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THE DOOB-MEYER DECOMPOSITION REVISITED

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ABSTRACT. A new proof is given of the Doob-Meyer decomposition of a supermartingale into martingale and decreasing parts. Although not the most concise proof, the proof is elementary in the sense that nothing more sophisticated than Doob's inequality is used. If the supermartingale is bounded and the jump times are totally inaccessible, then it is shown that discrete time approximations converge to the decreasing part in L^2. The general case is handled by reduction to the above special case.

1. **Introduction.** The Doob-Meyer decomposition says that under mild integrability conditions, a supermartingale can be decomposed into the difference of a martingale and an increasing process. This was first proved by Doob [D] in the discrete time case and Meyer [M1, M2] in the much harder continuous time case. There are a number of other proofs, including those of Doléans-Dade [DD] and Rao [R]. Doléans-Dade uses the notions of predictable projections and dual predictable projections. Rao's proof is probably the simplest; he uses a discrete time approximation, Doob's decomposition for discrete time supermartingales, and a limit procedure. Despite this, the Doob-Meyer result is still considered a hard theorem, most likely because the limit procedure uses convergence in the topology $\sigma(L^1, L^\infty)$, which in turn uses the Dunford-Pettis compactness criterion.

In this paper we give a new proof of the Doob-Meyer decomposition. Our proof, although not the most concise, is completely elementary in the sense that the most sophisticated technique we use is Doob's inequality. We also start with a discrete time approximation, but now the convergences are in probability. In fact, when the supermartingale is bounded with totally inaccessible jump times, we show directly that the discrete approximations converge in L^2.

In Section 2 we look at the case where the jump times are totally inaccessible. The general case, which involves a reduction to the case of Section 2, is done in Section 3.

When the supermartingale is continuous, our proof can be made much more straightforward; this case has been presented in Bass [B]. For more information on predictable and totally inaccessible stopping times, see [DM1, DM2].

After this paper was submitted, we learned that T. Brown had earlier found an elementary proof of this theorem, but his proof was never published.

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2. **The totally inaccessible case.** In this section we do the case when the jumps of the supermartingale are totally inaccessible. This is the heart of the matter and also the most interesting and useful situation. Recall that a stopping time S is predictable if there exist stopping times S_n strictly less than S which increase up to S, a.s., on $(S < \infty)$. A stopping time S is totally inaccessible if $\mathbb{P}(S = T < \infty) = 0$ for each predictable stopping time T.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{F}_t$ be a right continuous filtration, and let Z_t be a supermartingale. Without loss of generality we may suppose Z has paths that are right continuous with left limits and that $Z_0 = 0$. If Z_{s-} denotes the left hand limit of Z at s, the jump at time s is $\Delta Z_s = Z_s - Z_{s-}$. In this section we assume that the jumps of Z_t are totally inaccessible. That is, if $S_{n,\varepsilon}$ is the n-th time that Z_t jumps more than ε in absolute value, then $S_{n,\varepsilon}$ is totally inaccessible for each n and ε. Another way of phrasing this is to say that whenever S_n are stopping times increasing up to S, then $Z_{S_n} \longrightarrow Z_S$, a.s. The Markov theory literature calls this property quasi-left continuity. A supermartingale is said to be of class D if the set of random variables $\{Z_T : T \text{ a stopping time}\}$ is uniformly integrable.

We prove

THEOREM 2.1. *Let Z_t be a supermartingale of class D with $Z_0 = 0$ where the paths are right continuous with left limits. Suppose the jumps of Z_t are totally inaccessible. Then there exists a continuous increasing process A_t such that $M_t = Z_t + A_t$ is a uniformly integrable martingale. The decomposition $Z_t = M_t - A_t$ is unique.*

REMARK 2.2. The proof of the uniqueness is easy (see, for example, [P], [IW], or [B]) and we have nothing to add here. In the remainder of the section we concentrate on the existence.

LEMMA 2.3. *Suppose $\{C_k, k = 0, 1, \ldots\}$ is an increasing sequence of random variables and $\mathcal{F}_k$ is an increasing sequence of σ-fields such that $C_0 = 0$, C_k is $\mathcal{F}_{k-1}$ measurable, and there exists $N \in (0, \infty)$ such that for all k,*

$$\mathbb{E}[C_\infty - C_k \mid \mathcal{F}_k] \le N, \qquad a.s.$$

Then $\mathbb{E}C_\infty^2 \le 2N^2$.

PROOF. We have

$$\mathbb{E}C_\infty = \mathbb{E}\big[\mathbb{E}[C_\infty - C_0 \mid \mathcal{F}_0]\big] \le N.$$

Let $c_k = C_{k+1} - C_k \ge 0$. Some algebra shows that

$$C_\infty^2 = 2\sum_{k=0}^{\infty}(C_\infty - C_k)c_k - \sum_{k=0}^{\infty} c_k^2.$$

Then

$$\mathbb{E}C_\infty^2 \le 2\mathbb{E}\Big[\sum_{k=0}^{\infty}\mathbb{E}[C_\infty - C_k \mid \mathcal{F}_k]c_k\Big] \le 2N\mathbb{E}\sum_{k=0}^{\infty} c_k = 2N\mathbb{E}C_\infty \le 2N^2,$$

as required. ∎

LEMMA 2.4. *Suppose $C_k^{(1)}$, $C_k^{(2)}$ are two increasing sequences, each satisfying the hypotheses of Lemma 2.3. Let $D_k = C_k^{(1)} - C_k^{(2)}$. Suppose there exists $Y \geq 0$ with $\mathbb{E}Y^2 < \infty$ such that for all k,*

$$|\mathbb{E}[D_\infty - D_k \mid \mathcal{F}_k]| \leq \mathbb{E}[Y \mid \mathcal{F}_k], \qquad a.s.$$

