hence the proof is complete. □

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