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BROWNIAN MOTION WITH SINGULAR DRIFT

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We consider the stochastic differential equation

$$dX_t = dW_t + dA_t,$$

where W_t is d-dimensional Brownian motion with $d \geq 2$ and the ith component of A_t is a process of bounded variation that stands in the same relationship to a measure π^i as $\int_0^t f(X_s)\,ds$ does to the measure $f(x)\,dx$. We prove weak existence and uniqueness for the above stochastic differential equation when the measures π^i are members of the Kato class $\mathbf{K}_{d-1}$. As a typical example, we obtain a Brownian motion that has upward drift when in certain fractal-like sets and show that such a process is unique in law.

1. Introduction. To introduce the subject of our paper, first consider the stochastic differential equation (SDE)

$$dX_t = dW_t + f(X_t)\,dt, \qquad X_0 = x_0, \tag{1.1}$$

where W_t is a d-dimensional Brownian motion, X_t is a d-dimensional semimartingale and $f : \mathbb{R}^d \to \mathbb{R}^d$. When f is bounded, weak existence and uniqueness of (1.1) are easily proved using the Girsanov transformation (see [20], Section 6.4).

If f_i is the ith component of f and we let

$$A_t^i = \int_0^t f_i(X_s)\,ds, \qquad A_t = (A_t^1, \dots, A_t^d),$$

then (1.1) can be written as

$$dX_t = dW_t + dA_t, \qquad X_0 = x_0. \tag{1.2}$$

In the terminology of Markov processes, A_t^i is the additive functional whose Revuz measure is $f_i(x)\,dx$. The solutions to (1.2) with starting points $x_0 \in \mathbb{R}^d$ form a strong Markov process whose infinitesimal generator is $\mathcal{L} = \frac{1}{2}\Delta + f \cdot \nabla$, where Δ is the Laplacian on $\mathbb{R}^d$ and $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_d)$.

In this paper we want to consider the SDE (1.2) with the $f_i(x)\,dx$ replaced by more general signed measures π^i on $\mathbb{R}^d$, which may not be absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$. This extension is motivated

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by considering examples such as Brownian motion, which drifts upward when penetrating fractal-like sets. For simplicity, let us assume throughout the paper that dimension $d \geq 3$, although our main result holds for $d = 2$ as well (see Remark 2.8). The one-dimensional case has been understood for some time; see [16].

For $\alpha > 0$, define the Kato class

$$\mathbf{K}_\alpha = \left\{ \pi(dx) : \limsup_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,\varepsilon)} |x-y|^{-\alpha} |\pi|(dy) = 0 \right\}, \tag{1.3}$$

where $|\pi|$ stands for the total variation of the signed measure π and $B(x, \varepsilon)$ denotes the ball in $\mathbb{R}^d$ centered at x with radius ε. We say a function $f \in \mathbf{K}_\alpha$ if $f(x)\,dx \in \mathbf{K}_\alpha$.

An illuminating prototype is the following. Suppose $d = 3$ and $\Gamma = A \times \mathbb{R}$, where A is the Sierpinski gasket in $\mathbb{R}^2$. It is well known that Γ is a $(1 + \log 3/\log 2)$-set in $\mathbb{R}^3$. Here a Borel-measurable set $\Gamma \subset \mathbb{R}^d$ is called a λ-set (cf. [15]) for some $0 < \lambda \leq d$ if there exist positive constants c_1 and c_2 such that, for all $x \in \Gamma$ and $r \in (0, 1]$,

$$c_1 r^\lambda \leq \mathcal{H}^\lambda(\Gamma \cap B(x,r)) \leq c_2 r^\lambda,$$

where $\mathcal{H}^\lambda$ denotes λ-dimensional Hausdorff measure in $\mathbb{R}^d$. It can be shown (see Proposition 2.1) that $\mathcal{H}^\lambda$ restricted to a λ-set in $\mathbb{R}^d$ is in the Kato class $\mathbf{K}_{d-1}$ if $\lambda > d - 1$.

QUESTION. Can one construct a diffusion process in $\mathbb{R}^3$ that behaves like Brownian motion outside Γ but drifts upward when it filters through the set Γ? If such a process exists, is it unique in law?

The main result of this paper says that there is a unique weak solution to the SDE (1.2) when $\pi^i \in \mathbf{K}_{d-1}$ for $1 \leq i \leq d$ and therefore gives an affirmative answer to the above question.

When $|f|^2 \in \mathbf{K}_{d-2}$, it is well known (e.g., see [8]) that

$$\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} \mathbb{E}^x \int_0^t f(W_s)^2\, ds = 0$$

and

$$M_t = \exp\left(\int_0^t f(W_s)\, dW_s - \frac{1}{2} \int_0^t f(W_s)^2\, ds \right), \qquad t \geq 0, \tag{1.4}$$

is a nonnegative martingale. The last fact can be proved similarly to arguments in [6]. In this case, weak existence and uniqueness for (1.1) can be obtained from the Girsanov transform (1.4) for standard Brownian motion W_t (see, e.g., [7] and [12]). On the other hand, when $|f|^2 \in \mathbf{K}_{d-2}$, the following Kato-type inequality holds: for any $\varepsilon > 0$ there exists $A_\varepsilon > 0$ so that, with $\pi(dx) = f(x)\,dx$

$$\begin{aligned}&\int_{\mathbb{R}^d} \psi(x)\nabla\phi(x)\cdot\pi(dx)\\ &\qquad\le \left(\int_{\mathbb{R}^d} |\nabla\phi(x)|^2\,dx\right)^{1/2}\left(\varepsilon\int_{\mathbb{R}^d} |\nabla\psi(x)|^2\,dx + A_\varepsilon\int_{\mathbb{R}^d}\psi(x)^2\,dx\right)^{1/2}\end{aligned} \tag{1.5}$$

for any $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$ (see [8]). Therefore, the bilinear form associated with the generator $(\mathcal{L}, C_c^\infty(\mathbb{R}^d))$, where $\mathcal{L} = \frac{1}{2}\Delta + f\cdot\nabla$, is lower semibounded, closable, Markovian and satisfies Silverstein's sector condition (cf. [7]). So there is a minimal diffusion process X_t associated with $\mathcal{L}$. It was proved in [7] that this diffusion process coincides with the diffusion obtained through the Girsanov transform (see also [17]).

At first glance, one might think that it would be easy to extend the above results with $\pi(dx) = f(x)\,dx$ to singular measures $\pi = (\pi^1, \dots, \pi^d)$ in some Kato class, using the above-mentioned Girsanov transform or Dirichlet form methods. But one quickly realizes that there are enormous difficulties in trying to use either of these two approaches. When π is not absolutely continuous with respect to Lebesgue measure, it is not at all clear how to interpret the condition that the square of $d\pi/dx$ is in $\mathbf{K}_{d-2}$ nor the meaning of the Girsanov transform (1.4). It is also not clear whether inequality (1.5) holds.

The results of this paper indicate that $\mathbf{K}_{d-1}$ is the right class to consider. As mentioned above, the Girsanov transform is not suitable for this class. We do not know whether inequality (1.5) holds for $\pi \in \mathbf{K}_{d-1}$ (we suspect in general it does not). In fact, it is not clear whether the model we consider in this paper can even be covered under the "generalized Dirichlet form" framework of Stannat [19]. So a new approach is needed.

We will give a number of examples of measures in $\mathbf{K}_{d-1}$ in Section 2. Surface measure on a $(d-1)$-dimensional hypersurface barely misses being in $\mathbf{K}_{d-1}$, while Hausdorff measure $\mathcal{H}_\lambda$ on a λ-set for $\lambda \in (d-1, d]$ is in $\mathbf{K}_{d-1}$. We note here that when σ is the surface measure of a $(d-1)$-dimensional hypersurface, $\mathbf{n}$ denotes its inward normal vector field and $\pi = \mathbf{n}\sigma$, then the corresponding SDE (1.2) gives a Brownian motion that is reflected on the hypersurface along the normal direction. There is quite an extensive literature on the study of reflecting Brownian motions. However, the approach in this paper does not cover this case. It follows from Hölder's inequality that $L^p(\mathbb{R}^d) \subset \mathbf{K}_{d-1} \cap \{f : f^2 \in \mathbf{K}_{d-2}\}$ for $p > d$; this is not true when $p = d$. While there are examples where f is not in $\mathbf{K}_{d-1}$ but f^2 is in $\mathbf{K}_{d-2}$, there are many $f \in \mathbf{K}_{d-1}$ where f^2 is not in $\mathbf{K}_{d-2}$. So even in the absolutely continuous case, while our approach cannot recover all the previously known results, our results include most of them and many more that are not covered by earlier results.

We now describe the approach of our paper. Given $\pi = (\pi^1, \dots, \pi^d)$ in $\mathbf{K}_{d-1}$ (by which we mean that each component π^i is a signed measure in $\mathbf{K}_{d-1}$), we approximate each π^i by $\pi_n^i(dx) = G_n^i(x)\,dx$, where $G_n^i(x)$ is a bounded continuous function, and we say that we have a solution to (1.2) if

$$X_t = x_0 + W_t + \lim_{n\to\infty} \int_0^t G_n(X_s)\,ds.$$

Here $G_n = (G_n^1, \dots, G_n^d)$ and we want the convergence to be uniform over t in finite intervals. We prove that there exists a weak solution in this sense and that the weak solution is unique.

Our method is essentially a perturbation one in the space of bounded continuous functions $C_b(\mathbb{R}^d)$ (rather than in some L^p-space or Sobolev space). For $\lambda > 0$, let R^λ be the resolvent operator for Brownian motion. We show (see Proposition 4.6) that

$$S^\lambda = R^\lambda\left(\sum_{j=0}^{\infty}(BR^\lambda)^j\right) \tag{1.6}$$

converges as a bounded operator on the space $C_{\mathrm{b}}(\mathbb{R}^d)$ equipped with the uniform norm $\|\cdot\|_\infty$, where $B = \pi\cdot\nabla$ denotes the operator that maps a C^1 function ϕ into the measure

$$B\phi(dx) = \sum_{i=1}^{d} \frac{\partial\phi}{\partial x_i}(x)\pi^i(dx).$$

Intuitively speaking, we construct S^λ as the λ-resolvent of $\frac{1}{2}\Delta + \pi\cdot\nabla$. Furthermore, $\{S^\lambda, \lambda > 0\}$ is the family of resolvent operators of the diffusion process, which is the unique weak solution to the SDE (1.2). The key is that if $\mu \in \mathbf{K}_{d-1}$, then, for each fixed x, $\nabla R^\lambda\mu(x)$ is continuous in μ with respect to the weak convergence topology on bounded measures (see Proposition 3.9). When $\pi^i(x) = f_i(x)\,dx$ with $f_i^2 \in \mathbf{K}_{d-2}$, the resolvent identity (1.6) is proved in [7] for diffusion processes obtained through Dirichlet form techniques but the convergence is in the Sobolev space $W^{1,2}$ of order $(1, 2)$. Thus, the weak solution or diffusion process constructed in this paper coincides with the previous known ones when $\pi(dx) = f(x)\,dx$ is in $\mathbf{K}_{d-1}$ with $|f|^2 \in \mathbf{K}_{d-2}$.