Then

$$\mathbb{E}\sup_k D_k^2 \leq 8\mathbb{E}Y^2 + 32\sqrt{2}N(\mathbb{E}Y^2)^{1/2}.$$

PROOF. Let $d_k = D_{k+1} - D_k$ and $c_k^{(i)} = C_{k+1}^{(i)} - C_k^{(i)}$. As above

$$D_\infty^2 = 2\sum_{k=0}^{\infty}(D_\infty - D_k)d_k - \sum_{k=0}^{\infty} d_k^2.$$

Then

$$\begin{aligned}\mathbb{E}D_\infty^2 &\leq 2\mathbb{E}\Big|\sum_{k=0}^{\infty}\mathbb{E}[D_\infty - D_k \mid \mathcal{F}_k]d_k\Big| \leq 2\mathbb{E}\Big[\sum_{k=0}^{\infty}\mathbb{E}[Y \mid \mathcal{F}_k](c_k^{(1)} + c_k^{(2)})\Big] \\ &= 2\mathbb{E}\Big[\sum_{k=0}^{\infty} Y(c_k^{(1)} + c_k^{(2)})\Big] = 2\mathbb{E}[Y(C_\infty^{(1)} + C_\infty^{(2)})].\end{aligned}$$

The Cauchy-Schwarz inequality and the bounds for $\mathbb{E}(C_\infty^{(i)})^2$ show $\mathbb{E}D_\infty^2 \leq 4\sqrt{2}N(\mathbb{E}Y^2)^{1/2}$.

To get the supremum over k, let $M_k = \mathbb{E}[D_\infty \mid \mathcal{F}_k]$, $N_k = \mathbb{E}[Y \mid \mathcal{F}_k]$, and $X_k = M_k - D_k$. Since $|X_k| = |\mathbb{E}[D_\infty - D_k \mid \mathcal{F}_k]| \leq N_k$, by Doob's inequality

$$\mathbb{E}\sup_k X_k^2 \leq \mathbb{E}\sup_k N_k^2 \leq 4\mathbb{E}N_\infty^2 = 4\mathbb{E}Y^2.$$

Another use of Doob's inequality shows that

$$\mathbb{E}\sup_k M_k^2 \leq 4\mathbb{E}M_\infty^2 = 4\mathbb{E}D_\infty^2.$$

Since $\sup_k |D_k| \leq \sup_k |X_k| + \sup_k |M_k|$, the result follows. ∎

REMARK 2.4. We will use the following observation several times.

$$\text{If } Y_n \longrightarrow 0 \text{ in } L^2, \text{ then } \sup_t \mathbb{E}[Y_n \mid \mathcal{F}_t] \longrightarrow 0 \text{ in } L^2.$$

This follows from the fact that $M_n(t) = \mathbb{E}[Y_n \mid \mathcal{F}_t]$ is a martingale, so by Doob's inequality,

$$\mathbb{E}\sup_t M_n(t)^2 \leq 4\mathbb{E}M_n(\infty)^2 = 4\mathbb{E}Y_n^2 \longrightarrow 0.$$

The following lemma is of interest in its own right. Let v be a positive integer and let $E_n = \{k/2^n : 0 \leq k/2^n \leq v\}$.

LEMMA 2.5. *Suppose T is a totally inaccessible stopping time. For $\delta > 0$ let*

$$R(\delta) = \sup_{t \le v} \mathbb{P}(t \le T \le t+\delta \mid \mathcal{F}_t).$$

Then $R(\delta) \longrightarrow 0$ in probability as $\delta \longrightarrow 0$.

PROOF. Let $a > 0$ and let

$$S_n(\delta) = \inf\{t \in E_n : \mathbb{P}(t \le T \le t+\delta \mid \mathcal{F}_t) > a\} \wedge v.$$

We first show $S_n(\delta) < T$, a.s. Note $S_n(\delta)$ takes on only the values $k/2^n$; T cannot take on any of the values $k/2^n$ with positive probability, or else part of T could be predicted by the stopping times $k/2^n - 1/m$. Hence $\mathbb{P}\big(S_n(\delta) = T\big) = 0$. If $A \subseteq (T < t)$ and A is $\mathcal{F}_t$ measurable, then

$$\mathbb{E}[\mathbb{P}(t \le T \le t+\delta \mid \mathcal{F}_t); A] = \mathbb{P}(t \le T \le t+\delta; A) = 0.$$

If T were less than $S_n(\delta)$ with positive probability, then for some $t \in E_n$, we have $\mathbb{P}\big(T < t,\ S_n(\delta) = t\big) > 0$. Let $A = \big(T < t,\ S_n(\delta) = t\big)$. Observe that $\mathbb{P}(t \le T \le t+\delta \mid \mathcal{F}_t) > a$ on the set $\big(S_n(\delta) = t\big)$, hence on the set A, which is a contradiction. We conclude that $S_n(\delta) < T$.

