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The Term Structure with Semi-credible Targeting

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ABSTRACT

The Federal Reserve sets targets for interest rates which it enforces through direct market intervention. These targets are changed periodically. In this paper, we develop a term structure model in which the short rate is subject to a control which keeps it close to a target which changes from time to time. The probability of target changes is not constant in the model, but changes as a function of observables. The model performs well at explaining the shifts in the yield curve that accompany target changes.

IN U.S. FINANCIAL MARKETS, the Federal Reserve Bank (the Fed) is a unique player. Usually we assume that market participants are price takers in financial markets. However, there is evidence that this is not true of the Fed. In fact, the Fed sets targets for short-term rates which it attempts to maintain by market intervention, injecting or withdrawing funds from the market to keep the rates near their targets. Targets are changed from time to time and market participants know when such a target change occurs. They may, in fact, have some limited ability to forecast such target changes.

Several recent papers have provided evidence that models which allow the short rate to revert toward a mean, which is itself a random process, perform better than models in which the mean is constant. Jegadeesh and Pennacchi (1996) suggest the relation between such a model and the interest rate targeting of the Fed. However their model assumes a continuously varying mean that is unobservable rather than the Fed targets, which are observable and change infrequently. Balduzzi, Das, and Foresi (1998b) also assume such a mean process and construct a proxy for this unobserved mean process using two bonds. By using this proxy, they find evidence that the mean is not constant, but do not explicitly relate it to observed Fed targets.

Past research on the effect of Fed policy on the term structure focused on the relation between money growth and short-term interest rates. By and large, the

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evidence of these studies was that the Fed had little control over interest rates (see Reichenstein (1987) for a survey). More recent papers have focused on the effect of federal funds rate targets on market interest rates. Cook and Hahn (1989) is the first such study. They find that during the 1970s, target changes caused significant movements in yields. They also document that market participants are aware of target changes by comparing reports in *The Wall Street Journal* to a record of actual target changes later published by the Fed.

As a result of this evidence, several term structure models have been advanced which take account of these targets. Rudebusch (1995), Balduzzi, Bertola, and Foresi (1997), and Roley and Sellon (1997) all rely on the expectations hypothesis to relate longer-term yields to short-term rates. While expectations of future changes in the target play a major role in each of these models, it is unclear why targets are changed. Balduzzi et al. (1998a) suggest a similar model with the extension that targets are somewhat predictable in the sense that changes follow an AR(1) process. This is consistent with the observation that the direction of target changes is highly persistent.

In addition to these papers, there have been several recent papers which examine the effect of central bank rate setting policies on bond prices in countries besides the United States. Babbs and Webber (1994) examine a model of interest rates in the United Kingdom in which the short rate is taken to be equal to the "Band 1 rate" (the rate analogous to the federal funds target rate). The short rate is modeled as a pure jump process with time-varying jump intensities. Bonds and other interest rate contingent claims can be priced using this model. The obvious shortcoming is that the true short rate does not always equal the Band 1 rate set by the Bank of England. Babbs and Webber (1997) present several models for various countries which focus on the rates at which banks can borrow from the central bank and which serve as a bound on the short rate.[1] They point out that in the United States, borrowing from the Fed is discouraged and that some change to their model would be necessary to capture the effect of the federal funds target rate. Piazzesi (2001), in a recent working paper, improves upon these models by modeling the probability of target changes as varying with several macroeconomic factors which determine the "desired rate" of the Fed.[2]

The purpose of this paper is to cast doubt on models which assume that the short rate mean reverts toward the target. We document that such models perform poorly empirically in explaining the nature of yield curve shifts associated with target changes. In addition, mean reversion lacks any theoretical justification as a model of how the short rate would behave in response to enforcement actions by a monetary authority. We propose instead a model in which short rates are subject to a controller such as the Fed who wants to keep the rate near a target. We show that this model does a much better job explaining yield curves

[1] Honoré (1998) has developed econometric tools for dealing with such models and analyzes the case of the German terms structure.

[2] One peculiar feature of this model is that when the Fed changes a target, it does not change it to the desired rate. So it is unclear what is really meant by the desired rate in this model or what keeps the Fed from achieving its desire.

shifts around target changes. This model also has a mathematical feature which is unique in the literature. In the simplest version of the model where there are three factors (the short rate, upward target changes, and downward target changes), claims can be hedged using only two bonds. This is due to a new type of target process which we introduce and which we designate a "semi-credible" target.

I. Targeting by the Fed

Banks and other financial institutions which accept deposits against which payments can be made are required to keep a certain proportion of these deposits in reserve in the form of vault cash or deposits at Federal Reserve Banks. Reserve requirements must be met over a statement period ending every other Wednesday (called settlement Wednesday). The required reserve is based on averages of deposits over another two-week period called the reserve computation period which ends previous to settlement Wednesday.[3] Banks who need extra reserves may borrow from other banks or directly from the Federal Reserve at the "Discount Window." The total supply of reserves thus has two components: (1) nonborrowed reserves, meaning reserves provided by banks in the system, and (2) borrowed reserves, reserves supplied by the Fed to banks via the discount window. Excess reserves are traded between banks in the Federal funds (fed funds) market, the Eurodollar market (for banks with offshore branches), or the repo market (for banks with unpledged securities which can be used as collateral).

While it is clear that the Fed sets targets for the fed funds rate, the reason for such targeting is less obvious. It would be desirable to have a model of the Fed which delivers fed funds rate targeting as an optimal policy. Any such model would have to specify the objective function of the Fed as well as the constraints (political and otherwise) under which the Fed operates. Such a model is beyond the scope of the current paper.[4] Conventional wisdom suggests that the Fed's main concern is controlling inflation. Further, it appears that Fed policy makers look to changes in long-term rates to signal changes in expectations of inflation. Given these observations, we can say a few things about what properties any optimal policy would have to have without fully specifying the problem which the optimal policy satisfies.

To be successful at affecting longer term rates, a targeting policy for the short rate must be informative about the future paths of the short rate. Hence a targeting policy must satisfy at least two conditions. First, there must be a certain level of commitment to a stated target.[5] To see why, suppose that the targets were subject to frequent changes. Then the current target would not tell very much

[3] From 1968 to 1982 the reserve computation period ended two weeks before settlement Wednesday. This was called lagged reserve accounting. In 1982 this was changed so that the computation period ended two days before settlement Wednesday. This system is referred to as quasi-contemporaneous reserve accounting or just contemporaneous reserve accounting.

[4] Dybvig (1994) contains a brief but thoughtful discussion of what such a model would involve.

[5] Actually a targeting policy need not correspond to a constant target level. One could imagine target policies which are deterministic functions of time. The Fed does not appear to

about the future paths of the short rate process and so would have only a minimal effect on longer term yields.[6] Secondly there must be some enforcement mechanism which keeps short-term rates near the target. If this were not the case, then targets would have little relation to either current or future short rates and so long-term rates would be unaffected by changes in the target. Interestingly these two functions, target setting and target enforcement, are handled by two separate bodies. The Federal Open Market Committee (FOMC) sets targets and delegates the enforcing of the target to the Trading Desk of the Federal Reserve Bank of New York (the Desk). Although we may not know the objective function for the FOMC, we may say with a fair degree of certainty that the objective function of the Desk is to minimize the "distance" between the fed funds rate and the current target.

The Desk enforces the target by trading, using the Fed's portfolio. Generally the Desk trades only in Treasury securities with very short maturities (often less than one week) or Repo's on these securities. To plan the trading which may be necessary to enforce the current target, the Desk estimates the demand and supply for reserves. Each day new information is gathered from financial markets and is used to update not only the estimates for demand and supply for the current day but the forecasts of demand and supply for coming days and weeks. The Desk calculates what quantity of reserves would need to be supplied in the market to make the market clear at the desired rate. If the supply of reserves predicted to be available for lending in the market is low, then the Desk will undertake open market purchases of Treasury securities. Proceeds from these purchases will end up on the balance sheets of banks, thus increasing the supply of reserves. If the shortfall in supply is predicted to be temporary, then the open market purchases will be in the form of overnight repurchase agreements (Repo's) rather than an outright purchase. Open market sales of securities drain reserves and tend to increase the fed funds rate. Open market operations take place each trading day unless the Desk predicts that the market will clear that day at a rate near the target rate without intervention.

During the 1970's the Fed followed a policy of targeting the overnight fed funds rate. The practice was discontinued after September 1979. There are two data sets of target changes for this early period. The most complete is the one used in Rudebusch (1995), which begins with the target change of September 13, 1974, and ends with the target change of September 19, 1979. During this period, there were 99 target changes, of which 56 were upward and 43 were downward. Rudebusch obtained this data series from notes of the FOMC. The other data set was collected by Cook and Hahn (1989) based on *Wall Street Journal* reports of what traders thought the Fed was doing. This second data set has 79 target changes, of which 50 were upward and 29 were downward. The reason that the two data series are not the same is that during this period, the Fed did not announce target

operate this way, and so for the remainder of the paper, we will only consider targeting policies which are constant targets between policy changes.

