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EIGENVALUE EXPANSIONS FOR BROWNIAN MOTION WITH AN APPLICATION TO OCCUPATION TIMES

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EIGENVALUE EXPANSIONS FOR BROWNIAN MOTION WITH AN APPLICATION TO OCCUPATION TIMES [1]

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Abstract. Let B be a Borel subset of $\mathbf{R}^d$ with finite volume. We give an eigenvalue expansion for the transition densities of Brownian motion killed on exiting B. Let A_1 be the time spent by Brownian motion in a closed cone with vertex 0 until time one. We show that $\lim_{u\to 0} \log P^0(A_1 < u)/\log u = 1/\xi$ where ξ is defined in terms of the first eigenvalue of the Laplacian in a compact domain. Eigenvalues of the Laplacian in open and closed sets are compared.

1. Introduction. It is well-known that the transition densities of Brownian motion killed on exiting a bounded open domain in $\mathbf{R}^d$ have an expansion in terms of the eigenvalues and eigenfunctions of the Laplacian on the domain. One of the purposes of this paper is to point out that there exists an eigenvalue expansion for the transition densities of Brownian motion killed on exiting an arbitrary Borel subset B of $\mathbf{R}^d$, provided only that the Lebesgue measure of B is finite; see Theorem 1.1. The notion of eigenvalues of the Dirichlet Laplacian in non-open sets seems to a large extent not to have been considered in analysis.

As a consequence of this expansion, we get some continuity results on the first eigenvalue. If $\lambda(B)$ denotes the first eigenvalue of a set B and J_n are sets of finite volume decreasing to a compact set K, we show in Theorem 1.2 that $\lambda(J_n) \to \lambda(K)$.

Another of the results of this paper is concerned with the amount of time Brownian motion X_t spends in a cone up to time $t = 1$; see Theorem 1.3. The formula we prove is the same as one of the formulae in Meyre and Werner [MW]. Our contribution consists of extending the result to a much larger family of cones.

Theorems 1.1 and 1.2 are proved in Section 2, while Theorem 1.3 is proved in Section 3. In the last section of the paper we compare the first eigenvalue of an open domain with the first eigenvalue of its closure, and illustrate by means of some examples.

We start by defining the eigenvalues of the Laplacian in arbitrary Borel sets of finite

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volume. Multidimensional Brownian motion will be denoted X_t. For a Borel set $B \in \mathbf{R}^d$, let

$$\tau(B) = \tau_B = \inf\{t > 0 : X_t \notin B\}$$

and

$$T(B) = T_B = \inf\{t > 0 : X_t \in B\}.$$

Let $p_B(t, x, y)$ be the transition densities for Brownian motion killed on exiting B, let $G_B(x, y)$ be the corresponding Green function, and set $P_t^B f(x) = \int_B p_B(t, x, y) f(y)\, dy$. For definitions and further information, see Bass [B], Sections III.3, III.4. We use $\langle f, g\rangle$ to denote $\int f(x) g(x)\, dx$. The sphere in $\mathbf{R}^d$ with center y and radius r will be denoted $S(y, r)$, while the corresponding open ball will be denoted $B(y, r)$.

Theorem 1.1. *Suppose $B \subset \mathbf{R}^d$ is a Borel set whose Lebesgue measure is finite and positive and let μ denote the restriction of the Lebesgue measure to B. There exist reals $0 < \lambda_1 \le \lambda_2 \le \cdots < \infty$ and a complete orthonormal system φ_i for $L^2(B)$ such that*

(i) the sequence $\{\lambda_i\}$ has no subsequential limit point other than ∞,

(ii) for each t we have $P_t^B \varphi_i = e^{-\lambda_i t}\varphi_i$, μ-a.e.,

(iii) if $f \in L^2(B)$, then

$$P_t^B f = \sum_{i=1}^{\infty} e^{-\lambda_i t} \langle f, \varphi_i\rangle \varphi_i, \qquad \mu\text{-a.e.},$$

the convergence is absolute, and the convergence takes place in $L^\infty(B)$,

(iv) we have the expansion

$$p_B(t, x, y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y)$$

for μ^2-almost every pair (x, y), the convergence is absolute, and the convergence takes place in $L^\infty(B \times B)$, and

(v) if for some $t > 0$ we have $p_B(t, x, y) > 0$ for μ^2-almost every pair (x, y), then $\lambda_1 < \lambda_2$ and $\varphi_1 > 0$, μ-almost everywhere.

Let $\lambda(B)$ denote the first eigenvalue, i.e., λ_1 from Theorem 1.1.

Theorem 1.2. *Let J_n be Borel subsets of $\mathbf{R}^d$ with finite volume decreasing to a compact set K. The eigenvalues $\lambda(J_n)$ converge to $\lambda(K)$ as $n \to \infty$.*

It is standard to adapt the proof of Theorem 1.1 to see that a result analogous to Theorem 1.1 holds for a compact subset K of a sphere $S(y, r)$ such that $S(y, r) \setminus K$ has a non-empty interior relative to $S(y, r)$. In this case, the transition probabilities $p_K(t, x, y)$

are those of the Brownian motion on $S(y,r)$. Suppose that K is a subset of $S(0,1)$ and let $\lambda(K)$ be the first eigenvalue (i.e., λ_1). Then let $v = d/2 - 1$ and

$$\xi(K) = \frac{(v^2 + 2\lambda(K))^{1/2} + v}{\lambda(K)}.$$

Let C be a closed cone in $\mathbf{R}^d$ with non-empty interior and vertex $(0,\ldots,0)$ and let J be the closure of $S(0,1) \setminus C$. We will write

$$A_t = \int_0^t 1_{(X_s \in C)} ds,$$

i.e., A_t is the amount of time spent by X inside C before time t.

