A Fatou theorem for α-harmonic functions in Lipschitz domains

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Abstract

We study α-harmonic functions in Lipschitz domains. We prove a Fatou theorem when the boundary function is bounded and L^p-Hölder continuous of order β with $\beta p > 1$.

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1 Introduction

Let D be a bounded Lipschitz domain in $\mathbb{R}^d$ and let $\alpha \in (0, 2)$. Our purpose in this paper is to establish a Fatou theorem for bounded functions which are α-harmonic in D. An α-harmonic function is one which is harmonic with respect to a symmetric stable process of index α, or equivalently, with respect to the operator $-(-\Delta)^{\alpha/2}$. There has been a great deal of recent work on the properties of α-harmonic functions; see [5]–[15], [18], [20], and [23].

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To be more precise, let f be a bounded function on D^c, let $(X_t, \mathbb{P}^x)$ be a symmetric stable process of index $\alpha \in (0, 2)$, let $\tau_D = \inf\{t : X_t \notin D\}$, and let

$$u_f(x) = \mathbb{E}^x f(X_{\tau_D}). \tag{1.1}$$

Then u_f is what is known as a regular α-harmonic function. It is not the case that u_f must have nontangential limits a.e. with respect to surface measure on the boundary of D. This was shown in Section 5 of [3] in the case of a half space; the example can be easily modified to hold in bounded Lipschitz domains. Therefore we assume in addition that f is L^p-Hölder continuous. That is, we let $\Lambda_\beta^{p,\infty}(\mathbb{R}^d)$ be the collection of functions in $L^p(\mathbb{R}^d)$ such that the norm

$$\|f\|_p + \sup_{|t|>0} \frac{\|f(\cdot + t) - f(\cdot)\|_p}{|t|^\beta}$$

is finite (the notation agrees with that of [21]). Let $B(x, \zeta)$ be the open ball with radius ζ and center x. We let $\Lambda_{loc}^b(\beta, p, D^c)$ be the collection of bounded functions f such that for each $x \in \partial D$ there exists ζ depending on x such that f agrees on $D^c \cap B(x, \zeta)$ with a bounded function in $\Lambda_\beta^{p,\infty}(\mathbb{R}^d)$.

Next we explain what nontangential convergence means. Define the truncated cone $C_{a,\theta} = \{(\widetilde{z}, z_d) : 0 < z_d < a, |\widetilde{z}| < \theta z_d\}$, where $\widetilde{z} \in \mathbb{R}^{d-1}$. Choose $a < a'$ and $\theta < \theta'$ and for each $x \in \partial D$ a rotation R_x such that $x + R_x(C_{a',\theta'}) \subset D$. Then let $C(x) = x + R_x(C_{a,\theta})$. We need to choose R_x in a measurable way, but since each rotation is associated with a $d \times d$ orthogonal matrix, this is routine.

We then say that u_f converges nontangentially at $x \in \partial D$ if

$$\lim_{z \in C(x), z \to x} u_f(x)$$

exists.

Our main theorem is

Theorem 1.1. *Suppose D is a bounded Lipschitz domain, $\beta \in (0, 1), p \in (1, \infty]$, and $\beta p > 1$. Suppose f is a function in $\Lambda_{loc}^b(\beta, p, D^c)$. Then u_f defined by (1.1) converges nontangentially at all points of ∂D except for a set of zero surface measure.*

It is routine to extend this result to unbounded Lipschitz domains. The example in Section 5 of [3] shows that our condition $\beta p > 1$ is sharp. We discuss the identification of the limit in the remark at the end of the paper.

Theorem 1.1 is in sharp contrast to the Fatou theorem for functions that are harmonic in D (with respect to the Laplacian). The Fatou theorem for harmonic functions says that if D is a Lipschitz domain and $f \in L^p(\partial D)$, then u_f converges nontangentially a.e. with respect to surface measure on ∂D. See [1], Section III.4, for details and some applications. While most results that are true for harmonic functions are also true for α-harmonic functions, often with weaker assumptions, the Fatou theorem is an exception in that extra regularity is needed.

Our method is to first obtain an estimate on the Poisson kernel. Unlike the half space case, our estimate is not at all sharp, and for $\alpha \in [1, 2)$ is not even locally integrable. We therefore have to derive some hitting probability estimates for symmetric stable processes in Lipschitz domains. Nevertheless, the theorem we obtain is as sharp as the one we obtained in [3] in the half space case, where an explicit formula for the Poisson kernel was available. We point out that our results complement those of [18]: see the remark following the proof of Proposition 3.5.