Weak existence and weak uniqueness of (1.1) is well known to be equivalent to the martingale problem for the operator $\mathcal{L}$ being well posed, where $\mathcal{L} = \frac{1}{2}\Delta + f\cdot\nabla$. One could view our results as extending existence and uniqueness of the martingale problem to $\mathcal{L} = \frac{1}{2}\Delta + \pi\cdot\nabla$ with $\pi \in \mathbf{K}_{d-1}$. The model studied in this paper raises interesting questions in PDE about the potential theory for the differential operator $\mathcal{L}$ with $\pi \in \mathbf{K}_{d-1}$ and warrants additional study. For example, do the Harnack principle and boundary Harnack principle hold for such operators $\mathcal{L}$?

As mentioned previously, the one-dimensional case has been understood for some time; see [16]. In higher dimensions, there is some previous work along these lines by [7] (cf. also [12]). In this connection, we would like to mention that if f in (1.1) is $\nabla \log \psi$, where $\psi > 0$ a.e. on $\mathbb{R}^d$ and is locally in $W^{1,2}(\mathbb{R}^d)$, then there is a conservative diffusion X that solves (1.1) (see [1, 5, 10, 11, 18]). This process is called distorted Brownian motion and has relationships to Euclidean field theory, generalized Schrödinger operators and stochastic mechanics.

It is possible to extend our results to the case where Brownian motion is replaced by a diffusion on $\mathbb{R}^d$ with sufficiently smooth coefficients (see Remark 6.1). However, our methods rely on a gradient estimate for the λ-resolvent density, and it is not at all clear that the analogue of our results holds for Markov processes corresponding to more general Dirichlet forms or infinitesimal generators.

In this paper we consider only weak solutions, that is, existence and uniqueness of a suitable probability measure. It would be interesting to know if strong solutions exist to (1.2), that is, where X_t is measurable with respect to the filtration generated by the Brownian motion; if so, this would imply a pathwise uniqueness result for (1.2) and vice versa. In this regard, see [21] for the existence and uniqueness of strong solutions to (1.1) when f is bounded and see also [9].

In Section 2 we define our notation and give a precise statement of our main theorem. Section 3 is devoted to some estimates on Brownian resolvents. Section 4 proves existence of a weak solution and Section 5 weak uniqueness, both under the assumption of bounded support. This additional assumption is removed in Section 6.

We will use $C_b(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$ to denote the space of bounded continuous functions on $\mathbb{R}^d$ and the space of continuous functions on $\mathbb{R}^d$ that vanish at ∞, respectively.

2. Preliminaries. We first give some examples of measures in the Kato class $\mathbf{K}_{d-1}$. To do this, we introduce a class of measures $\mathcal{M}(\gamma, \kappa)$.

Let $B(x,r)$ denote the open ball of radius r with center x. We let the letter c with subscripts denote finite positive constants whose exact value is unimportant. If μ is a signed measure, let μ^+ and μ^- be the positive and negative parts, respectively, and let $|\mu| = \mu^+ + \mu^-$. Let $\gamma, \kappa > 0$ and set

$$\mathcal{M}(\gamma,\kappa) = \{\mu : \mu \text{ is a signed measure and } \\ |\mu|(B(x,r)) \le \kappa r^{d-1+\gamma} \text{ for all } x \in \mathbb{R}^d, r \in (0,1]\}.$$

If Γ is a λ-set with $\lambda \in (d-1, d]$, then Hausdorff measure $\mathcal{H}^\lambda$ restricted to the set Γ as well as $g(x)\mathbb{1}_\Gamma(x)\mathcal{H}^\lambda(dx)$ when g is bounded are measures in the class $\mathcal{M}(\lambda + 1 - d, \kappa)$ for some $\kappa > 0$.

PROPOSITION 2.1. *Suppose $\mu \in \mathcal{M}(\gamma,\kappa)$ is a positive measure and $x_1 \in \mathbb{R}^d$. There exists c_1 not depending on x_1 or μ such that, for each x and $\rho \le 1$,*

$$\int_{B(x_1,\rho)} \frac{1}{|x-y|^{d-1}} \mu(dy) \le c_1 \kappa \rho^\gamma.$$

In particular, $\mathcal{M}(\gamma,\kappa) \subset \mathbf{K}_{d-1}$.

PROOF. Clearly, the integral is largest when $x_1 = x$. Since $B(x, 2)$ can be covered by a finite number of balls of radius 1, then $\mu(B(x, 2)) \le c_2\kappa$. Suppose $2^{m-1} < \rho \le 2^m$. We have

$$\begin{aligned}\int_{B(x_1,\rho)} \frac{1}{|x-y|^{d-1}}\mu(dy) &\le \sum_{k=-\infty}^{m} \int_{B(x,2^k)-B(x,2^{k-1})} \frac{1}{|x-y|^{d-1}}\mu(dy)\\ &\le \sum_{k=-\infty}^{m} \frac{1}{(2^{k-1})^{d-1}}\mu\big(B(x,2^k)\big)\\ &\le \sum_{k=-\infty}^{m} c_3 2^{-k(d-1)}\kappa(2^k)^{d-1+\gamma}\\ &\le c_4\kappa 2^{m\gamma} \le c_5\kappa\rho^{\gamma}.\end{aligned}$$

□

Let us give some examples of measures lying in $\mathcal{M}(\gamma, \kappa)$.

EXAMPLE 2.2. Suppose $d = 2$, let μ_0 be a Cantor–Lebesgue measure on $[0, 1]$ and let $\mu(dx\,dy) = \mu_0(dx) \times dy$. It is easy to see that $\mu(B(x, r)) \le c_1 r^{1+(\log 2/\log 3)}$. Hence $\mu \in \mathcal{M}(\gamma, \kappa)$ with $\gamma = \log 2/\log 3$. This example can be generalized to higher order fractal sets lying in Euclidean space; for example, one could take the Hausdorff measure on the Sierpinski carpet in $\mathbb{R}^2$ times the Lebesgue measure along the z-axis to get a measure in $\mathbb{R}^3$.

EXAMPLE 2.3. Suppose $f \in L^p$ for some $p > d$. Then $\mu(dx) = f(x)\,dx \in \mathcal{M}(\gamma, \kappa)$ for some positive γ and κ. In fact, by Hölder's inequality,

$$|\mu|(B(x,r)) = \int_{B(x,r)} |f(y)|\,dy \le c_1\|f\|_p (r^d)^{1/q},$$

where $p^{-1} + q^{-1} = 1$. Since $p > d$, then $1/q > (d-1)/d$ and so $d/q > d-1$. In this case, $\gamma = (d/q) - (d-1)$.

EXAMPLE 2.4. Let $d \ge 2$, $\gamma \in (0, 1)$ and $g : \mathbb{R}^{d-1} \to \mathbb{R}$. Suppose

$$f(x_1,\dots,x_d) = \big(|x_d - g(x_1,\dots,x_{d-1})| \wedge 1\big)^{\gamma-1}$$

and $\mu(dx) = f(x)\,dx$. It is easy to check that $\mu \in \mathcal{M}(\gamma, \kappa)$. What is interesting in this example is that $|f|^2$ is not even locally L^1-integrable when $\gamma < 1/2$, so $|f|^2$ cannot be in the Kato class $\mathbf{K}_{d-2}$. Thus, in general, drifts with this kind of singularity cannot be handled by Girsanov's theorem. The above also gives an example of an f that is in $\mathbf{K}_{d-1}$ but $f^2 \notin \mathbf{K}_{d-2}$.

Let $\psi(x)$ be a nonnegative C^∞ function on $\mathbb{R}^d$ with compact support such that $\int \psi(x)\,dx = 1$. Let $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(x/\varepsilon)$. For signed Radon measures π^i on $\mathbb{R}^d$

with $1 \le i \le d$, define

$$G_n^i(x) = \int \psi_{2^{-n}}(x-y)\pi^i(dy) \tag{2.1}$$

and set

$$\pi_n^i(dx) = G_n^i(x)\,dx. \tag{2.2}$$

We write $G_n(x)$ for $(G_n^1(x), \dots, G_n^d(x))$. The stochastic differential equation we consider is

$$X_t = x_0 + W_t + A_t, \tag{2.3}$$

where A_t is the limit of $\int_0^t G_n(X_t)\,dt$. More precisely, let Ω be the set of continuous functions mapping $[0,\infty)$ to $\mathbb{R}^d$, let $X_t(\omega) = \omega(t)$ and let $\mathcal{F}_t$ be the usual cylindrical σ-field generated by $\{X_s, s \le t\}$.

DEFINITION 2.5. Given Radon measures $\{\pi^i, 1 \le i \le d\}$ on $\mathbb{R}^d$, a weak solution for Brownian motion with generalized drift $\pi = (\pi^1, \dots, \pi^d)$ is a probability measure on $(\Omega, \mathcal{F}_\infty)$ such that

$$X_t = x_0 + W_t + A_t, \tag{2.4}$$

where:

(a) $A_t = \lim_{n\to\infty} \int_0^t G_n(X_s)\,ds$ uniformly over t in finite intervals, where the convergence is in probability and G_n is defined by (2.1);

(b) there exists a subsequence $\{n_k\}$ such that $\sup_k \int_0^t |G_{n_k}(X_s)|\,ds < \infty$ a.s. for each $t > 0$;

(c) W_t with $W_0 = 0$ is a d-dimensional Brownian motion under $\mathbb{P}$ with respect to the σ-fields $\mathcal{F}_t$.

Our main theorem is as follows.