Theorem 1.3. *Assume that the d-dimensional Lebesgue measure of the boundary of C is zero and that both C and its complement have non-empty interiors. Then*

$$\lim_{u\to 0} \frac{\log P^0(A_1 < u)}{\log u} = 1/\xi(J).$$

It seems that the only case when an explicit formula for $P^0(A_1 < u)$ is known is when C is a half-space, where the distribution is known as the "arc-sine law." See Bingham and Doney [BD] for related results.

Meyre and Werner [MR] proved that $P^0(A_1 < u)$ is comparable to $u^{1/\xi(J)}$, but they had to assume that C is convex. Meyre [M] considered more general cones but made a strong assumption of regularity on the boundary.

Our paper is inspired by a question posed by W. Werner. We are grateful for discussions with P. Baxendale, E.B. Davies, R. Howard, P. March, M. van den Berg, W. Werner, and Z. Zhao.

2. Eigenvalue expansions. In this section we prove Theorem 1.1, 1.2, and also give a result (Proposition 2.1) on hitting times which is of independent interest.

Proof of Theorem 1.1. For each t, $p_B(t,x,y) \le p(t,x,y)$, the density of unkilled Brownian motion, which is bounded by a constant depending only on t. Since B has finite volume, $\int_B \int_B p_B(t,x,y)^2\, dx\, dy < \infty$. By Riesz and Sz.-Nagy [RN], page 179, P_t^B is a completely continuous operator, that is, the image of the unit ball in $L^2(B)$ under P_t^B is a set whose closure in L^2 is compact. By the Hilbert-Schmidt expansion theorem (Riesz and Sz.-Nagy [RN] or Bass [B], Section III.4), P_t^B has an eigenvalue expansion as in (iii) and (iv). The fact that the eigenvalues of P_t^B are of the form $e^{-\lambda_i t}$, that the λ_i and φ_i do not depend on t, and that (i) holds may be proved as in Bass [B] or Port and Stone [PS], using the complete continuity of P_t^B in L^2 in place of the equicontinuity of $\{P_t^B f : \|f\|_\infty \le 1\}$.

Because

$$e^{-\lambda_i}\varphi_i(x) = P_1^B\varphi_i(x) = \int p_B(1,x,y)\varphi_i(y)\,dy$$
$$\le \Big(\int_B p(1,x,y)^2\,dy\Big)^{1/2}\Big(\int_B \varphi_i(y)^2\,dy\Big)^{1/2} \le c_1, \qquad \text{a.e.,}$$

then

$$\varphi_i(x) \le e^{\lambda_i} c_1, \qquad \text{a.e.} \tag{2.1}$$

Let $\|f\|_\infty$ denote the $L^\infty(B)$ norm of f. By the semigroup property and Parseval's identity applied to $f(y) = p_B(t/2,x,y)$,

$$\begin{aligned} p_B(t,x,x) &= \int_B p_B(t/2,x,y)^2\,dy = \langle p_B(t/2,x,\cdot), p_B(t/2,x,\cdot)\rangle \\ &= \sum_{i=1}^\infty \langle p_B(t/2,x,\cdot),\varphi_i\rangle^2 = \sum_{i=1}^\infty (P_{t/2}^B\varphi_i(x))^2 \\ &= \sum_{i=1}^\infty e^{-\lambda_i t}\varphi_i(x)^2. \end{aligned}$$

Integrating over B,

$$\sum_{i=1}^\infty e^{-\lambda_i t} = \int_B p_B(t,x,x)\,dx < \infty \tag{2.2}$$

for all $t > 0$. Since (2.2) holds for all t, this and (2.1) imply that the convergence in (iii) and (iv) is absolute and takes place in $L^\infty(B)$ and $L^\infty(B\times B)$, respectively.

Assertion (v) is an immediate consequence of the Krein-Rutman theorem [KR]. □

The hypothesis of finite volume is sufficient, but not necessary and sufficient; see, e.g., [vdB]. On the other hand, some restriction on B is required for the conclusion of Theorem 1.1 to hold, as the case $B = \mathbf{R}^d$ shows.

Proposition 2.1. *Suppose $K \subset \mathbf{R}^d$ is a compact set.*

(i) There exist $\lambda, c_1 \in (0,\infty)$ such that $P^x(\tau_K \ge t) \le c_1 e^{-\lambda t}$ for all $x \in K$ and all $t > 0$;

(ii) There exists $c_2 \in (0,\infty)$ and a Borel set L contained in K such that $P^x(\tau_K > t) \ge c_2 e^{-\lambda t}$ for all $x \in L$ and all $t > 0$, where λ is the same constant as in (i).

(iii) Suppose there exists a ball $B = B(x_0, r)$ contained in the interior of K such that for almost every x that is not regular for K^c,

$$P^x(T_B < \tau_K) > 0.$$

Then there exists $c_3 \in (0,\infty)$ and an open set M contained in K such that $P^x(\tau_K > t) \geq c_3 e^{-\lambda t}$ for all $x \in M$ and all $t > 0$, where λ is the same constant as in (i) and (ii).

Proof of Proposition 2.1. Let us apply Theorem 1.1(iii) with the function $f = 1_K$ to get for a.e. x,

$$\begin{aligned} P_t^K 1_K(x) &= \sum_{\{i:\lambda_i=\lambda_1\}} e^{-\lambda_1 t}\langle \mathbf{1}_K, \varphi_i\rangle \varphi_i(x) + e^{-\lambda_1 t} \sum_{\{i:\lambda_i>\lambda_1\}} e^{-(\lambda_i-\lambda_1)t}\langle \mathbf{1}_K, \varphi_i\rangle \varphi_i(x) \\ &= e^{-\lambda_1 t}\Phi(x) + e^{-\lambda_1 t}\Psi(t,x). \end{aligned} \tag{2.3}$$

Let $|K|$ denote the Lebesgue measure of K. Note that

$$\langle \mathbf{1}_K, \varphi_i\rangle \leq \Big(\int (\mathbf{1}_K(x)^2 dx\Big)^{1/2}\Big(\int (\varphi_i^2(x)dx\Big)^{1/2} = |K|^{1/2}.$$