The paper is organized as follows. In Section 2 we compute an upper bound of the Poisson kernel for Lipschitz domains. Section 3 contains some estimates for α-harmonic functions in Lipschitz domains. In Section 4 we introduce a maximal function similar to the one in [3] and establish an estimate for it, while in Section 5 we prove Theorem 1.1.

We use the letter c, with or without subscripts, to denote positive finite constants whose exact value is unimportant. Let $B(x_0, r) = \{x \in \mathbb{R}^d : |x - x_0| < r\}$ be the open ball centered at x_0 with radius r. Given a Borel subset D of $\mathbb{R}^d$, let $|D|$ denote the Lebesgue measure of D and let $\delta_D(x)$ be the distance between x and ∂D, where ∂D is the boundary of D. We sometimes write points $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$ as $z = (\widetilde{z}, z_d)$ with $\widetilde{z} \in \mathbb{R}^{d-1}$. Given a Borel set A, we use $A - x = \{z - x : z \in A\}$, $aA = \{az : z \in A\}$, and $\Psi(A) = \{\Psi(z) : z \in A\}$ when Ψ is a map from $\mathbb{R}^d$ to $\mathbb{R}^d$. The paths of X_t are right continuous with left limits; we use $X_{t-} = \lim_{s\uparrow t, s<t} X_s$ and $\Delta X_t = X_t - X_{t-}$. If A is a Borel set, we define

$$\tau_A = \inf\{t : X_t \notin A\}, \qquad T_A = \inf\{t : X_t \in A\}$$

for the first exit and first entrance of A, respectively. A Lipschitz function Γ with Lipschitz constant λ is a map $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying

$$|\Gamma(\widetilde{x}) - \Gamma(\widetilde{y})| \leq \lambda |\widetilde{x} - \widetilde{y}|, \qquad \widetilde{x}, \widetilde{y} \in \mathbb{R}^{d-1}.$$

A Lipschitz domain is one where for each $x \in \partial D$ there is $r_x > 0$ and a coordinate system depending on x such that $D \cap B(x, r_x)$ agrees with the intersection of $B(x, r_x)$ with the region above the graph of a Lipschitz function.

2 Poisson kernel

The Green function G_D for X_t is

$$G_D(x,y) = c(\alpha,d)\left[\frac{1}{|x-y|^{d-\alpha}} - \int_{D^c} \frac{1}{|y-z|^{d-\alpha}}\,\omega^x(dz,D)\right]$$

where $\omega^x(\cdot, D)$ is the α-harmonic measure on D^c given by

$$\omega^x(A,D) = \mathbb{P}^x(X_{\tau_D} \in A), \quad A \subset D^c.$$

It is known (see [13], [17]) that the distribution of X_{τ_D} under $\mathbb{P}^x$ has a density with respect to d-dimensional Lebesgue measure on D^c given by

$$P_D(x,z) = c(\alpha,d)\int_D \frac{G_D(x,y)}{|y-z|^{d+\alpha}}dy, \qquad x \in D,\ z \in \overline{D}^c, \tag{2.1}$$

when D is a domain satisfying a uniform exterior cone condition. In particular, in this case $\mathbb{P}^x(X_{\tau_D} \in \partial D) = 0$. P_D is called the Poisson kernel for D.

In the case of a ball,

$$P_{B(0,r)}(x,y) = c_1\left[\frac{r^2 - |x|^2}{|y|^2 - r^2}\right]^{\alpha/2} |x-y|^{-d}; \tag{2.2}$$

see [19], pp. 121–122. We deduce from this that

$$\mathbb{P}^x(|X_{\tau_{B(x,r)}} - x| > K) \le c_2 (r/K)^\alpha. \tag{2.3}$$

Another consequence of (2.2) is that u_f is C^∞ in D if f is bounded, although we do not use this fact.

The following proposition gives an upper bound for the Poisson kernel in a Lipschitz domain.