THEOREM 2.6. *Suppose $x_0 \in \mathbb{R}^d$, $d \ge 3$ and $\pi^i \in \mathbf{K}_{d-1}$ for $i = 1, \dots, d$. Then*:

(a) *There exists one and only one weak solution to* (2.4). *This unique solution is conservative.*

(b) *Let $(X, \mathbb{P}^x)$ denote the unique solution in* (a) *with $X_0 = x$. Then the collection $(X, \mathbb{P}^x, x \in \mathbb{R}^d)$ forms a strong Markov process. Furthermore, each component A^i of A is a continuous additive functional of X of finite variation.*

REMARK 2.7. As we mentioned earlier, the existing Dirichlet form literature allows the construction of a solution to (1.1) provided $|f|^2 \in \mathbf{K}_{d-2}$. Although it is not true that $|f|^2 \in \mathbf{K}_{d-2}$ implies $f \in \mathbf{K}_{d-1}$ (e.g., $f(x) = [|x|\log^{3/4}(1/|x|)]^{-1}$

is such a counterexample), a simple application of Cauchy–Schwarz shows that if, for some $\delta > 0$,

$$\limsup_{\varepsilon\to 0} \sup_{x\in\mathbb{R}^d} \int_{B(x,\varepsilon)} |f(y)|^2 \big|\log^{1+\delta}(1/|x-y|)\big| |x-y|^{2-d}\, dy = 0,$$

then $f \in \mathbf{K}_{d-1}$. Thus, the condition $f \in \mathbf{K}_{d-1}$ includes the vast majority (but not all) of what can be done using Dirichlet form theory. On the other hand, $f \in \mathbf{K}_{d-1}$ is, in general, a much less restrictive condition than $|f|^2 \in \mathbf{K}_{d-2}$.

REMARK 2.8. We prove the theorem only for $d \geq 3$. However, it is also true for $d = 2$. In the proofs it is necessary to replace $p_t(x)$ by the transition density for two-dimensional Brownian motion killed on exiting a large square $[-M, M]^2$ and then to let $M \to \infty$.

We will need the following technical lemma. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}_\infty$. A regular conditional probability for $\mathbb{P}(\cdot \mid \mathcal{G})$ is a kernel $\mathbb{Q}(\omega, d\omega')$ such that (1) $\mathbb{Q}(\omega, \cdot)$ is a probability measure for each ω, (2) $\mathbb{Q}(\cdot, A)$ is a measurable random variable for each $A \in \mathcal{F}_\infty$ and (3) if $A \in \mathcal{F}_\infty$, then $\mathbb{P}(A \mid \mathcal{G}) = \mathbb{Q}(\omega, A)$ for almost every ω.

Let θ_t be the usual shift operators on Ω so that $\theta_t(\omega)(s) = \omega(s+t)$. Let S be a bounded stopping time and let $\mathbb{P}_S(A) = \mathbb{P}(A \circ \theta_S)$.

PROPOSITION 2.9. *Suppose $\mathbb{P}$ is a solution to* (2.4) *and S is a bounded stopping time. If $\mathbb{Q}_S$ is a regular conditional probability for $\mathbb{P}_S(\cdot \mid \mathcal{F}_S)$, then for almost every ω the probability $\mathbb{Q}_S(\omega, \cdot)$ is a solution to* (2.4) *starting at $X_S(\omega)$. The same is true if $\mathbb{Q}_S$ is replaced by $\overline{\mathbb{Q}}_S$, a regular conditional probability for $\mathbb{P}_S(\cdot \mid X_S)$.*

The proof of this is very similar to [3], Proposition 6.2.1, and is left to the reader.

3. Estimates.

Throughout we assume $d \geq 3$. Let

$$p_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/2t), \qquad R^\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(x)\, dx.$$

Define

$$R^\lambda f(x) = \int f(y) R^\lambda(x-y)\, dy = \int_0^\infty e^{-\lambda t} \int f(y) p_t(x-y)\, dy\, dt.$$

We also write $R^\lambda \mu(x) = \int R^\lambda(x-y)\mu(dy)$ when $\int R^\lambda(x-y)|\mu|(dy)$ is finite. Let $B = \pi \cdot \nabla$ and $B_n = \pi_n \cdot \nabla$ be the operators that map a C^1 function ϕ into a measure

$$B\phi(dx) = \sum_{i=1}^d \frac{\partial \phi}{\partial x_i}(x) \pi^i(dx) \quad \text{and} \quad B_n\phi(dx) = \sum_{i=1}^d \frac{\partial \phi}{\partial x_i}(x) \pi_n^i(dx),$$

respectively, where π_n^i is given by (2.2). For a signed Radon measure μ, define

$$m_\mu(r) = \sup_{x\in\mathbb{R}^d} \int_{B(x,r)} |x-y|^{1-d}|\mu|(dy).$$

That μ is in $\mathbf{K}_{d-1}$ is equivalent to $\lim_{r\to 0} m_\mu(r) = 0$. It follows from Proposition 2.1 that $m_\mu(r) \le c\kappa r^\gamma$ if $\mu \in \mathcal{M}(\gamma,\kappa)$, where c is independent of κ, γ and μ.

REMARK 3.1. Clearly, if $\mu \in \mathbf{K}_{d-1}$, then

$$\int_{B(x_1,\rho)} \frac{1}{|x-y|^{d-2}}|\mu|(dy) \le \rho m_\mu(\rho).$$

REMARK 3.2. Since $|R^\lambda(x)| \le c_1|x|^{2-d}$, if the measure $\mu \in \mathbf{K}_{d-1}$ is supported in $B(x_1,\rho)$ for some $x_1 \in \mathbb{R}^d$ and $\rho \in (0,1]$, then

$$|R^\lambda \mu(x)| \le R^\lambda|\mu|(x) \le c_2\rho m_\mu(\rho).$$

PROPOSITION 3.3. *If $\mu \in \mathbf{K}_{d-1}$ is supported in $B(x_1,\rho)$ for some $x_1 \in \mathbb{R}^d$ and $\rho \in (0,1]$, then for all $\lambda \ge 0$, $R^\lambda\mu$ is a C^1 function with*

$$|\nabla R^\lambda \mu(x)| \le K m_\mu(\rho),$$

where K does not depend on x, x_1 or μ.

PROOF. From the definition of R^λ, we have

$$\frac{\partial R^\lambda}{\partial x_i}(x) = -\int_0^\infty e^{-\lambda t}\frac{x_i}{t}(2\pi t)^{-d/2}e^{-|x|^2/2t}\,dt. \tag{3.1}$$

This is bounded in absolute value by

$$c_1\int_0^\infty \frac{|x|}{t}t^{-d/2}e^{-|x|^2/2t}\,dt \le c_2|x|^{1-d}.$$

So

$$\left|\frac{\partial R^\lambda\mu}{\partial x_i}(x)\right| \le c_2\int_{B(x_1,\rho)} \frac{1}{|x-y|^{d-1}}|\mu|(dy) \le c_2 m_\mu(\rho). \qquad \square$$

PROPOSITION 3.4. *If g is bounded, and $\lambda > 0$, then there exists c_1 depending only on λ such that*

$$|\nabla R^\lambda g(x)| \le c_1\|g\|_\infty.$$

PROOF. As in (3.1),

$$\begin{aligned}|\nabla R^\lambda g(x)| &\le c_2 \int_0^\infty \int e^{-\lambda t} \frac{|x-y|}{t} t^{-d/2} e^{-|x-y|^2/2t} |g(y)|\,dy\,dt \\ &\le c_3 \|g\|_\infty \int_0^\infty e^{-\lambda t} t^{-1/2} \int t^{-d/2} e^{-|x-y|^2/4t}\,dy\,dt \\ &\le c_4 \|g\|_\infty \int_0^\infty e^{-\lambda t} t^{-1/2}\,dt \\ &\le c_5 \|g\|_\infty. \qquad \square\end{aligned}$$

We now impose a condition on our π^i's that we will remove in Section 6. Recall that K is the constant in Proposition 3.3.

ASSUMPTION 3.5. There exist $x_1 \in \mathbb{R}^d$ and $\rho > 0$ such that for each i the measure π^i is in $\mathbf{K}_{d-1}$, π^i has support in $B(x_1, \rho)$ and $m_{\pi^i}(\rho) < (2dK)^{-1}$.

PROPOSITION 3.6. *If $\mu \in \mathbf{K}_{d-1}$ with μ supported in $B(x_1, \rho)$, $x_1 \in \mathbb{R}^d$ and $\rho > 0$ is such that $m_\mu(\rho) < \kappa(2dK)^{-1}$ for $\kappa \in [0, 1]$, then $\nu = BR^\lambda\mu$ is in $\mathbf{K}_{d-1}$, supported in $B(x_1, \rho)$ and $m_\nu(r) \le (\kappa/2d)\sum_{i=1}^d m_{\pi^i}(r)$ for $r > 0$. In fact, we have*

$$|\nu|(dx) \le \frac{\kappa}{2d} \sum_{i=1}^d |\pi^i|(dx).$$

The same is true if ν is replaced by $\nu_n = B_n R^\lambda \mu$.

PROOF. We have

$$BR^\lambda\mu(dx) = \sum_{i=1}^d \frac{\partial R^\lambda \mu}{\partial x_i}(x)\pi^i(dx).$$

By Proposition 3.3, the right-hand side is bounded by

$$\sum_{i=1}^d |\nabla R^\lambda \mu(x)||\pi^i|(dx) \le K m_\mu(\rho) \sum_{i=1}^d |\pi^i|(dx) \le \frac{\kappa}{2d} \sum_{i=1}^d |\pi^i|(dx).$$

The result now follows by our assumptions on π^i.

Similarly,

$$|B_n R^\lambda \mu(dx)| \le K m_\mu(\rho) \sum_{i=1}^d |\pi_n^i|(dx) \le \frac{\kappa}{2d} \sum_{i=1}^d |\pi_n^i|(dx). \tag{3.2}$$

Note that

$$\begin{aligned} m_{\pi_n^i}(r) &\le \int \sup_{x \in \mathbb{R}^d} \int \psi_{2^{-n}}(x-y) \biggl(\int_{B(x,r)} |y-z|^{1-d} |\pi^i|(dz) \biggr)\,dy \\ &\le \sup_{x \in \mathbb{R}^d} \int \psi_{2^{-n}}(x-y) m_\mu(r)\,dy = m_\mu(r), \end{aligned} \tag{3.3}$$

so, in particular, π_i^n is in $\mathbf{K}_{d-1}$. Combining with (3.2) proves the proposition. □

PROPOSITION 3.7. *Let $x_1 \in \mathbb{R}^d$, $\rho \in (0, 1]$ and μ be a signed measure in $\mathbf{K}_{d-1}$ having support in $B(x_1, \rho)$. Given $\varepsilon > 0$, there exists $\lambda_0 > 0$ that depends only on the pointwise bound of the function $m_\mu(r)$ such that if $\lambda \ge \lambda_0$, then*

$$|\nabla R^\lambda \mu(x)| \le \varepsilon.$$

PROOF. By the change of variables $s = |x|^2/t$, we have

$$\begin{aligned}\left|\frac{\partial R^\lambda}{\partial x_i}(x)\right| &\le \int_0^\infty e^{-\lambda t}\left|\frac{\partial p_t}{\partial x_i}\right|(x)\,dt \\ &\le c_1 \int_0^\infty e^{-\lambda t}\frac{|x|}{t} t^{-d/2} e^{-|x|^2/2t}\,dt \\ &= c_2 \int_0^\infty e^{-\lambda |x|^2/s}|x|^{1-d} s^{d/2-1} e^{-s/2}\,ds.\end{aligned}$$

We will choose $\beta > 0$ in a moment. If $|x| \le \beta$, this is less than $c_3|x|^{1-d}$. If $|x| > \beta$, this is less than $c_4|x|^{1-d}\varphi(\lambda, \beta)$, where

$$\varphi(\lambda, \beta) = \int_0^\infty e^{-\lambda\beta^2/s} s^{d/2-1} e^{-s/2}\,ds.$$

Note that, for each β, $\varphi(\lambda, \beta) \to 0$ as $\lambda \to \infty$.