We next define a stopping time S. Let

$$\bar{S}(\delta) = \inf_n S_n(\delta), \qquad S = \sup_n \bar{S}(1/n).$$

Since $S_n(\delta) < T$, then $\bar{S}(\delta) < T$, a.s. Since T is totally inaccessible, we must have $\mathbb{P}(S = T) = 0$. This implies

$$\mathbb{P}(S = T \mid \mathcal{F}_{S-}) = 0, \qquad \text{a.s.} \tag{2.1}$$

(If U is a predictable stopping time predicted by the stopping times U_n, then $\mathcal{F}_{U-}$ is the σ-field generated by the sets in $\cup_n \mathcal{F}_{U_n}$.)

We now complete the proof of the lemma. Suppose there exists $\varepsilon > 0$ such that

$$\mathbb{P}\big(R(\delta) > a\big) > \varepsilon, \tag{2.2}$$

no matter how small δ. Let $\beta > 0$ and take $\delta < \beta$. For n sufficiently large,

$$\mathbb{P}\Big(\mathbb{P}\big(S_n(\delta) \le T \le S_n(\delta) + \delta \mid \mathcal{F}_{S_n(\delta)}\big) > a\Big) \ge \varepsilon.$$

T cannot equal $\bar{S}(\delta) + \delta$ with positive probability, or else part of T could be predicted by the stopping times $\bar{S}(\delta) + \delta - 1/m$. So the probability of the symmetric difference of the set $\big(S_n(\delta) \le T \le S_n(\delta) + \delta\big)$ and the set $\big(\bar{S}(\delta) \le T \le \bar{S}(\delta) + \delta\big)$ tends to 0 as $n \longrightarrow \infty$. Using Remark 2.4,

$$\mathbb{P}\Big(\mathbb{P}\big(\bar{S}(\delta) \le T \le \bar{S}(\delta) + \delta \mid \mathcal{F}_{\bar{S}(\delta)}\big) \ge a\Big) \ge \varepsilon,$$

hence $\mathbb{P}\big(\bar{S}(\delta) \le T \le \bar{S}(\delta) + \beta \mid \mathcal{F}_{\bar{S}(\delta)}\big)$ is greater than or equal to a with probability at least ε. Next we let $\delta = 1/n$ and let $n \longrightarrow \infty$. Repeating the argument we just gave, we get that

$$\mathbb{P}\big(\mathbb{P}(S \le T \le S+\beta \mid \mathcal{F}_{S-}) \ge a\big) \ge \varepsilon.$$

Letting $\beta \longrightarrow 0$, $\mathbb{P}(S = T \mid \mathcal{F}_{S-}) \ge a$ with positive probability, which contradicts (2.1). Hence (2.2) fails and the lemma is proved. ∎

LEMMA 2.6. *Suppose Z satisfies the hypotheses of Theorem 2.1 and in addition $|Z|$ is bounded by N and the paths of Z are constant after time v. Let*

$$W(\delta) = \sup_{t \le u \le t+\delta} \mathbb{E}[Z_t - Z_u \mid \mathcal{F}_t]. \tag{2.3}$$

Then $W(\delta) \longrightarrow 0$ in L^2 as $\delta \longrightarrow 0$.

PROOF. Since $|Z|$ is bounded by N, then $W(\delta)$ is bounded by $2N$, so it suffices to show $W(\delta) \longrightarrow 0$ in probability.

Let ε, $a > 0$ and $b = a\sqrt{\varepsilon}$. Let

$$Z_t^P = \sum_{s \le t} \Delta Z_s 1_{(b < \Delta Z_s)}, \qquad Z_t^M = \sum_{s \le t} \Delta Z_s 1_{(b < -\Delta Z_s)},$$
$$Z_t^S = Z_t - (Z_t^P + Z_t^M).$$

Let

$$W^S(\delta) = \sup_{t \le u \le t+\delta} |\mathbb{E}[Z_t^S - Z_u^S \mid \mathcal{F}_t]|,$$

with $W^P(\delta)$ and $W^M(\delta)$ defined analogously.

Note

$$W^S(\delta) \le \sup_t \mathbb{E}\Big[\sup_{r \le s \le r+\delta} |Z_r^S - Z_s^S| \mid \mathcal{F}_t\Big].$$

By Doob's inequality,

$$\mathbb{P}\big(W^S(\delta) > a\big) \le \mathbb{E}\Big[\sup_{r \le s \le r+\delta} |Z_r^S - Z_s^S|^2\Big]/a^2.$$

Since Z^S is right continuous with left limits and the jumps of Z^S are bounded by b in absolute value, the lim sup of the right hand side as $\delta \longrightarrow 0$ is less than or equal to $b^2/a^2 = \varepsilon$ by Fatou's lemma.

Since Z is right continuous with left limits, there are only finitely many jumps of size larger than b. Let $T_1 = \inf\{t : \Delta Z_t > b\}$, and for $i \ge 1$, $T_{i+1} = \inf\{t > T_i : \Delta Z_t > b\}$. These are the times when Z has a jump of size larger than b. Since $|Z|$ is bounded by N, then $|\Delta Z_{T_i}|$ is bounded by $2N$. Choose K such that $\mathbb{P}(T_K \le v) < \varepsilon$. Then

$$\mathbb{P}\big(W^P(\delta) > a\big) \le \mathbb{P}(T_K \le v) + \sum_{i=1}^K \mathbb{P}\Big(\sup_t \mathbb{E}[\Delta Z_{T_i} 1_{(t < T_i \le t+\delta)} \mid \mathcal{F}_t] > a/K\Big)$$
$$\le \varepsilon + \sum_{i=1}^K \mathbb{P}\Big(\sup_t \mathbb{P}(t \le T_i \le t+\delta \mid \mathcal{F}_t) > a/2KN\Big). \tag{2.4}$$

By Lemma 2.5, the right hand side can be made less than 2ε if δ is small enough.