By (2.1) and (2.2) of the proof of Theorem 1.1 it follows that

$$\begin{aligned} \|\Psi(t,\cdot)\|_\infty &\leq \sum_{\{i:\lambda_i>\lambda_1\}} |K|^{1/2} c_3 e^{-(\lambda_i-\lambda_1)t} e^{\lambda_i} \\ &\leq \sum_{\{i:\lambda_1<\lambda_i<2\lambda_1\}} |K|^{1/2} c_3 e^{-(\lambda_i-\lambda_1)t} e^{\lambda_i} + \sum_{\{i:2\lambda_1\leq\lambda_i\}} |K|^{1/2} c_3 e^{-\lambda_i(t/2-1)} \\ &\to 0 \end{aligned}$$

as $t \to \infty$ by dominated convergence. Similarly,

$$\|\Phi(\cdot)\|_\infty \leq c_4 |K|^{1/2} e^{\lambda_1}.$$

From these estimates, there exists c_5 such that

$$\|P_t^K 1_K(\cdot)\|_\infty \leq c_5 e^{-\lambda_1 t}$$

for t large. Then if $x \in K$,

$$\begin{aligned} P^x(\tau_K > t) &= P_t^K 1_K(x) = P_1^K P_{t-1}^K 1_K(x) \\ &= \int p_K(1,x,y) P_{t-1}^K 1_K(y)\, dy \\ &\leq c_6 c_5 e^{-\lambda_1(t-1)} = c_7 e^{-\lambda_1 t} \end{aligned}$$

for t large, with c_7 independent of t and x. Property (i) follows easily from this with $\lambda = \lambda_1$.

We turn next to the proof of (iii). There are at most finitely many i such that $\lambda_i = \lambda_1$ because $\{\lambda_i\}$ has only ∞ as a subsequential limit point. Φ cannot be equal to 0, a.e., because that would contradict the linear independence of the φ_i. Since $P_t^K 1_K \geq 0$ for all t and $\Psi(t,x) = o(\Phi(x))$ for almost every x on the set where $\Phi \neq 0$, we must have $\Phi \geq 0$, a.e. Hence Φ must be positive on a set of positive measure.

Note $P_t^K \Phi = e^{-\lambda_1 t}\Phi$, a.e. Therefore

$$G_K \Phi = \int_0^\infty P_t^K \Phi \, dt = \lambda_1^{-1} \Phi, \qquad \text{a.e.} \tag{2.4}$$

If x is regular for K^c, then

$$\Phi(x) = \lambda_1 G_K \Phi(x) = \lambda_1 E^x \int_0^{\tau_K} \Phi(X_s)\, ds = 0, \qquad \text{a.e.}$$

We conclude that $\{x \in K : x \text{ is not regular for } K^c, \Phi(x) > 0\}$ has positive measure.

Suppose x is not regular for K^c and $P^x(T_B < \tau_K) > 0$. By the strong Markov property and the support theorem for Brownian motion, $G_K 1_B(x) > 0$. If $x \notin B$, then $G_K(x,y)$ must be positive for some $y \in B$, and hence for all $y \in B$ by Harnack's inequality. If $x \in B$, then $G_K(x,y) \geq G_B(x,y) > 0$ for all $y \in B$. So for all $y \in B$, $G_K(x,y) > 0$ for almost all x that are not regular for K^c. Therefore for $y \in B$,

$$G_K \Phi(y) = \int_K G_K(x,y)\Phi(x)\, dx > 0.$$

The function $G_K\Phi$ is continuous in the interior of K, hence in B. So if $M = B(x_0, r/2)$, the ball with the same center as B but half the radius, there exists $\delta > 0$ such that $G_K \Phi > \delta$ in M. Since $\Phi = \lambda_1 G_K \Phi$ (see (2.4)), then $\Phi > \lambda_1 \delta$ almost everywhere in M. Recall that $\|\Psi(t,\cdot)\|_\infty \to 0$. So, using (2.3), there exist $c_8 > 0$ and t_0 such that for $t \geq t_0 - 1$,

$$P_t^K 1_K(x) \geq c_8 e^{-\lambda_1 t} \qquad \text{for almost every } x \in M.$$

If $t \geq t_0$ and $x \in M$,

$$\begin{aligned} P^x(\tau_K > t) &= P_t^K 1_K(x) = P_1^K P_{t-1}^K 1_K(x) \\ &= \int_K p_K(1,x,y) P_{t-1}^K 1_K(y)\, dy \\ &\geq \int_M p_B(1,x,y) P_{t-1}^K 1_K(y)\, dy \\ &\geq c_9 c_8 e^{-\lambda_1(t-1)} = c_9 c_8 e^{\lambda_1} e^{-\lambda_1 t}, \end{aligned}$$

since $p_B(1,x,y)$ is bounded below for $x,y \in M$. Property (iii) follows for $t \geq t_0$ with the same λ as in (i), namely, $\lambda = \lambda_1$. To remove the restriction involving t_0, note that if $t < t_0$, then

$$P^x(\tau_K > t) \geq P^x(\tau_K > t_0) \geq c_{10} e^{-\lambda t_0} \geq \left(c_{10} e^{-\lambda t_0}\right) e^{-\lambda t}.$$

Finally, we show (ii). By the first paragraph of the proof of (iii), Φ is positive on a set of positive measure. Hence there exists $c_{11} > 0$ and a Borel set L contained in K such that L has positive measure and $\Phi > c_{11}$ on L. The conclusion (ii) follows from this similarly to the above. □

Proof of Theorem 1.2. Applying (2.3) with $t - n^{-1}$ and letting $n \to \infty$, we get

$$P^x(\tau_K \geq t) = P^x(\tau_K > t), \qquad \text{a.e.},$$

by (2.1), (2.2), and dominated convergence. Hence, for all x and t,

$$P^x(\tau_K = t) = 0. \tag{2.5}$$

Since the events $\{\tau_{J_n} \geq t\}$ are decreasing in n, the same is true for the probabilities $P^x(\tau_{J_n} \geq t)$. By (2.3)

$$\lambda_1(B) = -\lim_{t\to\infty} \log P^x(\tau_B > t)/t, \qquad \text{a.e.},$$

from which it follows that $\lambda(J_n)$ is an increasing function of n.