Proposition 2.1. *Let $d \geq 2$. Let D be either a bounded Lipschitz domain in $\mathbb{R}^d$ or else the region above the graph of a Lipschitz function with Lipschitz constant λ. There exists a constant c_1 depending only on α, λ, and d such that*

$$P_D(x,z) \leq c_1 \frac{1}{|x-z|^{d-\alpha}\delta_D(z)^\alpha}, \qquad x \in D, z \in \overline{D}^c.$$

Proof. Let $r = |x-z|$. We write $D = A_1 \cup A_2 \cup A_3$, where

$$\begin{aligned} A_1 &= D \cap B(x, r/2), \\ A_2 &= D \cap [B(x, 6r) \setminus B(x, r/2)], \\ A_3 &= D \cap B(x, 6r)^c. \end{aligned}$$

Then by (2.1)

$$P_D(x,z) \leq c_2 \int_D \frac{1}{|x-y|^{d-\alpha}} \frac{1}{|y-z|^{d+\alpha}} dy = c_2 \Big[\int_{A_1} + \int_{A_2} + \int_{A_3} \Big].$$

For $y \in A_1$ we have $|y-z| \geq r/2$, and so the integral over A_1 is bounded by

$$c_3 r^{-d-\alpha} \int_{B(x,r/2)} \frac{1}{|x-y|^{d-\alpha}} dy = c_4 r^{-d}. \tag{2.4}$$

For $y \in A_2$ we have $|x-y| \geq r/2$ and $|y-z| \geq \delta_D(z)$, and so the integral over A_2 is bounded by

$$r^{-d+\alpha} \int_{\{|y-z| \geq \delta_D(z)\}} \frac{1}{|y-z|^{d+\alpha}} dy \leq c_5 r^{-d+\alpha} \delta_D(z)^{-\alpha}. \tag{2.5}$$

For $y \in A_3$ we have $|y-z| \geq |x-y|/2$, and so the integral over A_3 is bounded by

$$c_6 \int_{B(x,6r)^c} \frac{1}{|x-y|^{2d}} dy \leq c_7 r^{-d}. \tag{2.6}$$

If we combine (2.4)–(2.6) and note that $r \geq \delta_D(z)$, we obtain our result. □

3 Estimates

Our estimate for the Poisson kernel given in the previous section is satisfactory when $\alpha \in (0,1)$ because the kernel is locally integrable. However, when $\alpha \in [1,2)$, more information is needed.

Let us suppose throughout this section that Γ is a bounded Lipschitz function with Lipschitz constant λ and that D is the region above Γ, that is, $D = \{z = (\widetilde{z}, z_d) : z_d > \Gamma(\widetilde{z})\}$. Define for $z \in \mathbb{R}^d$

$$v(z) = z_d - \Gamma(\widetilde{z}).$$

Note $v(z)$ and $\delta_D(z)$ are comparable. Let

$$W(b) = \{z : 0 < v(z) < b\}.$$

Let $\rho \in (0, \frac{1}{4})$ be chosen later, let

$$W_i = W(\rho^i), \qquad \widetilde{W}_i = W(2\rho^i),$$

and let

$$T_i = T_{W_i}, \qquad \widetilde{T}_i = T_{\widetilde{W}_i},$$

where $T_A = \inf\{t : X_t \in A\}$ for any Borel set A.

Because we are in the region above a Lipschitz function, the exterior cone condition holds. If $z \in D$ and $\gamma < 1$, the distribution of $\mathbb{P}^z(X_{T_{B(z,\gamma\delta_D(z))^c}} \in dy)$ is given by (2.2). Because of the exterior cone condition, we see that there exists $a \in (0, 1)$, depending on γ, such that

$$\mathbb{P}^z(X_{T_{B(z,\gamma\delta_D(z))^c}} \in D^c) \geq a. \tag{3.1}$$

Lemma 3.1. *There exists a constant c_1 not depending on ρ such that if $i > j$ and $z \in D \setminus W_j$, then*

$$\mathbb{P}^z(X_{T_j} \in \widetilde{W}_i) \leq c_1 \rho^{i-j}.$$

Proof. Set $n(x, h) = |h|^{-d-\alpha}$; although n does not depend on x, we include x in the notation to conform to [2], which we will use later on. We begin by claiming that if $x \in D \setminus W_j$, then

$$\int_{\widetilde{W}_i - x} n(x, h)\, dh \leq c_2 \rho^i v(x)^{-1-\alpha}. \tag{3.2}$$

To see this, note that the distance from x to $\widetilde{W}_i$ is at least $c_3(v(x) - 2\rho^i) \geq c_4 v(x)$ since $x \notin W_j$ and $j < i$. So

$$\int_{(\widetilde{W}_i - x) \cap B(x, v(x))} n(x, h)\, dh \leq c_5 \frac{\rho^i v(x)^{d-1}}{v(x)^{d+\alpha}} \tag{3.3}$$

and for $k \geq 0$

$$\int_{(\widetilde{W}_i - x)\cap(B(x,2^{k+1}v(x))\setminus B(x,2^k v(x)))} n(x,h)\,dh \leq c_6 \frac{\rho^i (2^{k+1}v(x))^{d-1}}{(2^k v(x))^{d+\alpha}}. \tag{3.4}$$

Summing (3.4) over k from 0 to ∞ and adding (3.3) yields (3.2).