[6] We are assuming here that the target process has some long-run mean from which the Fed is loath to stray far. If the target process were a martingale, then, of course, the current target would be the best forecast of any future target and so would have a large impact on the yield curve even if it were changed frequently.

changes at the time they occurred.[7] When examining the effects of target changes on Treasury securities, we use the Cook and Hahn data for this period because we are interested in market reactions to target changes and the Cook and Hahn data contains only those changes that market participants knew about.[8] During this period, target changes were generally 12.5 basis points in size, although some other sizes did occur (12.5 basis points was the median target change during this period).

The more intervention is done, the more reserve levels (and money growth) will fluctuate. In fact, it is not possible to simultaneously control interest rates and reserves (at least not perfectly). In October 1979, the Fed, under new chairman Paul Volcker, changed to a policy of targeting nonborrowed reserves[9] in an effort to control money growth. Not surprisingly, the volatility of interest rates increased greatly during this time. At its October 1982 meeting, the Federal Open Market Committee decided to abandon these targets.

After this period, the Fed changed its official policy to one of targeting borrowed reserves. However, there is disagreement as to whether this was really the policy the Fed was following. Transcripts of meetings make it clear that the Fed was setting fed funds targets as early as 1983 (see Thornton (2000)). At the very least, there was (as Meulendyke puts it) an "informal move away from borrowed reserve targets" during the decade of the 1980s and the Fed became more and more explicit about their fed funds targets. Meulendyke (1998) describes this process as being speeded by the stock market crash of 1987. Since December 1984, we have data on 71 target changes up to the target change of November 16, 1999. Following Rudebusch (1995), we have elected to use only those following the target change of November 4, 1987. This leaves 52 target changes, of which 20 were upward and 32 were downward. During this later time period, target changes were usually 25 basis points in size with a few exceptions (25 basis points was the median target change).

The fed funds data used in this paper are the rates on overnight fed funds. All data on fed funds rate and yields were obtained from the web site of the Federal Reserve Bank of St. Louis. The fed funds rate is collected at the end of each day and is the average from trades made during the day through five brokers who report to the Fed. The fed funds rate and targets are plotted for each of these periods in Figures 1 and 2. There are a few striking details apparent in these figures. The first is that the fed funds rate stays very close to the target during both periods. The second is that there are some significant deviations from the target, but these are remarkably short-lived (they show up as one-day "spikes" in the data). Most of these correspond to calendar events such as the end of the reserve maintenance period or the end of the calendar year.[10]

[7] See Thornton (2000) for a detailed discussion of the differences between these series.

[8] The results we present here are actually not much affected by which data set we use.

[9] These historical details are found in Meulendyke (1998, Chap. 2).

[10] These spikes are more pronounced in the later period because of the move from lagged to contemporaneous reserve accounting mentioned in Footnote 3.

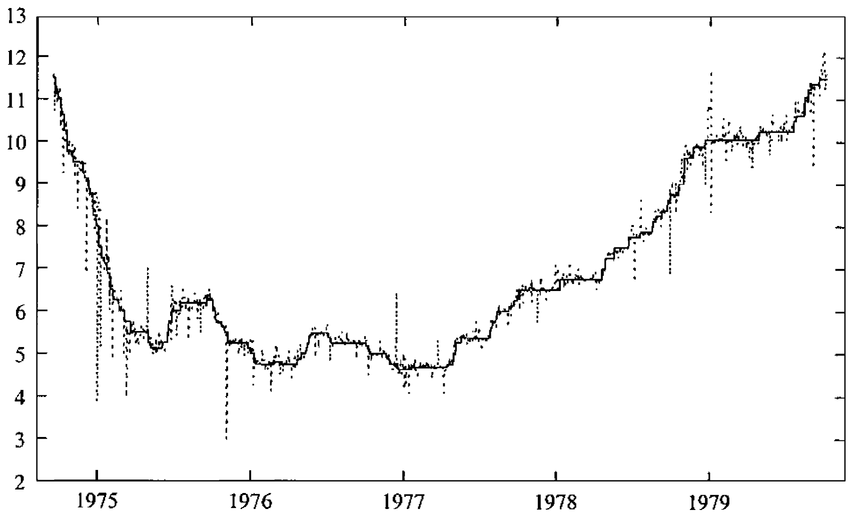

Figure 1. Funds rate and targets in the 1970s. This figure plots the overnight fed funds rate and its target for the period September 1974 to September 1979.

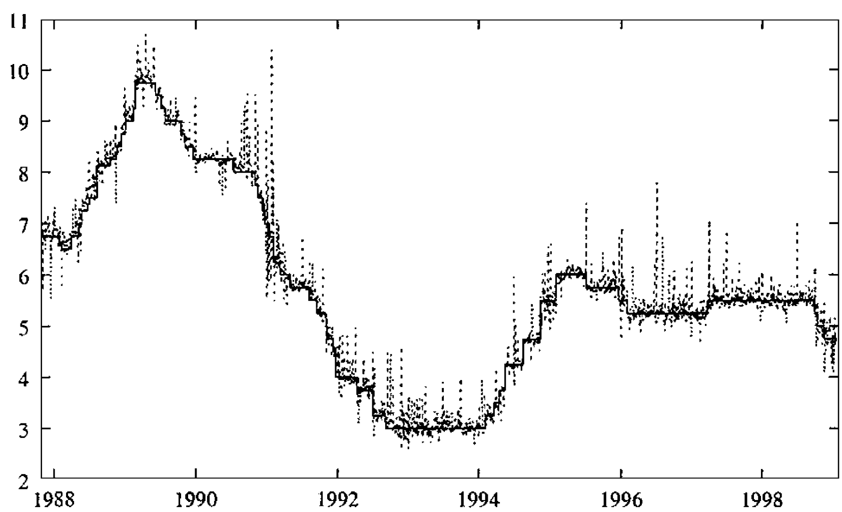

Figure 2. Funds rate and targets since 1987. This figure plots the overnight fed funds rate and its target for the period November 1987 to December 1998.

Before moving on to modeling, it is important to recognize the limitations inherent in using fed funds targets as the basis for a term-structure model. In term-structure modeling, one typically begins with a model for the short rate. As mentioned previously, the fed funds rate contains high-frequency noise which makes it unsuitable as a short rate proxy. Even if this noise could be filtered out, we would not have an adequate short rate proxy because the fed funds rate is not a true riskless interest rate. The parties who trade in the fed funds market are banks, and so some default risk exists. The fed funds rate is higher than the true riskless rate would be for this reason. In fact, general collateral overnight repo rates (which are quoted from the fed funds rate) are typically less than the fed funds rate.[11] The target must therefore reflect this risk premium as well. Our solution to this issue will be to focus on changes in targets and yields rather than levels. We assume that corresponding to the unobservable short rate there is a target which is related to the observable target in that jumps happen at the same time and are of the same magnitude.

The first study of the effect of target changes on the yield curve was Cook and Hahn (1989). Using data from the 1974 to 1979 period, they regressed daily changes in yields of bonds of various maturities on changes in the target. In Table I, we have performed similar regressions for both periods. In doing so, we have dropped three observations because they correspond with some of the large spikes in the data. Our criterion for identifying outliers is that a one-day change in the fed funds rate of more than 100 basis points which immediately reverses itself will be called an outlier. Accordingly, we dropped the target changes of January 2, 1975, February 1, 1991, and July 6, 1995. The smallest one-day move for the fed funds rate was 135 basis points on the last of the four dates. The largest move was on the first date and the fed funds rate moved by over 450 basis points that day.[12] Since target changes are separated by several weeks to several months, we are not surprised to find that Durbin–Watson tests show no serial correlation in these regressions.

There are several things to notice about Table I. First, note that the slopes are all positive and significantly different from zero except the very long maturities in the later period. The second thing to notice is that all the slopes are much less than unity. This means that while the yields on bonds of all maturities respond to a target change, the magnitude of the change is less than the amount of the target change. Finally, there is a distinct pattern to the magnitude of these responses. To illustrate this more clearly, the slopes from Table I are plotted in Figure 3. Notice that the response to a target change is greatest for maturities of about three months and much less for the very short and very long maturities. Also from Figure 3, note that the responses are much less for all maturities in the later time period, when the size of target changes was greater. These observations will guide us in developing a term-structure model which incorporates targets.

II. Exponential Affine Models and Targets

One of the first attempts at a term structure model which incorporates targets is Balduzzi et al. (1997). This model is perhaps the simplest imaginable. Because

[11] The natural question this raises is why not use the repo rate rather than the fed funds rate. One reason is that the fed funds rate data is much more readily available. Also, the repo rate exhibits the same high-frequency noise that the fed funds rate does, although to a somewhat lesser extent.

[12] The fed funds rate had dropped from 8.75 percent to 3.87 percent on the last day of 1974. On January 2 of 1975, it went back to 8.55 percent.

Table I

Response of the yield curve to target changes.