Let $\mu_\beta = \mu|_{B(x,\beta)}$. Then, by Proposition 2.1,

$$\left|\frac{\partial R^\lambda \mu_\beta}{\partial x_i}(x)\right| \le c_5 m_\mu(\beta)$$

and

$$\left|\frac{\partial}{\partial x_i} R^\lambda(\mu - \mu_\beta)(x)\right| \le c_6 m_\mu(\rho)\varphi(\lambda, \beta).$$

If we first choose β small so that $c_5 m_\mu(\beta) < \varepsilon/(2d)$ and then choose λ large so that $c_6 m_\mu(\rho)\varphi(\lambda, \beta) < \varepsilon/(2d)$, our proof is complete. □

Let

$$\mathcal{L} = \{f : f \text{ maps } \mathbb{R}^d \text{ to } [-1, 1],\ f \text{ is Lipschitz with Lipschitz constant } 1\}.$$

Define $d_{\mathcal{L}}(\mu, \nu) = \sup\{|\int f\,d\mu - \int f\,d\nu| : f \in \mathcal{L}\}$. The distance $d_{\mathcal{L}}$ is a metric for the topology of weak convergence for finite measures.

PROPOSITION 3.8. *Suppose $\mu \in \mathbf{K}_{d-1}$ with support in $B(x_1, \rho)$, where $\rho < 1$. Then $R^\lambda \mu$ is Lipschitz:*

$$|R^\lambda \mu(x) - R^\lambda \mu(y)| \le c_1(|x - y| \wedge 1),$$

where c_1 depends only on $m_\mu(\rho)$.

PROOF. This is immediate from Proposition 3.3 and Remark 3.2. □

PROPOSITION 3.9. *If μ and ν are in $\mathbf{K}_{d-1}$ and have support in $B(x_1,\rho)$ with $\rho<1$, then there is a decreasing function $\phi>0$ with $\lim_{r\downarrow 0}\phi(r)=0$ such that*

$$|\nabla R^\lambda(\mu-\nu)(x)|\le \phi(d_{\mathcal{L}}(\mu,\nu)).$$

The function ϕ depends only on the bounds on $m_\mu(r)$ and $m_\nu(r)$.

PROOF. Fix i and define

$$H_\delta(x)=\int_\delta^\infty e^{-\lambda t}\frac{\partial p_t}{\partial x_i}(x)\,dt=-\int_\delta^\infty e^{-\lambda t}\frac{x_i}{t}(2\pi t)^{-d/2}e^{-|x|^2/2t}\,dt.$$

Then

$$|H_\delta(x)|\le c_2\int_\delta^\infty \frac{|x|}{t}t^{-d/2}e^{-|x|^2/2t}\,dt\le c_3\int_\delta^\infty t^{-(d+1)/2}\,dt=c_4\delta^{-(d-1)/2}. \tag{3.4}$$

Similarly, we compute $\partial H_\delta/\partial x_j$ and we see that

$$|\nabla H_\delta(x)|\le c_5\delta^{-d/2}. \tag{3.5}$$

We next look at $(H-H_\delta)(x)$. Similarly to the above, we see that

$$|(H-H_\delta)(x)|\le c_6\int_0^\delta t^{-(d+1)/2}\frac{|x|}{t^{1/2}}e^{-|x|^2/2t}\,dt. \tag{3.6}$$

We will choose $\beta>0$ in a moment. If $|x|\le\beta$, we have

$$|(H-H_\delta)(x)|\le c_7|x|^{1-d}. \tag{3.7}$$

If $|x|>\beta$, we have

$$|(H-H_\delta)(x)|\le c_8\int_0^\delta t^{-(d+1)/2}e^{-\beta^2/4t}\,dt\le c_9\beta^{-d-1}\delta\, e^{-\beta^2/16\delta}. \tag{3.8}$$

From (3.4) and (3.5), we have

$$|H_\delta(\mu-\nu)(x)|\le c_{10}\delta^{-d/2}d_{\mathcal{L}}(\mu,\nu), \tag{3.9}$$

where we write $H_\delta\mu(x)=\int H_\delta(x-z)\mu(dz)$ and c_{10} depends on κ,γ,ρ. From Proposition 2.1 and (3.7), we have

$$|(H-H_\delta)\mu_\beta(x)|\le c_{11}m_\mu(\beta), \tag{3.10}$$

where $\mu_\beta=\mu|_{B(x,\beta)}$. By (3.8), we have

$$|(H-H_\delta)(\mu-\mu_\beta)(x)|\le c_{12}\delta\beta^{-d-1}e^{-\beta^2/16\delta}\le c_{13}\delta\beta^{-d-3}. \tag{3.11}$$

If we choose β so that $\beta^{d+3}=\delta^{1/2}$, then combining (3.10) and (3.11) yields

$$|(H-H_\delta)\mu(x)|\le c_{14}\big(m_\mu(\delta^{1/(2(d+3))}+\delta^{1/2})\big). \tag{3.12}$$

We have a similar estimate with μ replaced by ν. Combining with (3.9) and choosing δ so that $\delta^{d/2} = (d_{\mathcal{L}}(\mu, \nu))^{1/2}$, we obtain our result. $\square$

Given a process A_t of bounded variation, let A_t^+ and A_t^- denote its positive and negative variation, respectively.

LEMMA 3.10. *Assume that X is a strong Markov process. Let A_t^1 and A_t^2 be two continuous additive functionals of X having bounded variations and let $B_t = A_t^1 - A_t^2$. Suppose* $\sup_x \mathbb{E}^x \int_0^\infty e^{-\lambda t}\, d((A_t^j)^+ + (A_t^j)^-) \le N,\ j = 1, 2,$ *and* $\sup_x |\mathbb{E}^x \int_0^\infty e^{-\lambda t}\, dB_t| \le \varepsilon$. *Then*

$$\mathbb{E}^x\left(\sup_t \left|\int_0^t e^{-\lambda s}\, dB_s\right|\right)^2 \le c_1 \varepsilon N.$$

PROOF. Let $C_t^1 = \int_0^t e^{-\lambda s}\, d((A_s^1)^+ + (A_s^2)^-)$ and $C_t^2 = \int_0^t e^{-\lambda s}\, d((A_s^2)^+ + (A_s^1)^-)$. Let $D_t = \int_0^t e^{-\lambda s}\, d(A_s^1 - A_s^2)$. Then, by the Markov property,

$$\mathbb{E}[C_\infty^j - C_t^j \mid \mathcal{F}_t] = e^{-\lambda t}\mathbb{E}^{X_t} \int_0^\infty e^{-\lambda s}\, dC_s^j \le 2N, \qquad j = 1, 2,$$

and

$$\big|\mathbb{E}[D_\infty - D_t \mid \mathcal{F}_t]\big| = e^{-\lambda t}\left|\mathbb{E}^{X_t} \int_0^\infty e^{-\lambda s}\, dD_s\right| \le \varepsilon.$$

By [2], Proposition I.6.14, $\mathbb{E}^x(\sup_t D_t^2) \le c_2 \varepsilon N$, which is what we wanted. $\square$

4. Existence. Throughout this section we assume Assumption 3.5 holds. Let $G_n^i(x)dx$ be approximations to $\pi^i(dx)$ as in (2.1). Let X_t^n be the solution to the stochastic differential equation

$$dX_t^n = dW_t + G_n(X_t^n)\, dt, \qquad X_0^n = x_0,$$

where W_t is a d-dimensional Brownian motion. Let $\mathbb{P}_n^{x_0}$ be the probability on Ω induced by the law of X_n under $\mathbb{P}$ when $X_0^n = x_0$. Define

$$S_n^\lambda f(x) = \mathbb{E}_n^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt$$

and

$$B_n f(dx) = \sum_{i=1}^d \frac{\partial f}{\partial x_i}(x)\pi_n^i(dx) = \sum_{i=1}^d \frac{\partial f}{\partial x_i}(x) G_n^i(x)\, dx.$$

LEMMA 4.1. *If g is bounded,*

$$S_n^\lambda g(x) = \sum_{j=0}^\infty R^\lambda (B_n R^\lambda)^j g(x). \tag{4.1}$$

PROOF. If $f \in C^2$ is bounded, then by Itô's formula

$$f(X_t^n) - f(X_0^n) = \int_0^t \nabla f(X_s^n) \cdot dW_s + \int_0^t \nabla f(X_s^n) \cdot G_n(X_s^n)\, ds + \tfrac{1}{2} \int_0^t \Delta f(X_s^n)\, ds.$$

Taking $\mathbb{P}^x$ expectations, multiplying by $e^{-\lambda t}$ and integrating over t from 0 to ∞, we obtain

$$\begin{aligned} &\mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t^n)\, dt - \frac{1}{\lambda} f(x) \\ &\quad = \mathbb{E}^x \int_0^\infty e^{-\lambda t} \int_0^t \left[\nabla f(X_s^n) \cdot G_n(X_s) + \frac{1}{2}\Delta f(X_s^n)\right] ds\, dt \\ &\quad = \frac{1}{\lambda}\mathbb{E}^x \int_0^\infty e^{-\lambda s}\left[\nabla f(X_s^n) \cdot G_n(X_s) + \frac{1}{2}\Delta f(X_s^n)\right] ds \end{aligned} \tag{4.2}$$

or

$$\lambda S_n^\lambda f(x) = f(x) + S_n^\lambda[\nabla f \cdot G_n + \tfrac{1}{2}\Delta f](x). \tag{4.3}$$

Suppose $g \in C^2$ is bounded and set $f = R^\lambda g$. Then $f \in C^2$ and $\frac{1}{2}\Delta f = \frac{1}{2}\Delta R^\lambda g = \lambda R^\lambda g - g$. Substituting in (4.3), we have

$$S_n^\lambda g(x) = R^\lambda g(x) + S_n^\lambda B_n R^\lambda g(x).$$

This holds for $g \in C^2$ if g is bounded. By taking limits and using Proposition 3.4, this holds for all bounded continuous g. Taking further limits, we have, by (3.3) and Proposition 3.9,

$$S_n^\lambda \mu(x) = R^\lambda \mu(x) + S_n^\lambda B_n R^\lambda \mu(x)$$

for $\mu \in \mathbf{K}_{d-1}$ such that μ has support in $B(x_1, \rho)$ for some ρ. We have finiteness by Proposition 3.3.

