Countable Systems of Degenerate Stochastic Differential Equations with Applications to Super-Markov Chains

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Abstract

We prove well-posedness of the martingale problem for an infinite-dimensional degenerate elliptic operator under appropriate Hölder continuity conditions on the coefficients. These martingale problems include large population limits of branching particle systems on a countable state space in which the particle dynamics and branching rates may depend on the entire population in a Hölder fashion. This extends an approach originally used by the authors in finite dimensions.

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1. Introduction.

We prove existence and uniqueness of the martingale problem for the infinite-dimensional degenerate operator

$$\mathcal{L}f(x) = \sum_{i\in S}[x_i\gamma_i(x)f_{ii}(x) + b_i(x)f_i(x)]$$

under suitable Hölder continuity assumptions on the coefficients γ_i and b_i. Here S is a countably infinite discrete set, we write $x = (x_i)_{i\in S}$ with $x_i \ge 0$ for each i, $\mathcal{L}$ operates on the class of finite-dimensional cylindrical functions, and f_i and f_{ii} denote the first and second partials of f in the direction x_i.

In the last ten years there has been considerable interest in infinite-dimensional operators whose coefficients are only Hölder continuous rather than Lipschitz continuous. See [CD96], [D96], [L96], [Z00], and [DZ02], for example, which consider operators that are perturbations of either the infinite-dimensional Laplacian or of the infinite-dimensional Ornstein-Uhlenbeck operator. The operator $\mathcal{L}$ given above is not only infinite-dimensional, but also degenerate, due to the x_i factor in the second order term. This degeneracy also means that the diffusion coefficient will not have a Lipschitz square root even for smooth γ_i, invalidating the standard fixed point approaches.

The principal motivation for this work is the question of uniqueness for measure-valued diffusions which behave locally like a superprocess. In general assume S is a Polish space and let $M_F(S)$ denote the space of finite measures on S with the weak topology. Write $m(f) = \int f\,dm$ for $m \in M_F(S)$ and an appropriate $\mathbb{R}$-valued f on S. Assume $\{A_x : x \in M_F(S)\}$ is a collection of generators, all defined on an appropriate domain D_0 of bounded continuous functions on S, and $\gamma : S \times M_F(S) \to \mathbb{R}_+$. Let Ω be $C(\mathbb{R}_+, M_F(S))$, equipped with the topology of uniform convergence on bounded intervals, its Borel σ-field $\mathcal{F}$, canonical right-continuous filtration $\mathcal{F}_t$, and coordinate maps $X_t(\omega) = \omega_t$.

For each law μ on $M_F(S)$, a probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a solution of the martingale problem associated with A, γ and initial law μ, written $MP(A, \gamma, \mu)$, if for each $f \in D_0$,

$$X_t(f) = X_0(f) + M_t^f + \int_0^t X_s(A_{X_s}f)\,ds,$$

where M^f is a continuous $\mathcal{F}_t$-martingale such that

$$\langle M^f\rangle_t = \int_0^t X_s(2\gamma(\cdot, X_s)f^2)\,ds.$$

Under appropriate continuity conditions on (A_x, γ) one can usually construct solutions to $MP(A, \gamma, \mu)$ as the weak limit points of large population (N), small mass (N^{-1}) systems of branching particle systems. In these approximating systems a particle at x

in population X_t branches into a mean 1 number of offspring with rate $N\gamma(x, X_t)$, and between branch times particles evolve like a Markov process with generator A_{X_t} (see e.g. [MR92]). The main difficulty lies in questions of uniqueness of solutions to $MP(A, \gamma, \mu)$. A case of particular interest is $S = \mathbb{R}^d$ and

$$A_X f(y) = \sum_{i,j \le d} a_{ij}(y, X) f_{ij}(y) + \sum_{i \le d} b_i(y, X) f_i(y),$$

in which particles evolve according to a state dependent Itô equation between branch times. For $\gamma = \gamma_0$ constant, uniqueness is proved in [DK98] under appropriate Lipschitz conditions on a, b, using methods in [P95]. The latter also effectively handles the case $\gamma(y, X) = \gamma(y)$ (and some other special cases of X-dependence) by proving uniqueness for an associated strong equation and historical martingale problem.

Even in the case where S is finite, the problem of handling general γ was only recently solved in [ABBP02] and [BP03]. If $S = \{1, \ldots, d\}$, then $M_F(S) = \mathbb{R}^d_+$ and $A_x f(i) = \sum_{j=1}^d q_{ij}(x) f(j)$, where for each $x \in \mathbb{R}^d_+$, $(q_{ij}(x))$ is a Q-matrix of a Markov chain on S, that is, $q_{ij} \ge 0$ for $i \ne j$ and $q_{ii} = -\sum_{j \ne i} q_{ij}$. Then X solves $MP(A, \gamma, \mu)$ if and only if $X_t \in \mathbb{R}^d_+$ solves the degenerate stochastic differential equation

$$X^i_t = X^i_0 + \int_0^t \sum_{j=1}^d X^j_s q_{ji}(X_s)\, ds + \int_0^t (2\gamma_i(X_s) X^i_s)^{1/2}\, dB^i_s, \qquad i = 1 \ldots d. \tag{1.1}$$

Here $\gamma_i : \mathbb{R}^d_+ \to [0, \infty)$, $i = 1, \ldots, d$, $B^1, \ldots, B^d$ are independent one-dimensional Brownian motions, and X_0 has law μ for a given probability measure μ on $\mathbb{R}^d_+$. More generally, consider the generator

$$\mathcal{L} f(x) = \sum_{i=1}^d [x_i \gamma_i(x) f_{ii}(x) + b_i(x) f_i(x)]$$

for $f \in C^2_b(\mathbb{R}^d_+)$, the space of bounded continuous functions on $\mathbb{R}^d_+$ whose first and second partials are also bounded and continuous; $b_i(x) = \sum_j x_j q_{ji}(x)$ would correspond to (1.1). If μ is a law on $\mathbb{R}^d_+$, a probability $\mathbb{P}$ on $C(\mathbb{R}_+, \mathbb{R}^d_+)$ solves $MP(\mathcal{L}, \mu)$ if and only if for all $f \in C^2_b(\mathbb{R}^d_+)$,

$$M^f_t = f(X_t) - f(X_0) - \int_0^t \mathcal{L} f(X_s)\, ds$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}$ and X_0 has law μ.

Theorem A (Corollary 1.3 of [BP03]). *Assume $\gamma_i : \mathbb{R}^d_+ \to (0, \infty)$, $b_i : \mathbb{R}^d_+ \to \mathbb{R}$ are α-Hölder continuous on compact sets and satisfy*

$$b_i(x) \ge 0 \text{ on } \{x_i = 0\}, \tag{1.2}$$

$$|b_i(x)| \le c(1+|x|). \tag{1.3}$$

Then there is a unique solution to $MP(\mathcal{L},\mu)$ for each law μ on $\mathbb{R}_+^d$.

A similar existence and uniqueness theorem was proved in [ABBP02] (see Theorem A of [BP03]) assuming only continuity of γ_i and b_i but with (1.2) strengthened to

$$b_i(x) > 0 \text{ on } \{x_i = 0\}. \tag{1.4}$$

A simple one-dimensional example shows these results are sharp in the sense that uniqueness fails if only continuity and (1.2) are assumed (see Section 8 of [ABBP03]). Clearly (1.2) is needed to ensure solutions remain in the positive orthant.

In this work we extend the method of [BP03] to the case where S is a countably infinite discrete set and hence take a step towards resolving the general uniqueness problem described above. Both [ABBP02] and [BP03] adapt the perturbation approach of [SV79] to this setting by considering $\mathcal{L}$ as a perturbation of $\mathcal{L}_0 f = \sum_i x_i \gamma_i^0 f_{ii} + b_i^0 f_i$ for some constants $\gamma_i^0 > 0$ and $b_i^0 \ge 0$. If R_λ is the resolvent associated with $\mathcal{L}^0$, the key step is to show that on a suitable Banach space $(B, \|\ \cdot\|)$, one has

$$\|(R_\lambda f)_i\| + \|x_i (R_\lambda f)_{ii}\| \le C\|f\| \text{ for all } i. \tag{1.5}$$

In [ABBP02] the space B is $L^2(\mathbb{R}_+^d, \prod_1^d x_i^{b_i^0/\gamma_i^0 - 1} dx_i)$ (here $b_i^0 > 0$), while in [BP] the space B is a weighted Hölder space ((4.9) below gives the precise norm). In both cases the constant C in (1.5) is independent of d. The L^2 setting in [ABBP02], however, does not appear to extend readily to infinite dimensions. The measures become infinite, there are problems extending the Krylov-Safanov type theorems on regularity of the resolvents which are required to handle all starting points as opposed to almost all starting points, and, as in the finite-dimensional setting, (1.4) will not hold for the most natural Q-matrices such as nearest random walk on the discrete circle. We therefore will extend the weighted Hölder approach in [BP03]. This approach has also been effective in other (non-singular) infinite-dimensional settings ([ABP03]).

Our main result (Theorem 2.7) states that the natural infinite dimensional analogue of $MP(\mathcal{L},\nu)$ has a unique solution when S is a discrete countably infinite set and X takes on values in an appropriate space of measures. To understand the nature of the assumptions made on the coefficients b_i and γ_i, consider the Corollaries 2.10–2.12 when $S = \mathbb{Z}^d$. Basically, we require b_i and γ_i to be Hölder continuous in the j^{th} variable, where the Hölder constant approaches 0 at a certain polynomial rate as $|i-j|$ approaches ∞. The state space of X will be measures $x(\cdot)$ satisfying $\sum_i |i|^q x(i) < \infty$ where $q > 0$ may approach zero for α close to 1 but becomes large as α gets small. There are cases where infinite measures are allowed but they require stronger Hölder conditions on the coefficients as the mass gets large (see Remark 2.13).

The main result and a number of corollaries are stated in Section 2. In Section 3 we prove a more general existence theorem (Theorem 2.4) by truncating to a finite-dimensional system and taking weak limits. Although these type of arguments are well-known (see [SS80]), we could not find the particular result we needed in the literature and have included the proof for completeness: in addition, there is an unexpected mild condition needed. The weighted Hölder spaces are introduced in Section 4 where the infinite dimensional analogues of (1.5) are derived. Since the constants in [BP03] are independent of dimension this should be easy, but some complications arise in infinite dimensions since boundedness of the weighted Hölder norms does not imply continuity, in contrast to the case of finite dimensions. We must establish uniform convergence of the appropriate derivatives of the resolvent by the corresponding quantities for a sequence of approximating finite-dimensional functions to carry over the finite-dimensional estimates from [BP03] and obtain continuity (Proposition 4.8). The key bounds on the weighted Hölder norm then follow from the finite-dimensional result in [BP] (Corollary 4.10). This approximation is also used to derive the perturbation equation for the resolvent of strong Markov solutions of $MP(\mathcal{L},\mu)$ in terms of $R_\lambda = (\lambda - \mathcal{L}_0)^{-1}$ (Proposition 5.4).

In Section 5 local uniqueness is established (Theorem 5.5), i.e., if γ_i and b_i are sufficiently close to constant functions, uniqueness is shown. In this setting our state space may include counting measure, but as the coefficients become asymptotically constant this is not surprising. In Section 6 we use the local uniqueness and a localization argument to prove Theorem 2.7. Localization in infinite dimensions still seems to be an awkward process and our arguments here are surely not optimal–we believe some of the additional continuity conditions in Assumption 2.6 may be weakened. Still it is important to note that the weighted Hölder spaces at least allow for localization. In their ground-breaking paper [DM95], Dawson and March establish a quite general uniqueness result in the Fleming-Viot setting but were are unable to carry out the localization step. Nonetheless [DM95] still represents the best available uniqueness result in general infinite dimensional settings albeit in the Fleming-Viot setting and for close to constant coefficients. Finally in Section 7 we prove the various corollaries to Theorem 2.7.

2. Notation and statement of results.

We will use the letter c with or without subscripts to denote positive finite constants whose exact value does not matter and which may change from line to line. We use κ with subscripts to denote positive finite constants whose value does matter.

Let S be a countable set equipped with a map $|\cdot| : S \to [0,\infty)$ such that $S_n = \{i \in S : |i| < n\}$ is finite for all $n \in \mathbb{Z}^+$. ($S_0 = \emptyset$). Our prototype is of course $S = \mathbb{Z}^d$ with $|\cdot|$ equal to the usual Euclidean length of i. Let $\nu : S \to (0,\infty)$ and for $x \in \mathbb{R}^S$ let

$$|x|_\nu = \sum_{i\in S} |x_i|\nu_i \in [0,\infty].$$

ν will be called a weight functon. We will use both ν_i and $\nu(i)$ for the i^{th} coordinate of ν and similarly for other maps defined on S.

We let

$$M_\nu(S) = \{x \in \mathbb{R}_+^S : |x|_\nu < \infty\}.$$

and consider elements of $M_\nu(S)$ as measures on S with

$$\langle x, \varphi \rangle = \sum_{i\in S} x_i \varphi(i) \qquad \text{when} \qquad \varphi : S \to \mathbb{R}, \qquad \sum_{i\in S} x_i |\varphi(i)| < \infty.$$

It is easy to see that $M_\nu(S)$ is a Polish space when equipped with the distance $|x - x'|_\nu$. If $\nu_i \equiv 1$, it is easy to check that $M_\nu(S) = M_F(S)$ is the usual space of finite measures on S equipped with the topology of weak convergence for the discrete topology on S.

$\mathbb{R}_+ = [0, \infty)$ and $C_b^2(\mathbb{R}_+^{S_n})$ is the set of bounded continuous functions $f : \mathbb{R}_+^{S_n} \to \mathbb{R}$ whose first and second partial derivatives f_i, f_{ij} are bounded and continuous. If $x_i = 0$, then the partials f_i, f_{ij} are interpreted as right-hand derivatives.

Define the projection operator $\pi_n : \mathbb{R}_+^S \to \mathbb{R}_+^{S_n}$ by

$$\pi_n x(i) = x(i), \qquad i \in S_n, \tag{2.1}$$

and define the lift operator $\Pi_n : \mathbb{R}_+^{S_n} \to \mathbb{R}_+^S$ by

$$\Pi_n x(i) = 1_{(i\in S_n)} x(i). \tag{2.2}$$

Let

$$C_{b,F}^2(M_\nu(S)) = \{f : M_\nu(S) \to \infty : \exists n, f_n \in C_b^2(\mathbb{R}_+^{S_n}) \text{ such that } f(x) = f_n(\pi_n x)\}.$$

These are the functions which only depend (in a C^2 way) on the coordinates x_i with $i \in S_n$. Clearly $C_{b,F}^2(M_\nu(S)) \subset C_b(M_\nu(S))$, the space of bounded continuous maps from $M_\nu(S)$ to $\mathbb{R}$.

For $i \in S$ assume $\gamma_i : M_\nu(S) \to [0, \infty)$, $b_i : M_\nu(S) \to \mathbb{R}$, and for $f : M_\nu(S) \to \mathbb{R}$ define

$$\mathcal{L}f(x) = \sum_{i\in S} \big[x_i \gamma_i(x) f_{ii}(x) + b_i(x) f_i(x)\big], \qquad x \in M_\nu(S), \tag{2.3}$$

provided these partial derivatives exist and the above series is absolutely convergent. Note that this is the case if $f \in C_{b,F}^2(M_\nu(S))$. Let Ω_ν equal $C(\mathbb{R}_+, M_\nu(S))$, equipped with the topology of uniform convergence on bounded intervals. Let $X_t(\omega) = \omega(t)$ for $\omega \in \Omega_\nu$, let $\mathcal{F}_u^0$ be the universal completion of $\sigma(X_s : s \le u)$, and set $\mathcal{F}_t = \cap_{u>t} \mathcal{F}_u^0$, $\mathcal{F} = \cup_{t\ge 0} \mathcal{F}_t^0$.

Definition 2.1. *Let μ be a probability on $M_\nu(S)$. A probability $\mathbb{P}$ on $(\Omega_\nu, \mathcal{F})$ solves $\mathcal{MP}(\mathcal{L}, \mu)$, the martingale problem for $\mathcal{L}$ in $M_\nu(S)$ started at μ, if $\mathbb{P}(X_0 \in \cdot) = \mu(\cdot)$ and for any $f \in C^2_{b,F}(M_\nu(S))$*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$$

is an $(\mathcal{F}_t)$-local martingale under $\mathbb{P}$. $\mathcal{MP}(\mathcal{L})$ is well-posed in $M_\nu(S)$ if and only if there is a unique solution to $\mathcal{MP}(\mathcal{L}, \mu)$ in $M_\nu(S)$ for every initial law μ on $M_\nu(S)$.

Note that $s \to \mathcal{L}f(X_s)$ and $t \to f(X_t)$ and hence $t \to M_t^f$ are all necessarily continuous functions.

Remark 2.2 (a) As in Remark 1.1(d) of [BP03] one could also consider test functions $f(x) = f_n(\pi_n x)$ for some $f_n \in C_b^2(\mathbb{R}^{S_n})$, (i.e., those which extend in a C^2 manner to all of $\mathbb{R}^{S_n}$ instead of $\mathbb{R}_+^{S_n}$).