$W^M(\delta)$ is treated similarly. Since $W(\delta) \le W^S(\delta) + W^P(\delta) + W^M(\delta)$, the result follows. ∎

PROPOSITION 2.7. *Suppose Z_t satisfies the assumptions of Theorem 2.1 and in addition $|Z|$ is bounded by N and the paths of Z are constant after time v. Then the conclusion of Theorem 2.1 holds.*

PROOF. Fix n for the moment and let $\mathcal{F}_n^k = \mathcal{F}_{k/2^n}$. Let

$$a_k^n = \mathbb{E}[Z_{(k-1)/2^n} - Z_{k/2^n} \mid \mathcal{F}_{k-1}^n].$$

Since Z_t is a supermartingale, the a_k^n are nonnegative. Note the a_k^n are $\mathcal{F}_{k-1}^n$ measurable. Let $A_k^n = \sum_{j=1}^{k-1} a_j^n$. It is trivial to check that $Z_{k/2^n} + A_k^n$ is a discrete time martingale with respect to $\mathcal{F}_k^n$. Let $B_t^n = A_k^n$ if $(k-1)/2^n < t \le k/2^n$.

We will show the B_t^n converge in L^2 as $n \to \infty$, uniformly over t, by showing

$$\mathbb{E}\big[\sup_t |B_t^n - B_t^m|^2\big] \to 0 \qquad \text{as } n, m \to \infty. \tag{2.5}$$

Suppose $m \ge n$. Since B_t^m and B_t^n are constant over intervals $(k/2^m, (k+1)/2^m]$, the supremum of the difference will take place at some $k/2^m$. We will apply Lemma 2.3 (with $C_k^{(1)} = A_k^m$, $C_k^{(2)} = B_{k/2^m}^n$, and Y equal to $W(2^{-n})$, and with respect to the σ-fields $\mathcal{F}_k^m$).

Fix $t = k/2^m$ and let u be the smallest element of E_n bigger than or equal to t.

$$\mathbb{E}[C_\infty^{(1)} - C_k^{(1)} \mid \mathcal{F}_k^m] = \mathbb{E}[A_\infty^m - A_k^m \mid \mathcal{F}_k^m] = \mathbb{E}[Z_t - Z_\infty \mid \mathcal{F}_t], \tag{2.6}$$

which is bounded by $2N$. On the other hand,

$$\begin{aligned} \mathbb{E}[C_\infty^{(2)} - C_k^{(2)} \mid \mathcal{F}_k^m] &= \mathbb{E}[A_\infty^n - B_t^n \mid \mathcal{F}_t] = \mathbb{E}\big[\mathbb{E}[A_\infty^n - B_u^n \mid \mathcal{F}_u] \mid F_t\big] \\ &= \mathbb{E}\big[\mathbb{E}[Z_u - Z_\infty \mid \mathcal{F}_u] \mid \mathcal{F}_t\big] = \mathbb{E}[Z_u - Z_\infty \mid F_t], \end{aligned} \tag{2.7}$$

which is also bounded by $2N$.

Taking the difference of (2.6) and (2.7),

$$\mathbb{E}[(B_\infty^m - B_\infty^n) - (B_t^m - B_t^n) \mid \mathcal{F}_t] = \mathbb{E}[Z_t - Z_u \mid \mathcal{F}_t].$$

The right hand side is nonnegative and bounded by $W(2^{-n})$. Since the right hand side is $\mathcal{F}_t$ measurable, it is also bounded by $\mathbb{E}[W(2^{-n}) \mid \mathcal{F}_t]$. So by Lemmas 2.3 and 2.6, we get (2.5). Let us denote the limit of the B_t^n by A_t.

Next we want to show that A_t is continuous. The jumps of B_t^n are

$$\Delta B_t^n = \mathbb{E}[Z_{(k-1)/2^n} - Z_{k/2^n} \mid \mathcal{F}_{(k-1)/2^n}], \qquad t = k/2^n,$$

which are bounded by $W(2^{-n})$. Hence $\sup_t |\Delta B_t^n| \to 0$ in L^2. By looking at a suitable subsequence n_j, $\sup_t |\Delta B_t^{n_j}| \to 0$, a.s., and so the limit is continuous.

Finally we show that $Z_t + A_t$ is a uniformly integrable martingale. Since Z_t is right continuous and A_t is continuous and both are square integrable, it suffice to show that for $s, t \in E_n$, $s < t$, and $B \in \mathcal{F}_s$,

$$\mathbb{E}[Z_t + A_t; B] = \mathbb{E}[Z_s + A_s; B].$$

This follows readily by a passage to the limit from the corresponding equation for $Z_t + B_t^n$. The uniform integrability follows since $|Z|$ is bounded and A is square integrable. ∎

COROLLARY 2.8. *Suppose Z_t satisfies the hypotheses of Theorem 2.1, but in addition the jumps of Z_t are bounded. Then the conclusion of Theorem 2.1 holds.*

PROOF. Let $T_N = \inf\{t : |Z_t| \geq N\} \wedge N$ and let $Z_t^N = Z_{t\wedge T_N}$. Since the jumps of Z_t are bounded, Z^N will be bounded, and by Proposition 2.7, there exist a continuous increasing process A_t^N and a martingale M_t^N such that $Z_t^N = M_t^N - A_t^N$.