Because D satisfies a uniform exterior cone condition, there exists a cone V_x with vertex at x and with axis in the $(0,\ldots,0,-1)$ direction and a constant c_7 such that $B(x, c_7\delta_D(x))^c \cap V_x \subset D^c$; we may take the aperture of V_x to be independent of x. Then

$$\int_{D^c - x} n(x,h)\,dh \geq c_8 \int_{B(x,c_7\delta_D(x))^c \cap V_x} \frac{1}{|h|^{d+\alpha}} dh \geq c_9 v(x)^{-\alpha}. \tag{3.5}$$

Combining (3.2) and (3.5) and using the fact that $v(x) \geq \rho^j$ we obtain

$$\begin{aligned} \int_{\widetilde{W}_i - x} n(x,h)\,dh &\leq c_2 \rho^i v(x)^{-1-\alpha} \\ &\leq c_{10} \frac{\rho^i}{v(x)} \int_{D^c - x} n(x,h)\,dh \\ &\leq c_{10} \rho^{i-j} \int_{D^c - x} n(x,h)\,dh. \end{aligned} \tag{3.6}$$

If there is a jump from $D \setminus W_j$ to $\widetilde{W}_i$ before or at time $T_j \wedge \tau_D$, then

$$\sum_{t \leq T_j \wedge \widetilde{T}_i \wedge \tau_D} 1_{(\Delta X_t \neq 0)} 1_{(X_t \in \widetilde{W}_i)} \geq 1.$$

Therefore

$$\mathbb{P}^z(X_{T_j} \in \widetilde{W}_i) \leq \mathbb{E}^z \sum_{t \leq T_j \wedge \widetilde{T}_i \wedge \tau_D} 1_{(\Delta X_t \neq 0)} 1_{(X_t \in \widetilde{W}_i)}. \tag{3.7}$$

Using the Lévy system formula (see [2], Proposition 2.3) and (3.6), if

$t_0 > 0$ we have

$$\begin{aligned}
\mathbb{E}^z & \sum_{t \le T_j \wedge \widetilde{T}_i \wedge \tau_D \wedge t_0} 1_{(\Delta X_t \neq 0)} 1_{(X_t \in \widetilde{W}_i)} \\
&= \mathbb{E}^z \int_0^{T_j \wedge \widetilde{T}_i \wedge \tau_D \wedge t_0} \int_{\widetilde{W}_i - X_s} n(X_s, h)\, dh\, ds \\
&\le c_{10} \rho^{i-j} \mathbb{E}^z \int_0^{T_j \wedge \widetilde{T}_i \wedge \tau_D \wedge t_0} \int_{D^c - X_s} n(X_s, h)\, dh\, ds \\
&= c_{10} \rho^{i-j} \mathbb{E}^z \sum_{t \le T_j \wedge \widetilde{T}_i \wedge \tau_D \wedge t_0} 1_{(\Delta X_t \neq 0)} 1_{(X_t \in D^c)} \\
&\le c_{10} \rho^{i-j}.
\end{aligned}$$

The last inequality follows because $\sum_{t \le \tau_D} 1_{(\Delta X_t \neq 0)} 1_{(X_t \in D^c)}$ is at most 1. If we now let $t_0 \uparrow \infty$, use monotone convergence, and combine with (3.7), we obtain our result. □

We next prove

Proposition 3.2. *There exist $\rho \in (0, \frac{1}{4})$ and $\sigma \in (\frac{1}{2}, 1)$ such that if $v(x_0) \ge 2$, then*

$$\mathbb{P}^{x_0}(T_i < \tau_D) \le 2\sigma^i. \tag{3.8}$$

Proof. Because we are in a Lipschitz domain with Lipschitz constant λ, there exists $\gamma < 1$ independent of i such that if $z \in W_i \setminus \widetilde{W}_{i+1}$, then $B(z, \gamma \delta_D(z)) \subset W_i \setminus W_{i+1}$. Let a be chosen as in (3.1). We will choose suitable $\rho \in (0, \frac{1}{4})$ and $\sigma \in (\frac{1}{2}, 1)$ later. Let $W_0 = D \setminus W_1$. We want to show that (3.8) holds. We will prove this by induction. Since $\sigma > \frac{1}{2}$, the case $i = 1$ is obvious. We now suppose that (3.8) holds for all $j \le i$ and we will prove it holds for $i+1$.