(b) Changing the class of test functions changes the martingale problem. The smaller the class of test functions for which one establish uniqueness, the stronger the theorem. $C^2_{b,F}(M_\nu(S))$ is a reasonably small class.

(c) Let $\{B_i : i \in S\}$ be a sequence of independent one-dimensional adapted Brownian motions on a filtered probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{F}_t, \overline{\mathbb{P}})$ and consider the stochastic differential equation

$$Y_i(t) = Y_i(0) + \int_0^t (2Y_i(s))\gamma_i(Y_s))^{1/2} dB_s^i + \int_0^t b_i(Y_s)ds, \qquad i \in S, t \geq 0. \tag{2.4}$$

Here $Y(0)$ is an $\overline{\mathcal{F}}_0$-measurable random vector in $M_\nu(S)$. If Y is a continuous $M_\nu(S)$-valued solution to (2.4), a simple application of Itô's formula shows that the law $\mathbb{P}_Y$ of Y is a solution to $\mathcal{MP}(\mathcal{L}, \mu)$ where μ is the law of $Y(0)$. Conversely, given a solution $\mathbb{P}$ of $\mathcal{MP}(\mathcal{L}, \mu)$, a standard construction allows one to build Brownian motions $\{B^i : i \in S\}$ and a solution Y to (2.4) on some $\overline{\Omega}$ such that $\mathbb{P}$ is the law of Y.

We introduce conditions on b_i, the first of which we assume throughout this work:

Assumption 2.3. *(a) There exists a constant $\kappa_{2.3a}(b)$ such that*

$$\sum_{i \in S} |b_i(x)|\nu_i \leq \kappa_{2.3a}(b)(|x|_\nu + 1), \qquad x \in M_\nu(S), \tag{2.5}$$

(b) There exists a constant $\kappa_{2.3b}(b)$ such that

$$b_i(x) \geq -\kappa_{2.3b}(b)x_i, \qquad i \in S, \quad x \in M_\nu(S). \tag{2.6}$$

Assumption 2.3(a) will avoid explosions in finite time while a condition such as Assumption 2.3(b) is needed to ensure that our solutions have non-negative components (although a weaker condition $b_i(x) \geq 0$ if $x_i = 0$ sufficed in finite dimensions – see [BP03].)

Our focus is on uniqueness in law of solutions to $\mathcal{MP}(\mathcal{L}, \delta_{x_0})$, but as our setting is slightly different from that considered in the literature (e.g., Shiga and Shimizu [SS80]), we state a general existence result. The proof is given in Section 3.

Theorem 2.4. *Assume there exists $\beta : S \to (0, \infty)$ such that $\lim_{|i|\to\infty} \beta(i) = 0$ and for all $i \in S$, b_i, γ_i have (necessarily unique) continuous extensions $b_i : M_{\beta\nu}(S) \to \mathbb{R}$, $\gamma_i : M_{\beta\nu}(S) \to [0, \infty)$. In addition to (2.5) and (2.6) assume there exists a constant $\kappa_{2.4}(\gamma)$ such that*

$$\sup_i \|\gamma_i\|_\infty = \kappa_{2.4}(\gamma) < \infty. \tag{2.7}$$

Then for any $x_0 \in M_\nu(S)$ there is a solution to $\mathcal{MP}(\mathcal{L}, \delta_{x_0})$.

Remark 2.5. Note the above continuity condition is trivially satisfied if γ_i, b_i are given as continuous functions on $\mathbb{R}_+^S$ with the product topology (as in Shiga and Shimizu [SS80]). This condition is only needed to obtain a compact containment condition in the usual tightness proof and will be easy to verify in the examples of interest. If γ_i, b_i on $M_\nu(S)$ are uniformly continuous with respect to $|\cdot|_{\beta\nu}$, the above extensions exist.

Here is our key Hölder continuity hypothesis on γ_i and b_i:

Assumption 2.6. *For some $\beta : S \to (0, \infty)$ satisfying $\lim_{|i|\to\infty} \beta(i) = 0$ and $\beta(i) \leq \kappa_{2.6a}\nu(i)^{-\alpha/2}$, for each $i \in S$, b_i and γ_i have (necessarily unique) continuous extensions $b_i : M_{\beta\nu}(S) \to \mathbb{R}$, $\gamma_i : M_{\beta\nu}(S) \to (0, \infty)$. For any $x_0 \in M_\nu(S)$, $\eta > 0$, there exists $\delta_0 > 0$ and a $\kappa_{2.6b} > 0$ such that if $x \in M_\nu(S)$ and $|x - x_0|_{\beta\nu} < \delta_0$, then (a) holds and either (b) or (c) holds, where*

(a)

$$\sum_{i\in S} \frac{|\gamma_i(x) - \gamma_i(x_0)|}{\gamma_i(x_0)} + \frac{|b_i(x) - b_i(x_0)|}{\gamma_i(x_0)} < \eta \tag{2.8}$$

and

(b) for any $j \in S, h > 0$,

$$\sum_i \frac{|\gamma_i(x + he_j) - \gamma_i(x)|}{\gamma_i(x_0)} + \frac{|b_i(x + he_j) - b_i(x)|}{\gamma_i(x_0)} \leq \kappa_{2.6b}\gamma_j(x_0)^{-\alpha/2} x_j^{-\alpha/2} h^\alpha$$

or

(c) $\lim_{|i|\to\infty} \nu(i) = \infty$ and for any $j \in S$, $h \in (0, 1]$,

$$\sum_i \frac{|\gamma_i(x + he_j) - \gamma_i(x)|}{\gamma_i(x_0)} + \frac{|b_i(x + he_j) - b_i(x)|}{\gamma_i(x_0)} \leq \kappa_{2.6b}\gamma_j(x_0)^{-\alpha/2}(1 + x_j^{-\alpha/2})h^\alpha.$$

Our main result is the following.

Theorem 2.7. *Suppose Assumptions 2.3(a) and 2.6 hold, (2.7) holds, and either*

$$b_i(x) \geq 0 \text{ for all } i \in S \text{ and } x \in M_\nu(S), \tag{2.9}$$

or

$$\lim_{|i|\to\infty} \nu(i) = \infty \qquad \text{and} \qquad b_i(x) \geq -\kappa_{2.7} x_i \gamma_i(x), \text{ for all } i \in S \text{ and } x \in M_\nu(S). \tag{2.10}$$

Then $\mathcal{MP}(\mathcal{L})$ is well-posed in $M_\nu(S)$.

We state some corollaries to Theorem 2.7, the proofs of which are given in Section 7.

Corollary 2.8. *Assume $b_i : M_\nu(S) \to \mathbb{R}, \gamma_i : M_\nu(S) \to \mathbb{R}_+$ are continuous maps for all $i \in S$, where $\nu : S \to (0,\infty)$ satisfies $\lim_{|i|\to\infty} \nu(i) = \infty$. Assume Assumption 2.3(a)-(b), (2.7),*

$$\inf_{i,x} \gamma_i(x) = \varepsilon_0 > 0, \tag{2.11}$$

there exists non-negative constants $\{C(i,j) : i,j \in S\}$ such that

$$|\gamma_i(x + he_j) - \gamma_i(x)| + |b_i(x + he_j) - b_i(x)| \leq C(i,j)h^\alpha, \qquad h > 0, \quad x \in M_\nu(S), \tag{2.12}$$

and if $\overline{C}(j) = \sum_i C(i,j)$, then

$$\Big[\sup_j \overline{C}(j)\Big] + 1_{(\alpha<1)} \sum_j \overline{C}(j)^{1/(1-\alpha)} \nu(j)^{-\alpha(1-\frac{\alpha}{2})/(1-\alpha)} < \infty. \tag{2.13}$$

Then the hypotheses of Theorem 2.7 are valid and so $\mathcal{MP}(\mathcal{L})$ is well-posed in $M_\nu(S)$.

Recall that $Q = (q_{ij})_{i,j\in S}$ is a Q-matrix on S if $q_{ij} \geq 0$ for all $i \neq j$ and $\sum_{j\neq i} q_{ij} = -q_{ii}$ for all $i \in S$.

Corollary 2.9. *Let (q_{ij}) be a Q-matrix satisfying*

$$\overline{q} = \sup_i |q_{ii}| < \infty \tag{2.14}$$

$$\sum_i 1(i \neq j) q_{ji} \nu_i \leq \kappa_{2.9} \nu(j), \qquad j \in S, \tag{2.15}$$

where $\nu : S \to (0,\infty)$ satisfies $\lim_{|i|\to\infty} \nu(i) = \infty$. Suppose $(\widehat{b}_i, \gamma_i)_{i\in S}$ satisfies the hypotheses of Corollary 2.8 (or more generally of Theorem 2.7 and (2.11) holds). If

$$b_i(x) = \widehat{b}_i(x) + \sum_j x_j q_{ji},$$

then (b_i, γ_i) satisfies the hypotheses of Theorem 2.7 and so $\mathcal{MP}(\mathcal{L})$ is well posed in $M_\nu(S)$.

Consider now the case when $S = \mathbb{Z}^d$ and $|i|$ is the usual Euclidean norm.

Corollary 2.10. *Let $\nu(i) = (|i| + 1)^q$ for some $q > d(1 - \alpha)(\alpha(1 - \frac{\alpha}{2}))^{-1}$, and, for some $c_0 > 0$, $p > d$, set $C(i, j) = c_0(|i - j| + 1)^{-p}$ for $i, j \in \mathbb{Z}^d$. Let $b_i : M_\nu(\mathbb{Z}^d) \to \mathbb{R}$ and $\gamma_i : M_\nu(\mathbb{Z}^d) \to (0, \infty)$ be continuous maps satisfying (2.5), (2.6), (2.7), (2.11), and (2.12). Then $\mathcal{MP}(\mathcal{L})$ is well-posed in $M_\nu(\mathbb{Z}^d)$.*

Corollary 2.11. *Let $p : \mathbb{Z}^d - \{0\} \to [0, 1]$ be a probability on $\mathbb{Z}^d - \{0\}$ such that $m_q = \sum_i |i|^q p(i) < \infty$, where q is as in Corollary 2.10. Let $q_{ij} = \lambda p(j - i)$ $(i \neq j)$ be the Q-matrix of a random walk which takes steps distributed as p with rate $\lambda > 0$. Let $\nu, (\widehat{b}_i)$, and (γ_i) satisfy the hypotheses of Corollary 2.10, and let $b_i(x) = \widehat{b}_i(x) + \sum_j x_j q_{ji}$. Then $\mathcal{MP}(\mathcal{L})$ is well-posed in $M_\nu(\mathbb{Z}^d)$.*

Corollary 2.12. *Let $R > 0$, $N = \{i \in \mathbb{Z}^d : |i| \leq R\}$ and $\gamma : \mathbb{R}_+^N \to [\varepsilon, \varepsilon^{-1}]$ be Hölder continuous of index $\alpha \in (0, 1]$. Assume $q > d(1 - \alpha)(\alpha(1 - \frac{\alpha}{2}))^{-1}$ and set $\nu(i) = (|i| + 1)^q$, $i \in \mathbb{Z}^d$. Let $q_{ij} = \lambda p(j - i)$, $(i \neq j)$ where p is a probability on $\mathbb{Z}^d - \{0\}$ with finite q^{th} moment. If $b_i(x) = \sum_j x_j q_{ji}$ and $\gamma_i(x) = \gamma((x_{i+j} : j \in N))$, then $\mathcal{MP}(\mathcal{L})$ is well-posed in $M_\nu(\mathbb{Z}^d)$.*

Corollary 2.8 is a version of Theorem 2.7 where the hypotheses are given in terms of the Hölder constants of the γ_i and b_i; Corollary 2.9 applies Corollary 2.8 to the case of super-Markov chains. Corollaries 2.10–2.12 are the application of Corollaries 2.8 and 2.9 to the case where $S = \mathbb{Z}^d$ and an explicit bound is given for the Hölder constants of the γ_i and b_i.

Remark 2.13. If we assume Assumptions 2.6 (a),(b) and (2.9) we may take $\nu_i \to 0$ so that $M_\nu(S)$ will contain some infinite measures, that is, points x such that $\sum_{i \in S} x_i = \infty$. In this case the Hölder condition Assumption 2.6 (b) becomes rather strong if x_j gets large.

3. Existence.

If $\varepsilon = \{\varepsilon_n\}$ is a sequence in $(0, \infty)$ decreasing to 0 and $S' = \{S'_n\}$ is a sequence of finite subsets of S which increases to S with $S'_0 = \emptyset$, let

$$K_{\varepsilon, S'} = \{x \in M_\nu(S) : \sum_{i \notin S'_n} x_i \nu_i \leq \varepsilon_n \text{ for all } n \in \mathbb{Z}_+\}.$$

Write K_ε for the above in the case when $S'_n = S_n$ for all $n \in \mathbb{Z}_+$.

Lemma 3.1. *(a) For any ε, S' as above, $K_{\varepsilon, S'}$ is a compact subset of $M_\nu(S)$.*
(b) If K is a compact subset of $M_\nu(S)$, there is a sequence ε_n decreasing to 0 such that $K \subset K_\varepsilon$.

Proof. The standard proof is left for the reader. □

Remark 3.2. If $\sigma_N : M_\nu(S) \to M_\nu(S)$ is defined by $\sigma_N(x)(i) = 1_{(i\in S_N)}x(i)$, then for any sequence ε_n decreasing to 0, it is easy to check that $\sigma_N(K_\varepsilon) \subset K_\varepsilon$.

Proof of Theorem 2.4. First, let X_t^n be the solution to

$$X_t^{n,i} = x_0^i + 1_{(i\in S_n)}\Big[\int_0^t b_i((X_s^n)^+)ds + \int_0^t \Big(\gamma_i((X_s^n)^+)|X_s^{n,i}|\Big)^{1/2} dB_s^i\Big]. \tag{3.2}$$

Here $(X_s^n)^+ = ((X_s^{n,i})^+ : i \in S)$ and $\{B^i\}$ is a sequence of independent one-dimensional Brownian motions on some filtered probability space. Note that

$$\widehat{b}_i((x_j)_{j\in S_n}) = b_i((x_j^+)_{j\in S_n}, (x_0^j)_{j\in S_n^c})$$

is a continuous function on $\mathbb{R}^{S_n}$ with linear growth (by (2.5) and the continuity assumptions on b_i). The same is true of

$$\widehat{\sigma}_i((x_j)_{j\in S_n}) = \Big(\gamma_i((x_j^+)_{j\in S_n}, (x_0^j)_{j\in S_n^c})|x_i|\Big)$$

and so the existence of X^n follows from Skorokhod's existence theorem for finite-dimensional SDEs. Using $L_t^0(X^{n,i}) = 0$ and $b_i(x) \geq 0$ if $x_i = 0$ (by (2.6)), one can use Tanaka's formula to see that $X_t^{n,i} \geq 0$ for all $t \geq 0$ and for all i almost surely, and one may therefore remove the superscript $^+$'s in (3.2). Let $T_k^n = \inf\{t : |X_t^n|_\nu \geq k\}$. For each n, $T_k^n \uparrow \infty$ as $k \to \infty$, since $x_0 \in M_\nu(S)$. Define $M_t^n = \sum_{i\in S_n} \nu_i \int_0^t (\gamma_i(X_s^n)X_s^{n,i})^{1/2} dB_s^i$, and note that if $\overline{\nu}_n = \max_{i\in S_n} \nu_i$, then

$$\begin{aligned}\langle M_t^n\rangle &= \int_0^t \sum_{i\in S_n} \nu_i^2 \gamma_i(X_s^n)X_s^{n,i} ds \\ &\leq \kappa_{2.4}\overline{\nu}_n \int_0^t |X_s^n|_\nu ds \\ &\leq \kappa_{2.4}\overline{\nu}_n kt \end{aligned} \tag{3.3}$$

for $t \leq T_k^n$. Let

$$\begin{aligned}\overline{X}_t^n &= |X_t^n|_\nu + \kappa_{2.3b}\int_0^t |X_s^n|_\nu ds \\ &= |x_0|_\nu + \int_0^t \sum_{i\in S_n} \nu_i b_i(X_s^n) + \kappa_{2.3b}|X_s^n|_\nu ds + M_t^n. \end{aligned} \tag{3.4}$$

$M_{t\wedge T_k^n}^n$ is a martingale by (3.3) and so (2.5) implies

$$\begin{aligned}\mathbb{E}\,\overline{X}_{t\wedge T_k^n}^n &\leq |x_0|_\nu + \int_0^t \mathbb{E}\,(\kappa_{2.3a}(|X_{s\wedge T_k^n}^n|_\nu + 1) + \kappa_{2.3b}|X_{s\wedge T_k^n}^n|_\nu)ds \\ &\leq |x_0|_\nu + \kappa_{2.3a}t + c_1\int_0^t \mathbb{E}\,(|\overline{X}_{s\wedge T_k^n}^n|_\nu)ds.\end{aligned}$$

The left hand side is clearly finite by the definition of T_k^n, Gronwall's lemma implies

$$\mathbb{E}\,(\overline{X}^n_{t\wedge T_k^n}) \leq [|x_0|_\nu + \kappa_{2.3a}t]e^{c_1 t}, \qquad t \geq 0,$$

and so Fatou's lemma gives

$$\mathbb{E}\,(\overline{X}^n_t) \leq [|x_0|_\nu + \kappa_{2.3a}t]e^{c_1 t}, \qquad t \geq 0. \tag{3.5}$$