Suppose $L \geq N$. Then $Z_t^N = Z_{t\wedge T_N}^L = M_{t\wedge T_N}^L - A_{t\wedge T_N}^L$ is another decomposition of Z^N. By the uniqueness result (Remark 2.2), $A_{t\wedge T_N}^L = A_t^N$. Thus if we define A_t to be A_t^N for $t \leq T_N$, the definition of A_t is unambiguous.

By monotone convergence and the fact that Z is of class D,

$$\mathbb{E}A_\infty \leq \lim \mathbb{E}A_{T_N} = -\lim \mathbb{E}Z_{T_N} < \infty.$$

Since A_t is increasing in t, this implies the uniform integrability of M_t. ∎

PROOF OF THEOREM 2.1. By the proof of Corollary 2.8, it suffices to obtain a decomposition of $Z_{t\wedge T}$, where $N \geq 0$ and $T = \inf\{t : |Z_t| \geq N\} \wedge N$. We may thus suppose that $|Z_{t-}|$ is bounded and that Z has at most a single jump larger than $2N$, occurring at time T.

Let

$$Z_t^P = \Delta Z_T 1_{(T\leq t)} 1_{(\Delta Z_T > 1)}, \qquad Z_t^J = \Delta Z_T 1_{(T\leq t)} 1_{(-\Delta Z_T > 1)}.$$

Z_t^J and $-Z_t^P$ are both supermartingales since they are decreasing processes. Suppose we can find a Doob-Meyer decomposition for each: $Z_t^J = M_t^J - A_t^J$ and $-Z_t^P = M_t^P - A_t^P$. Then note that

$$Z_t^C = Z_t - (Z_t^P - A_t^P) - (Z_t^J + A_t^J) = Z_t + M_t^P - M_t^J$$

will be a supermartingale with jumps bounded by $2N + 1$, and by Corollary 2.8 will have a decomposition $M_t^C - A_t^C$. $Z_t = (M_t^C - M_t^P + M_t^J) - A_t^C$ will then be our desired decomposition for Z_t.

We proceed to decompose Z_t^J, the decomposition of $-Z_t^P$ being similar. Note $|\Delta Z_T| \leq |Z_{T-}| + |Z_T| \leq N + |Z_T|$, so $|\Delta Z_T|$ is integrable. Let $a, \varepsilon > 0$. Choose $R > 1$ large enough so that $\mathbb{E}[|\Delta Z_T|; |\Delta Z_T| \geq R] \leq \varepsilon a$. Let

$$Z_t^L = \Delta Z_T 1_{(T\leq t)} 1_{(-\Delta Z_T > R)}, \qquad Z_t^S = Z_t^J - Z_t^L.$$

Define $B_t^{J,n}$, $B_t^{L,n}$, and $B_t^{S,n}$ in terms of Z^J, Z^L, and Z^S in exactly the same way B_t^n was defined in terms of Z in the proof of Proposition 2.7.

We show $B_t^{J,n}$ converges in probability, uniformly in t, by showing it is a Cauchy sequence. We have

$$\begin{aligned}(2.8)\qquad \mathbb{P}(\sup_t |B_t^{J,n} - B_t^{J,m}| > a) \leq\ & \mathbb{P}(\sup_t |B_t^{S,n} - B_t^{S,m}| > a/3) \\ & + \mathbb{P}(\sup_t |B_t^{L,n}| > a/3) \\ & + \mathbb{P}(\sup_t |B_t^{L,m}| > a/3).\end{aligned}$$

The second term is small since

$$\begin{aligned}\mathbb{P}(\sup_t |B_t^{L,n}| > a/3) = \mathbb{P}(B_\infty^{L,n} > a/3) &\le (3/a)\mathbb{E}B_\infty^{L,n}\\ &\le (3/a)\mathbb{E}|Z_\infty^L| \le (3/a)\mathbb{E}\big[|\Delta Z_T|; |\Delta Z_T| > R\big]\\ &\le 3\varepsilon,\end{aligned}$$

and similarly for the third term. $|Z_t^L|$ is bounded by R, so the first term on the right of (2.8) can be made small by taking m and n large as in the proof of Proposition 2.7.

Therefore $B_t^{J,n}$ converges, uniformly in t, as $n \to \infty$. Let the limit be denoted by A_t^J. The continuity of A_t^J is exactly as in Proposition 2.7. For each n, $\mathbb{E}B_\infty^{J,n} = -\mathbb{E}Z_\infty^J$, so by Fatou's lemma, $\mathbb{E}A_\infty^J$ is integrable. With this fact it is not hard to see that $Z_t^J + A_t^J$ is a martingale, and that this martingale is uniformly integrable. ∎

3. **The general case.** In this section we prove the general case of the Doob-Meyer decomposition. If R and S are stopping times, let $[R, S] \subseteq \Omega \times [0, \infty]$ denote $\{(\omega, s) : R(\omega) \le s \le S(\omega)\}$. The graph of a stopping time S is the set $[S, S]$; the finite part of the graph of S will be defined to be $\{(\omega, S(\omega)) : S(\omega) < \infty\}$. A process is predictable if, considered as a map from $\Omega \times [0, \infty]$, it is measurable with respect to the σ-field generated by the sets $\{[0, S] : S \text{ a predictable stopping time}\}$.

THEOREM 3.1. *Suppose Z_t is a supermartingale of class D with $Z_0 = 0$ and with paths that are right continuous with left limits. Then there exists a predictable increasing process A_t such that $M_t = Z_t + A_t$ is a uniformly integrable martingale. The decomposition $Z_t = M_t - A_t$ is unique.*

REMARK 3.2. Again, the uniqueness is not difficult—see [IW].