Suppose that $\lambda^* = \lim_{n\to\infty} \lambda(J_n)$ is strictly smaller than $\lambda(K)$. So there exists $\varepsilon > 0$ such that $\lambda(J_n) < \lambda(K) - \varepsilon$ for all n. We will show that this assumption leads to a contradiction.

Let φ_1^n be the eigenfunction corresponding to the first eigenvalue in J_n. Recall from the proof of Theorem 1.1 that

$$\varphi_1^n(x) \leq e^{\lambda(J_n)} \left(\int_{J_n} p(1,x,y)^2 dy \right)^{1/2}$$

and so we can find a constant c_1 such that

$$\varphi_1^n(x) \leq c_1$$

for all $x \in J_n$ and all n sufficiently large.

Since $\varphi_1^n(x) \leq c_1$ but the $L^2(J_n)$ norm of φ_1^n is equal to 1, for large n we can find $c_2, c_3 > 0$ and a set $M = M(n) \subset K$ such that the Lebesgue measure of M is greater than c_3 and $\varphi_1^n(x) > c_2$ for every $x \in M$.

Let $|K|$ denote the Lebesgue measure of K and let μ be the probability measure on K obtained by renormalizing the Lebesgue measure restricted to K. Then, for n large,

$$\int_K \varphi_1^n(x)\,\mu(dx) \geq c_2 c_3/|K| = c_4.$$

If n is sufficiently large, for all t,

$$\begin{aligned} E^\mu(\varphi_1^n(X_t);\tau_{J_n} > t) &= \int P_t^{J_n}\varphi_1^n(x)\,\mu(dx) = e^{-\lambda(J_n)t}\int \varphi_1^n(x)\,\mu(dx) \\ &\geq c_4 e^{-\lambda(J_n)t} \geq c_4 e^{-\lambda(K)t} e^{\varepsilon t}. \end{aligned}$$

On the other hand, if n is sufficiently large,

$$E^\mu(\varphi_1^n(X_t);\tau_K \geq t) \leq c_1 P^\mu(\tau_K \geq t) \leq c_1 c_5 e^{-\lambda(K)t}$$

by Proposition 2.1(i), where c_5 is the constant c_1 of that proposition.

Now take t large so that $c_4 e^{\varepsilon t} > c_1 c_5$. The set K is contained in the sets J_n, so $\{\tau_K > t\} \subset \{\tau_{J_n} > t\}$. Using (2.5),

$$\begin{aligned} 0 &< c_4 e^{-\lambda(K)t} e^{\varepsilon t} - c_1 c_5 e^{-\lambda(K)t} \\ &\leq E^\mu((\varphi_1^n(X_t);\tau_{J_n} > t) - E^\mu(\varphi_1^n(X_t);\tau_K \geq t) \\ &= E^\mu(\varphi_1^n(X_t);\tau_K < t < \tau_{J_n}) \leq c_1 P^\mu(\tau_K < t < \tau_{J_n}). \end{aligned}$$

Since the events $\{\tau_K < t < \tau_{J_n}\}$ decrease to $\emptyset$, the right hand side tends to 0 as $n \to \infty$, a contradiction. □

3. Time spent in a cone. This section is devoted to the proof of Theorem 1.3.

In this section we want to consider eigenvalue expansions for Brownian motion on $S(y,r)$ killed on exiting a subset B of $S(y,r)$. The proofs of Section 2 are easily adapted to this situation; we leave it to the reader to supply the details, and we apply the results of Section 2 without further mention.

Let $C_\delta = \{x \in \mathbf{R}^d : \operatorname{dist}(x, C^c) \geq \delta\}$ and J_δ be the closure of $S(0,1)\setminus C_\delta$. Fix some small $\delta^* > 0$ so that C_{δ^*} has a non-empty interior.

Lemma 3.1. *Let C be a closed cone in $\mathbf{R}^d$ such that C and its complement have non-empty interiors. Let J be the closure of $S(0,1)\setminus C$. Recall that $v = d/2-1$ and*

$$\xi(J) = \frac{(v^2 + 2\lambda(J))^{1/2} + v}{\lambda(J)}.$$

(i) There is an open set $M \subset S(0,1)$ and a constant $c_1 > 0$ such that for every x with $x/|x| \in M$ we have for $t \geq |x|^2$,

$$P^x(T_C > t) \geq c_1 (t/|x|^2)^{-1/\xi(J)}.$$

(ii) There exists $c_2 < \infty$ such that for all $x \neq 0$,

$$P^x(T_C > t) \leq c_2 (t/|x|^2)^{-1/\xi(J)}.$$

Proof. Lemma 3.1 follows from Proposition 2.1 in the same way as Proposition 2.3 follows from Proposition 2.2 in Meyre [M]. □

Proof of Theorem 1.3. *Step 1.* First we will show that there is $c_1 > 0$ such that for all cones $\widetilde{C}$ with $C_{\delta^*} \subset \widetilde{C} \subset C$ we have

$$P^x(|X(T_{\widetilde{C}})| > |x|/2) > c_1. \tag{3.1}$$

In other words, with probability greater than c_1 the cone is hit at a place at least half as far from the origin as the starting point. The constant c_1 may depend on C_{δ^*} and C but does not otherwise depend on $\widetilde{C}$.