Therefore, using (2.6) in (3.4), we see that $\overline{X}^n_t$ is a submartingale. The weak L^1 inequality and (3.5) imply

$$\mathbb{P}(\sup_{t\leq T}|X_t^n|_\nu \geq k) \leq \mathbb{P}(\sup_{t\leq T}\overline{X}^n_t \geq k) \leq k^{-1}[|x_0|_\nu + \kappa_{2.3a}T]e^{c_1 T}, \tag{3.6}$$

which implies

$$\lim_{k\to\infty}\sup_n \mathbb{P}(T_k^n \leq T) = 0, \qquad T > 0. \tag{3.7}$$

Define $X_t^n(\varphi) = \sum_{i\in S} X_t^{n,i}\varphi(i)$ for $\varphi : S \to \mathbb{R}_+$ and $\overline{\beta}_m = \sup_{i\in S_m^c}\beta(i) \downarrow 0$. Then (recall $S_0^c = S$)

$$\begin{aligned}\mathbb{P}(\sup_{t\leq T} X_t^n(1_{S_m^c}\nu\beta) > \varepsilon_0) &\leq \mathbb{P}(\sup_{t\leq T}\sum_{i\in S_m^c}\nu(i)X_t^{n,i} > \varepsilon_0/\overline{\beta}_m) \\ &\leq \frac{\overline{\beta}_m}{\varepsilon_0}[|x_0|_\nu + \kappa_{2.3a}T]e^{c_1 T}.\end{aligned} \tag{3.8}$$

Let $\varepsilon, T > 0$. Choose $m_k \uparrow \infty$ and $K_0 > 0$ such that

$$\sum_{k=1}^\infty \left(\overline{\beta}_{m_k}\right)^{1/2}[|x_0|_\nu + \kappa_{2.3a}T]e^{c_1 T} + K_0^{-1}[|x_0|_\nu + \kappa_{2.3a}T]e^{c_1 T} < \varepsilon, \tag{3.9}$$

and define

$$K = \{x \in M_{\nu\beta}(S) : \sum_{i\in S_{m_k}^c} x_i\nu_i\beta_i \leq (\overline{\beta}_{m_k})^{1/2} \text{ for all } k \in \mathbb{N}, |x|_\nu \leq K_0\}.$$

Then K is compact in $M_{\nu\beta}(S)$ by Lemma 3.1. By (3.8) with $\varepsilon_0 = \left(\overline{\beta}_{m_k}\right)^{1/2}$ and $m = m_k$ and (3.6) with $k = K_0$ we get that for each n

$$\begin{aligned}&\mathbb{P}(X_t^n \in K \text{ for all } t \leq T) \qquad (3.10)\\ &\quad\geq 1 - \Big[\sum_{k=1}^\infty (\overline{\beta}_{m_k})^{1/2}[|x_0|_\nu + \kappa_{2.3a}T]e^{c_1 T} + K_0^{-1}[|x_0|_\nu + \kappa_{2.3a}T]e^{c_1 T}\Big] \\ &\quad\geq 1 - \varepsilon,\end{aligned}$$

by (3.9). This will give us the compact containment required for the tightness of $\{\mathbb{P}(X^n_\cdot \in \cdot) : n \in \mathbb{N}\}$.

We claim next that if $i \in S$ is fixed, then

$$\{X^{n,i} : n \in \mathbb{N}\} \text{ is a tight sequence of processes in } C([0,\infty),\mathbb{R}). \tag{3.11}$$

By (3.7) it suffices to show

$$\{X^{n,i}(\cdot \wedge T^n_k) : n \in \mathbb{N}\} \text{ is tight in } C([0,\infty),\mathbb{R}) \text{ for each } k \in \mathbb{N}. \tag{3.12}$$

Let $M^{n,i}_t$ denote the stochastic integral on the right hand side of (3.2). Then for $s \le t$,

$$\begin{aligned}\mathbb{E}\left(\left(\int_s^t 1_{(r\le T^n_k)} dM^{n,i}_r\right)^4\right) &\le c_2\mathbb{E}\left(\left(\int_s^t 1_{(r\le T^n_k)} X^{n,i}_r \gamma_i(X^n_r)dr\right)^2\right)\\ &\le c_3(\kappa_{2.4})^2(k/\nu_i)^2(t-s)^2.\end{aligned} \tag{3.13}$$

In addition by (2.5),

$$\begin{aligned}\int_s^t 1_{(r\le T^n_k)}|b_i(X^n_r)|dr &\le \frac{\kappa_{2.3a}}{\nu_i}\int_s^t 1_{(r\le T^n_k)}(|X^n_r|_\nu + 1)dr\\ &\le \frac{\kappa_{2.3a}}{\nu_i}(k+1)(t-s).\end{aligned}$$

This, (3.13), (3.2), and standard arguments now imply (3.12).

(3.10) and (3.11) imply $\{X^n_\cdot : n \in \mathbb{N}\}$ is a tight sequence in $C([0,\infty), M_{\nu\beta}(S))$. If $\nu\beta(i) \equiv 1$, this is standard, as then $M_{\nu\beta}(S) = M_F(S)$ (see, e.g., Theorem II.4.1 in [Pe02]). In general, define $\Phi : M_{\nu\beta}(S) \to M_F(S)$ by

$$\Phi(x)(i) = x(i)\nu_i\beta_i.$$

Then Φ is an isometry and the above result for $M_F(S)$ gives the required implication.

By Skorokhod's theorem we may first extract a weakly convergent subsequence $\{X^{n_m}\}$ and then assume $X^{n_m} \to X$ a.s. in $C([0,\infty), M_{\nu\beta}(S))$. It is easy to use the continuity of b_i, γ_i on $M_{\nu\beta}(S)$ to let $n = n_m \to \infty$ in (3.2) and conclude

$$X^i_t = x^i_0 + \int_0^t b_i(X_s)ds + \int_0^t \sqrt{\gamma_i(X_s)X^i_s}dB^i_s \tag{3.14}$$

for all $t \ge 0$ and all $i \in S$, a.s. Fatou's lemma implies for any $t > 0$

$$|X_t|_\nu = \sum_{i\in S}\liminf_{m\to\infty} X^{n_m,i}_t \nu_i \le \liminf_{m\to\infty}|X^{n_m}_t|_\nu,$$

and so an elementary argument implies

$$\begin{aligned}\mathbb{P}(\sup_{t\le T}|X_t|_\nu > k) &\le \mathbb{P}(\liminf_{m\to\infty}\sup_{t\le T}|X_t^{n_m}|_\nu > k)\\ &\le \liminf_{m\to\infty}\mathbb{P}(\sup_{t\le T}|X_t^{n_m}|_\nu > k)\\ &\le k^{-1}[|x_0|_\nu + \kappa_{2.3a}T]e^{c_1T},\end{aligned}$$

the last by (3.6). This proves

$$\sup_{t\le T}|X_t|_\nu < \infty, \qquad T>0, \quad a.s., \tag{3.15}$$

and so $X.$ has $M_\nu(S)$ valued paths a.s.

To show that $X.$ has continuous $M_\nu(S)$-valued paths a.s. we use the following lemma, whose elementary proof is left to the reader.

Lemma 3.3. *Suppose $x : [0,\infty) \to M_\nu(S)$ is such that $x_t(i), i \in S$, and $|x_t|_\nu$ are all continuous. Then x is continuous.*

Clearly $X_t(i)$ is continuous for all $i \in S$ a.s. since it is continuous in $M_{\nu\beta}(S)$. From (3.14) we see that if $M_n(t) = \sum_{i\in S_n}\nu_i\int_0^t\sqrt{\gamma_i(X_s)X_s^i}dB_s^i$, then

$$\sum_{i\in S_n}X_t(i)\nu_i = \sum_{i\in S_n}x_0(i)\nu_i + \int_0^t\sum_{i\in S_n}\nu_ib_i(X_s)ds + M_n(t). \tag{3.16}$$

(3.15) and (2.5) show that

$$\sum_{i\in S_n}\nu_i|b_i(X_s)| \to \sum_{i\in S}\nu_i|b_i(X_s)|,$$

which is bounded uniformly on compact time intervals a.s. as $n\to\infty$, and so

$$\begin{aligned}&\sup_{t\le T}\Big|\int_0^t\sum_{i\in S_n}\nu_ib_i(X_s)ds - \int_0^t\sum_{i\in S}\nu_ib_i(X_s)ds\Big|\\ &\le \int_0^T\sum_{i\in S-S_n}\nu_i|b_i(X_s)|ds\\ &\to 0\end{aligned} \tag{3.17}$$

as $n\to\infty$ by dominated convergence. By (3.16) and (2.5),

$$\sup_{t\le T}|M_n(t)| \le \sup_{t\le T}|X_t|_\nu + |x_0|_\nu + \int_0^T\kappa_{2.3a}(|X_s|_\nu+1)ds.$$

By (3.15) and the Dubins-Schwartz theorem, this means $\{\langle M^n\rangle_T\}$ remains bounded in probability as $n \to \infty$. Therefore

$$\int_0^T \sum_{i\in S} \nu_i^2 \gamma_i(X_s)X_s(i)ds = \lim_{n\to\infty} \langle M^n\rangle_T < \infty, \qquad a.s., \quad T>0.$$

A standard square function inequality now implies

$$\sup_{t\le T} |M_n(t) - M_m(t)| \to 0$$

in probability as $m, n \to 0$ for all $T > 0$, and so we may take a subsequence such that M_{n_k} converges uniformly on compact time intervals a.s. Let $n = n_k \to \infty$ in (3.16). The above and (3.17) show that the right hand side of (3.16) converges uniformly on compact time intervals a.s. to a necessarily continuous process. As the left hand side converges to $|X_t|_\nu$ for all $t \ge 0$, a.s., it follows that $t \to |X_t|_\nu$ is continuous. Lemma 3.3 therefore shows $t \to X_t$ is a continuous $M_\nu(S)$-valued process. By (3.14) and Remark 2.2(c), the law of X is a solution of the martingale problem for $\mathcal{L}$ starting at x_0. □

4. Estimates.

We first obtain some key analytic estimates for the special case when $\gamma_i = \gamma_i^0$ and $b_i = b_i^0$ are constants. Assume

$$0 < \gamma_i^0 \le \sup_{i'} \gamma_{i'}^0 = \kappa_{4.1}(\gamma^0) < \infty, \qquad i \in S, \tag{4.1}$$

$$0 \le b_i^0, \qquad i\in S, \qquad \text{and} \qquad |b^0|_\nu = \sum_{i\in S} b_i^0 \nu(i) < \infty. \tag{4.2}$$

Let

$$\mathcal{L}^0 f(x) = \sum_{i\in S} \gamma_i^0 x_i f_{ii}(x) + b_i^0 f_i(x).$$

By Theorem 2.4 there is a solution $\mathbb{P}_{x_0}^0$ to $\mathcal{MP}(\mathcal{L}, \delta_{x_0})$ in $M_\nu(S)$ for each $x_0 \in M_\nu(S)$. In fact it is easy to see that under $\mathbb{P}_{x_0}^0$, the processes $\{X_i : i \in S\}$ are independent diffusions and each X^i is a suitably scaled squared Bessel process whose law is that of the pathwise unique solution to

$$X^i(t) = x_0(i) + \int_0^t (X^i(s)\gamma_i^0)^{1/2} dB_s^i + b_0^i t, \tag{4.3}$$

where the B_s^i are independent one dimensional Brownian motions. (Theorem 2.4 is only needed here to ensure X has paths in Ω_ν.) An explicit formula for the transition kernel of $p_t^i(x_i, dy_i)$ of X_i is given in (2.2) and (2.4) of [BP03]. Let $(P_t)_{t\ge0}$ and $(R_\lambda)_{\lambda\ge0}$ be the semigroup and resolvent, respectively, of the $M_\nu(S)$-valued diffusion $X_t = (X_t^i)_{i\in S}$.

Lemma 4.1. *For any compact set $K \in M_\nu(S)$, $T > 0$, and $\varepsilon > 0$, there is a sequence $\eta = \{\eta_n\}$ decreasing to zero such that*

$$\sup_{x_0 \in K} \mathbb{P}^0_{x_0}(X_t \in K_\eta \text{ for } t \le T) \ge 1 - \varepsilon.$$

Proof. By Lemma 3.1(b) we may assume $K = K_\delta$ for some sequence δ_n decreasing to zero. Set $B(N) = \sum_{i \notin S_N} \nu(i) b^0_i$. If

$$Z_{n,N}(t) = \sum_{i \in S_n - S_N} \nu(i) X_i(t)$$

for $n > N \ge 0$ and

$$Z_N(t) = \sum_{i \notin S_N} \nu(i) X_i(t),$$

then for $x_0 \in K_\delta$

$$\mathbb{E}^{\,0}_{x_0}[Z_N(t)] = \sum_{i \notin S_N} x_0(i)\nu(i) + b^0_i \nu(i) t \le \delta(N) + B(N)t,$$

where $\delta(N), B(N) \downarrow 0$ by (4.2). Since $Z_{n,N}(t)$ is a submartingale, the weak L^1 inequality implies that for any $x_0 \in K_\delta$

$$\begin{aligned}
\mathbb{P}^0_{x_0}(\sup_{t \le T} Z_N(t) > A) &= \lim_{n \to \infty} \mathbb{P}^0_{x_0}(\sup_{t \le T} Z_{n,N}(t) > A) \\
&\le \lim_{n \to \infty} A^{-1} \mathbb{E}^{\,0}_{x_0}[Z_{n,N}(t)] \\
&\le A^{-1}(\delta(N) + B(N)T). \qquad (4.4)
\end{aligned}$$

Choose $N_k \uparrow \infty$ such that

$$\sup_{x_0 \in K_\delta} \mathbb{P}^0_{x_0}(\sup_{t \le T} Z_{N_k}(t) > 2^{-k}) \le 2^{-k-1}\varepsilon \qquad (4.5)$$

and then $\overline{\eta} > 1$ sufficiently large so that

$$\sup_{x_0 \in K_\delta} \mathbb{P}^0_{x_0}(\sup_{t \le T} |X_t|_\nu > \overline{\eta}) \le \varepsilon/2. \qquad (4.6)$$

The latter is possible by (4.4) with $N = 0$ since $Z_0(t) = |X_t|_\nu$. Now define

$$\eta_n = \begin{cases} \overline{\eta}, & n \le N \\ 2^{-k}, & N_k \le n < N_{k+1}. \end{cases}$$

Then for any $x_0 \in K_\delta$

$$\mathbb{P}^0_{x_0}(X_t \in K^c_\eta \text{ for some } t \le T) \le \mathbb{P}^0_{x_0}(\sup_{t\le T}|X_t|_\nu > \overline{\eta}) + \sum_{k=1}^\infty \mathbb{P}^0_{x_0}(\sup_{t\le T} Z_{N_k}(t) > 2^{-k}) < \varepsilon$$

by (4.5) and (4.6). □

Define $e_i \in M_\nu(S)$ by $e_i(j) = 1_{(i=j)}$. Let $\alpha \in (0,1]$ and for $f : M_\nu(S) \to \mathbb{R}$ and $i \in S$ define

$$|f|_{\alpha,i} = \sup\{|f(x+he_i) - f(x)|x(i)^{\frac{\alpha}{2}}h^{-\alpha} : x \in M_\nu(S), h > 0\}. \tag{4.7}$$

Set

$$|f|_\alpha = \sup_i (\gamma^0_i)^{\frac{\alpha}{2}}|f|_{\alpha,i}, \qquad \|f\|_\alpha = \|f\|_\infty + |f|_\alpha. \tag{4.8}$$

If

$$\mathcal{C}_\alpha = \{f : M_\nu(S) \to \mathbb{R} : f \text{ continuous }, \|f\|_\alpha < \infty\},$$

then it is easy to check that $(\mathcal{C}_\alpha, \|\cdot\|_\alpha)$ is a Banach space.

Remark 4.2. One difference between our infinite dimensional setting and the finite dimensional setting in [BP03] is that $\sup_\alpha |f|_{\alpha,i} < \infty$ does not imply that f is uniformly continuous on $I = \{x \in M_\nu(S) : x(i) > 0 \text{ for all } i \in S\}$ and hence has a continuous extension to $M_\nu(S)$. This is true on $\mathbb{R}^d_+$ (see Lemma 2.2 of [BP03]). Suppose we define $f(x)$ to be 1 if infinitely many of the $x(i) \ne 2^{-i}/\nu(i)$ and 0 otherwise. Then $|f|_{\alpha,i} = 0$ but f is discontinuous on I. This complicates things a bit when checking whether various operators preserve $\mathcal{C}_\alpha$.