LEMMA 3.3. *Suppose R and S are predictable stopping times. Let*

$$S'(\omega) = \begin{cases} S(\omega) & \text{if } S(\omega) \ne R(\omega)\\ \infty & \text{if } S(\omega) = R(\omega).\end{cases}$$

Then S' is a predictable stopping time.

PROOF. Let R_i, S_i be stopping times predicting R and S, respectively. Define

$$\bar S_i^A = \begin{cases} S_i & \text{if } S_i > R\\ \infty & \text{otherwise}\end{cases}, \qquad S_i^A = \inf_{j \ge i} \bar S_j^A, \qquad S^A = \sup_i S_i^A.$$

It is easy to see that $\bar S_i^A$ is a stopping time, hence so is S^A. If $S > R$, then for all i sufficiently large, $S_i > R$. Hence for i large, $\bar S_i^A = S_i$, so $S_i^A = S_i$, and thus $S^A = S$. If $S \le R$, then $S_i < R$ for all i, or $\bar S_i^A = \infty$ for all i, hence $S^A = \infty$. Thus $S^A = S$ if $S > R$ and equals ∞ otherwise. If $S^A < \infty$, then $S > R$, hence $S_i^A = S_i < S = S^A$ for i large. Therefore S^A is predictable.

On $(S = R)$ for each i we have $\sup_j S_j > R_i$. Pick $j_i > j_{i-1}$ so that

$$\mathbb{P}(S = R, S_{j_i} \le R_i) < 2^{-i}.$$

Define

$$\bar{S}_i^B = \begin{cases} S_{j_i} & \text{if } S_{j_i} < R_i \\ \infty & \text{otherwise} \end{cases}, \qquad S_i^B = \inf_{k\geq i} \bar{S}_k^B, \qquad S^B = \sup_i S_i^B.$$

If $S < R$, then for i large, $S_{j_i} < S < R_i$, or $\bar{S}_i^B = S_{j_i}$. So for i large, $S_i^B = S_{j_i}$, and thus $S^B = S$. If $S > R$, then for i large, $S_{j_i} > R > R_i$, so $\bar{S}_i^B = \infty$, hence $S^B = \infty$. By our choice of j_i, we have $\mathbb{P}(S = R, \bar{S}_k^B \neq \infty) < 2^{-k}$, hence $\mathbb{P}(S = R, S_i^B \neq \infty) < \sum_{k=i}^{\infty} 2^{-k} = 2^{-i+1}$, hence $\mathbb{P}(S = R, S^B \neq \infty) = 0$. Thus S^B equals S if $S < R$ and equals ∞ otherwise. That S^B is a predictable stopping time is proved just as for S^A.

Since $S' = S^A \wedge S^B$ and the minimum of two predictable stopping times is predictable, the assertion is proved. ∎

REMARK 3.4. Given predictable stopping times $R_1, R_2, \ldots, R_i$, and S, iterating Lemma 3.3 shows that if S' is defined to be ∞ if $S = R_i$ for some i and equal to S otherwise, then S' is predictable.

LEMMA 3.5. *Suppose T is a stopping time. There exist stopping times U and V_1, $V_2, \ldots$ such that U is totally inaccessible, each V_i is predictable, the finite parts of the graphs of the V_i are disjoint, and the finite part of the graph of T is contained in the union of the graphs of U and the V_i.*

PROOF. Let $T_1 = T$ and

$$f_1 = \sup\{\mathbb{P}(S = T_1 < \infty) : S \text{ predictable}\}.$$

Choose S_1 predictable so that $\mathbb{P}(S_1 = T_1 < \infty) > f_1/2$ and let $V_1 = S_1$. Define T_2 to equal T_1 if $V_1 \neq T_1$ and to equal ∞ otherwise.

We continue by induction. We let $f_i = \sup\{\mathbb{P}(S = T_i < \infty) : S \text{ predictable}\}$. If $f_i = 0$, we stop; if not, we choose S_i predictable so that $\mathbb{P}(S_i = T_i < \infty) > f_i/2$. We use Remark 3.4 to let V_i equal S_i on the set where S_i is not equal to any of $V_1, V_2, \ldots, V_{i-1}$ and equal to ∞ otherwise. We let T_{i+1} equal T_i on the set where $V_i \neq T_i$ and otherwise set it equal to ∞.

Note the finite parts of the graphs of the V_i are disjoint by construction. Also, the sets $(T_{i+1} \neq T_i, T_i < \infty)$ are disjoint. Then

$$\sum f_i/2 \leq \sum \mathbb{P}(T_{i+1} \neq T_i, T_i < \infty) \leq 1,$$

or $f_i \longrightarrow 0$. The T_i increase, and we let $U = \lim_i T_i$. U must be totally inaccessible: if not, there exists S predictable such that $\mathbb{P}(U = S) > 0$. Then $\mathbb{P}(U = S) > f_i$ for some i, contradicting the fact that we chose S_i at the i-th stage, not S. ∎

REMARK 3.6. Suppose $S_1, S_2, \ldots$ are predictable stopping times. Set $V_1 = S_1$ and use Remark 3.4 to let V_i equal S_i on the set where S_i is not equal to any of $V_1, V_2, \ldots, V_{i-1}$ and equal to ∞ otherwise. Then the V_i are predictable stopping times, the union of the finite parts of their graphs is the same as the union of the finite parts of the graphs of the S_i, and the finite parts of the graphs of the V_i are disjoint.