Let T be the hitting time of the set $S(0,|x|/2) \cup S(0,2|x|) \cup \widetilde{C}$. The process $R_t = |X_t|$ is a submartingale and T is a stopping time so $E^x R_T \geq |x|$. Since $R_T \in [|x|/2, 2|x|]$, a.s., we must have

$$P^x(R_T > |x|/2) \geq 1/4.$$

The cone C_{δ^*} has non-empty interior so it is easy to see that for every point $y \in S(0,2|x|)$, Brownian motion starting from y will hit C_{δ^*} before hitting $S(0,|x|) \cup S(0,3|x|)$ with probability greater than $c_2 > 0$. If $R_T > |x|/2$ then either $X_T \in \widetilde{C}$ or $X_T \in S(0,2|x|)$. By applying the strong Markov property at T we conclude that

$$P^x(|X(T_{\widetilde{C}})| > |x|/2) \geq c_2/4$$

which proves (3.1).

Step 2. Suppose that $0 < a < r$. Assume that $\widetilde{C}$ is a cone with $C_{\delta^*} \subset \widetilde{C} \subset C$ and

$$P^x(T_{S(0,r)} < T_{\widetilde{C}}) \leq \rho$$

for all $x \in S(0,a)$. Let $\widehat{C} = \{y \in \widetilde{C} : |y| \geq a/2\}$. We will prove that

$$P^x(T_{S(0,r)} < T_{\widehat{C}}) \leq c_3 \rho \tag{3.2}$$

for all $x \in S(0,a)$ where c_3 may depend on C and δ^* but does not otherwise depend on $\widetilde{C}$, a or r.

Let $T_0 = 0$,

$$S_k = \inf\{t > T_k : X_t \in \widetilde{C} \cup S(0,a/2)\}, \quad k \geq 0,$$
$$T_k = \inf\{t > S_{k-1} : X_t \in S(0,a)\}, \quad k > 0.$$

Typically, S_k and T_k are finite for small k and infinite for large k in dimensions higher than 2. For the event $\{T_k < T_{\widehat{C}}\}$ to happen, the process would have to return k times to $S(0,a)$ and after each return it would have to hit $S(0,a/2)$ before hitting $\widetilde{C}$. A repeated application of the strong Markov property and (3.1) yield for $x \in S(0,a)$,

$$P^x(T_k < T_{\widehat{C}}) \leq (1-c_1)^k.$$

Hence

$$\begin{aligned} P^x(T_{S(0,r)} < T_{\widehat{C}}) &\leq \sum_{k\geq 0} E^x[1_{\{T_k < T(\widehat{C})\}} P^{X(T_k)}(T_{S(0,r)} < T(\widetilde{C} \cup S(0,a/2)))] \\ &\leq \sum_{k\geq 0} (1-c_1)^k \rho \leq c_3 \rho \end{aligned}$$

and the proof of (3.2) is complete.

Step 3. Fix some small $\alpha > 0$ and for small $s > 0$ let $a = s^{1/2-\alpha}$. Recall the truncated cone $\widehat{C}$ from the previous step and suppose that $x \in S(0,a)$. We have

$$P^x(T_{\widehat{C}} > 1/4) \leq P^x(T_{\widehat{C}} > T_{S(0,s^\alpha)}) + P^x(T_{S(0,s^\alpha)} > 1/4).$$

A standard estimate gives

$$P^x(T_{S(0,s^\alpha)} > 1/4) \leq \exp(-s^{-\alpha})$$

for small s and so

$$P^x(T_{\widehat{C}} > 1/4) \leq P^x(T_{\widehat{C}} > T_{S(0,s^\alpha)}) + \exp(-s^{-\alpha}).$$

We also have

$$P^x(T_{\widehat{C}} > T_{S(0,s^\alpha)}) \leq P^x(T_{\widehat{C}} > s^{3\alpha}) + P^x(T_{S(0,s^\alpha)} < s^{3\alpha}).$$

It is easy to see that

$$P^x(T_{S(0,s^\alpha)} < s^{3\alpha}) \leq \exp(-s^{-\alpha/2})$$

for small s, so it follows that

$$P^x(T_{\widehat{C}} > 1/4) \le P^x(T_{\widehat{C}} > s^{3\alpha}) + 2\exp(-s^{-\alpha/2}). \tag{3.3}$$

Recall that $\delta^* > 0$ is small and let $\xi^* = \xi(J_{\delta^*})$. According to Lemma 3.1 (ii) there exists c_4 such that for all $\delta \in (0, \delta^*)$, $\widetilde{C} = C_\delta$, all $t > 0$ and all $x \in S(0,a)$,

$$P^x(T_{\widetilde{C}} > t) \le P^x(T_{C_{\delta^*}} > t) \le c_4(t/a^2)^{-1/\xi^*}.$$

We obtain from (3.2),

$$P^x(T_{\widehat{C}} > t) \le c_3c_4(t/a^2)^{-1/\xi^*}$$

for all $x \in S(0,a)$. We apply this formula with $t = s^{3\alpha}$ and combine it with (3.3) to obtain

$$\begin{aligned} P^x(T_{\widehat{C}} > 1/4) &\le c_3c_4(s^{3\alpha}/a^2)^{-1/\xi^*} + 2\exp(-s^{-\alpha/2}) \\ &= c_3c_4(s^{3\alpha}/s^{1-2\alpha})^{-1/\xi^*} + 2\exp(-s^{-\alpha/2}) \\ &\le c_5 s^{(1-5\alpha)/\xi^*} \end{aligned}$$

for small s.