Remark 4.3. A key fact in our argument is that the estimates on $(R_\lambda f)_i$ and $x_i(R_\lambda f)_{ii}$ from [BP03] are independent of the dimension of the space. Recall that the way we obtained the estimates in [BP03] was to first consider the one-dimensional case. If P^i_t denotes the semigroup corresponding to the operator $x\gamma^0_i f''(x) + b^0_i f'(x)$, then we obtained bounds on $|(P^i_t f)'(x)|$ and on $|(P^i_t f)'(x+\Delta) - (P^i_t f)'(x)|$ in terms of constants depending only on γ^0_i and b^0_i; see Lemmas 4.3, 4.4, 4.6, and 4.7 of [BP03]. If P_t denotes the semigroup corresponding to $\mathcal{L}^0 f(x) = \sum_{i=1}^d [x_i\gamma^0_i f_{ii}(x) + b^0_i f_i(x)]$, we then derived bounds on $|(P_t f)_i(x)|$ and on $|(P_t f)_i(x+\Delta e_j) - (P_t f)_i(x)|$ with the *same* constants (Propositions 5.1 and 5.2 of [BP03]); hence the constants did not depend on the dimension d of the underlying space. We then deduced estimates on $(R_\lambda f)_i$. The same reasoning was applied for $x_i(R_\lambda f)_{ii}$.

Lemma 4.4. *Let $f \in \mathcal{C}_\alpha$, $\lambda > 0$, and $i \in S$. There is a $\kappa_{4.4} = \kappa_{4.4}(\alpha)$ independent of f, i, λ such that the following hold:*
(a) The partial derivative $(R_\lambda f)_i(x)$ exists for every $x \in M_\nu(S)$ and satisfies

$$\|(R_\lambda f)_i\|_\infty \le \kappa_{4.4}(\gamma^0_i)^{\frac{\alpha}{2}-1}\lambda^{-\frac{\alpha}{2}}|f|_{\alpha,i}. \tag{4.9}$$

(b) The second order derivative $(R_\lambda f)_{ii}(x)$ exists on $\{x \in M_\nu(S) : x_i > 0\}$ and satisfies

$$|x_i(R_\lambda f)_{ii}(x)| \le \kappa_{4.4}(\gamma_i^0)^{-1}|f|_{\alpha,i}\Big[\Big(\frac{\gamma_i^0}{\lambda}\Big)^{\frac{\alpha}{2}} \wedge x_i^{\frac{\alpha}{2}}\Big]. \tag{4.10}$$

In particular, $\lim_{x_i\to 0} x_i(R_\lambda f)_{ii}(x) = 0$ uniformly on $M_\nu(S)$ and if $x_i(R_\lambda f)_{ii}(x)$ is set to be this limit on $\{x \in M_\nu(S) : x_i = 0\}$, then

$$\|x_i(R_\lambda f)_{ii}(x)\|_\infty \le \kappa_{4.4}(\gamma_i^0)^{\frac{\alpha}{2}-1}\lambda^{-\frac{\alpha}{2}}|f|_{\alpha,i}. \tag{4.11}$$

Proof. We only prove (b) as (a) is similar but easier. Let $t > 0$. Then argue as in the finite-dimensional argument (Proposition 5.1 of [BP03]), noting the constants there are independent of dimension, to see that $(P_t f)_{ii}$ exists on $M_\nu(S)$ (in fact on $\mathbb{R}_+^S$) and satisfies

$$|x_i(P_t f)_{ii}(x)| \le c_1|f|_{\alpha,i}(\gamma_i^0 t)^{\frac{\alpha}{2}-1}\Big(\frac{x_i}{\gamma_i^0 t} \wedge 1\Big). \tag{4.12}$$

If $x_i > 0$, this allows one to differentiate through the time integral (by the dominated convergence and the mean value theorem) and conclude for $x_i > 0$ that $(R_\lambda f)_{ii}$ exists and satisfies

$$x_i(R_\lambda f)_{ii}(x) = \int_0^\infty e^{-\lambda t} x_i(P_t f)_{ii}(x)dt.$$

A simple calculation using (4.12) leads to (4.10) for $x_i > 0$. The fact that $x_i(R_\lambda f)_{ii}$ approaches 0 uniformly as $x_i \downarrow 0$ is then immediate, as is (4.11). □

Lemma 4.5. *If $f \in C_b(M_\nu(S))$ and $f_n = f \circ \pi_n$, then for any compact subset K of $M_\nu(S)$*

$$\lim_{n\to\infty} \sup_{x\in K} |f(x) - f_n(x)| = 0.$$

Proof. By Lemma 3.1(b) we may assume $K = K_\eta$ for some sequence $\eta = \{\eta_n\}$ decreasing to 0. Then

$$\sup_{x\in K_\eta} |x - \pi_n x|_\nu = \sup_{x\in K_\eta} \sum_{i\in S_n^c} x_i\nu_i \le \eta_n \to 0.$$

Since $\pi_n(K_\eta) \subset K_\eta$ (recall Remark 3.2) and f is uniformly continuous on K_η, the result follows. □

Corollary 4.6. *If $f \in C_b(M_\nu(S))$ and $f_n = f \circ \pi_n$, then for any $\lambda > 0$, $R_\lambda f_n \to R_\lambda f$ uniformly on compact subsets of $M_\nu(S)$.*

Proof. Let K be a compact subset of $M_\nu(S)$ and $\varepsilon > 0$. Lemma 4.1 shows there is a compact K_η such that

$$\sup_{x\in K}(R_\lambda 1_{K_\eta^c})(x) < \varepsilon.$$

Lemma 4.5 implies that $\|R_\lambda f 1_{K_\eta} - R_\lambda f_n 1_{K_\eta}\|_\infty \to 0$ as $n \to \infty$ and so

$$\begin{aligned}\limsup_{n\to\infty}\sup_{x\in K}|R_\lambda f_n(x) - R_\lambda f(x)| &\le \limsup_{n\to\infty}\sup_{x\in K}|R_\lambda f_n 1_{K_\eta^c}(x) - R_\lambda f 1_{K_\eta^c}(x)|\\ &\le 2\varepsilon\|f\|_\infty.\end{aligned}$$

□

Let R_λ^n denote the resolvent of the finite-dimensional diffusion $(X_i)_{i\in S_n}$ under $\{\mathbb{P}_{x_0}\}$. Then R_λ^n is a Feller resolvent (i.e., it maps $C_b(\mathbb{R}_+^{S_n})$ to itself) and so if $\widetilde{f}_n((x_i)_{i\in S_n}) = f(\Pi_n(x))$ for $f \in C_b(M_\nu(S))$, then $\widetilde{f}_n \in C_b(\mathbb{R}_+^{S_n})$ and $R_\lambda f_n(x) = R_\lambda^n \widetilde{f}_n((x_i)_{i\in S_n})$ is continuous on $M_\nu(S)$. (Π_n is defined in (2.2).) The convergence in Corollary 4.6 therefore shows

$$R_\lambda : C_b(M_\nu(S)) \to C_b(M_\nu(S)). \tag{4.13}$$

Our immediate goal is to extend the continuity on $M_\nu(S)$ to $(R_\lambda f)_i$ and $x_i(R_\lambda f)_{ii}$ for $f \in \mathcal{C}_\alpha$. As explained earlier, this is more delicate in our infinite-dimensional setting.

Lemma 4.7. *There is a $\kappa_{4.7} \ge 1$ such that*
(a) if $0 < r \le R/2$, then

$$\int_R^\infty e^{-z}\frac{z^{r-1}}{\Gamma(r)}\frac{|z-r|}{r}dz \le \frac{\kappa_{4.7}}{R+1};$$

(b) for any $r > 0$

$$\int_0^\infty e^{-z}\frac{z^{r-1}}{\Gamma(r)}\frac{|z-r|}{r}dz \le \kappa_{4.7}(r+1)^{-1/2}.$$

Proof. (a) In view of (b) we may assume $R \ge 1$. The integral in (a) is bounded by

$$\int_R^\infty e^{-z}\frac{z^{r-1}}{\Gamma(r)}\frac{(z-r)^2}{rR/2}dz,$$

which is bounded by $cR^{-1}r^{-1}(r+1)$; see, e.g., Lemma 3.2(a) of [BP03]. This gives the required bound if $r \ge 1$ (recall $R \ge 1$). Assume now that $0 < r < 1$. The integral in (a) is at most

$$\begin{aligned}\int_R^\infty e^{-z}\frac{z^r}{\Gamma(r+1)}dz &\le \int_0^\infty e^{-z}\frac{z^{r+1}}{\Gamma(r+1)R}dz\\ &= R^{-1}\Big[\frac{\Gamma(r+2)}{\Gamma(r+1)}\Big] = R^{-1}(r+1) \le 2R^{-1} \le 4(R+1)^{-1}.\end{aligned}$$

(b) If $r \ge 1$ this is immediate from Lemma 3.2(a) of [BP03]. If $0 < r < 1$, then the required integral is at most

$$\int_0^\infty e^{-z}\frac{z^r}{\Gamma(r+1)}dz + \int_0^\infty e^{-z}\frac{z^{r-1}}{\Gamma(r)}dz = 2.$$

□

Proposition 4.8. *Let $f \in \mathcal{C}_\alpha$ and $f_n = f \circ \pi_n$. If $i \in S$ and $\lambda > 0$, then for any compact set $K \subset M_\nu(S)$*

$$\lim_{n\to\infty} \sup_{x\in K} \Big[|R_\lambda f_n(x) - R_\lambda f(x)| + |(R_\lambda f_n)_i(x) - (R_\lambda f)_i(x)| \\ + |x_i(R_\lambda f_n)_{ii}(x) - x_i(R_\lambda f)_{ii}(x)|\Big] = 0.$$

Proof. Note first that $f \in \mathcal{C}_\alpha$ implies $f_n \in \mathcal{C}_\alpha$ and so the existence of the above partial derivatives follows from Lemma 4.4. We focus on the convergence of the second order derivatives as the first order derivatives are handled in a similar and slightly simpler way, while the resolvents themselves were handled in Corollary 4.6. Fix $f \in \mathcal{C}_\alpha$.

If $y \in M_\nu(S)$, write $\widehat{y}_i = y|_{S-\{i\}}$ and define $Y(v,i) \in M_\nu(S)$ by setting $Y(v,i)(j) = y(j)$ if $j \neq i$ and $Y(v,i)(i) = v$; in other words, $Y(v,i)$ is the point which has the same coordinates as y except that the i^{th} coordinate is equal to v instead of y_i. We may then define $d_n(v;\widehat{y}_i) = f(\pi_n Y(v,i)) - f(Y(v,i))$ and

$$h_n(\widehat{y}_i;t,x_i) = x_i \frac{\partial^2}{\partial x_i^2} \int d_n(y_i;\widehat{y}_i) p_t^i(x_i, dy_i).$$

If $|d_n(\cdot;\widehat{y}_i)|_\alpha$ denotes the $|\cdot|_{\alpha,i}$ norm of $d_n(\cdot;\widehat{y}_i)$ with $S = \{i\}$, then $|d_n(\cdot;\widehat{y}_i)|_\alpha \le 2|f|_{\alpha,i}$, and so the above derivative exists and satisfies (see Lemmas 4.1, 4.3, 4.5, and 4.6 of [BP03])

$$\|h_n(\cdot;t,x_i)\|_\infty \le c_\alpha |f|_{\alpha,i} (\gamma_i^0 t)^{\frac{\alpha}{2}-1} \Big(\frac{x_i}{\gamma_i^0 t} \wedge 1\Big). \tag{4.14}$$

Note that

$$x_i(P_t f_n)_{ii}(x) - x_i(P_t f)_{ii}(x) = \int h_n(\widehat{y}_i;t,x_i) \prod_{j\neq i} p_t^j(x_j, dy_j), \tag{4.15}$$

where differentiation through the integrals is justified by the above bound and dominated convergence.

If $g : \mathbb{R}_+ \to \mathbb{R}$, let $\|g\|_R = \sup\{|g(y)| : y \le R\}$. Assume first $b_i^0 > 0$ and use (4.14) in [BP03] to conclude that if $w = x_i/\gamma_i^0 t$ and $r_i = b_i^0/\gamma_i^0$, then

$$|h_n(\widehat{y}_i;t,x_i)| \le \sum_{k=0}^\infty e^{-w} \frac{w^k}{k!} |w-k| \int_0^\infty |d_n(z\gamma_i^0 t;\widehat{y}_i)| e^{-z} \frac{z^{k+r_i-1}}{\Gamma(k+r_i)} \frac{|z-(k+r_i)|}{k+r_i} \frac{dz}{\gamma_i^0 t}. \tag{4.16}$$

By Lemma 4.7, if $R > 0$,

$$\begin{aligned} &\int_{R/\gamma_i^0 t}^\infty e^{-z} \frac{z^{k+r_i-1}}{\Gamma(k+r_i)} \frac{|z-(k+r_i)|}{k+r_i} dz \\ &\quad\le 1_{(k+r_i \le R/2\gamma_i^0 t)} \frac{\kappa_{4.7}}{(R/\gamma_i^0 t)+1} + 1_{(k+r_i > R/2\gamma_i^0 t)} \frac{\kappa_{4.7}}{\sqrt{r_i+k+1}} \\ &\quad\le \frac{\kappa_{4.7}}{\sqrt{r_i+k+1}} \Big[1_{(k+r_i \le R/2\gamma_i^0 t)} \frac{\sqrt{\gamma_i^0 t}}{\sqrt{R}} + 1_{(k+r_i > R/2\gamma_i^0 t)}\Big]. \end{aligned}$$

Use this and Lemma 4.7(b) in (4.16) to see that

$$
\begin{aligned}
&|h_n(\widehat{y}_i;t,x_i)| \\
&\quad \le (\gamma_i^0 t)^{-1} \sum_{k=0}^{\infty} e^{-w} \frac{w^k}{k!} |w-k| \Big[\|d_n(\cdot;\widehat{y}_i)\|_R \int_0^{R/\gamma_i^0 t} e^{-z} \frac{z^{k+r_i-1}}{\Gamma(k+r_i)} \frac{|z-(k+r_i)|}{k+r_i} dz \\
&\qquad + 2\|f\|_\infty \kappa_{4.7} (r_i+k+1)^{-1/2} \Big[1_{(k+r_i \le R/2\gamma_i^0 t)} \frac{\sqrt{\gamma_i^0 t}}{\sqrt{R}} + 1_{(k+r_1 > R/2\gamma_i^0 t)} \Big]\Big] \\
&\quad \le (\gamma_i^0 t)^{-1} \kappa_{4.7} \sum_{k=0}^{\infty} e^{-w} \frac{w^k}{k!} \frac{|w-k|}{(r_i+k+1)^{1/2}} \Big[\|d_n(\cdot;\widehat{y}_i)\|_R + \Big(\frac{\gamma_i^0 t}{R}\Big)^{1/2} 2\|f\|_\infty \Big] \\
&\qquad + 2\|f\|_\infty (\gamma_i^0 t)^{-1} \kappa_{4.7} \sum_{k=0}^{\infty} 1_{(k+r_i > R/2\gamma_i^0 t)} e^{-w} \frac{w^k}{k!} \frac{|w-k|}{(r_i+k+1)^{1/2}} \\
&\quad = I_1 + I_2. \qquad (4.17)
\end{aligned}
$$

A simple calculation (see Lemma 3.3(b) of [BP03]) shows that

$$
\begin{aligned}
I &\le c_1 (\gamma_i^0 t)^{-1} \Big(\frac{x_i}{\gamma_i^0 t} \wedge 1\Big) \Big[\|d_n(\cdot;\widehat{y}_i)\|_R + \Big(\frac{\gamma_i^0 t}{R}\Big)^{1/2} \|f\|_\infty \Big] \qquad (4.18) \\
&\le c_2((\gamma_i^0 t)^{-1} + 1) \Big[\|d_n(\cdot;\widehat{y}_i)\|_R + R^{-1/2} \|f\|_\infty \Big].
\end{aligned}
$$

If N is a Poisson random variable with mean w, the series in I_2 is

$$
\mathbb{E}\Big[|N-w| (r_i+N+1)^{-1/2} 1_{(N+r_i > R/2\gamma_i^0 t)} \Big]. \qquad (4.19)
$$

Assume now

$$
\frac{R}{8} > \max(1, x_i^2, \gamma_i^0 t). \qquad (4.20)
$$

If $r_i < R/(4\gamma_i^0 t)$, recalling $x_i < R/8$ by (4.20), the expectation in (4.19) is at most

$$
\begin{aligned}
&\mathbb{E}\Big[|N-w| (r_i+N+1)^{-1/2} 1_{(N-w > \frac{R-4x_i}{4\gamma_i^0 t})} \Big] \\
&\qquad \le \mathbb{E}\Big[|N-w| (N+1)^{-1/2} 1_{(N-w > R/8\gamma_i^0 t)} \Big] \\
&\qquad \le \frac{\mathbb{E}(N-w)^2}{(R/8\gamma_i^0 t)^{3/2}} \\
&\qquad = c_3 \frac{x_i}{R^{3/2}} (\gamma_i^0 t)^{1/2} \le \frac{c_4}{R} (\gamma_i^0 t)^{1/2} \le c_4 R^{-1/2}.
\end{aligned}
$$

If $r_i \ge R/(4\gamma_i^0 t)$, then (4.19) is bounded by

$$
\frac{\mathbb{E}|N-w|}{\sqrt{r_i}} \le \frac{c_5\sqrt{w}}{\sqrt{R/\gamma_i^0 t}} \le \frac{c_6\sqrt{x_i}}{\sqrt{R}} \le c_7 R^{-1/4},
$$

the last by (4.20). Therefore under (4.20)

$$I_2 \le \|f\|_\infty c_8((\gamma_i^0 t)^{-1}+1)R^{-1/4}.$$

Now use the above bounds in (4.17) to see that for $R \ge 8\max(1,\gamma_i^0 t)$,

$$\sup_{x_i \le \sqrt{R}/4} |h(\widehat{y}_i;t,x_i)| \le c_9((\gamma_i^0 t)^{-1}+1)[\|d_n(\cdot;\widehat{y}_i)\|_R + \|f\|_\infty R^{-1/4}]. \tag{4.21}$$

If $b_i^0 = 0$, a slightly simpler argument starting with (4.6) in [BP03] will lead to the same bound.