We now iterate and obtain

$$S_n^\lambda \mu(x) = R^\lambda \mu(x) + R^\lambda B_n R^\lambda \mu(x) + S_n^\lambda B_n R^\lambda B_n R^\lambda \mu(x).$$

Continuing to substitute for S_n^λ on the right-hand side, we have

$$S_n^\lambda \mu(x) = \sum_{j=0}^k R^\lambda (B_n R^\lambda)^j \mu(x) + S_n^\lambda (B_n R^\lambda)^{k+1} \mu(x).$$

By Proposition 3.6, $G_n(x)(\nabla R^\lambda (B_n R^\lambda)^k \mu)(x)$ is a function bounded by $c_1 2^{-k}\kappa$ (c_1 may depend on n), so $S_n^\lambda (B_n R^\lambda)^k \mu(x) \to 0$ as $k \to \infty$. Similarly, $\sum_{j=k+1}^\infty R^\lambda (B_n R^\lambda)^j \mu(x) \to 0$ as $k \to \infty$. Therefore,

$$S_n^\lambda \mu(x) = \sum_{j=0}^\infty R^\lambda (B_n R^\lambda)^j \mu(x).$$

If g is bounded with support in $B(x_1, \frac{1}{2})$ for some x_1, setting $\mu(dx) = g(x)\,dx$ establishes (4.1) for such g. A function that is bounded with compact support can be written as the sum of finitely many bounded functions, each of which has support in some ball of radius $\frac{1}{2}$. Thus, by linearity, (4.1) holds for bounded functions with compact support.

Finally, for g bounded, let $g_m(x) = g(x)\mathbb{1}_{\{|x|\le m\}}$. Clearly, $R^\lambda g_m \to R^\lambda g$. Note that, under Assumption 3.5, π_n^i has support in some ball $B(x_1, \rho)$ for each i. In view of the proof of Proposition 3.4, $\nabla g_m \to \nabla g$ uniformly and boundedly in $B(x_1, \rho)$. Consequently, $d_{\mathcal{L}}(B_n R^\lambda g_m, B_n R^\lambda g) \to 0$ as $m \to \infty$. Hence, by (3.3) and Propositions 3.6 and 3.9, $\sum_{j=0}^\infty R^\lambda (B_n R^\lambda)^j g_m(x) \to \sum_{j=0}^\infty R^\lambda (B_n R^\lambda)^j g(x)$ as $m \to \infty$. Now applying (4.1) to g_m and letting $m \to \infty$, establishes (4.1) for bounded functions g. $\square$

As a corollary we have the following.

THEOREM 4.2. *The collection of functions* $\{S_n^\lambda g : n \ge 1, \|g\|_\infty \le 1\}$ *is equicontinuous.*

PROOF. By the preceding proposition,

$$S_n^\lambda g = R^\lambda \left(\sum_{j=0}^\infty (B_n R^\lambda)^j \right) g.$$

Note that, for every $\phi \in C^1$, the support of $B_n \phi$ is contained in $B(x_1, 2\rho)$ for n large. By Proposition 3.6, there is $0 < \kappa \le 1$ such that, for any $j \ge 1$,

$$|(B_n R^\lambda)^j g| \le \frac{\kappa}{2^j d} \sum_{i=1}^d |\pi^i|.$$

Therefore, $\sum_{j=1}^\infty |(B_n R^\lambda)^j g| \le (\kappa/d) \sum_{i=1}^d |\pi^i|$. The result now follows by (3.3) and Proposition 3.3. $\square$

Next we show that for each x the sequence $\mathbb{P}_n^x$ is tight. In fact, we have a uniformity over x as well.

THEOREM 4.3. *Let* $\beta, \varepsilon > 0, T > 0$. *There exists* δ *not depending on* x *or* n *such that*

$$\mathbb{P}_n^x \left(\sup_{s,t \le T,\, |t-s| < \delta} |X_t - X_s| > \beta \right) < \varepsilon.$$

PROOF. By the Markov property and standard arguments, it is enough to show that there exists δ such that

$$\mathbb{P}_n^x \left(\sup_{t \le \delta} |X_t - x| > \beta \right) < \varepsilon.$$

By standard estimates on Brownian motion, it is well known that we can make

$$\mathbb{P}_n^x\Big(\sup_{t\le\delta}|W_t|>\frac{\beta}{2}\Big)<\frac{\varepsilon}{2}$$

if we take δ small enough. Let $H_n(x)=\sum_{i=1}^d|G_n^i(x)|$. Therefore, it suffices to show that

$$\mathbb{P}^x\Big(\int_0^\delta H_n(X_s^n)\,ds>\frac{\beta}{2}\Big)<\frac{\varepsilon}{2} \tag{4.4}$$

uniformly in x and n if we take δ small enough.

By Chebyshev's inequality, the probability in (4.4) is bounded by

$$\frac{2}{\beta}\mathbb{E}^x\int_0^\delta H_n(X_s^n)\,ds.$$

If we set $\theta=1/\delta$, this in turn is bounded by

$$\frac{2e}{\beta}\mathbb{E}^x\int_0^\delta e^{-\theta t}H_n(X_s^n)\,ds\le\frac{2e}{\beta}S^\theta H_n(x). \tag{4.5}$$

By (3.3), $m_{G_n^i(x)\,dx}(r)\le m_{\pi^i}(r)$ and therefore $m_{H_n(x)\,dx}(r)\le\sum_{i=1}^d m_{\pi^i}(r)$. By Proposition 3.7, if we take δ sufficiently small, then

$$\|S^\theta H_n(x)\|_\infty\le\varepsilon(1\wedge\beta)/4e\qquad\text{for every } n\ge1.$$

With (4.5), this yields the desired estimate. □

COROLLARY 4.4. *Let $\beta\in(0,1]$. There exists $\delta<1$ such that if $\tau=\inf\{t:|X_t-X_0|>\beta\}$, then $\sup_x\mathbb{E}_n^x e^{-\tau}\le\delta$.*

PROOF. By Theorem 4.3, there exists ε such that $\mathbb{P}_n^x(\tau<\varepsilon)<\frac{1}{2}$. Then

$$\begin{aligned}\mathbb{E}_n^x e^{-\tau}&\le e^{-\varepsilon}\mathbb{P}_n^x(\tau\ge\varepsilon)+\mathbb{P}_n^x(\tau<\varepsilon)\\&=e^{-\varepsilon}[1-\mathbb{P}_n^x(\tau<\varepsilon)]+\mathbb{P}_n^x(\tau<\varepsilon)\\&=\mathbb{P}_n^x(\tau<\varepsilon)(1-e^{-\varepsilon})+e^{-\varepsilon}\\&\le\frac{1-e^{-\varepsilon}}{2}+e^{-\varepsilon}=\frac{1+e^{-\varepsilon}}{2}.\end{aligned}$$

Now set $\delta=(1+e^{-\varepsilon})/2$. □

THEOREM 4.5. *There exists a subsequence n_m such that $\mathbb{P}_{n_m}^x$ converges weakly, say to $\mathbb{P}^x$, for each x. The collection $(X_t,\mathbb{P}^x)$ forms a strong Markov process having the strong Feller property.*

PROOF. Let $\{g_i\}$ be a countable collection of continuous functions that are dense in $C_0(\mathbb{R}^d)$ in the uniform topology. Let $\{\lambda_k\}$ be a set of positive reals that is dense in some interval $[a,b] \subset (0,\infty)$. By Theorem 4.2 and a diagonalization procedure, we can extract a subsequence $\{n_m\}$ such that $S_{n_m}^{\lambda_k} g_i$ converges for each i and k as $m \to \infty$, say, to $S^{\lambda_k} g_i$ uniformly on every compact set. Since $\|S^{\lambda_k}\|_\infty \le 1/\lambda_k$, it follows that along this sequence $S_{n_m}^{\lambda_k} g$ converges for each continuous g in $C_0(\mathbb{R}^d)$. Moreover, by Theorem 4.2, the modulus of continuity depends only on a, b and $\|g\|_\infty$. By the resolvent identity, $\|S_n^p - S_n^q\|_\infty \le c_1|p-q|$ if $p, q \in [a,b]$, where c_1 depends only on a and b. Therefore, $S_{n_m}^\lambda g$ converges, say, to $S^\lambda g$, for every $\lambda \in [a,b]$.