LEMMA 3.7. *Suppose Z_t is a supermartingale such that $Z_0 = 0$, the paths of Z_t are right continuous with left limits, Z_t is bounded by N and constant after time v, and furthermore*

$$\mathbb{E}[\Delta Z_S \mid \mathcal{F}_{S-}] = 0, \qquad a.s. \tag{3.1}$$

whenever S is a predictable stopping time. Let $W(\delta)$ be defined as in (2.3). Then $W(\delta) \longrightarrow 0$ in probability as $\delta \longrightarrow 0$.

PROOF. The proof is a modification of the proof of Lemma 2.6. If T_i is the i-th time that Z_t jumps more than b, then provided we can show that

$$\mathbb{P}\Big(\sup_t \mathbb{E}[\Delta Z_{T_i} 1_{(t<T_i\le t+\delta)} \mid \mathcal{F}_t] > a/K\Big) \longrightarrow 0$$

as $\delta \longrightarrow 0$, the remainder of the proof is exactly as in Lemma 2.6.

Let $\varepsilon > 0$. We decompose T_i into stopping times U and $V_1, V_2, \ldots$ as in Lemma 3.5.

$$\begin{aligned}\mathbb{P}\Big(\sup_t \mathbb{E}[\Delta Z_U 1_{(t<U\le t+\delta)} \mid \mathcal{F}_t] > a/K\Big)& \\ \le \mathbb{P}\Big(\sup_t \mathbb{P}(t \le U \le t+\delta \mid \mathcal{F}_t) > a/2KN\Big) &\longrightarrow 0\end{aligned} \tag{3.2}$$

just as in the proof of Lemma 2.6.

Fix t, δ for the moment, and let $V_j' = (t \vee V_j) \wedge (t+\delta)$. Then

$$\begin{aligned}\mathbb{E}[\Delta Z_{V_j} 1_{(t<V_j\le t+\delta)} \mid \mathcal{F}_t] &= \mathbb{E}[\Delta Z_{V_j'} \mid \mathcal{F}_t] \\ &= \mathbb{E}\big[\mathbb{E}[\Delta Z_{V_j'} \mid \mathcal{F}_{V_j'-}] \mid \mathcal{F}_t\big] = 0, \qquad j = 1, 2, \ldots\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) and using the boundedness of $|Z|$, we get our result. ■

PROPOSITION 3.8. *Suppose Z_t satisfies the conditions of Theorem 3.1 and in addition (3.1) holds. Then the conclusion of Theorem 3.1 holds.*

PROOF. We follow the proof of Theorem 2.1, using Lemma 3.7 in place of Lemma 2.6. ■

REMARK 3.9. Let $S_1, \ldots, S_n$ be a sequence of predictable stopping times such that the finite parts of their graphs are disjoint. Let $V_1 = S_1 \wedge \cdots \wedge S_n$,

$$S^*_{i_1,\ldots,i_j} = \min\{S_i : 1 \le i \le n, i \ne i_1, \ldots, i_j\},$$

and

$$V_j = \max\{S^*_{i_1,\ldots,i_{j-1}} : i_1 < \cdots < i_{j-1}\}.$$

It is not hard to check that $V_j(\omega)$ is the j-th smallest of the $S_i(\omega)$, so $V_1 \le \cdots \le V_n$. Since the maximum and minimum of a finite number of predictable stopping times is a predictable stopping time, then each V_i is a predictable stopping time. And notice that the finite parts of the graphs of the V_i are still disjoint and their union is the same as the union of the finite parts of the graphs of the S_i.

PROOF OF THEOREM 3.1. Let T_{nj} be the j-th time $|\Delta Z_t|$ is in $(2^{-n}, 2^{-n+1}]$. Using Lemma 3.5 we decompose each T_{nj} into predictable and totally inaccessible parts. Let us relabel the collection of such stopping times, n an integer, j a positive integer, by S_i, $i = 1, 2, \ldots$, so that each S_i is either totally inaccessible or predictable, for each i there exists a b_i such that $b_i < |\Delta Z_{S_i}| \leq 2b_i$, and the set of jump times of Z_t is contained in the union of the graphs of the S_i. By means of Remark 3.6 we may assume without loss of generality that the finite parts of the graphs of the S_i are all disjoint.

Let $Z_0(t) = Z_t$. If S_i is totally inaccessible, let $Z_{i+1}(t) = Z_i(t)$ and $A_i(t) = 0$. If S_i is predictable, let

$$A_i(t) = -\mathbb{E}[\Delta Z_{S_i} \mid \mathcal{F}_{S_i-}]1_{(S_i \leq t)}, \qquad Z_{i+1}(t) = Z_i(t) + A_i(t).$$

We will show (a) each $A_i(t)$ is increasing, (b) each $Z_i(t)$ is a supermartingale, and (c) $\mathbb{E}\sum_{j=1}^{i} A_j(\infty) \leq C$, where C is a constant not depending on i. Once we show these three facts, the proof is quick. In view of (a) and (c), $\sum_{i=1}^{I} A_i(t)$ converges, uniformly in t, a.s., as $I \longrightarrow \infty$. Call the limit $A_\infty(t)$. By (c) and Fatou's lemma, $A_\infty(\infty)$ will be integrable. It is easy to see that each $A_i(t)$ is predictable, hence so is $A_\infty(t)$. Let

$$Z_\infty = Z_t + A_\infty(t) = \lim_i Z_i(t).$$

It follows from (b) and the uniform convergence of $\sum_{i=1}^{I} A_i(t)$ that $Z_\infty(t)$ is a supermartingale. From our construction, each Z_i will have paths that are right continuous with left limits; using the uniform convergence of $\sum_{i=1}^{I} A_i(t)$, we see that Z_∞ will, too. Because of (c), Z_∞ will be of class D. By our construction of the S_i, $\mathbb{E}[\Delta Z_\infty(T) \mid \mathcal{F}_{T-}] = 0$ for all predictable stopping times T. By applying Proposition 3.8 to Z_∞, we get $Z_\infty(t) = M_t - A^R(t)$. Setting $A_t = A_\infty(t) + A^R(t)$ then completes the proof.