Now choose a compact set K in $M_\nu(S)$, $T > 1$ and $\varepsilon > 0$. Assume R is large enough so that $R \ge 8\max(1,\gamma_i^0 t)$,

$$\text{if } x \in K, \text{ then } x_i \le \sqrt{R}/4, \tag{4.22}$$

and $\|f\|_\infty R^{-1/4} < \varepsilon$. Let η_n be such that K_η is a compact set satisfying the conclusion of Lemma 4.1. Let $\widehat{\pi}_i(y) = (y(j), j \in S - \{i\})$ be the projection of $y \in M_\nu(S)$ onto $M_{\nu_i}(S-\{i\})$, $\nu_i = \nu|_{S-\{i\}}$, and let

$$\widehat{K}_\eta = \{y \in M_\nu(S) : \widehat{\pi}_i(y) \in \widehat{\pi}_i(K_\eta), y(i) \le R\}.$$

Then it is easy to use Lemma 3.1 to check that $\widehat{K}_\eta$ is compact, and so by Lemma 4.5

$$\lim_{n\to\infty} \sup_{y \in \widehat{K}_\eta} |f_n(y) - f(y)| = 0.$$

This implies

$$\lim_{n\to\infty} \sup_{\widehat{y}_i \in \widehat{\pi}_i(K_\eta)} \|d_n(\cdot;\widehat{y}_i)\|_R = 0.$$

Choose N such that $n \ge N$ implies

$$\sup_{\widehat{y}_i \in \widehat{\pi}_i(K_\eta)} \|d_n(\cdot;\widehat{y}_i)\|_R < \varepsilon.$$

Use this with (4.14), (4.21), and (4.22) in (4.15) and conclude that for $n \ge N$ and $t \in [0,T]$

$$\begin{aligned}
&\sup_{x\in K} |x_i(P_t f_n)_{ii}(x) - x_i(P_t f)_{ii}(x)| \qquad (4.23)\\
&\quad \le c_\alpha |f|_{\alpha,i}(\gamma_i^0 t)^{\frac{\alpha}{2}-1} \sup_{x\in K} \mathbb{P}_x^0(\widehat{\pi}_i(X_t) \notin \widehat{\pi}_i(K_\eta))\\
&\qquad + c_9((\gamma_i^0 t)^{-1}+1)[\varepsilon + \|f\|_\infty R^{-1/4}]\\
&\quad \le c_\alpha |f|_{\alpha,i}(\gamma_i^0 t)^{\frac{\alpha}{2}-1}\varepsilon + c_9((\gamma_i^0 t)^{-1}+1)2\varepsilon\\
&\quad \le c_{10}(|f|_{\alpha,i}+1)((\gamma_i^0 t)^{-1}+1)\varepsilon.
\end{aligned}$$

Use the above for $t \in [T^{-1}, T]$ and (4.14) for $t \in [T^{-1}, T]^c$ to see that for $x \in K$ and $n \in \mathbb{N}$

$$\begin{aligned} |x_i(R_\lambda f_n)_{ii}(x) - & x_i(R_\lambda f)_{ii}(x)| \qquad (4.24) \\ &\leq c_\alpha |f|_{\alpha,i} (\gamma_i^0 t)^{\frac{\alpha}{2}-1} \Big[\int_0^{1/T} t^{\frac{\alpha}{2}-1} dt + \int_T^\infty e^{-\lambda t} dt \Big] \\ &\quad + c_{11}(|f|_{\alpha,i} + 1)((\gamma_i^0)^{-1} T + 1)\varepsilon \int_{1/T}^T e^{-\lambda t} dt \\ &\leq c_\alpha |f|_{\alpha,i} (\gamma_i^0 t)^{\frac{\alpha}{2}-1} [T^{-\alpha/2} + e^{-\lambda T}\lambda^{-1}] \\ &\quad + c_{12}(|f|_{\alpha,i} + 1)((\gamma_i^0)^{-1} T + 1)\varepsilon \lambda^{-1}. \end{aligned}$$

As $T > 1$ and $\varepsilon > 0$ are arbitrary, this gives

$$\lim_{n\to\infty} \sup_{x \in K} |x_i(R_\lambda f_n)_{ii}(x) - x_i(R_\lambda f)_{ii}(x)| = 0.$$

The differentiation through the time integral in (4.24) does require $x_i > 0$ as in the proof of Lemma 4.4, but that result shows the left-hand side is 0 if $x_i = 0$. □

Corollary 4.9. *If $f \in \mathcal{C}_\alpha$, then for any $i \in S$ and $\lambda > 0$, $R_\lambda f, (R_\lambda f)_i$, and $x_i(R_\lambda f)_{ii}$ are all continuous bounded functions on $M_\nu(S)$.*

Proof. Fix i and consider n large enough so that $i \in S_n$. Recall R_λ^n is the resolvent of $(X_i)_{i\in S_n}$ and $\widetilde{f}_n(x) = f \circ \Pi_n(x)$ on $\mathbb{R}_+^{S_n}$. Then $R_\lambda f_n(x) = R_\lambda^n \widetilde{f}_n(\pi_n x)$ and so by Proposition 5.3 of [BP03] $(R_\lambda f_n)_i(x) = (R_\lambda^n)_i \widetilde{f}_n(\pi_n x)$ and $x_i(R_\lambda f_n)_{ii}(x) = x_i(R_\lambda^n \widetilde{f}_n)_{ii}(\pi_n(x))$ are bounded continuous functions on $M_\nu(S)$. The uniform convergence in Proposition 4.8 and the bound in Lemma 4.4 show that $(R_\lambda f)_i$ and $x_i(R_\lambda f)_{ii}$ are in $C_b(M_\nu(S))$. (4.13) already gave the result for $R_\lambda f$. □

Corollary 4.10. *There is a $\kappa_{4.10} > 0$ depending only on α such that for all $f \in \mathcal{C}_\alpha$, $\lambda > 0$, and $i, j \in S$,*

$$|(R_\lambda f)_i|_{\alpha,j} + |x_i(R_\lambda f)_{ii}|_{\alpha,j} \leq \kappa_{4.10} |f|_{\alpha,i}^{1-\alpha} |f|_{\alpha,j}^{\alpha} (\gamma_i^0)^{-1} (\gamma_i^0/\gamma_j^0)^{(1-\alpha)\alpha/2}.$$

Proof. The proof is almost the same as that of Proposition 5.3 of [BP03] for the finite-dimensional case – again the constants given there are independent of dimension; cf. Remark 4.3. The only change is that once the bounds on the increments of $x_i(R_\lambda f)_{ii}$ are established for $x_i > 0$, they follow for $x_i = 0$ by the continuity established in Corollary 4.9; this is in place of the use of Lemma 2.2 in [BP03]. □

5. Local uniqueness.

We make the following assumption.

Assumption 5.1. *Assume $\gamma_i : M_\nu(S) \to (0,\infty)$, $b_i : M_\nu(S) \to \mathbb{R}$ are continuous and $(\gamma_i^0)_{i\in S}, (b_i^0)_{i\in S}$ satisfy (4.1) and (4.2). Assume also*

(a)

$$\sup_{x\in M_\nu(S)} \sum_{i\in S} \left[\frac{|\gamma_i(x)-\gamma_i^0|}{\gamma_i^0} + \frac{|b_i(x)-b_i^0|}{\gamma_i^0} \right] \equiv \rho < \infty. \tag{5.1}$$

(b) For all $x \in M_\nu(S)$

$$\lim_{x'\to x} \sum_{i\in S} \left[\frac{|\gamma_i(x')-\gamma_i^0|}{\gamma_i^0} + \frac{|b_i(x')-b_i^0|}{\gamma_i^0} \right] = \sum_{i\in S} \left[\frac{|\gamma_i(x)-\gamma_i^0|}{\gamma_i^0} + \frac{|b_i(x)-b_i^0|}{\gamma_i^0} \right]. \tag{5.2}$$

(c) For all $j \in S$, $0 < h$, $x \in M_\nu(S)$

$$\sum_{i\in S} \left[\frac{|\gamma_i(x+he_j)-\gamma_i(x)|}{\gamma_i^0} + \frac{|b_i(x+he_j)-b_i(x)|}{\gamma_i^0} \right] \le \kappa_{5.1} h^\alpha x_j^{-\alpha/2} (\gamma_j^0)^{-\alpha/2}. \tag{5.3}$$

The following result uses only Assumption 5.1(a)-(b).

Lemma 5.2. *For any $\lambda > 0$, $(\mathcal{L}-\mathcal{L}^0)R_\lambda : \mathcal{C}_\alpha \to C_b(M_\nu(S))$, and there is a $\kappa_{5.2} = \kappa_{5.2}(\alpha)$ such that*

$$\|(\mathcal{L}-\mathcal{L}^0)R_\lambda f\|_\infty \le \kappa_{5.2}\rho |f|_\alpha \lambda^{-\alpha/2}, \qquad f \in \mathcal{C}_\alpha, \lambda > 0.$$

Proof. Lemma 4.4 shows that for $f \in \mathcal{C}_\alpha$

$$\begin{aligned}
&\sum_{i\in S} |\gamma_i(x)-\gamma_i^0|\, |x_i (R_\lambda f)_{ii}(x)| + |b_i(x)-b_i^0|\, |(R_\lambda f)_i(x)| \\
&\quad \le \sum_{i\in S} \Big[\frac{|\gamma_i(x)-\gamma_i^0|}{\gamma_i^0} + \frac{|b_i(x)-b_i^0|}{\gamma_i^0} \Big] c_\alpha (\gamma_i^0)^{\alpha/2} |f|_{\alpha,i} \lambda^{-\alpha/2} \\
&\quad \le c_\alpha \rho |f|_\alpha \lambda^{-\alpha/2},
\end{aligned}$$

by Assumption 5.1(a). This gives the required bound on $\|(\mathcal{L}-\mathcal{L}^0)R_\lambda f\|_\infty$.

Note that Lemma 4.4 implies

$$\begin{aligned}
&|\gamma_i(x)-\gamma_i^0|\, |x(i)(R_\lambda f)_{ii}(x)| + |b_i(x)-b_i^0|\, |(R_\lambda f)_i(x)| \\
&\qquad \le c_\alpha \Big[\frac{|\gamma_i(x)-\gamma_i^0|}{\gamma_i^0} + \frac{|b_i(x)-b_i^0|}{\gamma_i^0} \Big] |f|_\alpha \lambda^{-\alpha/2}.
\end{aligned} \tag{5.4}$$

Let $x_n \to x$ in $M_\nu(S)$. Assumptions 5.1(a),(b) imply that if

$$f_n(i) = \frac{|\gamma_i(x_n)-\gamma_i^0|}{\gamma_i^0} + \frac{|b_i(x_n)-b_i^0|}{\gamma_i^0},$$

then $\{f_n\}$ is uniformly integrable with respect to counting measure on S and hence by (5.4) so is

$$g_n(i) = |\gamma_i(x_n) - \gamma_i^0|\,|x_n(i)(R_\lambda f)_{ii}(x_n)| + |b_i(x_n) - b_i^0|\,|(R_\lambda f)_i(x_n)|.$$

This allows us to take the limit as $n \to \infty$ through the summation and conclude by the continuity of

$$(\gamma_i(x) - \gamma_i^0)\,(x(i)(R_\lambda f)_{ii}(x)) + (b_i(x) - b_i^0)\,((R_\lambda f)_i(x))$$

(see Corollary 4.9) that

$$\lim_{n\to\infty} (\mathcal{L} - \mathcal{L}^0)R_\lambda f(x_n) = (\mathcal{L} - \mathcal{L}^0)R_\lambda f(x).$$

This proves $(\mathcal{L} - \mathcal{L}^0)R_\lambda f$ is continuous. $\square$

We let $\mathcal{B}_\lambda$ denote the operator

$$\mathcal{B}_\lambda = (\mathcal{L} - \mathcal{L}^0)R_\lambda = \sum_i \left[(\gamma_i - \gamma_i^0)x(i)(R_\lambda)_{ii} + (b_i - b_i^0)(R_\lambda)_i\right].$$

Proposition 5.3. *Assume Assumption 5.1(a)-(c). For any* $\lambda > 0$, $\mathcal{B}_\lambda : \mathcal{C}_\alpha \to \mathcal{C}_\alpha$ *is a bounded operator. Moreover there exist* $\lambda_0 = \lambda_0(\alpha)$ *and* $\rho_0 = \rho_0(\alpha) > 0$ *such that if* $\lambda \geq \lambda_0$ *and* $\rho \leq \rho_0$, *then*

$$\|\mathcal{B}_\lambda\|_{\mathcal{C}_\alpha} < 1/2.$$

Proof. Use Lemma 4.4 and Corollary 4.10 to see that for $f \in \mathcal{C}_\alpha$, $j \in S$, and $h > 0$,

$$\begin{aligned}
&|\mathcal{B}_\lambda f(x + he_j) - \mathcal{B}_\lambda f(x)| \\
&\quad \leq \sum_i |\gamma_i(x + he_j) - \gamma_i(x)|\,|(x + he_j)(i)(R_\lambda f)_{ii}(x + he_j)| \\
&\qquad\qquad + |b_i(x + he_j) - b_i(x)|\,|(R_\lambda f)_i(x + he_j)| \\
&\qquad + \sum_i |\gamma_i(x) - \gamma_i^0|\,|(x + he_j)(i)(R_\lambda f)_{ii}(x + he_j) - x(i)(R_\lambda f)_{ii}(x)| \\
&\qquad\qquad + |b_i(x) - b_i^0|\,|(R_\lambda f)_i(x + he_j) - (R_\lambda f)_i(x)| \\
&\quad \leq \sum_i [|\gamma_i(x + he_j) - \gamma_i(x)| + |b_i(x + he_j) - b_i(x)|](\gamma_i^0)^{-1}\lambda^{-\alpha/2}\kappa_{4.4} \sup_{i'} |f|_{\alpha,i'}(\gamma_{i'}^0)^{\alpha/2} \\
&\qquad + \sum_i [|\gamma_i(x) - \gamma_i^0| + |b_i(x) - b_i^0|](\gamma_i^0)^{-1}\kappa_{4.10}|f|_{\alpha,i}^{1-\alpha}|f|_{\alpha,j}^{\alpha}(\gamma_i^0/\gamma_j^0)^{(1-\alpha)\alpha/2}h^\alpha x_j^{-\alpha/2}.
\end{aligned}$$

The first summation is bounded by (use Assumption 5.1(c))

$$\kappa_{4.4}\lambda^{-\alpha/2}|f|_\alpha(\gamma_j^0)^{-\alpha/2}\kappa_{5.1}h^\alpha x_j^{-\alpha/2}.$$

The second summation is bounded by (use Assumption 5.1(a))

$$\begin{aligned}&\kappa_{4.10}\rho \sup_i(|f|_{\alpha,i}(\gamma_i^0)^{\alpha/2})^{1-\alpha} \sup_{j'}(|f|_{\alpha,j'}(\gamma_{j'}^0)^{\alpha/2})^{\alpha}(\gamma_j^0)^{-\alpha/2}h^\alpha x(j)^{-\alpha/2}\\ &\quad\le \kappa_{4.10}\rho|f|_\alpha(\gamma_j^0)^{-\alpha/2}h^\alpha x_j^{-\alpha/2}.\end{aligned}$$

We may therefore conclude

$$|\mathcal{B}_\lambda f|_{\alpha,j}(\gamma_j^0)^{\alpha/2} \le [\kappa_{4.4}\kappa_{5.1}\lambda^{-\alpha/2}+\kappa_{4.10}\rho]\,|f|_\alpha$$

and so

$$|\mathcal{B}_\lambda f|_\alpha \le [\kappa_{4.4}\kappa_{5.1}\lambda^{-\alpha/2}+\kappa_{4.10}\rho]|f|_\alpha.$$

Combine this with Lemma 5.2 to see that

$$\|\mathcal{B}_\lambda f\|_\alpha \le [\kappa_{4.4}\kappa_{5.1}\lambda^{-\alpha/2}+(\kappa_{4.10}+\kappa_{5.2}\lambda^{-\alpha/2})\rho]\,|f|_\alpha.$$

This, together with Lemma 5.2, shows $\mathcal{B}_\lambda : \mathcal{C}_\alpha \to \mathcal{C}_\alpha$ is a bounded operator with $\|\mathcal{B}_\lambda\| \le 1/2$ for $\rho \le \rho_0(\alpha), \lambda \ge \lambda_0(\alpha)$. □

Let $\mathbb{P}_\mu$ be a solution of $\mathcal{MP}(\mathcal{L},\mu)$ for some law μ on $M_\nu(S)$ and for $\lambda > 0$, let $S_\lambda f = \mathbb{E}_\mu(\int_0^\infty e^{-\lambda t} f(X_t)dt)$.