Next fix x. Let $\mathbb{P}'$ and $\mathbb{P}''$ be any two subsequential limit points of the sequence $\mathbb{P}_{n_m}^x$ with respect to weak convergence. Since

$$U_1(\mathbb{P}', \lambda) = \mathbb{E}_{\mathbb{P}'} \int_0^\infty e^{-\lambda t} g(X_t)\, dt$$

will be the limit of some subsequence of $S_{n_m}^\lambda g$ if g is continuous and bounded, then $U_1(\mathbb{P}', \lambda) = S^\lambda g(x)$ if $\lambda \in [a,b]$. The same thing holds if we replace $\mathbb{P}'$ by $\mathbb{P}''$. By the uniqueness of the Laplace transform and the continuity of the paths of X_t, we have that $\mathbb{E}_{\mathbb{P}'} g(X_t) = \mathbb{E}_{\mathbb{P}''} g(X_t)$ for all t and all $g \in C_0(\mathbb{R}^d)$. To get the equality of higher order joint distributions, consider

$$U_2(\mathbb{P}', \lambda, \mu) = \mathbb{E}_{\mathbb{P}'}\left[\int_0^\infty \int_0^\infty e^{-\mu s} e^{-\lambda t} f(X_s) g(X_{s+t})\, ds\, dt\right],$$

where f and g are in $C_0(\mathbb{R}^d)$. This is the limit along an appropriate subsequence of

$$\mathbb{E}_{n_m}^x\left[\int_0^\infty \int_0^\infty e^{-\mu s} e^{-\lambda t} f(X_s) g(X_{s+t})\, ds\, dt\right].$$

By the Markov property, this equals

$$\begin{aligned}
&\mathbb{E}_{n_m}^x\left[\int_0^\infty e^{-\mu s} f(X_s) \mathbb{E}_{n_m}^{X_s}\left[\int_0^\infty e^{-\lambda t} g(X_t)\, dt\right] ds\right] \\
&\quad = \mathbb{E}_{n_m}^x\left[\int_0^\infty e^{-\mu s} f(X_s) S_{n_m}^\lambda g(X_s)\, ds\right] \\
&\quad = S_{n_m}^\mu (f S_{n_m}^\lambda g)(x).
\end{aligned}$$

Using the fact that $S_{n_m}^\lambda g$ converges to $S^\lambda g$ and the equicontinuity of $S_n^\lambda g$, we deduce that this expression converges to $S^\mu(f S^\lambda g)$. Therefore, $U_2(\mathbb{P}', \lambda, \mu) = S^\mu(f S^\lambda g)(x)$. The same is true for $U_2(\mathbb{P}'', \lambda, \mu)$. By the uniqueness of the multivariate Laplace transform, we conclude the two-dimensional joint distributions under $\mathbb{P}'$ and $\mathbb{P}''$ are the same. The higher order joint distributions are handled similarly. Therefore, we deduce that $\mathbb{P}' = \mathbb{P}''$.

We have thus shown that $\mathbb{P}^x_{n_m}$ converges weakly, say, to $\mathbb{P}^x$, and the resolvents $S^\lambda_{n_m} g$ converge uniformly on compacts to $S^\lambda g(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} g(X_t)\, dt$ if $\lambda \in [a, b]$ and $g \in C_0(\mathbb{R}^d)$. Clearly, $(X, \mathbb{P}_x, x \in \mathbb{R}^d)$ is a conservative Markov process with $P^x(X_0 = x) = 1$. Moreover, $S^\lambda g$ is continuous with a modulus of continuity that depends only on a, b and $\|g\|_\infty$.

By taking limits, we have $S^\lambda g$ is continuous if g is bounded. Let $P_t g(x) = \mathbb{E}^x g(X_t)$. For any $t > 0$,

$$P_t g(x) = \lim_{\lambda\to\infty} e^{-\lambda t} \sum_{n=0}^\infty \frac{(\lambda t)^n}{n!} (\lambda S^\lambda)^n g(x), \tag{4.6}$$

where the limit holds in the sup norm. Thus, $P_t g$ is continuous if g is bounded. That is, X has the strong Feller property. It is then standard (see Theorem I.8.11 of [4]) that $(X_t, \mathbb{P}^x)$ is a strong Markov process. $\square$

Let $S^\lambda g(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} g(X_t)\, dt$ and $S^\lambda_n g(x) = \mathbb{E}^x_n \int_0^\infty e^{-\lambda t} g(X_t)\, dt$.

PROPOSITION 4.6. *If $\lambda > 0$ and g is bounded, then*

$$S^\lambda g = R^\lambda \left(\sum_{j=0}^\infty (BR^\lambda)^j \right) g.$$

Furthermore, $S^\lambda g \in C_0(\mathbb{R}^d)$ if $g \in C_0(\mathbb{R}^d)$.

PROOF. Let g be continuous. We have $S^\lambda_n g = R^\lambda(\sum_{j=0}^\infty (B_n R^\lambda)^j) g$ and $S^\lambda_n g \to S^\lambda g$ provided, we take $n \to \infty$ along an appropriate subsequence. We will show $R^\lambda (B_n R^\lambda)^j g$ converges. In view of Proposition 3.4, Assumption 3.5 and Proposition 3.6,

$$\|R^\lambda (B_n R^\lambda)^j g\|_\infty \le c_1 2^{-j} \|g\|_\infty.$$

Using dominated convergence, we will then have the proposition for continuous g. The case of bounded g then follows by a limit argument.

Write

$$R^\lambda (B_n R^\lambda)^j g - R^\lambda (BR^\lambda)^j g = \sum_{k=0}^{j-1} R^\lambda (B_n R^\lambda)^k (B_n R^\lambda - BR^\lambda)(BR^\lambda)^{j-k-1} g.$$

By Propositions 3.4 and 3.6, $\nu_{jk}(g) = (BR^\lambda)^{j-k-1} g$ will be a measure in $\mathbf{K}_{d-1}$ whose total variation is bounded by $(\kappa/d) \sum_{i=1}^d |\pi^i|$ for some $\kappa > 0$. By Proposition 3.3,

$$(B_n R^\lambda - BR^\lambda)(BR^\lambda)^{j-k-1} g = \sum_{i=1}^d \nabla R^\lambda(\nu_{jk}(g))(x)(\pi^i_n - \pi^i)(dx)$$

is a measure in $\mathbf{K}_{d-1}$. By (3.3) and Proposition 3.9,

$$(B_n R^\lambda)(B_n R^\lambda - BR^\lambda)(BR^\lambda)^{j-k-1} g$$

will be a measure in $\mathbf{K}_{d-1}$ whose total variation is bounded by $c_3 \phi(\beta(n)) \sum_{i=1}^d |\pi^i|$ where $\beta(n) = \sum_{i=1}^d d_{\mathcal{L}}(\pi_n^i, \pi^i) c_3$ depends on $\|g\|_\infty$ but not on n, and ϕ is a decreasing function with $\lim_{r\downarrow 0} \phi(0) = 0$ that depends only on the upper bounds on $\sum_{i=1}^d m_{\pi^i}(r)$. By Proposition 3.3,

$$(B_n R^\lambda)^k (B_n R^\lambda - BR^\lambda)(BR^\lambda)^{j-k-1} g$$

will be a measure in $\mathbf{K}_{d-1}$ whose total variation is bounded by $c_4 \phi(\beta(n)) \sum_{i=1}^d |\pi^i|$ where c_4 does not depend on n, and by Remark 3.3,

$$R^\lambda (B_n R^\lambda)^k (B_n R^\lambda - BR^\lambda)(BR^\lambda)^{j-k-1} g$$

is a function whose sup norm is less than or equal to $c_5 \phi(\beta(n)) \sum_{i=1}^d m_{\pi^i}(\rho)$, where again the constant depends on $\|g\|_\infty$ but not on n. This implies that this term goes to 0 as $n \to \infty$.

Now suppose $g \in C_0(\mathbb{R}^d)$. Clearly, $R^\lambda g \in C_0(\mathbb{R}^d)$. As π has compact support, $R^\lambda (BR^\lambda)^j g \in C_0(\mathbb{R}^d)$. So, by Proposition 3.6 and Remark 3.1, $S^\lambda g \in C_0(\mathbb{R}^d)$. □

REMARK 4.7. The proof of Theorem 4.5 in fact shows that for every subsequence n_k there is a sub-subsequence n_{k_m} such that $\mathbb{P}^x_{n_{k_m}}$ converges for every $x \in \mathbb{R}^d$. The proof of Proposition 4.6 tells us that every subsequential limit of $\mathbb{P}^k_n$ has the same resolvent and therefore has the same law. This implies that $\mathbb{P}^x_n$ is convergent for each x.

THEOREM 4.8. *The strong Markov process* $(X, \mathbb{P}^x)$ *is a Feller process having the strong Feller property. In particular, it is a Hunt process.*

PROOF. It follows from (4.6) and Proposition 4.6 that $P_t g \in C_0(\mathbb{R}^d)$ if $g \in C_0(\mathbb{R}^d)$. So X is a Feller process. The strong Feller property was proved in Theorem 4.5. □

PROPOSITION 4.9. *Under* $\mathbb{P}^x$ *we have* $X_t = x + W_t + A_t$ *and* $\mathbb{P}^x(X_0 = x) = 1$, *where* W_t *is a* d*-dimensional Brownian motion such that* $A_t = \lim_{n\to\infty} \int_0^t G_n(X_s)\, ds$ *is an* $\mathbb{R}^d$*-valued continuous additive functional of* X *having finite variation, where the convergence is in the following sense: for any* $\varepsilon > 0$ *and* $t > 0$,

$$\lim_{n\to\infty} \mathbb{P}^x\left(\sup_{s\le t} \left| \int_0^s G_n(X_r)\, dr - A_s \right| > \varepsilon \right) = 0.$$

Furthermore, there exists a subsequence $\{n_k\}$ *such that* $\sup_k \int_0^t |G_{n_k}(X_s)|\, ds < \infty$ *a.s. for each* $t > 0$.

PROOF. Fix i and let us look at the ith component. Let $\varepsilon > 0$. From Proposition 3.6,

$$S^\lambda |\pi_n^i|(x) = R^\lambda \left(\sum_{j=0}^{\infty} (BR^\lambda)^j \right) |\pi_n^i|(x)$$

is bounded, independently of n. Define $\mu_n^{i,+} = \psi_{2^{-n}} * \pi^{i,+}$ and $\mu_n^{i,-} = \psi_{2^{-n}} * \pi^{i,-}$. Then $\pi_n^i = \mu_n^{i,+} - \mu_n^{i,-}$. Let

$$S^\lambda(\pi^{i,+})(x) = R^\lambda \left(\sum_{j=0}^{\infty} (BR^\lambda)^j \right) \pi^{i,+}(x).$$

Define $S^\lambda(\pi^{i,-})$ similarly. Set $S^\lambda(\pi^i)(x) = S^\lambda(\pi^{i,+})(x) - S^\lambda(\pi^{i,-})$. It follows from (3.3) and Propositions 3.9 and 3.6 that $S^\lambda(\mu_n^{i,+})(x) \to S^\lambda(\pi^{i,+})(x)$ and $S^\lambda(\mu_n^{i,-})(x) \to S^\lambda(\pi^{i,-})(x)$ uniformly in x. Hence, $S^\lambda(\pi^{i,+})$ and $S^\lambda(\pi^{i,-})$ are bounded continuous λ-potentials of the Feller process X. So there are two positive continuous additive functionals $A^{i,+}$ and $A^{i,-}$ of X such that $S^\lambda(\pi^{i,+})(x) = \mathbb{E}^x[\int_0^\infty e^{-\lambda t}\, dA_t^{i+}]$ and $S^\lambda(\pi^{i,-})(x) = \mathbb{E}^x[\int_0^\infty e^{-\lambda t}\, dA_t^{i,-}]$ (see Theorem IV.3.13 of [4]). Let $A^i = A^{i,+} - A^{i,-}$ and $A = (A^1, \dots, A^d)$. Note that A is an $\mathbb{R}^d$-valued continuous additive functional of X having finite variation whose signed Revuz measure is π. As $\sup_{x\in\mathbb{R}^d} |S^\lambda(\pi_n^i - \pi^i)(x)| \to 0$, it follows by Lemma 3.10 that $\mathbb{E}^x[\sup_t |B_t|^2] \to 0$ as $n \to \infty$, where

$$B_t = \int_0^t e^{-\lambda s}\, dA_s^i - \int_0^t e^{-\lambda s} G_n^i(X_s)\, ds.$$