We show (a), (b), and (c) by induction. Let us start with (a). There is nothing to prove when S_i is totally inaccessible. If S_i is a predictable stopping time, let S_{ij} be stopping times predicting S_i. Z_i is uniformly integrable by the induction hypothesis (c). Using (b) and the martingale convergence theorem,

$$\mathbb{E}[\Delta Z_i(S_i) \mid \mathcal{F}_{S_i-}] = \lim_n \mathbb{E}[\Delta Z_i(S_i) \mid \mathcal{F}_{S_{in}}].$$

But for each n,

$$\begin{aligned}\mathbb{E}[\Delta Z_i(S_i) \mid \mathcal{F}_{S_{in}}] &= \lim_m \mathbb{E}[Z_i(S_i) - Z_i(S_{im}) \mid \mathcal{F}_{S_{in}}] \\ &= \lim_m \mathbb{E}\Big[\mathbb{E}[Z_i(S_i) - Z_i(S_{im}) \mid \mathcal{F}_{S_{im}}]\mathcal{F}_{S_{in}}\Big] \leq 0.\end{aligned}$$

Next we look at (b). To show Z_{i+1} is a supermartingale, it suffices to show that

$$\mathbb{E}Z_{i+1}(U_1) \geq \mathbb{E}Z_{i+1}(U_2) \tag{3.4}$$

whenever U_1 and U_2 are stopping times with $U_1 \leq U_2$. If S_{ij} are stopping times predicting S_i,

$$\begin{aligned}\mathbb{E}Z_i(U_1) - \mathbb{E}Z_i(U_2) = &\Big[\mathbb{E}Z_i(U_1) - \mathbb{E}Z_i\big((U_1 \vee S_{ij}) \wedge U_2\big)\Big] \\ &+ \Big[\mathbb{E}Z_i\big((U_1 \vee S_{ij}) \wedge U_2\big) - \mathbb{E}Z_i\big((U_1 \vee S_i) \wedge U_2\big)\Big] \\ &+ \Big[\mathbb{E}Z_i\big((U_1 \vee S_i) \wedge U_2\big) - \mathbb{E}Z_i(U_2)\Big],\end{aligned}$$

and each of the summands on the right is nonnegative. Letting $j \longrightarrow \infty$,

$$\mathbb{E}Z_i(U_1) - \mathbb{E}Z_i(U_2) - \Big[\mathbb{E}Z_i\Big(\big((U_1 \vee S_i) \wedge U_2\big)-\Big) - \mathbb{E}Z_i\big((U_1 \vee S_i) \wedge U_2\big)\Big] \geq 0,$$

which is (3.4).

Lastly we look at (c). We need to get a bound on

$$\mathbb{E}\sum{}' \mathbb{E}[-\Delta Z_{S_j} \mid \mathcal{F}_{S_j -}] = -\mathbb{E}\sum{}' \Delta Z_{S_j},$$

where $\sum'$ denotes the sum over j's such that S_j is predictable and $j \leq i$. Since we have a finite sum, let us use Remark 3.9 and relabel the stopping times so that $S_1 < S_2 < \cdots$ on the set where they are finite. Let S_{jm} be stopping times predicting S_j. Since we have a finite sum and Z is a supermartingale,

$$\begin{aligned}
-\mathbb{E}\sum{}' \Delta Z_{S_j} &= \lim_m \sum{}' \mathbb{E}[Z_{S_{jm}} - Z_{S_j}] \\
&\leq \lim_m \sum{}' \Big[\mathbb{E}[Z_{S_{jm} \vee S_{j-1}} - Z_{S_j}] + \mathbb{E}[Z_{S_{j-1}} - Z_{S_{jm} \vee S_{j-1}}] \\
&\qquad + \mathbb{E}[Z_{S_i} - Z_\infty] + \mathbb{E}[Z_0 - Z_{S_{1m}}]\Big] \\
&= \mathbb{E}[Z_0 - Z_\infty],
\end{aligned}$$

which is bounded by a constant independent of i. ∎

A supermartingale is said to be regular if $\mathbb{E}Z_{S_n} \longrightarrow \mathbb{E}Z_S$ whenever $S_n \uparrow S$.

COROLLARY 3.8. *A_t is continuous if and only if Z is regular.*

PROOF. Clearly, if $Z_t = M_t - A_t$ is the decomposition of Z and A_t is continuous, then Z is regular. On the other hand, suppose A_t has a jump of size $b > 0$ with positive probability and let $S = \inf\{t : \Delta A_t > b\}$. Since A_t is predictable, it is easy to see that S is predictable. Let S_n be stopping times predicting S. Then by monotone convergence,

$$-\mathbb{E}Z_{S_n} = \mathbb{E}A_{S_n} \longrightarrow \mathbb{E}A_{S-} < \mathbb{E}A_S = -\mathbb{E}Z_S,$$

or Z is not regular. ∎

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