Estimates from a regression of daily yield changes on daily target changes. Only days with target changes are included in the regression. White standard errors are in parentheses. Data for the two year bond are not available for the earlier period

	1974–1979		post 1987	
Maturity	Intercept	Target Change	Intercept	Target Change
Fed funds	– 0.0085	0.3685	– 0.0284	0.0866
	(0.0206)	(0.0870)	(0.0419)	(0.1261)
3-month	0.0145	0.5381	– 0.0233	0.2489
	(0.0144)	(0.0677)	(0.0102)	(0.0371)
6-month	0.0123	0.5281	– 0.0276	0.2174
	(0.0109)	(0.0528)	(0.0109)	(0.0377)
1-year	0.0158	0.4710	– 0.0277	0.1861
	(0.0108)	(0.0492)	(0.0116)	(0.0421)
2-year			– 0.0276	0.1550
			(0.0112)	(0.0404)
3-year	0.0120	0.2847	– 0.0280	0.1172
	(0.0091)	(0.0457)	(0.0114)	(0.0389)
5-year	0.0049	0.1927	– 0.0246	0.0910
	(0.0089)	(0.0437)	(0.0116)	(0.0412)
7-year	0.0025	0.1737	– 0.0258	0.0643
	(0.0075)	(0.0387)	(0.0111)	(0.0409)
10-year	0.0057	0.1192	– 0.0205	0.0499
	(0.0063)	(0.0321)	(0.0103)	(0.0379)

the fed funds rate stays close to the target, the spread (fed funds rate minus target) is modeled as an AR(1) process with a long-run mean of zero, while the target is a pure jump process with fixed jump sizes. In that paper, the modeling is done in discrete time. The continuous-time version of this model is

$$\begin{aligned} s_t &= r_t - J_t \\ ds_t &= -\kappa s_t dt + \sigma dW_t \\ dJ_t &= b(dN_t^u - dN_t^d), \end{aligned} \tag{1}$$

where r_t is the short rate and J_t is the target. The process s_t is the spread between short rate and the target. The target process is driven by two jump processes, N^u and N^d, which represent upward and downward jumps, respectively. These are assumed to be independent Poisson processes and b is the jump size.

An appealing thing about this model is that it falls into the exponential-affine class of Duffie and Kan (1996). Most well-known continuous-time term-structure models are in this class. The appeal of models in this class lies in the fact that they all possess zero coupon bond pricing solutions of the form

$$P(x, y, \tau) = \exp(A(\tau) + B_1(\tau)x + B_2(\tau)y) \tag{2}$$

(for a two-factor version), where τ is the time to maturity of the bond and x and y are state variables. The functions A, B_1, and B_2 are functions of τ only and satisfy a particular ODE.

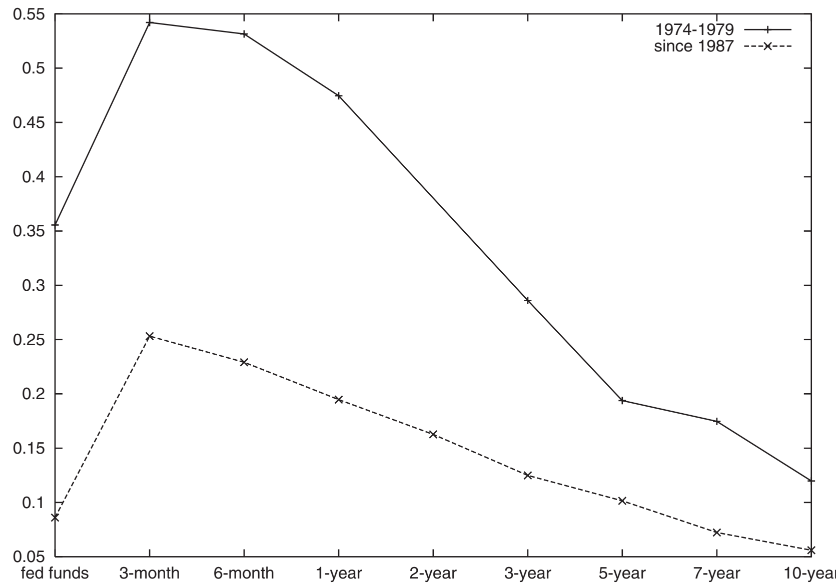

Figure 3. Estimated slopes from Table I. This figure plots the slope coefficients from Table I against the maturity of the asset in question. This gives us some idea of the response function of the yield curve as a function of maturity. The two curves correspond to the two time periods under consideration.

If we identify y with the target J, then the response to a change in the target as shown in Figure 3 is $-B_2(\tau)/\tau$. For the model above, it turns out that $B_2(\tau) = -\tau$, so the response to a target change is unity for all maturities. It seems quite clear from Table I and Figure 3 that this model is not consistent with the data.

Despite this obvious shortcoming, we need not abandon the exponential affine class without first seeing if there might not be a model in this class which has the right properties. Rather than assuming that the spread is a continuous process, which mean reverts toward zero, consider the following specification for the short rate:

$$dr_t = \kappa(J_{t-} - r_t)dt + \sigma dW_t, \tag{3}$$

with J_t as above. In this model, the short rate reverts toward the target as in the previous model, but the short rate does not jump in response to target changes, so the response at the short end is zero. However, this model gives too large a response at the long end of the curve. For this model, we have that

$$-B_2(\tau)/\tau = 1 + \frac{\exp(-\kappa\tau) - 1}{\kappa\tau}, \tag{4}$$

which is monotone and increases from zero at $\tau = 0$ to unity as τ gets large.[13]

[13] We estimated this model to verify our intuition that this model was inappropriate and found that the model is soundly rejected in both the early and late subperiods.

In order to have a response at the long end of the curve which is less than unity we need a stationary model. This can be accomplished by altering the intensities of the target process to keep the target in a certain range. For instance, investors probably do not believe that the Fed will ever set the target rate to a negative number. Hence, the probability of this event should be zero. Similarly, investors may believe that there is an upper limit above which the Fed will not raise rates. The highest the target has ever been is 11.5%, so we could model the upward jumps such that the probability of going higher than this level is zero.

This can be accomplished by letting the intensities of N^u and N^d depend on the current level of the target. For instance if we let the intensity of downward jumps be $\lambda^d J_{t-}$ (where λ^d is some constant), then downward jumps get less and less likely the lower the target gets. At a target level of zero, the probability that the next target change is downward is zero. Similarly, the intensity of upward jumps can be $\lambda^u(.12 - J_{t-})$. Fortunately, this structure keeps us within the exponential affine class. However, in this case, we cannot derive the response function, $-B_2(\tau)/\tau$ in closed form. However, we do know that $B_2(\tau)$ solves the following ODE:

$$B_2'(\tau) = \kappa B_1(\tau) - \lambda^u \exp(B_2(\tau)b) + \lambda^d \exp(-B_2(\tau)b), \tag{5}$$

where b is the size of the target change and $B_1(\tau)$ is the function

$$B_1(\tau) = \frac{-1}{\kappa} + \frac{\exp(-\kappa\tau)}{\kappa}. \tag{6}$$

The initial condition for this ODE is $B_2(0) = 0$. We can solve this numerically for each guess of the parameters in an estimation procedure. For simplicity, we assume that $\lambda^u = \lambda^d = \lambda$. The response function for this model is not monotone. Instead, the response function rises with maturity to a single peak and then declines for larger maturities. The larger the value of κ, the higher the peak and the faster the function rises to the peak. The larger is the value of λ, the lower is the peak and the steeper is the decline for larger maturities.

The estimation method we employ is related to the regression approach of Cook and Hahn (1989), but constrains the responses to agree with the model predicted response. Define $\eta_n \equiv \Delta y_n + B_2(\tau_n)/\tau \Delta J$ where Δy_n is the daily change on the bond which matures in τ_n periods. We use a GMM system with the following moments:

$$E(\eta_n) = 0 \tag{7}$$

$$E(\eta_n \Delta J) = 0.$$

The lack of serial correlation allows us to use Hansen's weighting matrix in the GMM estimation. We perform a separate estimation for each time period. The only difference in the two estimations is the different size of target change (b) used by the FOMC in each period. In the 1974 through 1979 period, b is 0.00125, and in the post 1987 period, it is 0.0025. For the 1974 through 1979 period, we estimate $\kappa = 2.5417$ and $\lambda = 703.37$. This model is rejected during this period with

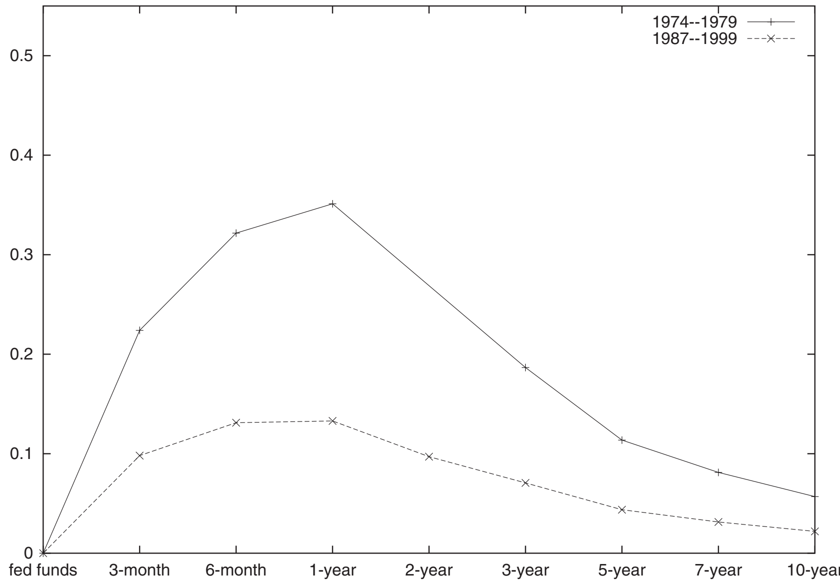

Figure 4. Response functions of the affine model. This figure plots the response function of the affine model as a function of maturity. The parameter values used are the estimated values.

a J-statistic of 52.389 ($p = 2.4 \times 10^{-6}$). In the post-1987 period, the model performs similarly. We estimate $\kappa = 1.2375$ and $\lambda = 911.54$. The model is rejected in this period with a J-statistic of 36.480 ($p = 0.0024$). The response function of this model is plotted in Figure 4 for both the 1974 through 1979 period and the post-1987 period for comparison with Figure 3.