If $H_t = A_t^i - \int_0^t G_n^i(X_s)\, ds$, then

$$H_t = \int_0^t e^{\lambda s}\, dB_s = e^{\lambda t} B_t - \int_0^t \lambda e^{\lambda s} B_s\, ds$$

by integration by parts. Therefore,

$$\mathbb{E}^x \left[\sup_{s\le t} \left| A_s^i - \int_0^s G_n^i(X_r)\, dr \right| \right]^2 = \mathbb{E}^x \sup_{s\le t} |H_s|^2 \le \varepsilon$$

if n is large enough. Therefore, $\int_0^t G_{n_k}^i(X_r)\, dr$ converges to A_t^i uniformly over t in finite intervals in probability. Consequently, there exists a subsequence $\{n_k\}$ such that $\int_0^t G_{n_k}^i(X_r)\, dr$ converges to A_t^i a.s. for each i, uniformly over t in finite intervals. Note also that we can choose the subsequence not depending on x so that we have this convergence $\mathbb{P}^x$-a.s. for each x.

If we let $\mu_n^i = \mu_n^{i,+} + \mu_n^{i,-}$, then μ_n^i converges weakly to $|\pi_n^i|$. By the same argument as in the preceding paragraph, there exists a subsequence $\{n_k\}$ such that $\int_0^t \mu_{n_k}^i(X_r)\, dr$ converges uniformly over t in finite intervals a.s. As $|G_n^i(x)| \le \mu_n^i(x)$ for each i, n and x, it follows that $\sup_k \int_0^t |G_{n_k}^i(X_r)|\, dr < \infty$ a.s.

As in the first paragraph of the proof, we have

$$S_m^\lambda |\pi_n^i|(x) = R^\lambda \Biggl(\sum_{j=0}^{\infty} (B_m R^\lambda)^j \Biggr) |\pi_n^i|$$

bounded, independently of m and n, and $S_m^\lambda(\pi_n^i - \pi^i)(x) \to 0$ at a rate that does not depend on x or m. Hence,

$$\mathbb{E}_m^x \biggl[\sup_t \biggl| \int_0^t e^{-\lambda s} G_p^i(X_s)\,ds - \int_0^t e^{-\lambda s} G_n^i(X_s)\,ds \biggr| \biggr]^2 \to 0$$

as $n, p \to \infty$, independently of m, and then

$$\mathbb{E}_m^x \biggl[\int_0^t G_p^i(X_s)\,ds - \int_0^t G_n^i(X_s)\,ds \biggr]^2 \le \varepsilon$$

if n and p are large enough.

Let $r_1 \le \cdots \le r_k \le s$, let $f_1, \dots, f_k$ be continuous functions on $\mathbb{R}^d$ with compact support and let $Y = \prod_{j=1}^k f_j(X_{r_j})$. By Cauchy–Schwarz, there exists n_0 such that if $n, p \ge n_0$, then

$$\biggl| \mathbb{E}^x[Y A_t^i] - \mathbb{E}^x \biggl[Y \int_0^t G_n^i(X_s)\,ds \biggr] \biggr| \le c_1 \varepsilon^{1/2}$$

and

$$\biggl| \mathbb{E}_m^x \biggl[Y \int_0^t G_p^i(X_s)\,ds \biggr] - \mathbb{E}_m^x \biggl[Y \int_0^t G_n^i(X_s)\,ds \biggr] \biggr| \le c_1 \varepsilon^{1/2}$$

for all m. By weak convergence, for each fixed n,

$$\mathbb{E}_m^x \biggl[Y \int_0^t G_n^i(X_s)\,ds \biggr] \to \mathbb{E}^x \biggl[Y \int_0^t G_n^i(X_s)\,ds \biggr]$$

as $m \to \infty$ along an appropriate subsequence, since G_n^i is continuous. Set $n = n_0$ and take $p = m$. If m is large enough,

$$\biggl| \mathbb{E}_m^x \biggl[Y \int_0^t G_m^i(X_s)\,ds \biggr] - \mathbb{E}^x[Y A_t^i] \biggr| \le 2c_1 \varepsilon^{1/2} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\mathbb{E}_m^x \biggl[Y \int_0^t G_m^i(X_s)\,ds \biggr] \to \mathbb{E}^x[Y A_t^i]$$

as $m \to \infty$ along an appropriate subsequence.

Now $X_m^i(t) - \int_0^t G_m^i(X_m(r))\,dr$ is a martingale. Then

$$\mathbb{E}_m^x \biggl[\biggl(X_t^i - \int_0^t G_m^i(X_r)\,dr \biggr) Y \biggr] = \mathbb{E}_m^x \biggl[\biggl(X_s^i - \int_0^s G_m^i(X_r)\,dr \biggr) Y \biggr].$$

Passing to the limit,

$$\mathbb{E}^x[(X_t^i - A_t^i)Y] = \mathbb{E}^x[(X_s^i - A_s^i)Y].$$

This proves that $X_t^i - A_t^i$ is a martingale.

We show similarly that $(X_t^i - A_t^i)(X_t^j - A_t^j) - \delta_{ij}t$ is a martingale, where δ_{ij} is 1 if $i = j$ and 0 otherwise. Hence, if $W_t^i = X_t^i - x^i - A_t^i$, then W_t is a continuous martingale with $\langle W^i, W^j\rangle_t = \delta_{ij}t$ and $W_0 = 0$. This implies that W_t is a standard d-dimensional Brownian motion. □

REMARK 4.10. It follows from the above proof that, given a subsequence $\{n_k\}$, there exists a further subsequence $\{n_{k_m}\}$ such that

$$\sup_m \int_0^t |G_{n_{k_m}}(X_s)|\,ds < \infty \qquad \text{a.s.}$$

and

$$\int_0^t G_{n_{k_m}}(X_s)\,ds$$

converges uniformly to A_t over t in finite intervals.

REMARK 4.11. Whenever $\nu_n^i(dx) = H^i(x)\,dx \in \mathbf{K}_{d-1}$ with $m_{\nu_n^i}(r) \le m_{\pi^i}(r)$ and $\nu_n^i(dx)$ converges weakly to $\pi^i(dx)$, then $S^\lambda H^i \to S^\lambda \pi^i$ uniformly, and so, by Lemma 3.10, $\int_0^t e^{-\lambda s} H^i(X_s)\,ds$ converges to $\int_0^t e^{-\lambda s}\,dA_s^i$ uniformly over $t \in [0,\infty)$.

5. Uniqueness. In this section we prove uniqueness of the solution to (2.4). We operate under Assumption 3.5 throughout this section.

The first step is to show that, in proving uniqueness, it suffices to consider solutions $\mathbb{P}$ for which there exists a subsequence $\{n_k\}$ with

$$\mathbb{E}_{\mathbb{P}} \sum_{i=1}^d \int_0^\infty e^{-\lambda t}\,d|A_t^i| < \infty \quad \text{and} \quad \mathbb{E}_{\mathbb{P}} \sup_k \sum_{i=1}^d \int_0^\infty e^{-\lambda t}|G_{n_k}^i(X_t)|\,dt < \infty. \tag{5.1}$$

To see this, let $\{\mathbb{P}^x,\ x \in \mathbb{R}\}$ be the solution constructed in Section 4 and let $\mathbb{Q}$ be any solution to (2.4). There is a subsequence $\{n_k\}$ associated to the solution $\mathbb{Q}$ and a subsequence $\{n_m\}$ associated to the solutions $\mathbb{P}^x$ (from the construction in the proof of Proposition 4.9, $\{n_m\}$ can be chosen independently of x). By using Remark 4.10, taking a subsequence of $\{n_k\}$ and relabeling if necessary, we may without loss of generality assume there is a single subsequence $\{n_k\}$ that can be used in the definition of weak solution for $\mathbb{Q}$ and for each $\mathbb{P}^x$ such that $\int_0^t G_{n_{k_m}}(X_s)\,ds$ converges uniformly over t in finite intervals a.s. under both $\mathbb{Q}$ and each $\mathbb{P}^x$. Let

$$T_N = \inf\left\{t : \sum_{i=1}^d \int_0^t d|A_t^i| + \sum_{i=1}^d \sup_{k\ge 1} \int_0^t |G_{n_k}^i(X_t)|\,dt \ge N\right\}.$$

Since A_t is locally of bounded variation and $\int_0^t G^i_{n_k}(X_s)\,ds$ converges to A^i_t uniformly over t in finite intervals, then $T_N < \infty$ a.s. We construct a new solution, $\widetilde{\mathbb{P}}$, that behaves according to $\mathbb{Q}$ up to time T_N and like $\mathbb{P}^{X_{T_N}}$ after that. We specify $\widetilde{\mathbb{P}}$ by setting

$$\widetilde{\mathbb{P}}\big(B \cap (C \circ \theta_{T_N})\big) = \mathbb{E}_{\mathbb{Q}}\big[\mathbb{P}^{X_{T_N}}(C);\, B\big],$$

whenever $B \in \mathcal{F}_{T_N}$ and $C \in \mathcal{F}_\infty$. It is easy to see that $\widetilde{\mathbb{P}}$ is again a solution to (2.4). Moreover,

$$\mathbb{E}_{\widetilde{\mathbb{P}}} \int_0^\infty e^{-\lambda t}\, d|A^i_t| = \mathbb{E}_{\mathbb{P}} \int_0^{T_N} e^{-\lambda t}\, d|A^i_t| + \mathbb{E}_{\mathbb{Q}}\bigg[e^{-\lambda T_N} \mathbb{E}_{\mathbb{P}^{X_{T_N}}} \int_0^\infty e^{-\lambda t}\, d|A^i_t|\bigg].$$

The first term is bounded by N and the second is less than $\mathbb{E}_{\mathbb{Q}}[S^\lambda(|\pi^i|)(X_{T_N})]$. Since $S^\lambda(|\pi^i|)$ is bounded, we conclude $\mathbb{E}_{\widetilde{\mathbb{P}}} \int_0^\infty e^{-\lambda t}\, d|A^i_t| < \infty$. Similarly,

$$\mathbb{E}_{\widetilde{\mathbb{P}}} \sup_k \int_0^\infty e^{-\lambda t} \big|G^i_{n_k}(X_t)\big|\, dt < \infty,$$

using Proposition 4.9. If we show $\widetilde{\mathbb{P}} = \mathbb{P}^{x_0}$, it follows that $\mathbb{Q}|_{\mathcal{F}_{T_N}} = \mathbb{P}^{x_0}|_{\mathcal{F}_{T_N}}$ for each N. Since we are supposing that A_t is locally of bounded variation, then $T_N \to \infty$ as $N \to \infty$ and we conclude that $\mathbb{Q} = \mathbb{P}^{x_0}$. It therefore suffices to consider only solutions $\mathbb{Q}$ for which (5.1) holds.