Now we let $\delta = s^{\alpha/2}$ (we consider only small s). Note that for small s, the distance between $\widehat{C}_\delta$ and ∂C is greater than $s^{1/2-\alpha}$. A standard estimate for Brownian motion shows that for $x \in \widehat{C}_\delta$ and small s,

$$P^x(T_{\partial C} < s^{1-\alpha}) \le \exp(-s^{-\alpha/2})$$

and so

$$P^x(A_{1/4} < s^{1-\alpha}) \le \exp(-s^{-\alpha/2}).$$

Another standard estimate gives for small a,

$$P^0(T_{S(0,a)} > 1/4) \le \exp(-a^{-1}).$$

We combine our estimates to see that for small s,

$$\begin{aligned} P^0(A_1 < s^{1-\alpha}) &\le P^0(T_{S(0,a)} > 1/4) \\ &\qquad + E^0 P^{X(T_{S(0,a)})}(T_{\widehat{C}_\delta} > 1/4) + E^0 P^{X(T(\widehat{C}_\delta))}(A_{1/4} < s^{1-\alpha}) \\ &\le \exp(-a^{-1}) + c_5 s^{(1-5\alpha)/\xi^*} + \exp(-s^{-\alpha/2}) \\ &\le c_6 s^{(1-5\alpha)/\xi^*}. \end{aligned}$$

If we substitute $u = s^{1-\alpha}$, we obtain

$$P^0(A_1 < u) \le c_6 u^{(1-5\alpha)/[(1-\alpha)\xi^*]}.$$

It follows that

$$\liminf_{u\to 0} \log P^0(A_1 < u)/\log u \ge (1-5\alpha)/[(1-\alpha)\xi^*].$$

We proved in Theorem 1.2 that $\lim_{\delta\to 0}\xi(J_\delta) = \xi(J)$. Hence, by choosing sufficiently small $\delta^* > 0$ we can assume that $\xi^* = \xi(J_{\delta^*})$ is arbitrarily close to $\xi(J)$. This and the fact that α may be chosen arbitrarily close to 0 show that

$$\liminf_{u\to 0} \log P^0(A_1 < u)/\log u \ge 1/\xi(J).$$

This proves the lower bound in Theorem 1.3.

Step 4. Next we prove the opposite inequality. Find a set M as in Lemma 3.1 (i) and let c_7 be equal to the c_1 in that same lemma. Let $M_1 = \{x : x/|x| \in M\}$ and $a = u^{1/2}$. The probability $p = P^0(A_{T_{S(0,a)}} < u,\, X(T_{S(0,a)}) \in M_1)$ does not depend on u, by scaling, and it is strictly positive.

Let C^o denote the interior of C. Recall that we have assumed that the boundary of C has zero d-dimensional Lebesgue measure. Hence,

$$\int_0^\infty 1_{(X_s\in\partial C)} ds = 0$$

and so

$$A_t = \int_0^\infty 1_{(X_s\in C^o)} ds.$$

This, the strong Markov property applied at $T_{S(0,a)}$ and Lemma 3.1 (i) imply that

$$\begin{aligned} P^0(A_1 < u) &\ge E^0[\mathbf{1}_{\{A_{T_{S(0,a)}} < u\}} \mathbf{1}_{\{X(T_{S(0,a)})\in M_1\}} P^{X(T_{S(0,a)})}(T_{C^o} > 1)] \\ &\ge pc_7(1/a^2)^{-1/\xi(J)} = pc_7(1/u)^{-1/\xi(J)}. \end{aligned}$$

It follows that

$$\limsup_{u\to 0} \log P^0(A_1 < u)/\log u \le 1/\xi(J).$$

The proof of Theorem 1.3 is complete. □

4. Eigenvalues of the Laplacian in compact and open sets. Classical spectral analysis of the Laplacian is limited to open domains. The Laplacian itself can be defined at every point of an open set using standard formulae for derivatives. Our Theorem 1.1

applies to many sets that have empty interior (see Example 4.1 below) and "Laplacian" has to be defined using, for example, Brownian transition probabilities. It is natural to ask about the relationship between eigenvalues in open and closed sets. It is easy to prove that the spectrum for a compact sets is the same as the spectrum for its interior if the common boundary of these sets is smooth. This section is devoted to a discussion of what happens for non-smooth sets.

We will be concerned only with the first eigenvalue and denote it λ.

We start with a simple example of a highly irregular set showing that the first eigenvalue for a compact set and its interior can be different.

Example 4.1. (Cheese set) Let $Q = \{(x_1, x_2) \in \mathbf{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$. Let $\{y_i\}_{i\geq 1}$ be an ordering of all points with rational coordinates in Q, except $(0,0)$. For each i, choose $r_i > 0$ so that $P^{(0,0)}(T_{B(y_i,r_i)} < T_{\partial Q}) < 2^{-(i+1)}$. Let $K = Q \setminus \bigcup_{i\geq 1} B(y_i, r_i)$. Then Brownian motion starting from $(0,0)$ does not hit K^c immediately a.s. and so the spectrum given in Theorem 1.1 is non-trivial. At the same time, the interior of K is empty.

□

For the rest of this section we will consider only compact sets K such that K is the closure of the interior D of K. It is clear from the above example that typically $\lambda(D) \neq \lambda(K)$ if we do not make this assumption.

For a set $D \in \mathbf{R}^3$ and $r \geq 0$, let $B(D,r) = \{x \in \mathbf{R}^3 : \mathrm{dist}(D,x) < r\}$ and $\overline{B}(D,r) = \{x \in \mathbf{R}^3 : \mathrm{dist}(D,x) \leq r\}$.

For a compact set K, the set of all $x \in \partial K$ such that

$$P^x(T_{K^c} = 0) = 0$$

will be denoted $\mathcal{I}(K)$. In other words, $\mathcal{I}(K)$ is the set of points which are not regular for K^c.

The following result has been inspired by discussions of eigenvalue continuity with R. Howard. We will only sketch its proof.