Even if this model had not been rejected by the data, we would still have serious doubts about this model based on the estimates. The speed of mean reversion toward the target is so low that large and persistent deviations from the target are very likely. This is not what Figures 1 and 2 would lead us to expect. The reason this model suggests lax target enforcement is that this model implies that the timing of target changes is not predictable by market participants. A target change which is a complete surprise can only have a small effect on yields if the target is not strictly enforced and so has little impact on future rates. To get better results than this model, we need to examine both of these issues: target enforcement and predictability of target changes.[14]

Even a casual observer in financial markets knows that target changes do not come as a complete surprise to investors. In the affine model, the probability of a target change in the next instant depends only on the current level of the target,

[14] We should mention at this point that we have not exhausted the set of two-factor affine models. In particular, we could have let the short rate have a square root volatility like the CIR model. This change does not change the qualitative performance of the model.

Table II
Effect of Rate Relative to Target on Subsequent Target Change

Below is a contingency table for target change direction and the position of the rate relative to its target on the day before the target change.

	Rate Above Target	Rate Below Target
Target Increase	52	35
Target Decrease	13	72

and so is constant for weeks or months at a time. A more realistic model would have this probability move through time.

In predicting Fed actions, a number of economic factors could be considered such as inflation reports, employment numbers, and consumer confidence. It may make sense to include these variables explicitly in a model.[15] We are reluctant to pursue this direction for several reasons. The first is parsimony. We want to try to explain the data with as simple a model as possible. The other reason is that explicitly adding economic variables does not really solve our problem. We would need to ask ourselves how much of the inflation announcement (for instance) was already anticipated by the market. Lastly, it may not be necessary to explicitly include these factors since market expectations of them are already present in the data we have. Instead we shall ask how we can use the interest rate data we have to predict target changes.

One interesting question would be whether investors can use the position of the fed funds rate relative to its target to forecast target changes. To investigate this, we performed the following analysis. We divided all the target changes in our data[16] into upward and downward target changes. Then we classified days previous to a target change according to whether the rate was above or below the target. The following contingency table contains the results. Table II uses data from both the 1974 through 1979 period as well as the post-1987 period. Examining these periods separately gives essentially the same result, so we do not present those numbers.

The models presented above all would predict no relation between these variables, but the above table shows that there is a strong and positive relation (the test for independence gives a chi-square value of 36.2). Upward target changes seem to follow times when the fed funds rate is above its target and the converse for downward changes.

It would be tempting to interpret this result as showing that the Fed reacts to market pressure, that is, the Fed will move the target in the direction the fed funds rate wants to go. There are some who believe that the Fed is more of a market follower than a market leader. Unfortunately, we cannot make any such conclusion on the basis of this evidence because other interpretations are possible.

[15] Some progress has been made in this direction in Piazzesi (2001).

[16] We include all the target changes in the early sample as reported by Rudebusch (1995) and not just those that were reported in *The Wall Street Journal*.

What does seem clear is that the position of the fed funds rate relative to its target captures some of the predictability of target changes. This might suggest that in a term-structure model, we would want the probability of upward target changes to be high when the short rate is high relative to the target and the reverse for downward target changes.

Unfortunately incorporating this into a term-structure model takes us outside the exponential affine class. To remain in this class, the intensities must be affine functions of the state variables. We would want the intensities to depend on the spread between the short rate and the target which cannot be accommodated in an affine function. The reason is that the spread can be both positive and negative and intensities must be positive to be well defined.

III. Target Zones and Semi-credible Targets

Stepping outside the exponential affine class does two things. First, it allows us to incorporate relationships which are economically or empirically motivated but not affine. Second, it increases the computational burden considerably, since there are not closed-form solutions available.

We noted in a previous section that the Desk probably minimizes an objective function which is something like the distance of the fed funds rate from the current target. However we note that the targeted interest rate is not always equal to the target, although it stays close to it, as shown in Figures 1 and 2. This would indicate that the costs of keeping the rate exactly equal to the target are prohibitive, and that the Fed instead allows the rate to vary but not drift too far from the target. In this regard, the findings of Cook and Hahn (1989) are interesting.

Cook and Hahn (1989) used reports from *The Wall Street Journal* to identify target changes. While these changes were not announced, they were quickly inferred by market participants from the Fed's actions as shown in the following section from an article quoted on page 347 of Cook and Hahn (italics added):

> "Friday's maneuver, dealers said, indicated the Fed may have lowered its *target range* on federal funds to the 11% to $11\frac{1}{2}$% vicinity … Over the past three weeks or so the Fed has used a rate of about 12% as a *trigger* to inject reserves and about an $11\frac{1}{2}$% rate as a trigger to absorb funds."

In fact most of the data Cook and Hahn were able to collect was about "ranges" of the fed funds rather than actual targets. This may be because, during the time period of their study, targets were not announced and market participants could only infer them imperfectly. A more recent study was done by Rudebusch (1995), who collected target data up through 1992. The primary source for target data was the weekly "Report of Open Market Operations and Money Market Conditions" from the Trading Desk of the Federal Reserve Bank of New York. In collecting this data he reports on page 252 (italics added):

> "In addition, a target *range* of about a quarter of a percentage point in size was sometimes specified rather than a precise level …"

indicating that the Fed itself may not care so much about a target level as about the neighborhood in which the fed funds rate will fluctuate.[17] This highlights an aspect of targeting not taken into account in existing models, namely that there is a range within which the targeted rate is allowed to fluctuate without triggering any response by the Fed. However, there are trigger levels which, when reached, evoke a reaction which will push the rate back inside this range.

Admittedly, the above evidence is only suggestive. However, there are good theoretical reasons to believe that the control policy of the Desk would take such a form. Stochastic control policies which involve intervention only when the object to be controlled has reached the edge of a "target band" have been studied by Harrison and Taksar (1983) and others. The control problem is one of a controller who monitors the level of some variable Z which in the absence of any control would evolve as the solution to $dX = \mu(X)dt + \sigma(X)dW$. The controller may affect the level of the process by *pushing* up or down. We define a control policy as a pair of nonnegative processes L_t^u and L_t^d, which we interpret as the cumulative amounts of pushing in each direction by the controller. Then we have

$$Z_t = X_t + L_t^d - L_t^u, \tag{8}$$

and we say that such a process is subject to a two-sided regulator. If the cost function is convex around the target level and the costs of intervention are proportional to the amount of "pushing," then the optimal policy for the controller is to push when the process reaches the edge of some (endogenously determined) band around the target level. The proper amount of control is just enough to keep the process inside this band. The controlled process will exhibit reflection at some points B^ℓ and B^u which are the lower and upper edges of the band, respectively. The cumulative "pushes," L_t^u and L_t^d, are the *local times* of the process at B^u and B^ℓ.[18]

The objective function of the Desk thus seems like that described above. There is some question as to what the costs of intervention might be. Normal transaction costs are of course a part of these. In addition, the desk may want to avoid destabilizing the money supply by whipsawing the market. So it intervenes only as much and as often as is necessary.

Such target bands have been studied in the foreign exchange target zone literature beginning with the seminal work of Krugman (1991). In that paper, there were no target changes. Other work which built on the Krugman paper incorporated target changes (called *realignments* in the exchange rate literature) of various kinds. In the model of Svensson (1991), realignments may come at any time and are unrelated to the position of the rate within the target band. In contrast, the model of Bertola and Caballero (1992) allows realignments to occur only when the rate is at the edge of the band. In this case, the rate process would not jump (unless the size of the realignment were greater than the width of the band).

[17] Subsequent researchers have adopted the convention of using the midpoint of this range as a target.

[18] For more information on reflected processes and local time we refer the reader to Bass (1997).