The following result uses only (2.5), (2.7), and Assumption 5.1(a) (it only requires the bound in Lemma 5.2 and so does not require Assumption 5.2(b)). Recall the constant $\kappa_{2.3a}$ in (2.5). We use $\xrightarrow{bp}$ to denote bounded pointwise convergence.

Proposition 5.4. *Assume (2.5), (2.7). If $f \in \mathcal{C}_\alpha$, then*

$$S_\lambda f = \int R_\lambda f(x)\mu(dx) + S_\lambda \mathcal{B}_\lambda f, \qquad \lambda > \kappa_{2.3a}.$$

Proof. Assume first that

$$\int |x|_\nu \mu(dx) < \infty. \tag{5.5}$$

Let $f(x) = f_0(\pi_n x)$ for some $f_0 : \mathbb{R}_+^{S_n} \to \mathbb{R}$, $f \in \mathcal{C}_\alpha$. Write $x^n = \pi_n(x)$ for $x \in M_\nu(S)$ and define

$$g_\delta(y) = \int_\delta^\infty e^{-\lambda t} P_t f(y)dt = \int_\delta^\infty e^{-\lambda t} P_t^n f_0(y^n)dt \equiv \widetilde{g}_\delta(y^n).$$

Here P_t^n is the semigroup of $\{X_i : i \in S_n\}$ under $\mathbb{P}_x^0$. The finite dimensional analysis in the proof of Lemma 6.1 of [BP03] shows that $\widetilde{g}_\delta \in C_b^2(\mathbb{R}_+^{S_n})$.

Use (2.4), (5.5), (2.5), a stopping time argument, and a Gronwall argument (cf. the proof of Theorem 2.4) to see that

$$\mathbb{E}_\mu(|X_t|_\nu) \le \Big(\int |x|_\nu d\mu(x) + \kappa_{2.3a}t\Big)e^{\kappa_{2.3a}t}, \qquad t \ge 0. \tag{5.6}$$

This and (2.7) shows that the stochastic integrals in (2.4) are square integrable martingales and by Itô's formula, the same is true of $M_t^{g_\delta}$, the martingale entering in $\mathcal{MP}(\mathcal{L},\mu)$. Take expectations in $\mathcal{MP}(\mathcal{L},\mu)$ to see that

$$\mathbb{E}_\mu(g_\delta(X_t)) = \int g_\delta d\mu + \int_0^t \mathbb{E}_\mu(\mathcal{L}g_\delta(X_s))ds.$$

Let $\lambda > \kappa_{2.3a}$, multiply the above by $\lambda e^{-\lambda t}$, and integrate over $t \in [0,\infty)$ to conclude

$$\lambda S_\lambda g_\delta = \int g_\delta d\mu + S_\lambda((\mathcal{L}-\mathcal{L}_0)g_\delta) + S_\lambda(\mathcal{L}_0 g_\delta). \tag{5.7}$$

Note here that (5.6) and $\lambda > \kappa_{2.3a}$ are needed to apply Fubini's theorem, since

$$\begin{aligned}\mathbb{E}_\mu\Big(\sum_{i\in S_n} X_s(i)\gamma_i(X_s)|(g_\delta)_{ii}(X_s)|\Big) &\le c_\delta \mathbb{E}\Big(\sum_{i\in S_n} X_s(i)\Big)\\ &\le c_{n,\delta}\mathbb{E}\left(|X_s|_\nu\right)\\ &\le c_{n,\delta} e^{\kappa_{2.3a}s}\Big[\int |x|_\nu d\mu + \kappa_{2.3a}s\Big].\end{aligned}$$

Now let $\delta \downarrow 0$ in (5.7). As $\delta \to 0$, $g_\delta \xrightarrow{bp} R_\lambda f$, and so $\lambda S_\lambda g_\delta \to \lambda S_\lambda R_\lambda f$ and $\int g_\delta d\mu \to \int R_\lambda f\, d\mu$ by dominated convergence. The finite-dimensional arguments in Lemma 6.1 of [BP03] show that as $\delta \downarrow 0$,

$$\mathcal{L}^0 g_\delta = \lambda g_\delta - e^{-\lambda\delta}P_\delta f \xrightarrow{bp} \lambda R_\lambda f - f \text{ and } (\mathcal{L}-\mathcal{L}^0)g_\delta \xrightarrow{bp} (\mathcal{L}-\mathcal{L}^0)R_\lambda f. \tag{5.8}$$

The latter implies that $S_\lambda((\mathcal{L}-\mathcal{L}^0)g_\delta) \xrightarrow{bp} S_\lambda \mathcal{B}_\lambda f$ and the former gives $S_\lambda(\mathcal{L}^0 g_\delta) \xrightarrow{bp} \lambda S_\lambda R_\lambda f$ $- S_\lambda f$. Therefore we may let $\delta \to 0$ in (5.7) to derive the required equality.

Now derive the result for a general $f \in \mathcal{C}_\alpha$ by approximation. Recall $f_n(x) = f \circ \pi_n(x)$, and so $|f_n|_{\alpha.i} \le |f|_{\alpha,i}$ implies $f_n \in \mathcal{C}_\alpha$. By the above

$$S_\lambda f_n = \int R_\lambda f_n d\mu + S_\lambda \mathcal{B}_\lambda f_n. \tag{5.9}$$

Now

$$\begin{aligned}|\mathcal{B}_\lambda f_n(x) - \mathcal{B}_\lambda f(x)| \le \sum_{i\in S} &|\gamma_i(x)-\gamma_i^0|\,|x(i)(R_\lambda f_n)_{ii}(x) - x(i)(R_\lambda f)_{ii}(x)|\\ &+ |b_i(x)-b_i^0|\,|(R_\lambda f_n)_i(x) - (R_\lambda f)_i(x)|.\end{aligned} \tag{5.10}$$

Proposition 4.8 shows that each of the summands approaches 0 as $n \to \infty$, while Lemma 4.4 and $|f_n|_{\alpha,i} \le |f|_{\alpha,i}$ show that the i^{th} summand is at most

$$[|\gamma_i(x)-\gamma_i^0| + |b_i(x)-b_i^0|](\gamma_i^0)^{-1}c_\alpha |f|_\alpha \lambda^{-\alpha/2}.$$

This is summable by Assumption 5.1(a) and we may use dominated convergence in (5.10) to see that $|\mathcal{B}_\lambda f_n(x) - \mathcal{B}_\lambda f(x)| \to 0$ as $n \to \infty$. The bound in Lemma 5.2 shows that the convergence is also bounded and so $S_\lambda \mathcal{B}_\lambda f_n \to S_\lambda \mathcal{B}_\lambda f$. Since $f_n \xrightarrow{bp} f$ (Lemma 4.5), we also have $S_\lambda f_n \to S_\lambda f$ and $\int R_\lambda f_n d\mu \to \int R_\lambda f\, d\mu$. Therefore we may let $n \to \infty$ in (5.9) to complete the proof under (5.5).

To remove (5.5), let $\mathbb{P}_N$ be the restriction of $\mathbb{P}_\mu$ to $\{\omega \in \Omega_\nu : |X_0(\omega)|_\nu \le N\}$. Note that $\mathbb{P}_N$ solves $\mathcal{MP}(\mathcal{L}, \mu_N)$, where $\mu_N = \mu(\cdot \mid |x|_\nu \le N)$. Here N is large enough so that $\mu(|x|_\nu \le N) > 0$. Let $H_{f,\lambda} = \int_0^\infty e^{-\lambda t} f(X_t) dt$. If $f \in \mathcal{C}_\alpha$, the previous case shows

$$\int H_{f,\lambda} d\mathbb{P}_N = \int R_\lambda f d\mu_N + \int H_{\mathcal{B}_\lambda f,\lambda} d\mathbb{P}_N.$$

Note $\mathcal{B}_\lambda f$ and hence $H_{\mathcal{B}_\lambda f,\lambda}$ are bounded by the upper bound in Lemma 5.2. Now let $N \to \infty$ in the above to finish the proof. □

Theorem 5.5. *Assume (2.5), (2.7) and Assumption 5.1 holds with $\rho \le \rho_0$ and ρ_0 is as in Proposition 5.3. For any probability μ on $M_\nu(S)$, there is at most one solution to $\mathcal{MP}(\mathcal{L}, \mu)$.*

Proof. Let λ_0 be as in Proposition 5.3 and assume $\lambda > \lambda_1 \equiv \max(\lambda_0, \kappa_{2.3a})$. Let $\mathbb{P}_\mu$ satisfy $\mathcal{MP}(\mathcal{L}, \mu)$. If $f \in \mathcal{C}_\alpha$, then $\mathcal{B}_\lambda f \in \mathcal{C}_\alpha$ by Proposition 5.3, and so iterating Proposition 5.4 gives

$$S_\lambda f = \int \sum_{k=0}^{n} R_\lambda \mathcal{B}_\lambda^k f\, d\mu + S_\lambda(\mathcal{B}_\lambda^{n+1} f). \tag{5.11}$$

By Proposition 5.3 $\|\mathcal{B}_\lambda^{n+1} f\|_\infty \le 2^{-(n+1)} \|f\|_\alpha$. This shows the last term in (5.11) converges to 0 as $n \to \infty$ and $\sum_{k=0}^\infty R_\lambda \mathcal{B}_\lambda^k f$ converges uniformly on $M_\nu(S)$ to a bounded continuous function (recall (4.13)). Therefore letting $n \to \infty$ in (5.11) we arrive at

$$S_\lambda f = \int \sum_{k=0}^{\infty} R_\lambda \mathcal{B}_\lambda^k f\, d\mu, \qquad \lambda > \lambda_1.$$

Inverting the Laplace transform ($t \to \mathbb{E}_\mu(f(X_t))$ is continuous) one sees that for any $t \ge 0$, $\mathbb{E}_\mu(f(X_t))$ is uniquely defined for all $f \in \mathcal{C}_\alpha$. This shows $\mathbb{P}_\mu(X_t \in \cdot)$ is unique ($\mathcal{C}_\alpha$ contains C^1 functions of finitely many coordinates with compact support). A standard result (see, e.g., Theorem 4.4.2 of Ethier-Kurtz [EK86]) now implies $\mathbb{P}_\mu$ is unique. Strictly speaking, the latter requires that $\mathcal{L}f$ be bounded for our test functions f and M_t^f should be a martingale. However the only test functions we actually used were the functions $g_\delta = \int_\delta^\infty e^{-\lambda t} P_t f\, dt$ with f a function in $\mathcal{C}_\alpha$ depending on finitely many coordinates. In the proof of Lemma 5.4, the boundedness of $\mathcal{L} g_\delta$ was made clear (see (5.8)), as was the fact that $M_t^{g_\delta}$ is then a martingale (which is also then immediate as it is bounded on bounded time intervals). $\square$

6. Uniqueness.

Proof of Theorem 2.7. A standard argument shows that it suffices to show that for each $z \in M_\nu(S)$ there is a unique solution to $\mathcal{MP}(\mathcal{L}, \delta_z)$ (see p. 136 of [Ba97].) Indeed, once this is established, Ex. 6.7.4 in [SV79] shows the laws of $\mathbb{P}_z$ are Borel measurable in z and then it is easy to see $\mathbb{P}_\mu(\cdot) = \int \mathbb{P}_z(\cdot)\, \mu(dz)$ is the unique solution to $\mathcal{MP}(\mathcal{L}, \mu)$.

Assumption 2.6 implies the continuity of b_i and γ_i on $M_{\beta\nu}(S)$. It is therefore easy to check that all the hypotheses of Theorem 2.4 are in force and hence existence holds.

Turning to uniqueness in $\mathcal{MP}(\mathcal{L}, \delta_z)$, let C be a compact set in $M_\nu(S)$ containing z. Assume the following:

For each $x_0 \in C$ there is a $\delta = \delta(x_0) > 0$ and coefficients $\widetilde{\gamma}_i, \widetilde{b}_i$, $i \in S$, agreeing with γ_i, b_i, respectively, on $B(x, \delta) \cap C = \{x \in C : |x - x_0|_\nu < \delta\}$ and such that if $\widetilde{\mathcal{L}}^{x_0} = \sum_i x(i)\widetilde{\gamma}_i f_{ii} + \widetilde{b}_i f_i$, then $\mathcal{MP}(\widetilde{\mathcal{L}}^{x_0}, \delta_y)$ is well-posed (i.e., has a unique solution) for all $y \in M_\nu(S)$. *(6.1)*

We first show that the theorem would then follow by a minor modification of the localization argument in [SV79] (Theorem 6.6.1). Let $\mathbb{P}$ be a solution of $\mathcal{MP}(\mathcal{L}, \delta_z)$ and let

$$T_C = \inf\{t : X_t \notin C\}.$$

The tightness of $\mathbb{P}$ on Ω_ν shows there are compact sets C_n in $M_\nu(S)$ increasing in n such that $T_{C_n} \uparrow \infty$ $\mathbb{P}$-a.s. It therefore suffices to show

$$\mathbb{P}(X(\cdot \wedge T_C) \in \cdot) \text{ is unique.} \tag{6.2}$$

If $\delta(x_0)$ is as in (6.1) we may choose a finite subcover $\{B(x_i, \delta(x_i))\}_{i=1}^N \equiv \{B_i\}_{i=1}^N$ of C. Let $\lambda > 0$ be a Lebesgue number for this cover, that is, a number λ such that for each $x \in C$ there is an i with $B(x, \lambda) \subset B_i$. Set $T_0 = 0$ and

$$T_{i+1} = \inf\{t > T_i : |X_t - X_{T_i}|_\nu > \lambda \text{ or } X_t \in C^c\}.$$

Note $T_i \uparrow T_C$ a.s. as $i \to \infty$ by the continuity of X in $M_\nu(S)$. Let $\{\widetilde{\mathbb{P}}_x^{x_0} : x \in M_\nu(S)\}$ be the unique solutions to $\mathcal{MP}(\widetilde{\mathcal{L}}^{x_0}, \delta_x)$ in (6.1). As noted above, $x \to \widetilde{\mathbb{P}}_x^{x_0}$ is Borel measurable. If $B(X_{T_i}, \lambda) \subset B_j$ (where we choose the minimal such $j = j(X_{T_i})$) and $\tau = \inf\{t : |X_t - X_0| > \lambda \text{ or } X_t \in C^c\}$, then the uniqueness of $\widetilde{\mathbb{P}}_{X(T_i)}^{x_{j(X_{T_i})}}$ shows that conditional on $\mathcal{F}_{T_i}$, $X((\cdot + T_i) \wedge T_{i+1})$ has law $\widetilde{\mathbb{P}}_{X(T_i)}^{x_{j(X_{T_i})}}(X(\cdot \wedge \tau) \in \cdot)$. As in the proof of Theorem 6.6.1 of [SV79], this easily gives (6.2).

It remains to establish (6.1), so fix $x_0 \in C$. Assume first $b_i \geq 0$ for all $i \in S$. For $r > 0$ let $\varphi_r : [0, \infty) \to [0, 1]$ be the map which is 1 on $[0, r]$, 0 on $[2r, \infty)$, and linear on $[r, 2r]$. Let ρ_0 be as in Proposition 5.3 and choose $\delta_0 = \delta_0(x_0) > 0$ as in Assumption 2.6 but with $\eta = \rho_0$. By Lemma 3.1 we may assume $C = K_\varepsilon$ for some $\varepsilon = \{\varepsilon_n\}$ decreasing to 0. If $\theta, x \in \mathbb{R}_+^S$, define $\theta \wedge x \in \mathbb{R}_+^S$ by $(\theta \wedge x)(i) = \theta(i) \wedge x(i)$. Define $\overline{\varepsilon} : S \to [0, \infty)$ by $\overline{\varepsilon}(i) = \varepsilon_n$ if $i \in S_{n+1} - S_n$, $n \geq 0$, and set $\theta(i) = \overline{\varepsilon}(i)/\nu(i)$. Let β be as in Assumption 2.6, where we may assume $\beta \leq 1$ without loss of generality, and set $\delta = \delta_0/3$, $\gamma_i^0 = \gamma_i(x_0)$, $b_i^0 = b_i(x_0)$. We define functions $\widetilde{\gamma}_i$, $\widetilde{b}_i$ in (6.1) as follows:

$$\widetilde{\gamma}_i(x) = \varphi_\delta(|(x \wedge \theta) - x_0|_{\beta\nu})\gamma_i(x \wedge \theta) + (1 - \varphi_\delta(|(x \wedge \theta) - x_0|_{\beta\nu}))\gamma_i^0,$$
$$\widetilde{b}_i(x) = \varphi_\delta(|(x \wedge \theta) - x_0|_{\beta\nu})b_i(x \wedge \theta) + (1 - \varphi_\delta(|(x \wedge \theta) - x_0|_{\beta\nu}))b_i^0.$$

If $x \in C$ and $i \in S_{n+1} - S_n$, then $x(i)\nu(i) \leq \varepsilon_n$ and so $x(i) \leq \theta(i)$, and hence $x \wedge \theta = x$. It follows easily that $\widetilde{\gamma}_i = \gamma_i$ and $\widetilde{b}_i = b_i$ on $B(x_0, \delta) \cap C$ (in fact we only need $|x - x_0|_{\beta\nu} \leq \delta$). We claim $(\widetilde{\gamma}_i, \widetilde{b}_i)$ satisfies the hypotheses of Theorems 2.4 and 5.5 and so (6.1) will follow from those results. (2.5) implies

$$\sum_i b_i^0 \nu(i) = \sum_i b_i(x_0)\nu(i) \leq \kappa_{2.3a}(|x_0|_\nu + 1) < \infty, \tag{6.3}$$

and so (4.2) holds for b^0. (4.1) is immediate from $\gamma_i > 0$ and (2.7). Clearly $\|\widetilde{\gamma}_i\|_\infty \leq \|\gamma_i\|_\infty$ and so (2.7) for γ implies (2.7) for $\widetilde{\gamma}$. Use (2.5) and (6.3) to see that

$$\begin{aligned}
&\sum_i |\widetilde{b}_i(x)|\nu(i) \\
&\quad \leq \varphi_\delta(|(x \wedge \theta) - x_0|_{\beta\nu}) \sum_i |b_i(x \wedge \theta)|\nu(i) + (1 - \varphi_\delta(|(x \wedge \theta) - x_0|_{\beta\nu})) \sum_i b_i^0 \nu(i) \\
&\quad \leq \kappa_{2.3a}(|x \wedge \theta|_\nu + 1) + \kappa_{2.3a}(|x_0|_\nu + 1) \\
&\quad \leq c_1(|x|_\nu + 1),
\end{aligned}$$

and hence derive (2.5) for $\widetilde{b}$. Note that

$$\varphi_\delta(|(x \wedge \theta) - x_0|_{\beta\nu}) = 0 \ \text{ if } \ |(x \wedge \theta) - x_0|_{\beta\nu} \geq 2\delta = 2\delta_0/3$$

and therefore

$$\begin{aligned}
&\sum_i \frac{|\widetilde{\gamma}_i(x) - \gamma_i^0|}{\gamma_i^0} + \frac{|\widetilde{b}_i(x) - b_i^0|}{\gamma_i^0} \\
&\qquad \leq \varphi_\delta(|x \wedge \theta - x_0|_{\beta\nu}) \sum_i \Big[\frac{|\gamma_i(x \wedge \theta) - \gamma_i(x_0)|}{\gamma_i(x_0)} + \frac{|b_i(x \wedge \theta) - b_i(x_0)|}{\gamma_i(x_0)}\Big] \\
&\qquad < \rho_0
\end{aligned} \tag{6.4}$$

by our choice of δ_0. Therefore $(\widetilde{\gamma}_i, \widetilde{b}_i)$ satisfies Assumption 5.1(a) with $\rho = \rho_0$.