Proposition 4.2. *Suppose that K is a compact set and K is the closure of its interior D. Then $\lambda(D) = \lambda(K)$ if and only if $\mathcal{I}(K)$ is polar.*

Proof. Let $D_n = B(D, 1/n)$ and $K_n = \overline{B}(K, 1/n)$. The sets D_n are open and K_n are compact. It is easy to check that $D_n \subset K_n \subset D_{n-1}$. Hence $\lambda(D_n) \leq \lambda(K_n) \leq \lambda(D_{n-1})$. The monotonicity of λ follows, for example, from the first paragraph of the proof of Theorem 1.2. We see that

$$\lim_{n\to\infty} \lambda(D_n) = \lim_{n\to\infty} \lambda(K_n).$$

By Theorem 1.2, $\lambda(K) = \lim_{n\to\infty} \lambda(K_n)$. Thus it will suffice to show that

$$\lim_{n\to\infty} \lambda(D_n) = \lambda(D)$$

if and only if $\mathcal{I}(K)$ is polar.

That the polarity of $\mathcal{I}(K)$ implies $\lim_{n\to\infty} \lambda(D_n) = \lambda(D)$ was proved by Le Gall [LG]. Now we will sketch how to prove the opposite implication. Assume that $\mathcal{I}(K)$ is not polar. We modify the proof of Theorem 1 in Gesztesy and Zhao [GZ] as follows.

We will discuss only the case when the boundary of D has zero d-dimensional Lebesgue measure. This is the case covered by the second part of the proof in [GZ]. The other case can be adapted to our purposes in an analogous manner.

Suppose that $x \in D$ and consider a Brownian bridge X starting at x at time 0 and returning to x at time 1. The path properties of a Brownian bridge are the same as those of Brownian motion away from the starting point and end point. Since $\mathcal{I}(K)$ is non-polar, there is a positive probability that X will hit $\mathcal{I}(K)$. The strong Markov property applied at the hitting time of $\mathcal{I}(K)$ may be used to show that X does not enter K^c just after hitting $\mathcal{I}(K)$ with positive probability. Then we can use time reversal to see that X may hit $\mathcal{I}(K)$ before hitting K^c with positive probability. Another application of the strong Markov property shows that $\{X_t, 0 \le t \le 1\}$ may hit $\mathcal{I}(K)$ but not hit K^c with positive probability. In terms of unconditioned Brownian motion, this implies that for $x \in D$,

$$P^x(T_{\mathcal{I}(K)} < T_{K^c}) = c(x) > 0.$$

Hence

$$P^x(T_{D^c} < T_{K^c}) \ge P^x(T_{\mathcal{I}(K)} < T_{K^c}) = c(x) > 0.$$

This is the analogue of (40) in [GZ]. The same argument as in [GZ] then implies

$$P^x(t < T_{K^c}) > P^x(t < T_{D^c}) + \varepsilon(x)$$

for some $t > 0$ and $\varepsilon(x) > 0$. This is the analogue of (43) in [GZ].

Next we obtain a formula corresponding to (44) in [GZ]. Let $\varphi(x)$ be the first eigenfunction in D and let us define $\varphi(x) = 0$ for $x \notin D$. Then

$$E^x(\varphi(X_t); t < T_{D_n^c}) \ge E^x(\varphi(X_t); t < T_{K^c}) > E^x(\varphi(X_t); t < T_{D^c}) + \delta(x).$$

It is shown in [GZ] (see (32), (33) and (45)) that this implies that $\lambda(D_n) < \lambda(D) - \eta$. Here $\eta > 0$ may be chosen independently of n because $\delta(x)$ does not depend on n. This completes the argument. $\square$

Can one describe in geometric terms all sets K with $\mathcal{I}(K)$ polar? We do not have a complete characterization but two partial results should shed some light on the problem. The first result is due to R. Howard who has an analytic proof. We supply our own proof which is probabilistic and seems to illustrate well the special role of the main assumption of the proposition.

Proposition 4.3. *(R. Howard (private communication)) If the boundary of K can be represented locally as the graph of a continuous function then $\mathcal{I}(K)$ is polar.*

We would like to point out that if $K \subset \mathbf{R}^d$, $d \geq 3$, and ∂K is locally the graph of a continuous function then the boundary of K may contain infinitely many points x which are irregular for $\overline{K^c}$, i.e., such that $P^x(T_{\overline{K^c}} = 0) = 0$. A standard example of such a point is the "Lebesgue thorn." If $d = 2$ then there are no such points.

Proof. Suppose that $f : \mathbf{R}^{d-1} \to \mathbf{R}$ is a continuous function, $D = \{(x^1, \dots, x^d) \in \mathbf{R}^d : x^d > f(x^1, \dots, x^{d-1})\}$ and $K = \overline{D}$. Suppose that $x \in \partial D$ is regular for D^c. Then P^x-a.s. there exist times t_n, $n \geq 1$, such that $t_n > 0$, $t_n \to 0$ and $X(t_n) \in D^c$ for every n. Let $Y(t) = X(t) - t \cdot (0, \dots, 0, 1)$, i.e., Y is a Brownian motion with drift. We have $Y(t_n) = X(t_n) - t_n \cdot (0, \dots, 0, 1) \in K^c$ since f is a continuous function and $X(t_n) \in D^c$. We see that Brownian motion with constant drift starting from x hits K^c immediately with probability 1. The distributions of Brownian motion and Brownian motion with constant drift are mutually absolutely continuous on the finite time interval $[0, 1]$ so we conclude that Brownian motion starting from x hits K^c immediately with probability 1. We have shown that if x is regular for D^c then it is also regular for K^c.

The set of all points in ∂D which are irregular for D^c is polar (Blumenthal and Getoor [BG]) so $\mathcal{I}(K)$ is polar.

It is easy to adapt the proof to the case when the boundary can be represented only locally as the graph of a function. □

The next example goes in the opposite direction to Proposition 4.3. If the boundary of K is represented locally by a bounded function, $\mathcal{I}(K)$ need not be polar.