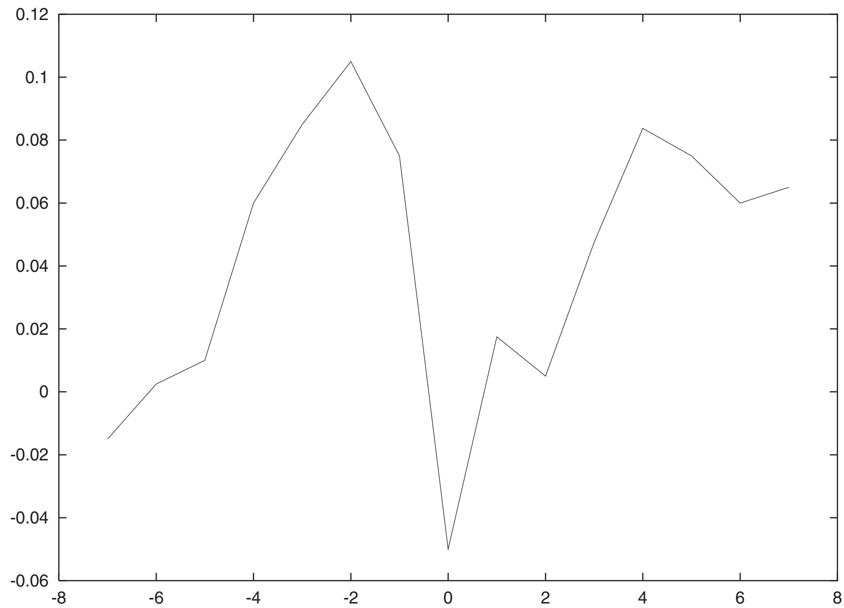

Figure 5. Fed funds rate minus target around an upward realignment. This figure plots the median of the fed funds rate minus its target against the days relative to a target change for every upward target change in the sample.

To illustrate let us construct a "typical" path for the fed funds rate relative to the target. Figure 5 plots the median of federal funds rate minus target for a window around upward target revisions. Note that at the beginning of the window, the rate tends to be near the target but moves further above the target until the realignment occurs. Since the target jumps up (over the current rate), we see that the realignment is followed by a period in which the rate is below the target. Figure 6 shows the same for downward realignments.

Given the evidence of Table II, we would prefer the Bertola and Caballero (1992) type realignment. However, there are technical problems with the Bertola and Cabellero model which make it unsuitable. In that model, the rate is taken to be a diffusion process which reflects inside a band. Each time the rate hits the edge of the band, a decision is made by the bank to either defend or realign. A realignment happens with probability p. Subsequent decisions are independent of past decisions. However, in such a continuous-time model, if the rate hits once, it hits infinitely many times in any Δt increment of time. Hence, the bank will never defend the target. Below, we derive a model with the behavior that Bertola and Cabellero intended but without this technical problem.

We wish the target to move upward (downward) when the short rate is at the top (bottom) of the band. We also know that it is precisely at these times that the controller is intervening. Therefore a natural way to accomplish this is to say that the Fed will change the target when the cumulative intervention by the controller has reached a certain threshold. Even if market participants know that this is the

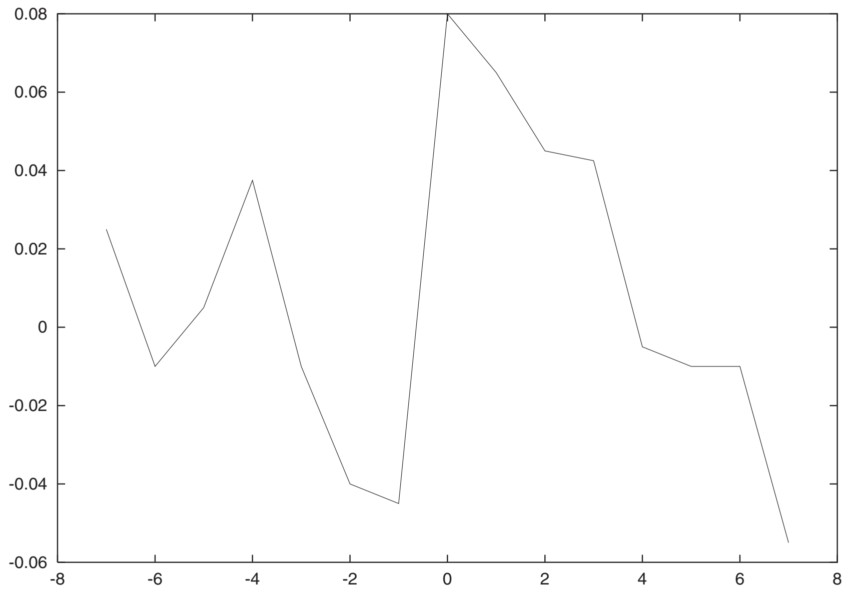

Figure 6. Fed funds rate minus target around a downward realignment. This figure plots the median of the fed funds rate minus its target against the days relative to a target change for every downward target change in the sample.

behavior to expect, they may not be able to observe the threshold. From the perspective of market participants, the threshold is a random variable. They have probabilistic beliefs about what the threshold may be.

We write (B_0^ℓ, B_0^u) for the initial (B^ℓ, B^u) and let T_1 be the first time the monetary authority decides to change the target. At time T_1 we shift the band (B_0^ℓ, B_0^u) upward or downward an amount $b > 0$ to (B_1^ℓ, B_1^u), depending on whether $r_{T_1} = B_0^\ell$ or B_0^u. The next realignment will result in setting the band to (B_2^ℓ, B_2^u) and so forth.

One way of modeling the change in the bands is to write $X_t = r_t - B_i^\ell$, $T_i \leq t < T_{i+1}$, and

$$dX_t = \mu(r_t)\,dt + \sigma(r_t)\,dW_t + dL_t^\ell - dL_t^u + dJ_t^u - dJ_t^\ell, \tag{9}$$

where now J_t^u and J_t^ℓ are both pure jump processes that both have jumps of size $+b$. The process J_t^u jumps only when r_t is at B^u and J_t^ℓ jumps only when r_t is at B^ℓ. By restricting the band to have width greater than or equal to b we guarantee that the process r_t so defined is continuous. If we assume that the random threshold is distributed exponentially with parameter λ_t^u, then J_t^u is a Poisson process when time is measured by L_t^u, and similarly for J_t^ℓ. To be more precise, J_t^u is a pure jump process with jumps of size b, and the compensator of J_t^u is $b\lambda_t^u L_t^u$, that is, $M_t^u = J_t^u - b\lambda_t^u L_t^u$ is a martingale, where λ^u is some fixed parameter. Similarly, $M_t^\ell = J_t^\ell + b\lambda_t^\ell L_t^\ell$ is a martingale. Both λ_t^u and λ_t^d can be functions of the current band location.

Note that although the rate is a continuous process, it is not a diffusion since it is not Markovian. Instead, if we let $J_t = J_t^u - J_t^\ell$, the vector process (r_t, J_t) is Markovian as is (X_t, J_t). When a realignment happens, there is no jump in the rate r (although X jumps, of course) but the stochastic behavior of the rate changes because the set of possible future paths of r changes.

The fact that the joint process (X_t, J_t) is Markovian may, at first, be surprising. One might expect that a knowledge of the past history of interventions (i.e., L_t^u and L_t^ℓ) might be informative as to the probability that the central bank will realign in the future. We have eliminated this by assuming that the realignment threshold is exponentially distributed, which implies a lack of memory. So when the rate is on the boundary and the Desk is intervening, there is a constant probability of a realignment in the next instant. This is another way to characterize the fact that the realignment process is Poisson when indexed by local time. Besides making the model more tractable, this assumption has the following economic interpretation. If the history of intervention was informative about future target changes, then that would mean that the FOMC reacts to market pressure: being more willing to change a target if the Desk is having to do a lot of intervention to defend the current target. Conventional wisdom suggests that this is not the case. This, together with the added tractability the memoryless property affords, leads us to chose an exponentially distributed threshold.

This provides us with a model for the stochastic behavior of the state variable. In the Appendix, we establish that the market is, in fact, complete under such a process. Since the market is complete, the absence of arbitrage implies that there is an (unique) equivalent martingale measure under which the deflated claim price must be a martingale. The drift of the process may be different under this equivalent martingale measure. However, now there is another difference. The jump intensities will also be different under the risk-neutral measure, although the jump sizes will, of course, be the same.

For the remainder of the section, we take the process parameters to be those which correspond to the risk-neutral measure. We shall derive the price of a claim C whose payoff depends on the future level of the short rate. If we set $J_0 = B_0^\ell$ (the initial lower edge of the band), then we have $r_t = X_t + J_t$. Under the risk-neutral measure we must have

$$C_t = \mathbb{E}[Y_T C_T | \mathcal{F}_t], \tag{10}$$

where $\mathcal{F}_t$ is the information set generated by the path of the vector process (X_s, J_s) up to time t, and $Y_t = \exp(-\int_0^t r_s ds)$ is the deflator. By the Markov property, we can write $C_t = C(X_t, J_t, t, L_t^l, L_t^u)$. Because of the stationary independent increments property of a Poisson process, we can, in fact, write $C(X_t, J_t, t)$. In the Appendix, we show that we must have the following holding in order for C to be a martingale:

$$\mathcal{A}C + \frac{\partial C}{\partial t} - r_t C = 0 \tag{11}$$

$$\frac{\partial}{\partial x}C(0, j, t) + \lambda_t^{\ell}[C(b, j-b, t) - C(0, j, t)] = 0 \tag{12}$$

$$-\frac{\partial}{\partial x}C(D, j, t) + \lambda_t^{u}[C(D-b, j+b, t) - C(D, j, t)] = 0 \tag{13}$$

where $D = B^u - B^\ell$, the width of the band and $\mathcal{A}$ is the operator which, when applied to a twice differentiable function gives $\mu(x)f'(x)+1/2\sigma(x)^2 f''(x)$. Now, adding the terminal value of the claim, we have a PDE which the price of any interest rate contingent claim must satisfy.