To check Assumption 5.1(c) note that

$$\begin{aligned}\widetilde{\gamma}_i(x+he_j) - \widetilde{\gamma}_i(x) &= (\widetilde{\gamma}_i(x+he_j) - \gamma_i^0) - (\widetilde{\gamma}_i(x) - \gamma_i^0) \\ &= \varphi_\delta(|((x+he_j)\wedge\theta) - x_0|_{\beta\nu})(\gamma_i((x+he_j)\wedge\theta) - \gamma_i^0) \\ &\quad - \varphi_\delta(|(x\wedge\theta) - x_0|_{\beta\nu})(\gamma_i(x\wedge\theta) - \gamma_i^0),\end{aligned}$$

and similarly for $\widetilde{b}_i$. Therefore, if $h > 0$, $j \in S$, and $x \in M_\nu(S)$,

$$\begin{aligned}&\sum_i |\widetilde{\gamma}_i(x+he_j) - \widetilde{\gamma}_i(x)|(\gamma_i^0)^{-1} + |\widetilde{b}_i(x+he_j) - \widetilde{b}_i(x)|(\gamma_i^0)^{-1} \\ &\quad \le \sum_i |\varphi_\delta(|((x+he_j)\wedge\theta) - x_0)|_{\beta\nu}) - \varphi_\delta(|(x\wedge\theta) - x_0|_{\beta\nu})| \\ &\qquad\quad \times [\gamma_i((x+he_j)\wedge\theta) - \gamma_i^0|(\gamma_i^0)^{-1} + |b_i((x+he_j)\wedge\theta) - b_i^0|(\gamma_i^0)^{-1}] \\ &\qquad\quad + \sum_i \varphi_\delta(|(x\wedge\theta) - x_0|_{\beta\nu})[|\gamma_i((x+he_j)\wedge\theta) - \gamma_i(x\wedge\theta)|(\gamma_i^0)^{-1} \\ &\qquad\qquad + |b_i((x+he_j)\wedge\theta) - b_i(x\wedge\theta)|(\gamma_i^0)^{-1}] \\ &\quad \equiv R_1 + R_2.\end{aligned}$$

Note that $(x+he_j)\wedge\theta = (x\wedge\theta) + h_je_j$, where $h_j = h\wedge(\theta_j - x_j)^+$. Therefore

$$\begin{aligned}&|\varphi_\delta(|((x+he_j)\wedge\theta) - x_0|_{\beta\nu}) - \varphi_\delta(|(x\wedge\theta) - x_0|_{\beta\nu})| \\ &\qquad \le c_\delta((h\wedge\theta_j)\beta(j)\nu(j))\wedge 1)1_{(x(j)<\theta(j))} \equiv c_\delta\delta_j(x).\end{aligned} \tag{6.5}$$

If $|((x+he_j)\wedge\theta) - x_0|_{\beta\nu} \le \delta_0$, this and Assumption 2.6(a) imply

$$R_1 \le c_\delta\delta_j(x)\rho_0. \tag{6.6}$$

If $|(x\wedge\theta) - x_0|_{\beta\nu} \le \delta_0$, then use Assumption 2.6(b) and Assumption 2.6(a) to see that

$$\begin{aligned}&\sum_i |\gamma_i((x+he_j)\wedge\theta) - \gamma_i^0|(\gamma_i^0)^{-1} + |b_i((x+he_j)\wedge\theta) - b_i^0|(\gamma_i^0)^{-1} \\ &\quad \le \sum_i |\gamma_i((x\wedge\theta) + h_je_j) - \gamma_i(x\wedge\theta)|(\gamma_i^0)^{-1} + |b_i((x\wedge\theta) + h_je_j) - b_i(x\wedge\theta)|(\gamma_i^0)^{-1} \\ &\qquad + \sum_i |\gamma_i(x\wedge\theta) - \gamma_i^0|(\gamma_i^0)^{-1} + |b_i(x\wedge\theta) - b_i^0|(\gamma_i^0)^{-1} \\ &\quad \le \kappa_{2.6b}(\gamma_j(x_0))^{-\alpha/2}(\theta_j\wedge x_j)^{-\alpha/2}h_j^\alpha) + \rho_0.\end{aligned} \tag{6.7}$$

This gives (recall $x_j < \theta_j$ or else $R_1 = \delta_j(x) = 0$)

$$R_1 \le c_2\delta_j(x)\Big[(\gamma_j^0)^{-\alpha/2}x_j^{-\alpha/2}(h\wedge\theta_j)^\alpha + \rho_0\Big]. \tag{6.8}$$

If Assumption 2.6(c) holds, note first that $\theta_j \to 0$ as $|j| \to \infty$, since $\nu(j) \to \infty$ as $|j| \to \infty$. Hence $\sup_j h_j < \infty$ and at the cost of increasing $\kappa_{2.6b}$ we may apply Assumption 2.6(c) with $h = h_j$ to get

$$R_1 \leq c_3 \delta_j(x)(\gamma_j^0)^{-\alpha/2}(h \wedge \theta_j)^\alpha(1 + (x_j \wedge \theta_j)^{-\alpha/2}) + \rho_0)$$

and so

$$R_1 \leq c_3 \delta_j(x)\Big[(\gamma_j^0)^{-\alpha/2}(h \wedge \theta_j)^\alpha x_j^{-\alpha/2} + \rho_0\Big],$$

the last because $\sup_j \theta_j < \infty$ and $x_j < \theta_j$ (or else $R_1 = 0$). Hence we get (6.8) in either case.

Next we claim

$$\delta_j(x) \leq c_4 h^\alpha x_j^{-\alpha/2}(\gamma_j^0)^{-\alpha/2}. \tag{6.9}$$

Assume this for the moment. Then we may use this and the trivial bound $\delta_j \leq 1$ in (6.8) to derive

$$R_1 \leq c_4 h^\alpha x_j^{-\alpha/2}(\gamma_j^0)^{-\alpha/2}. \tag{6.10}$$

To prove (6.9) use the bound on β in Assumption 2.6 and $x_j < \theta(j) \leq c_5 \nu(j)^{-1}$ to see

$$\begin{aligned} h^{-\alpha} x_j^{\alpha/2} \delta_j(x) &\leq c_6(h \wedge (\nu(j))^{-1}) h^{-\alpha} \theta(j)^{\alpha/2} \nu(j)^{1-\alpha/2} \\ &\leq c_6(h \wedge (\nu(j))^{-1}) h^{-\alpha} \nu(j)^{1-\alpha} \\ &= c_6((h\nu(j))^{1-\alpha} \wedge (h\nu(j))^{-\alpha}) \leq c_6. \end{aligned}$$

As $\sup_j \gamma_j^0 < \infty$, (6.9) follows and hence so does (6.10).

Next we show

$$R_2 \leq c_7 h^\alpha x_j^{-\alpha/2}(\gamma_j^0)^{-\alpha/2}. \tag{6.11}$$

If Assumption 2.6(b) holds this is immediate because $h_j \leq h$ and $R_2 = 0$ if $x_j \geq \theta_j$. Assume Assumption 2.6(c). Then as $h_j \leq \theta_j \leq \|\theta\|_\infty < \infty$, we may apply Assumption 2.6(c) with $h = h_j$ and assume $x_j \leq \theta_j \leq \|\theta\|_\infty$ to conclude

$$\begin{aligned} R_2 &\leq \kappa_{2.6b}(\gamma_j^0)^{-\alpha/2}(h \wedge \theta_j)^\alpha(1 + (x_j \wedge \theta(j))^{-\alpha/2}) \\ &\leq c_8(\gamma_j^0)^{-\alpha/2} h^\alpha x_j^{-\alpha/2}. \end{aligned}$$

Finally (6.10) and (6.11) show Assumption 5.1(c) holds for $(\tilde{\gamma}_j, \tilde{b}_j)$.

Next consider Assumption 5.1(b). Let $x \in M_\nu(S)$ and $\eta > 0$. If $|(x \wedge \theta) - x_0|_{\beta\nu} > \frac{2}{3}\delta_0$, assume $|x' - x|_{\beta\nu} < |(x \wedge \theta) - x_0|_{\beta\nu} - \frac{2}{3}\delta_0$, so that

$$\begin{aligned} |(x' \wedge \theta) - x_0|_{\beta\nu} &\geq |(x \wedge \theta) - x_0|_{\beta\nu} - |(x \wedge \theta) - (x' \wedge \theta)|_{\beta\nu} \\ &\geq |(x \wedge \theta) - x_0|_{\beta\nu} - |x - x'|_{\beta\nu} \\ &> \tfrac{2}{3}\delta_0 \end{aligned}$$

as well. Then $\widetilde{\gamma}_i(x) = \widetilde{\gamma}_i(x') = \gamma_i^0$ and $\widetilde{b}_i(x) = \widetilde{b}_i(x') = b_i^0$ and so Assumption 5.1(b) holds as both sides are zero. (Here we are taking limits in the weaker norm $|\cdot|_{\beta\nu}$, as will be the case below.) Assume therefore that $|(x\wedge\theta) - x_0|_{\beta\nu} \le \frac{2}{3}\delta_0 = 2\delta$. By Assumption 2.6(a) we may choose $\delta_1 < \eta$ so that

$$\text{if} \qquad |x'-x|_{\beta\nu} < \delta_1, \qquad \text{then} \qquad \sum_i \left[\frac{|\gamma_i(x') - \gamma_i(x)|}{\gamma_i(x)} + \frac{|b_i(x') - b_i(x)|}{\gamma_i(x)}\right] < \eta. \tag{6.12}$$

Suppose $|x'-x|_{\beta\nu} < \delta_1$. This implies $|(x'\wedge\theta) - (x\wedge\theta)|_{\beta\nu} < \delta_1$. Note that as $|(x\wedge\theta) - x_0|_{\beta\nu} \le \frac{2}{3}\delta_0 < \delta_0$,

$$\Big|\frac{\gamma_i(x\wedge\theta)}{\gamma_i(x_0)} - 1\Big| < \rho_0,$$

and so

$$\frac{\gamma_i(x\wedge\theta)}{\gamma_i(x_0)} < 1 + \rho_0. \tag{6.13}$$

Therefore

$$\begin{aligned}
&\Big|\sum_i \frac{|\widetilde{\gamma}_i(x') - \gamma_i^0|}{\gamma_i^0} + \frac{|\widetilde{b}_i(x') - b_i(0)|}{\gamma_i^0} - \sum_i \frac{|\widetilde{\gamma}_i(x) - \gamma_i^0|}{\gamma_i^0} + \frac{|\widetilde{b}_i(x) - b_i^0|}{\gamma_i^0}\Big| \\
&\quad \le \varphi_\delta(|(x'\wedge\theta) - x_0|_{\beta\nu})\Big(\sum_i \frac{|\gamma_i(x'\wedge\theta) - \gamma_i(x\wedge\theta)|}{\gamma_i^0} + \frac{|b_i(x'\wedge\theta) - b_i(x\wedge\theta)|}{\gamma_i^0}\Big) \\
&\qquad + |\varphi_\delta(|(x'\wedge\theta) - x_0|_{\beta\nu}) - \varphi_\delta(|(x\wedge\theta) - x_0|_{\beta\nu})| \\
&\qquad\qquad \times \Big(\sum_i \frac{|\gamma_i(x\wedge\theta) - \gamma_i^0|}{\gamma_i^0} + \frac{|b_i(x\wedge\theta) - b_i^0|}{\gamma_i^0}\Big) \\
&\quad \equiv S_1 + S_2.
\end{aligned}$$

Use (6.12) and (6.13) to see that

$$S_1 < \eta(1+\rho_0).$$

Use Assumption 2.6(a) and our choice of δ_0 to see that (recall $|(x\wedge\theta) - x_0|_{\beta\nu} < \delta_0$)

$$S_2 \le c_\delta |(x'\wedge\theta) - (x\wedge\theta)|_{\beta\nu}\rho_0 \le c_\delta |x'-x|_{\beta\nu}\rho_0 \le c_\delta \eta \rho_0.$$

These bounds verify Assumption 5.1(b) for $(\widetilde{\gamma}_i, \widetilde{b}_i)$ and complete the verification of the hypotheses of Theorem 5.5.

Now consider the conditions in Theorem 2.4. The continuity of $\widetilde{\gamma}_i$ and $\widetilde{b}_i$ on $M_{\beta\nu}(S)$ is clear. (2.6) for $\widetilde{b}$ is clear as $\widetilde{b}_i \ge 0$, and the other conditions have already been checked. This completes the proof of (6.1) and hence the theorem if $b_i \ge 0$.

Next assume (2.10), i.e., $\lim_{|i|\to\infty}\nu(i)=\infty$ and $b_i(x)\geq -\kappa_{2.7}x_i\gamma_i(x)$. Let $\underline{\nu}=\inf_i \nu(i)>0$. As increasing β only weakens the hypotheses and multiplying β by a constant will not change the conditions, we may assume $\beta(i)=\nu(i)^{-\alpha/2}$. Let $\widehat{b}_i(x)=b_i(x)+\kappa_{2.7}x_i\gamma_i(x)\geq 0$. We claim $(\widehat{b}_i,\gamma_i)$ satisfy the hypotheses of the previous case. As Assumption 2.6(c) is now a weaker condition than 2.6(b), we assume (b_i,γ_i) satisfies Assumption 2.6(a),(c). By (2.7) and (2.5)

$$\sum_i|\widehat{b}_i(x)|\nu(i)\leq\sum_i|b_i(x)|\nu(i)+\kappa_{2.7}\kappa_{2.4}\sum_i x(i)\nu(i)\leq c_9(|x|_\nu+1),$$

and so (2.5) holds for $\widehat{b}$. The continuity of $\widehat{b}_i$ on $M_{\beta\nu}(S)=M_{\nu^{1-\frac{\alpha}{2}}}(S)$ is clear. To check Assumption 2.6(a), let $x_0\in M_\nu(S)$ and $\eta>0$. Then

$$\begin{aligned}\sum_i\frac{|\widehat{b}_i(x)-\widehat{b}_i(x_0)|}{\gamma_i^0}&\leq\sum_i\frac{|b_i(x)-b_i(x_0)|}{\gamma_i^0}+\kappa_{2.7}\sum_i\frac{\gamma_i(x)}{\gamma_i^0}|x(i)-x_0(i)|\\&\quad+\kappa_{2.7}\sum_i\frac{|\gamma_i(x)-\gamma_i(x_0)|}{\gamma_i(x_0)}x_0(i)\\&\equiv T_1+T_2+T_3.\end{aligned}$$

By Assumption 2.6(a) we may choose δ_0 such that $|x-x_0|_{\beta\nu}<\delta_0$ implies $T_1<\eta$ and

$$T_3<\eta\|x_0\|_\infty\leq\eta|x_0|_\nu/\underline{\nu}.$$

For $|x-x_0|_{\beta\nu}<\delta_0$ we may bound T_2 by

$$\begin{aligned}&\kappa_{2.7}\sum_i\frac{|\gamma_i(x)-\gamma_i(x_0)|}{\gamma_i(x_0)}\|x-x_0\|_\infty+\kappa_{2.7}\sum_i|x(i)-x_0(i)|\\&\quad\leq c_{10}[\eta|x-x_0|_{\beta\nu}+|x-x_0|_{\beta\nu}].\end{aligned}$$

In the last line we used $\beta(i)\nu(i)=\nu(i)^{1-\alpha/2}\to\infty$, and so $\beta\nu$ is bounded below. This shows $(\widehat{b}_i,\gamma_i)$ satisfies Assumption 2.6(a).