Example 4.4. We will construct an open set D such that

(i) $D = \{(x^1, x^2, x^3) \in \mathbf{R}^3 : |x^1| < 1, |x^2| < 1, f(x^1, x^2) < x^3 < 1\}$ for some (discontinuous) bounded function f,

(ii) the volume (i.e., the three-dimensional Lebesgue measure) of the boundary ∂D is equal to zero,

(iii) D is the interior of the closure K of D,

(iv) $\lambda(D) \neq \lambda(K)$.

We will identify $\mathbf{R}^2$ with $\{(x^1, x^2, x^3) \in R^3 : x^3 = 0\}$ and similarly for subsets of $\mathbf{R}^2$. Let

$$Q = \{(x^1, x^2, x^3) \in \mathbf{R}^3 : |x^1| < 1, |x^2| < 1, |x^3| < 1\},$$
$$M = \{(x^1, x^2, x^3) \in Q : x^1 = 0, x^3 \leq 0\},$$

and let M_1 be the orthogonal projection of M on $\mathbf{R}^2$.

Fix a base point $z = (0, 0, 1/2) \in Q$. Let

$$p = P^z(T_M < T_{\partial Q}).$$

It is clear that p is strictly positive.

Choose a sequence of distinct points $y_k = (y_k^1, y_k^2) \in \mathbf{R}^2$, $k \geq 1$, such that each point of M_1 is an accumulation point of the sequence $\{y_k\}$ but there are no accumulation points outside the closure of M_1. Let

$$F_k(r) = \{(x^1, x^2, x^3) \in Q : x^3 \leq 0, |x^1 - y_k^1| \leq r, |x^2 - y_k^2| \leq r\}.$$

For a fixed k, $P^z(T_{F_k(r)} < T_{\partial Q})$ goes to zero as $r \to 0$. We choose $r_k > 0$ so small that

$$P^z(T_{F_k(r_k)} < T_{\partial Q}) < p/2^{k+1}$$

for every $k > 0$. Moreover we choose r_k so small that the sets $F_k(r_k)$, $k \geq 1$, are disjoint.

We let $D = Q \setminus \left(M \cup \bigcup_k F_k(r_k) \right)$.

We will now verify that properties (i)-(iv) hold for D.

(i) Let G be the projection of $M \cup \bigcup_k F_k$ on $\mathbf{R}^2$. Then let f be equal to 0 on G and equal to -1 otherwise. It is easy to see that (i) is satisfied by this function f and domain D.

(ii) The boundary of D is a subset of $\partial Q \cup M \cup \bigcup_k \partial F_k$. Hence, ∂D is a subset of the countable union of sets whose three-dimensional Lebesgue measure is zero.

(iii) Every open set is a subset of the interior of its closure. Recall that every point of M_1 is a cluster point of the sequence $\{y_k\}$. Hence every point of M is a cluster point of some points in the interiors of F_k's. This implies that no point of M may belong to the interior of K. It is evident that no other point of D^c may belong to the interior of K.

(iv) Note that

$$P^z(T_M < T_{\partial Q} < T_{K^c}) \geq P^z(T_M < T_{\partial Q}) - \sum_{k=1}^{\infty} P^z(T_{F_k(r_k)} < T_{\partial Q})$$
$$\geq p - \sum_{k=1}^{\infty} p/2^{k+1} = p/2 > 0.$$

By applying the strong Markov property at T_M we conclude that there is a non-polar subset $\widetilde{M}$ of $M \setminus \partial Q$ such that for every $x \in \widetilde{M}$ we have

$$P^x(T_{K^c} > 0) \geq P^x(T_{\partial Q} < T_{K^c}) > 0.$$

By Blumenthal's 0-1 law, $P^x(T_{K^c} > 0) = 1$ for such x. Hence, $\widetilde{M} \subset \mathcal{I}(K)$, and so $\mathcal{I}(K)$ is non-polar. Proposition 4.2 now implies that $\lambda(D) \neq \lambda(K)$. $\square$

The referee for this paper suggested the following two problems.

Problem 1. If K is compact and is equal to the closure of its interior, estimate $|\lambda(D) - \lambda(K)|$ in terms of the capacity of $\mathcal{I}(K)$.

Problem 2. Is it possible to define the Dirichlet Laplacian in terms of "generalized" derivatives at most points of a compact set and extend the theory of Sobolev spaces to such sets?

REFERENCES

[B] R.F. Bass, *Probabilistic Techniques in Analysis*, Springer, New York (1995)

[BD] N.H. Bingham and R.A. Doney, On higher-dimensional analogues of the arc-sine law, J. Appl. Probab. **25**, (1988) 120–131.

[BG] R.M. Blumenthal and R.K. Getoor, *Markov Processes and Potential Theory*, Academic Press, New York (1968)

[GZ] F. Gesztesy and Z. Zhao, Domain perturbations, Brownian motion, capacities, and ground states of Dirichlet Schrödinger operators, *Math. Z.* **215**, (1994), 143–150.

[KR] M. G. Krein and M. A. Rutman, Linear operators leaving invariant a cone in a Banach space. *Amer. Math. Soc. Sel. Transl. Ser. 1* **10**, (1962) 199–325.

[LG] J.-F. Le Gall, Mouvement brownien, cônes et processus stables, *Prob. Th. Rel. Fields* **76**, (1987) 587–627.

[M] T. Meyre, Etude asymptotique du temps passé par le mouvement brownien dans un cône, *Ann. Inst. Henri Poincaré* **27**, (1991) 107–124.

[MW] T. Meyre and W. Werner, On the occupation times of cones by Brownian motion, *Probab. Theory Rel. Fields* **101**, (1995) 409–419.

[PS] S.C. Port and C.J. Stone, *Brownian Motion and Classical Potential Theory*, Academic Press, New York (1978)

[RN] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Ungar, New York (1955)

[vdB] M. van den Berg, On the spectrum of the Dirichlet Laplacian for horn-shaped regions in $\mathbf{R}^n$, *J. Functl. Anal.* **58**, (1984) 150–156.

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