Note that if $\lambda_t^u = \lambda_t^\ell = 0$ for all t, then the bands never move, and the derivative of the claim price with respect to x must vanish at either edge of the band. This is related to discussion of "smooth pasting" in Krugman (1991) for a model in which targets are perfectly credible. Since targets do move in our model, we refer to them as "semi-credible."

The above model is interesting for purely theoretical reasons as well. In Jones (1984), the pricing of options is examined when the stock price process is driven by a jump diffusion. It was shown in that paper that it would take three assets to form a riskless hedge if there were jumps of only one size (only two assets are required for models with one state variable which has continuous paths). Here, we still only need two assets to form a riskless hedge. A close examination reveals the reason for this. Although asset prices do have jumps, we know that they can only occur when the rate is at the boundaries. Hence, within the target zone, we can hedge as though the process were continuous. At the edges, the above boundary conditions guarantee that the same hedge ratios will also hedge against jumps. Here, we have a model with three sources of risk that can be hedged with only two assets. As far as we know, this is unique in the literature on arbitrage pricing.

A simple version of the random threshold model, analogous to the affine model of the previous section, assumes that the drift of the short rate is $\kappa(\theta - r_t)$ and the diffusion function is σ, a constant. The parameter κ has a different interpretation in this model. In the affine model, a larger κ indicated more effective target enforcement. In this model, it is merely the degree to which the short rate tends toward θ. The intensities of the jump processes are taken to be the same as in the affine model, $\lambda_t^u = \lambda(.12 - J_{t-})$ and $\lambda_t^\ell = \lambda J_{t-}$, where in this case, J_{t-} is the lower edge of the band.[19] Analogous to the affine model, we wish to examine the response function of this model to target changes. We cannot solve this model in closed form, but we can solve it numerically. To estimate the response function for a given set of parameters, we calculate the yield curve with the rate at the top of the band and then calculate it again with the rate in the same place but the band shifted up by b. The response function is the difference between these yield curves divided by b.

[19] An interesting thing about this formulation is that it precludes negative interest rates. The reason is that when the lower edge of the band is at zero, it will never go down further. Since the rate is always constrained to be within the band, it can never go below zero but exhibits reflection at zero with probability 1.

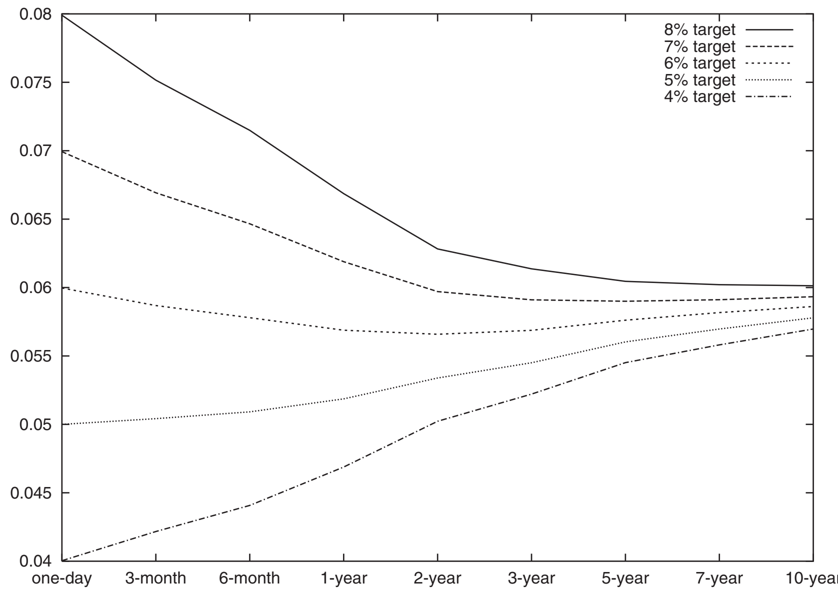

Figure 7. Effect of target level on the yield curve. This figure plots yield curves for various target levels with the short rate at the target (the midpoint of the band). Parameter values used are $k = 2$, $\theta = 0.05$, $\sigma = 0.02$, and $\lambda = 12{,}000$.

The response function of this model is hump shaped, similar to the affine model. It is tempting, therefore, to label the target as a "curvature factor," since the response function is a curve rather than a straight line. This is misleading. The reason is that two effects are combined to make this response function. Recall that the target only moves upward when the short rate is on the upper edge of the band, and that the short rate does not move at target changes. So an upward target change has a double effect; the bands move up and the short rate changes from being on the upper edge of the band to being somewhere inside the new band. To isolate the effect of the target, we solve the model for several different target levels where, in each case, the current rate is taken to be at the target (the midrange of the band). The results are plotted in Figure 7. Figure 8 shows the effect of changing the position of the short rate within a fixed target zone. In the figure, there are several yield curves, each corresponding to a different level of the short rate inside an initial band of [.0475, .0525]. From these figures, we see that the effect of a changing target and a movement of the rate within the target band both have an effect that looks like a rotation of the yield curve. However, when a target change occurs, both effects come into play but in opposite directions: If the target increases, then the rate moves to a lower point in the new target band than it occupied in the old target band. Taking these two effects together gives the hump shape of the response function.

The slope of the term structure behaves much the way it does in a one-factor model like Vasicek (1977) or Cox, Ingersoll, and Ross (1985). The term structure is

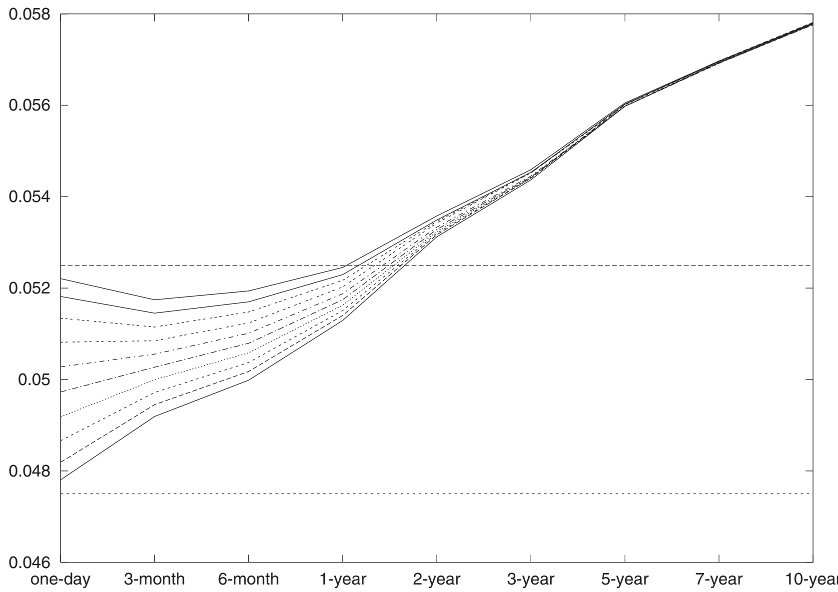

Figure 8. Effect of rate location inside the band. This figure plots yield curves with short rates set at evenly spaced positions within the band. Parameter values used are $k = 2$, $\theta = 0.05$, $\sigma = 0.02$, and $\lambda = 12{,}000$. The band is also shown for reference.

upward sloping if the current rate is below its long run mean and the converse if it is above its long-run mean. The difference is that here the long-run mean is not determined by the parameter θ but rather by the intensities of upward and downward jumps. Regardless of what θ is, the short rate is confined to stay within its current band until the band moves. Even if θ is very high, the rate may not often attain that level if the probability of the target moving to that level is very low. In this case, note that the long-run mean is really 0.06. This is because we defined the intensities such that a target of 0.06 has the lowest probability of a target change. Lower targets have a higher probability of moving up, and higher targets have a higher probability of moving down. Notice in Figure 8 how the term structure rises to 0.06 even though θ is set to 0.05. Notice the slight downward "bow" in all the yield curves of Figure 7 at maturities of one or two years. This is also caused by θ being below 0.06, the long-run mean determined by the intensities of the jump processes. If the short rate were above this long-run mean, then the bow would go the other direction.

The slope of the term structure is also affected by the credibility of the current target. By credibility, we mean how long we can expect the current target to be in force. Suppose the current short rate is below its long-run mean. If the target moves frequently, then the term structure will rise quickly to the long-run mean level. Credibility is determined by two parameters: λ and σ. Both of these parameters are inversely related to credibility. It should be fairly obvious that a higher λ leads to more frequent target changes, since λ determines the intensity of the target change process. But recall that the target change process counts time in terms of local time. A larger σ means that the rate will move around inside the target zone more and hit the edges more frequently. This means that local time will increase more quickly, and so the arrival rate (in calendar time) of target changes will increase.