By Assumption 2.6(c) (for (b_i,γ_i)), if $h\in(0,1]$, $j\in S$, and $|x-x_0|_{\beta\nu}<\delta_0$,

$$\begin{aligned}\sum_i\frac{|\widehat{b}_i(x+he_j)-\widehat{b}_i(x)|}{\gamma_i(x_0)}&\leq\sum_i\frac{|b_i(x+he_j)-b_i(x)|}{\gamma_i(x_0)}+\sum_i\frac{\gamma_i(x+he_j)h\delta_{ij}}{\gamma_i(x_0)}\\&\quad+\sum_i\frac{|\gamma_i(x+he_j)-\gamma_i(x)|}{\gamma_i(x_0)}x_i\qquad(6.14)\\&\leq\kappa_{2.6b}\gamma_j(x_0)^{-\alpha/2}h^\alpha(1+x_j^{-\alpha/2})(1+\|x\|_\infty)+\frac{\gamma_j(x+he_j)}{\gamma_j(x_0)}h.\end{aligned}$$

The last term is bounded by

$$\Big[\frac{|\gamma_j(x+he_j)-\gamma_j(x)|}{\gamma_j(x_0)}+\frac{|\gamma_j(x)-\gamma_j(x_0)|}{\gamma_j(x_0)}+1\Big]h$$
$$\le [\kappa_{2.6b}\gamma_j(x_0)^{-\alpha/2}h^\alpha(1+x_j^{-\alpha/2})+\eta+1]h$$
$$\le c_{11}(\gamma_j(x_0))^{-\alpha/2}h^\alpha(1+x_j^{-\alpha/2}),$$

where in the last line we use $\sup_j \gamma_j(x_0)<\infty$ and $h\le 1$. In addition, $\|x\|_\infty \le c_{12}|x|_{\beta\nu}\le c_{13}[|x_0|_{\beta\nu}+\delta_0]$. Put these bounds into (6.14) to see that $(\widehat{b}_i,\gamma_i)$ satisfies Assumption 2.6(c) and hence all the hypotheses of the previous case.

We now check (6.1) for (b_i,γ_i) by a Girsanov argument. Let $(\beta_i,\widetilde{\gamma}_i)$ denote the coefficients constructed above for which (6.1) holds; that is $\beta_i=\widetilde{(\widehat{b}_i)}$. Let $\{\widehat{\mathbb{P}}^{x_0}_x : x\in M_\nu(S)\}$ be the corresponding measurable (recall this is a consequence of uniqueness) unique solutions of $\mathcal{MP}(\widehat{\mathcal{L}},\delta_x)$ ($\widehat{\mathcal{L}}$ has coefficients $(\beta_i,\widetilde{\gamma}_i)$). Let $\widetilde{b}_i(x)=\beta_i-\kappa_{2.7}x_i\widetilde{\gamma}_i(x)$. Let $\widetilde{\mathcal{L}}$ be the generator with coefficients $(\widetilde{b}_i,\widetilde{\gamma}_i)$. Note that if $|x-x_0|_\nu<\delta$,

$$\widetilde{\gamma}_i(x)=\gamma_i(x)\ \text{ and }\ \widetilde{b}_i(x)=\widehat{b}_i(x)-\kappa_{2.7}x_i\gamma_i(x)=b_i(x).$$

Let $\mathbb{P}$ be a solution to $\mathcal{MP}(\widetilde{\mathcal{L}},\delta_z)$, $z\in M_\nu(S)$. Define $M^i_t=X_t(i)-x_0(i)-\int_0^t\widetilde{b}_i(X_s)ds$ and

$$R_t=\exp\Big(\sum_i\frac{\kappa_{2.7}}{2}M^i_t-\frac{\kappa^2_{2.7}}{4}\int_0^t X_s(i)\widetilde{\gamma}_i(X_s)ds\Big).$$

Under $\mathbb{P}$, $N^i_t=\kappa_{2.7}M^i_t/2$ is a collection of orthogonal continuous local martingales such that

$$\sum_i\langle N^i\rangle_t=\frac{\kappa^2_{2.7}}{2}\int_0^t\sum_i X_s(i)\widetilde{\gamma}_i(X_s)ds$$
$$\le\frac{\kappa^2_{2.7}}{2}\sup_i\|\widetilde{\gamma}_i\|_\infty\int_0^t\frac{|X_s|_\nu}{\underline{\nu}}ds<\infty,\qquad t>0,\quad a.s.$$

This shows that R_t is a well defined positive local martingale. Define $T_n=\inf\{t:|X_t|_\nu>n\}\uparrow\infty$ $\mathbb{P}$-a.s. It follows from the above that $(R_{t\wedge T_n}:t\ge 0)$ is a uniformly integrable positive martingale starting at 1 and so $d\mathbb{Q}_n=R_{T_n}d\mathbb{P}$ defines a probability on $(\Omega_\nu,\mathcal{F})$. Under $\mathbb{P}$,

$$\Big(M^i_{t\wedge T_n}-\int_0^{t\wedge T_n}\kappa_{2.7}X_s(i)\widetilde{\gamma}_i(X_s)ds\Big)R_{t\wedge T_n}$$

differs by a continuous local martingale from

$$-\kappa_{2.7}\int_0^{t\wedge T_n}X_s(i)\widetilde{\gamma}_i(X_s)R_sds+\langle M^i,R\rangle_{t\wedge T_n}$$
$$=-\kappa_{2.7}\int_0^{t\wedge T_n}X_s(i)\widetilde{\gamma}_i(X_s)R_sds+\frac{\kappa_{2.7}}{2}\int_0^{t\wedge T_n}R_s2\widetilde{\gamma}_i(X_s)X_s(i)ds$$
$$=0.$$

Therefore

$$X_t^i - x_0(i) - \int_0^t \widetilde{b}_i(X_s) + \kappa_{2.7} X_s(i)\widetilde{\gamma}_i(X_s)ds = X_t^i - x_0(i) - \int_0^t \beta_i(X_s)ds$$

is a local martingale under $\mathbb{Q}_n$. Let $\widehat{\mathbb{Q}}_n$ be the unique law on $(\Omega_\nu, \mathcal{F})$ such that $\widehat{\mathbb{Q}}_n|_{\mathcal{F}_{T_n}} = \mathbb{Q}_n|_{\mathcal{F}_{T_n}}$ and $\widehat{\mathbb{Q}}_n(X(T_n + \cdot) \in \cdot) \mid \mathcal{F}_{T_n}) = \mathbb{P}^{x_0}_{X(T_n)}(\cdot)$. Then it is easy to see from the above and Itô's formula that $\widehat{\mathbb{Q}}_n$ solves $\mathcal{MP}(\widehat{\mathcal{L}}, \delta_z)$ and so $\widehat{\mathbb{Q}}_n = \widehat{\mathbb{P}}^{x_0}_z$. This implies

$$\mathbb{Q}_n(T_n < t) = \widehat{\mathbb{P}}^{x_0}_z(T_n < t) \to 0$$

as $n \to \infty$. Therefore

$$\begin{aligned}\mathbb{P}(R_t) &\geq \mathbb{P}(R_{t\wedge T_n} 1_{(T_n \geq t)}) \\ &= \mathbb{P}(R_{t\wedge T_n}) - \mathbb{Q}_n(T_n < t) \\ &\to 1\end{aligned}$$

as $n \to \infty$. This proves $(R_t : t > 0)$ is a martingale under $\mathbb{P}$ and so $d\mathbb{Q}|_{\mathcal{F}_t} = R_t d\mathbb{P}|_{\mathcal{F}_t}$ defines a probability on $(\Omega_\nu, \mathcal{F})$. Now we repeat the above without the T_n's to see that $\mathbb{Q} = \widehat{\mathbb{P}}^{x_0}_z$. Therefore if $\widehat{M}_t^i = M_t^i - \kappa_{2.7}\int_0^t X_s(i)\widetilde{\gamma}_i(X_s)ds$, then

$$\begin{aligned}d\mathbb{P}|_{\mathcal{F}_t} &= R_t^{-1} d\widehat{\mathbb{P}}^{x_0}_z|_{\mathcal{F}_t} \\ &= \exp\Big(-\sum_{i\in S} \frac{\kappa_{2.7}}{2} M_t^i + \frac{\kappa_{2.7}^2}{4}\int_0^t X_s(i)\widetilde{\gamma}_i(X_s)ds\Big) d\widehat{\mathbb{P}}^{x_0}_z|_{\mathcal{F}_t} \\ &= \exp\Big(-\sum_{i\in S} \frac{\kappa_{2.7}}{2} \widehat{M}_t^i - \frac{\kappa_{2.7}^2}{4}\int_0^t X_s(i)\widetilde{\gamma}_i(X_s)ds\Big) d\widehat{\mathbb{P}}^{x_0}_z|_{\mathcal{F}_t}.\end{aligned}$$

This shows $\mathbb{P}$ is unique. Existence of solutions to $\mathcal{MP}(\widetilde{\mathcal{L}}, \delta_z)$ can be shown by either using the above formula or directly applying Theorem 2.4. For the latter note by (2.7) for $\widetilde{\gamma}$, that $\widetilde{b}_i \geq -\kappa_{2.7}\kappa_{2.4} x_i$. This verifies (6.1) for (b_i, γ_i) and so the proof is complete. □

7. Proofs of corollaries.

Proof of Corollary 2.8. Fix $x, y \in M_\nu(S)$ and define $z_n \in M_\nu(S)$ by

$$z_n(i) = \begin{cases} x(i), & i \in S_n \\ y(i), & i \notin S_n \end{cases}. \tag{7.1}$$

Then $|z_n - x|_\nu \to 0$ as $n \to \infty$ by dominated convergence. The continuity of γ_i implies

$$\begin{aligned}|\gamma_i(x) - \gamma_i(y)| &= \lim_{n\to\infty} |\gamma_i(z_n) - \gamma_i(z_0)| \\ &\leq \sum_{n=0}^{\infty} |\gamma_i(z_{n+1}) - \gamma_i(z_n)| \\ &= \sum_{n=0}^{\infty} \Big|\gamma_i\Big(z_n + \sum_{j\in S_{n+1}-S_n} (x_j - y_j)e_j\Big) - \gamma_i(z_n)\Big|.\end{aligned}$$

A similar bound on $|b_i(x) - b_i(y)|$ and an application of (2.12) leads to

$$\begin{aligned}\sum_{i\in S} |\gamma_i(x) - \gamma_i(y)| + |b_i(x) - b_i(y)| &\le \sum_{i\in S}\sum_{n=0}^{\infty}\sum_{j\in S_{n+1}-S_n} C(i,j)|x_j - y_j|^\alpha \\ &= \sum_{j\in S} \overline{C}(j)|x_j - y_j|^\alpha \end{aligned} \tag{7.2}$$

If $\alpha = 1$, thanks to (2.13), this leads to

$$\sum_{i\in S} |\gamma_i(x) - \gamma_i(y)| + |b_i(x) - b_i(y)| \le c_1|x - y|_1. \tag{7.3}$$

If $\alpha < 1$, set $\tau = \tau(\alpha) = \alpha(1 - \frac{\alpha}{2})$. Then Hölder's inequality and (2.13) bound (7.2) by

$$\begin{aligned}&\sum_j \overline{C}(j)\nu(j)^{-\tau}\nu(j)^{\tau}|x_j - y_j|^\alpha \\ &\quad\le \Big[\sum_j \overline{C}(j)^{1/(1-\alpha)}\nu(j)^{-\tau/(1-\alpha)}\Big]^{1-\alpha}\Big[\sum_j \nu(j)^{\tau/\alpha}|x(j) - y(j)|\Big]^\alpha \\ &\quad= c_2|x - y|^\alpha_{\nu^{1-\frac{\alpha}{2}}}. \end{aligned}\tag{7.4}$$

Assumption 2.6(a) with $\beta = \nu^{-\alpha/2}$ follows from (2.11), and (7.3) if $\alpha = 1$ or (7.4) if $\alpha < 1$. If $\alpha = 1$, note that $|\cdot|_{\beta\nu}$ is a stronger norm that $|\cdot|_1$. (7.3) and (7.4) also show that b_i and γ_i are uniformly continuous on $M_\nu(S)$ with respect to $|\cdot|_{\nu^{1-\alpha/2}}$ and so have unique continuous extensions to $M_{\nu^{1-\alpha/2}}(S)$. Assumption 2.6(c) is a simple consequence of (7.2), $\sup_j \overline{C}(j) < \infty$ (by (2.13)), (2.11), and (2.7). (2.10) follows from (2.11) and Assumption 2.3(b), and so Theorem 2.7 applies. □

Proof of Corollary 2.9. We verify the hypotheses of Theorem 2.7. Let

$$\widetilde{b}_i(x) = \sum_j x_j q_{ji}.$$

Then

$$\sum_i |\widetilde{b}_i(x)|\nu(i) = \sum_i\Big|\sum_j x_j q_{ji}\Big|\nu(i) \le \sum_j x_j[\kappa_{2.9}\nu(j) + \overline{q}\nu(j)] \le c_1|x|_\nu.$$

Hence (2.5) holds with b replaced by $\widetilde{b}$; it follows that Assumption 2.3(a) holds for b. By (2.14), (2.10) with b replaced by $\widetilde{b}$, and (2.11),

$$b_i(x) \ge \widehat{b}_i(x) + x_i q_{ii} \ge (-c_2\gamma_i(x) - q_{ii})x_i \ge -c_3\gamma_i(x)x_i.$$

Hence (2.10) holds with b. Next consider Assumption 2.6 with $\beta = \nu^{-\alpha/2}$. We may assume without loss of generality that $\widehat{b}_i$ satisfies Assumption 2.6(a) and (c) with $\beta = \nu^{-\alpha/2}$. Note that if $x, x_0 \in M_\nu(S)$, then

$$\begin{aligned}\sum_i |\widetilde{b}_i(x) - \widetilde{b}_i(x_0)| &\le \sum_i \sum_j |x(j) - x_0(j)|\,|q_{ji}| \le 2\overline{q} \sum_j |x(j) - x_0(j)| \\ &\le c_4 |x - x_0|_{\nu^{1-\alpha/2}}.\end{aligned}$$

Assumption 2.6(a) follows for $(b_i)_{i\in S}$ as does the fact that $\widetilde{b}_i$, and hence b_i, has a continuous extension to $M_{\nu^{1-\alpha/2}}(S)$. If $h \in (0,1]$ and $j \in S$, then

$$\begin{aligned}\sum_i |\widetilde{b}_i(x + he_j) - \widetilde{b}_i(x)| &= \sum_i \Big| \sum_\ell (x_\ell + h\delta_{ij}) q_{\ell i} - \sum_\ell x_\ell q_{\ell i} \Big| \\ &\le \sum_i h|q_{ji}| \le 2\overline{q} h^\alpha.\end{aligned}$$

In view of (2.7) and (2.11) this shows $\widetilde{b}_i$, and hence b_i, satisfies Assumption 2.6(c). This establishes the hypotheses of Theorem 2.7. □

Proof of Corollary 2.10. Our choice of p implies $\sup_j \overline{C}(j) < \infty$. The choice of q then easily gives (2.13). The required result now follows from Corollary 2.8. □

Proof of Corollary 2.11. By Corollary 2.10 (and its proof) $(\widehat{b}_i, \gamma_i)_{i\in S}$ satisfies the hypotheses of Corollary 2.9, and hence Theorem 2.7. Also (2.11) holds by hypothesis. The required result will therefore follow from Corollary 2.9 if we can show (2.14) and (2.15) hold. The former is trivial. For (2.15) note that

$$\begin{aligned}\sum_i 1(i \ne j) q_{ji} (|i| + 1)^q &= \lambda \sum_k p(k)(|j + k| + 1)^q \\ &\le c_1 \lambda \sum_k p(k)(|k|^q + (|j| + 1)^q) \\ &\le c_1 \lambda [m_q + (|j| + 1)^q] \le c_3 \lambda (|j| + 1)^q.\end{aligned}$$

This gives (2.15) and completes the proof. □

Proof of Corollary 2.12. We apply Corollary 2.11 with $\widehat{b}_i \equiv 0$. We only need to check that (γ_i) satisfies (2.12). Assume $|\gamma(x) - \gamma(y)| \le c_1 |x - y|^\alpha$, where $|\cdot|$ is the usual distance on $\mathbb{R}^N$. Then for any fixed $p > d$,

$$|\gamma_i(x + he_j) - \gamma_i(x)| \le 1_{(|i-j|\le R)} c_1 h^\alpha \le c_2 (|i - j| + 1)^{-p} h^\alpha.$$

and so (2.12) is valid. □

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