PROPOSITION 5.1. *If $\mathbb{Q}$ is a solution to* (2.4) *and* (5.1) *holds, then* $\mathbb{Q} = \mathbb{P}^{x_0}$.

PROOF. Let $\mathbb{Q}$ be such a solution and let $f \in C^2$. By Itô's formula,

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t \nabla f(X_s) \cdot dW_s + \int_0^t \nabla f(X_s) \cdot dA_s \\ &\quad + \tfrac{1}{2} \int_0^t \Delta f(X_s)\, ds. \end{aligned}$$

Let us take the expectation with respect to $\mathbb{Q}$, multiply by $e^{-\lambda t}$ and integrate over t from 0 to ∞. We then have

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} f(X_t)\, dt - \frac{1}{\lambda} f(x_0) \\ &\quad = \mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} \int_0^t \nabla f(X_s) \cdot dA_s\, dt + \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} \int_0^t \Delta f(X_s)\, ds\, dt \\ &\quad = \frac{1}{\lambda} \mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda s} \nabla f(X_s) \cdot dA_s + \frac{1}{\lambda} \mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda s} \Delta f(X_s)\, ds. \end{aligned}$$

Next multiply both sides by λ. If $g \in C^2$ and we set $f = R^\lambda g$, then $f \in C^2$ and $\frac{1}{2}\Delta f = \lambda R^\lambda g - g$. Substituting, we obtain

$$\mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} g(X_t)\, dt = R^\lambda g(x_0) + \mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} \nabla R^\lambda g(X_t) \cdot dA_t. \tag{5.2}$$

Define the linear functional V^λ by

$$V^\lambda h = \mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} h(X_t)\,dt.$$

We can then express (5.2) by

$$V^\lambda g = R^\lambda g(x_0) + \mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} \nabla R^\lambda g(X_t) \cdot dA_t. \tag{5.3}$$

We first have this for $g \in C^2$, but by taking limits, we have (5.3) for g continuous and bounded.

Since $\int_0^t G_{n_k}(X_s)\,ds$ converges to A_t uniformly over t in finite intervals, then, using (5.1), we have that $\int_0^\infty e^{-\lambda s} H_s \cdot G_{n_k}(X_s)\,ds$ converges to $\int_0^\infty e^{-\lambda s} H_s \cdot dA_s$ when $s \to H_s$ is piecewise linear and H takes only finitely many values. By approximating continuous functions by piecewise linear ones, we again have convergence if $s \to H_s$ is continuous. Therefore, the second term on the right-hand side of (5.3) is equal to

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} \nabla R^\lambda g(X_t) \cdot G_{n_k}(X_t)\,dt = \lim_{k\to\infty} V^\lambda\big((\nabla R^\lambda g) G_{n_k}\big).$$

But, using (5.3) again,

$$\begin{aligned} &V^\lambda\big((\nabla R^\lambda g)G_{n_k}\big) \\ &\qquad = R^\lambda\big((\nabla R^\lambda g)G_{n_k}\big)(x_0) + \mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} \nabla R^\lambda\big((\nabla R^\lambda g)G_{n_k}\big)(X_t) \cdot dA_t. \end{aligned} \tag{5.4}$$

The limit as $n \to \infty$ of the first term on the right-hand side is $R^\lambda(BR^\lambda g)(x_0)$. For the second term on the right-hand side of (5.4), the integral is dominated by

$$\int_0^\infty e^{-\lambda t} |\nabla R^\lambda (B_{n_k} R^\lambda g)(X_t)|\, d|A_t|,$$

and $|\nabla R^\lambda(B_{n_k} R^\lambda g)|$ is uniformly bounded by Proposition 3.3. Therefore, the limit of the second term is

$$\mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} (BR^\lambda g)(X_t) \cdot dA_t.$$

We thus have

$$V^\lambda g = R^\lambda g(x_0) + R^\lambda B R^\lambda g(x_0) + \mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} \nabla R^\lambda (BR^\lambda)(X_t) \cdot dA_t.$$

We continue by writing the last expectation as the limit of

$$\mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} \nabla R^\lambda (BR^\lambda)(X_t) \cdot G_{n_k}(X_t)\,dt = V^\lambda\big(\nabla R^\lambda (BR^\lambda) G_{n_k}\big).$$

After k steps we arrive at

$$V^\lambda g = R^\lambda \left(\sum_{j=0}^k (BR^\lambda)^j \right) g(x_0) + \mathbb{E}_{\mathbb{Q}} \int_0^\infty e^{-\lambda t} \nabla R^\lambda (BR^\lambda)^k g(X_t) \cdot dA_t.$$

The absolute value of the last term is bounded by $c_1\|g\|_\infty 2^{-k}\mathbb{E}_{\mathbb{Q}}\int_0^\infty e^{-\lambda t}\,d|A_t|$, which tends to 0 as $k\to\infty$. Since

$$\left|R^\lambda\left(\sum_{j=k+1}^{\infty}(BR^\lambda)^j\right)g(x_0)\right|\le c_2\|g\|_\infty\sum_{j=k+1}^{\infty}2^{-j}\to 0$$

as $k\to\infty$, we can pass to the limit and obtain

$$V^\lambda g(x_0)=R^\lambda\left(\sum_{j=0}^{\infty}(BR^\lambda)^j\right)g(x_0)=S^\lambda g(x_0).$$

By the uniqueness of the Laplace transform, we have $\mathbb{E}_{\mathbb{Q}}[g(X_t)]=\mathbb{E}_{\mathbb{P}^{x_0}}[g(X_t)]$ for all t, or the one-dimensional distributions of X_t under $\mathbb{Q}$ and $\mathbb{P}^{x_0}$ are the same. To obtain equality of all the finite-dimensional distributions and hence equality of $\mathbb{Q}$ and $\mathbb{P}^{x_0}$ is standard; see [3], Section 6.3. $\square$

6. Global results. In this section we sketch the rather routine argument that shows that Assumption 3.5 is not necessary, leaving the details to the reader.

PROOF OF THEOREM 2.6. We can find $\rho>0$ such that $\sum_{i=1}^d m_{\pi^i}(\rho)<(2dK)^{-1}$, where K is the constant in Proposition 3.3. Let $T_0=0$ and $T_{i+1}=\inf\{t>T_i:|X_t-X_{T_i}|\ge\rho\}$. Let $\mathcal{Q}^x$ be the solution to (2.4) with $X_0=x$ and π^i replaced by $\pi^i|_{B(x,\rho)}$. Let $\mathbb{Q}_1=\mathcal{Q}^x$ and define inductively

$$\mathbb{Q}_{i+1}\big(A\cap(B\circ\theta_{T_i})\big)=\mathbb{E}_{\mathbb{Q}_i}\big[\mathcal{Q}^{X_{T_i}}(B);A\big],\qquad A\in\mathcal{F}_{T_i},\ B\in\mathcal{F}_\infty.$$

It is obvious that $\mathbb{Q}_m|_{\mathcal{F}_{T_k}}=\mathbb{Q}_k|_{\mathcal{F}_{T_k}}$ if $m\ge k$. Define $\mathbb{P}^x(A)=\mathbb{Q}_k(A)$ if $A\in\mathcal{F}_{T_k}$.

Note

$$\mathbb{E}_{\mathbb{Q}_{i+1}}[e^{-T_{i+1}}]=\mathbb{E}_{\mathbb{Q}_{i+1}}[e^{-(T_{i+1}-T_i)}e^{-T_i}]=\mathbb{E}_{\mathbb{Q}_i}\Big[e^{-T_i}\mathbb{E}_{\mathcal{Q}^{X_{T_i}}}[e^{-T_1}]\Big].\tag{6.1}$$

Since, by Corollary 4.4, $\sup_x\mathbb{E}_{\mathcal{Q}^x}[e^{-T_1}]\le\delta<1$, by induction the left-hand side of (6.1) is less than δ^{i+1}. So $\mathbb{E}_{\mathbb{P}^x}[e^{-T_i}]\le\delta^i\to 0$, which implies that $T_i\to\infty$.

To show $\int_0^t G^i_{n_k}(X_s)\,ds\to A^i_t$ under $\mathbb{P}^x$, it is enough to show that $\int_{t\wedge T_j}^{t\wedge T_{j+1}}G^i_{n_k}(X_s)\,ds$ converges to $A^i_{t\wedge T_{j+1}}-A^i_{t\wedge T_j}$ for each j. By conditioning on $\mathcal{F}_{T_j}$, it is enough to show $\int_0^{t\wedge T_1}G^i_{n_k}(X_s)\,ds$ converges to $A^i_{t\wedge T_1}$ uniformly over t, and we have that by Proposition 4.9.

It is routine to check that $\mathbb{P}^x$ is a solution to (2.4) with $X_0=x$. By standard arguments (cf. [3], Section 6.3), we also have uniqueness.

Part (b) follows from the uniqueness in (a) and Proposition 4.9. $\square$

REMARK 6.1. One question that arises is whether we can replace Brownian motion in (1.2) by other processes. If we have a diffusion in $\mathbb{R}^d$, in either divergence or nondivergence form, whose coefficients are sufficiently smooth (see, e.g., [14]), the above proofs can be suitably modified. For example, using Schauder's estimate (cf. [13]) and the gradient estimates for the Green functions in [14], the main result, Theorem 2.6, of this paper holds if Brownian motion is replaced by a symmetric diffusion whose infinitesimal generator is a uniform elliptic operator in divergence form having C^1 coefficients. In general, however, the conditions of bounded and measurable coefficients together with uniform ellipticity are not enough to guarantee the necessary estimates.

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