Credibility also affects the size of the response to target changes. If targets are very credible, the response to a target change will be larger. However the parameter which has the most effect on the size of the response function is the size of the target change. It is true that, holding credibility constant, a larger target change will have a larger impact on the yield curve. However, since we measure the response function as the change in yield per unit of target change, the size of the response function is actually inversely related to the size of the target change. It is this comparative static which is of most interest in the current context, because both the size of target changes and the size of the response functions differ between the two periods of interest. In fact, this seems to be mostly responsible for the superior performance of this model over the affine model.

The estimation procedure is the same as in the affine case. In the early period, we estimate $\kappa = 4.50$, $\theta = 0.053$, $\sigma = 0.025$, and $\lambda = 9{,}396$, with the rate starting at the upper edge of the band at a level of .051. The jump intensity is much higher than in the affine model, but the two are not directly comparable, since in this model, the Poisson process of interest is indexed by local time, which moves much more slowly than calendar time, so a higher intensity is required to produce any jumps in a reasonable amount of calendar time. The J-statistic for this estimation is 13.355 ($p = 0.271$). In the later period, we estimate $\kappa = 4.75$, $\theta = 0.047$, $\sigma = 0.0155$, $\lambda = 16{,}500$, with the rate starting at the upper edge of the band at a level of 0.047. The J-statistic of this estimation is 14.145 ($p = 0.364$). So the model cannot be rejected in either subperiod. The estimated response functions are plotted in Figure 9. Notice the striking similarity with Figure 3.

IV. Conclusion

In this paper, we have presented a model which is stationary and allows target changes to depend on the level of the short rate relative to the target. We have shown how this model improves on both stationary and nonstationary exponential-affine models. In so doing, we have developed new tools for modeling in the form of semi-credible bands.

One issue which we have not addressed is how to accommodate target changes that can only happen at FOMC meeting dates. In the past, target changes could occur at any time, but since 1994, the Fed has tried to limit target changes to FOMC meeting dates. The Fed has not always kept to this commitment, and recent experience casts doubt on whether the Fed is indeed committed to this policy. For a model which assumes fixed dates for target changes, see Piazzesi (2001). One could certainly add a target band to such a model, but the target change process would likely have to be different than we have suggested here or

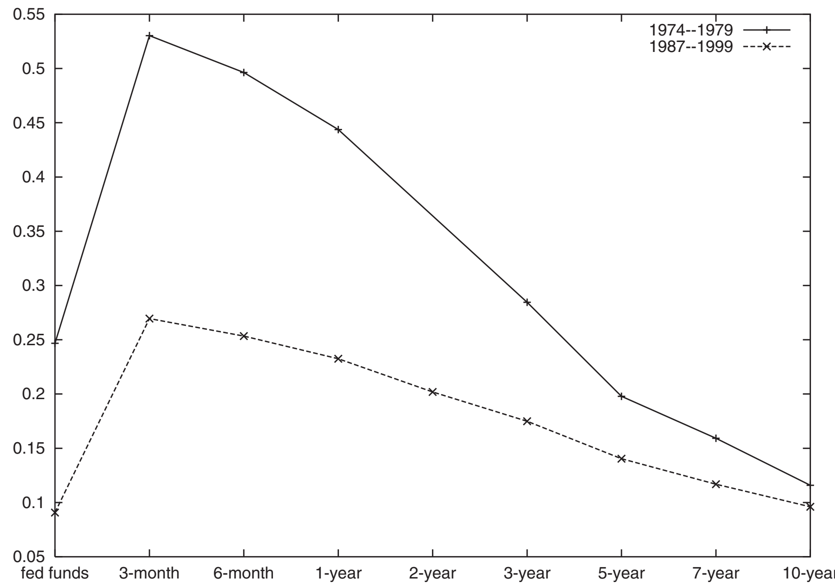

Figure 9. Estimated response functions from the random threshold model. This figure plots the response function for the random threshold model as a function of maturity for the two time periods under consideration. The parameter values are set to their estimated values.

else target changes would only occur on meeting dates when the short rate is also currently at the edge of the band.

Appendix

In all proofs, we assume that the jump intensity process is fixed through time to simplify notation. The extension to time-varying intensities is trivial.

A. Proof of Completeness

Completeness

Let r_t denote the interest rate process and let B_t^u and B_t^ℓ denote upper and lower band edge processes. Recall that upward jumps in B_t^u and B_t^ℓ are driven by a jump process J_t^u which is a Poisson process when indexed by local time on the upper band edge with jump size $b > 0$. More formally J_t^u is a jump process with compensator $b\lambda^u L_t^u$, that is, $M_t^u = J_t^u - b\lambda^u L_t^u$ is a martingale, where λ^u is some fixed parameter. Downward jumps are driven by J_t^ℓ, which is defined similarly so that $M_t^\ell = J_t^\ell + b\lambda^\ell L_t^\ell$ is a martingale. There are two steps in showing completeness. The first is to show that every martingale adapted to the filtration generated by $\mathcal{F}_t = \sigma(r_s, J_s^u, J_s^\ell; s \le t)$ can be written in terms of stochastic integrals with respect to W_t, M_t^u, and M_t^ℓ. See Meyer (1976) for information on stochastic integrals and stochastic calculus for not necessarily continuous processes.

THEOREM: *If $Y \in L^2$ is $\mathcal{F}_t$ measurable, then there exist I_s^ℓ, I_s^u, and I_s^W predictable such that*

$$Y = EY + \int_0^T I_s^\ell dM_s^\ell + \int_0^T I_s^u dM_s^u + \int_0^T I_s^W \, dW_s. \tag{A1}$$

Proof: It is well known that a Poisson process has the martingale representation property, that is, every L^2 random variable adapted to the filtration generated by a Poisson process can be represented by a stochastic integral with respect to a certain martingale. A simple time change shows that every L^2 random variable Y adapted to $\sigma(J_s^u; s \le T)$ can be written as $Y = EY + \int_0^T H_s dM_s^u$ for some predictable integrand H_s. In particular, this holds for $Y^u = \exp(i \sum_{j=1}^m v_j J_{s_j}^u)$ if $0 \le s_1 \le \cdots \le s_m \le T$. So $Y^u = EY^u + \int_0^T H_s^u dM_s^u$. Similarly, we have $Y^\ell = EY^\ell + \int_0^T H_s^\ell dM_s^\ell$, and $Y^W = EY^W + \int_0^T H_s^W dW_s$, where $Y^\ell = \exp\left(i \sum_{j=1}^m w_j J_{s_j}^\ell\right)$, $Y^W = \exp\left(i \sum_{j=1}^m x_j W_{s_j}\right)$. The representation for Y^W follows because W has the martingale representation property.

The martingales M^u and M^ℓ have no continuous parts and no jumps in common. So M^u, M^ℓ, and W are mutually orthogonal martingales, which means $[M^\ell, M^u]_t = 0$, $[M^\ell, W]_t = 0$, and $[M^u, W]_t = 0$ for all t. By the product formula,

$$\begin{aligned} Y^u Y^\ell = &(EY^u)(EY^\ell) + (EY^\ell) Y^u + (EY^u) Y^\ell \\ &+ \int_0^T \left(\int_0^{s-} H_s^u dM_s^u \right) H_s^\ell dM_s^\ell \\ &+ \int_0^T \left(\int_0^{s-} H_s^\ell dM_s^\ell \right) H_s^u dM_s^u \\ = &E(Y^u Y^\ell) + \int_0^T K_s^\ell dM_s^\ell + \int_0^T K_s^u dM_s^u \end{aligned} \tag{A2}$$

for some predictable K_s^ℓ, K_s^u. By a similar argument, $Y^u Y^\ell Y^W$ has the form (A2). So (A2) holds when $Y = \exp(i \sum_{j=1}^m (v_j J_{s_j}^u + w_j J_{s_j}^\ell + x_j W_{s_j}))$. Linearity shows that (A2) holds when Y is a linear combination of such random variables. Since such linear combinations are dense in L^2, a limit argument establishes the theorem.

Since M_t^u increases only when r_t is at B^u and similarly for M_t^ℓ, we can let $N_t = M_t^u + M_t^\ell + W_t$ and write

$$Y = EY + \int_0^T \left[I_s^u 1_{(r_s = B_{s-}^u)} + I_s^\ell 1_{(r_s = B_{s-}^\ell)} + I_s^W 1_{(r_s \in (B_{s-}^\ell, B_{s-}^u))} \right] dN_s. \tag{A3}$$

We now show there exists a claim C such that every other claim can be written in terms of a self-financing strategy with respect to C. Let $\{Y_n\}_{n=1}^\infty$ be a subset of $L^\infty(\mathbb{P})$ such that every random variable in $L^\infty(\mathbb{P})$ is the almost sure limit of a uniformly bounded subsequence of $\{Y_n\}$. Let $Y_n(t)$ be the price of Y_n at time t. There exists an equivalent martingale measure $\mathbb{Q}$ such that under $\mathbb{Q}$, each $Y_n(t)$ is a martingale.

If $Y \in L^\infty(\mathbb{Q})$, let $Y_t = E_{\mathbb{Q}}[Y|\mathcal{F}_t]$. Take a subsequence $\{Y_{n_j}\}$ of $\{Y_n\}$ that converges boundedly and almost surely to Y. Then, by the dominated convergence theorem $Y_{n_j}(t) \to Y_t$ almost surely. Since $Y_{n_j}(t)$ is the price of Y_{n_j} at time t, it is not hard to see that Y_t must be the price of Y at time t, or else an arbitrage opportunity exists.

By (A3), we know every random variable in $L^2(\mathbb{P})$ can be expressed as $Z = E_{\mathbb{P}}Z + \int_0^T H_s dN_s$ for some predictable process H_s. Since $\mathbb{Q}$ is equivalent to $\mathbb{P}$, it is known that every Z in $L^2(\mathbb{Q})$ can be expressed as

$$Z = E_{\mathbb{Q}}Z + \int_0^T K_s dM_s \tag{A4}$$

for some martingale M_t and some predictable process K_t. Choose $\tilde{M}_t$ to be a martingale that is uniformly bounded and such that if $\tilde{M}_t = \int_0^t I_s dM_s$, then I_t is never 0. This can be accomplished as follows. Let $R > b$, $S_0 = 0$, and $S_{i+1} = \inf\{t > S_i : |M_t - M_{S_i}| \geq R\}$. Since M_t has left limits and is right continuous, $S_i \to \infty$. If we let $I_s = 2^{-i}$ on $[S_i, S_{i+1})$, then $|\tilde{M}_t| \leq 4R$ for all t. Using (A4), we can write any Z as

$$Z = E_{\mathbb{Q}}Z + \int_0^T \left(\frac{H_s}{I_s}\right) d\tilde{M}_s \tag{A5}$$

Define the claim C by $C = \tilde{M}_t$. Since C is bounded, by the above, the price of C at time t is $E_{\mathbb{Q}}[C|\mathcal{F}_t] = \tilde{M}_t$. Equation (A5) then asserts that any claim Z can be attained by a self-financing strategy.

B. Derivation of the Pricing Equation

By the product formula and Ito's lemma

$$\begin{aligned} Y_T C(X_T, J_T, T) = {} & C(X_t, J_t, t) + \int_t^T C_{s-} dY_s + \int_t^T Y_{s-} dC_s \\ = {} & C(X_t, J_t, t) - \int_t^T Y_{s-} C_{s-} r_s ds + \int_t^T Y_{s-} \left(\mathcal{A}C_{s-} + \frac{\partial C_{s-}}{\partial s}\right) ds \\ & + \int_t^T Y_{s-} \frac{\partial C_{s-}}{\partial x} dL_s^\ell - \int_t^T Y_{s-} \frac{\partial C_{s-}}{\partial x} dL_s^u + \int_t^T Y_{s-} \frac{\partial C_{s-}}{\partial x} dJ_s \\ & + \int_t^T Y_{s-} \frac{\partial C_{s-}}{\partial J} dJ_s + MG \\ & + \sum_{s \leq T} Y_s \left(C(X_s, J_s, s) - C(X_{s-}, J_{s-}, s-) - \frac{\partial C_{s-}}{\partial x} \Delta X_s - \frac{\partial C_{s-}}{\partial J} \Delta \tilde{J}_s \right), \end{aligned} \tag{A6}$$

where $\Delta X_s = X_s - X_{s-}$. Now recall that all of the jumps in X come from J, and so we have that

$$\int_t^T Y_{s-} \frac{\partial C_{s-}}{\partial x} dJ_s = \sum_{s \leq T} Y_{s-} \frac{\partial C_{s-}}{\partial x} \Delta X_s \tag{A7}$$

and likewise

$$\int_t^T Y_{s-} \frac{\partial C_{s-}}{\partial J} dJ_s = \sum_{s \leq T} Y_{s-} \frac{\partial C_{s-}}{\partial J} \Delta J_s. \tag{A8}$$

So these terms cancel and we obtain

$$\begin{aligned} Y_T C(X_T, J_T, T) =& C(X_t, J_t, t) + \int_t^T C_{s-} dY_s + \int_t^T Y_{s-} dC_s \\ =& C(X_t, J_t, t) - \int_t^T Y_s C r_s ds + \int_t^T Y_s (\mathcal{A}C) ds + MG \\ &+ \int_t^T Y_s \frac{\partial C}{\partial x} dL_s^\ell - \int_t^T Y_s \frac{\partial C}{\partial x} dL_s^u \\ &+ \sum_{s \leq T} Y_{s-}(C(X_s, J_s, s) - C(X_{s-}, J_{s-}, s-)), \end{aligned} \tag{A9}$$

where we have dropped the subscript $s-$ from the integrals because dL_s and ds do not charge points. Now define $\Delta C_s = C(X_s, J_s, s) - C(X_{s-}, J_{s-}, s-)$ and note that

$$\sum_{s \leq T} \Delta C_s = \sum_{s \leq T} \Delta C_s 1_{\{\Delta J_s^\ell \neq 0\}} + \sum_{s \leq T} \Delta C_s 1_{\{\Delta J_s^u \neq 0\}}. \tag{A10}$$

If $\Delta J_s^\ell \neq 0$, then $X_{s-} = 0$ (since $r_{s-} = B^\ell$ and $X = r - B^\ell$), and so $C(X_s, J_s, s) - C(X_{s-}, J_{s-}, s-) = C(b, J_{s-} - b, s) - C(0, J_{s-}, s)$.

$$\sum_{s \leq T} \Delta C 1_{\{\Delta J_s^\ell \neq 0\}} = \frac{1}{b} \int_0^T [C(b, J_{s-} - b, s) - C(0, J_{s-}, s)]\, dJ_s^\ell \tag{A11}$$

For jumps at B^u, we have a similar expression.

In order for C to be a martingale, we must have

$$\mathcal{A}C + \frac{\partial C}{\partial t} - r_t C = 0 \tag{A12}$$

$$\frac{\partial}{\partial x} C(0, j, t) + \lambda^\ell [C(b, j - b, t) - C(0, j, t)] = 0 \tag{A13}$$

$$-\frac{\partial}{\partial x} C(D, j, t) + \lambda^u [C(D - b, j + b, t) - C(D, j, t)] = 0 \tag{A14}$$

where $D = B^u - B^\ell$, the width of the band. Now adding the terminal value of the claim, we have a PDE which the price of any interest rate contingent claim must satisfy.

C. Monte Carlo Method for Pricing Bonds

The price of a bond which pays one dollar at time T and makes no other payments is given by

$$P_t = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) | r_t, J_t\right] \tag{A15}$$

where the expectation is with respect to the risk-neutral measure. Consider N dates equally spaced between t and T. Let δ be the time between dates. For each date, we simulate an interest rate realization and approximate the integral inside the expectation as

$$\int_t^T r_s ds \approx \sum_{i=1}^{N} r_{t_i}\delta. \tag{A16}$$

Clearly as $\delta \to 0$, this approximation becomes better and better. We simulate M such paths, and the Monte Carlo estimate for the bond price is the mean of the exponential of minus this summation.

To simulate the interest rate process, begin with an initial value for r_t and initial band edges B_t^u and B_t^ℓ. Initialize L_t^u and L_t^ℓ to zero. Generate two independent exponential random variables, e^u and e^ℓ with parameters λ^u and λ^ℓ, respectively. Then use the following scheme to simulate an interest rate path. Begin with $i = 1$.

(1) Generate r_{t_i} from $r_{t_{i-1}}$ using an appropriate time discrete approximation.
(2) Set $B_{t_i}^u$, $B_{t_i}^\ell$, $L_{t_i}^u$ and $L_{t_i}^\ell$ to be equal to their values at the last iteration.
(3) If $r_{t_i} > B_{t_i}^u - L_{t_i}^\ell$ and $L_{t_i}^\ell - B_{t_i}^u + r_{t_i} > L_{t_i}^u$, then

(a) set $L_{t_i}^u = r_{t_i} + L_{t_i}^\ell - B_{t_i}^u$.
(b) If $L_{t_i}^u > e^u$, then

(i) $B_{t_i}^u = B_{t_i}^u + b$;
(ii) $B_{t_i}^u = B_{t_i}^u + b$;
(iii) $e^u = e^u +$ another exponential with parameter λ^u.

(4) If $r_{t_i} > B_{t_i}^\ell - L_{t_i}^u$ and $L_{t_i}^u + B_{t_i}^\ell - r_{t_i} > L_{t_i}^\ell$, then

(a) set $L_{t_i}^\ell = r_{t_i} + L_{t_i}^\ell - B_{t_i}^\ell$
(b) If $L_{t_i}^\ell > e^\ell$, then

(i) $B_{t_i}^\ell = B_{t_i}^\ell - b$;
(ii) $B_{t_j}^\ell = B_{t_i}^\ell - b$;
(iii) $e^\ell = e^\ell +$ another exponential with parameter λ^ℓ.

(5) Let $i = i+1$ and go back to 1 unless $i = N$, in which case exit.

This generates an uncontrolled process r and the controls L^u and L^ℓ. To get the controlled process, define $r^c = r + L^\ell - L^u$. To improve efficiency, the method of antithetic variables is used, which means that if 1,000 paths are to be simulated, we first generate a matrix Z of standard normal deviates which has 500 rows and a matrix U of uniform deviates. The first 500 paths are based on Z and U. The next 500 are generated using $-Z$ and $1-U$.

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