If $T_{i+1} < \tau_D$, then either

(a) the process X_t hits $W_i \setminus \widetilde{W}_{i+1}$ and then hits W_{i+1}, all before time τ_D, or else

(b) there is a largest $j < i$ such that X_t hits W_j but only hits W_{j+1} when it jumps into $\widetilde{W}_{i+1}$.

We first estimate the probability of (a). We need to bound

$$\mathbb{P}^{x_0}(T_i < \tau_D, X_{T_i} \in W_i \setminus \widetilde{W}_{i+1}, T_{i+1} < \tau_D).$$

Using the strong Markov property at time T_i, this is bounded by

$$\mathbb{E}^{x_0}[\mathbb{P}^{X_{T_i}}(T_{i+1} < \tau_D); X_{T_i} \in W_i \setminus \widetilde{W}_{i+1}, T_i < \tau_D]. \tag{3.9}$$

Using (3.1) we conclude that

$$\mathbb{P}^z(T_{i+1} < \tau_D) \leq \mathbb{P}^z(X_{T_{B(z,\gamma\delta_D(z))^c}} \in D) \leq 1 - a.$$

Therefore, using the induction hypothesis, (3.9) is bounded by

$$\begin{aligned}(1-a)\mathbb{P}^{x_0}&(X_{T_i} \in W_i \setminus \widetilde{W}_{i+1}, T_i < \tau_D)\\ &\leq (1-a)\mathbb{P}^{x_0}(T_i < \tau_D)\\ &\leq 2(1-a)\sigma^i.\end{aligned}$$

So the probability of (a) is bounded by

$$2(1-a)\sigma^i. \tag{3.10}$$

Next we bound the probability of the event in (b). When $j = 0$ we need to estimate

$$\mathbb{P}^{x_0}(X_{T_1} \in \widetilde{W}_{i+1}),$$

and by Lemma 3.1 this is bounded by

$$c_2\rho^i. \tag{3.11}$$

Now suppose $1 \leq j < i$. We need to bound

$$\mathbb{P}^{x_0}(T_j < \tau_D, X_{T_j} \in W_j \setminus W_{j+1}, X_{T_{j+1}} \in \widetilde{W}_{i+1}).$$

By the strong Markov property, Lemma 3.1, and the induction hypothesis, this is bounded by

$$\begin{aligned}\mathbb{E}^{x_0}[\mathbb{P}^{X_{T_j}}&(X_{T_{j+1}} \in \widetilde{W}_{i+1}); T_j < \tau_D, X_{T_j} \in W_j \setminus W_{j+1}]\\ &\leq c_3\rho^{i-j}\mathbb{P}^{x_0}(T_j < \tau_D)\\ &\leq 2c_3\rho^{i-j}\sigma^j.\end{aligned}$$

So if $1 \leq j < i$, the probability of (b) is bounded by

$$2c_3\rho^{i-j}\sigma^j. \tag{3.12}$$

If we add the estimates (3.10), (3.11), and (3.12), we have

$$
\begin{aligned}
\mathbb{P}^{x_0}(T_{i+1} < \tau_D) &\leq c_2\rho^i + 2\sum_{j=1}^{i-1} c_3\rho^{i-j}\sigma^j + 2(1-a)\sigma^i &(3.13)\\
&= c_2\rho^i + 2c_3\rho^i\frac{\sigma}{\rho}\,\frac{\sigma/\rho)^{i-1}-1}{(\sigma/\rho)-1} + 2(1-a)\sigma^i.
\end{aligned}
$$

Since $\sigma/\rho > 2$, this is bounded by

$$
\begin{aligned}
c_2\rho^i &+ c_4\rho^i\Big(\frac{\sigma}{\rho}\Big)^{i-1} + 2(1-a)\sigma^i\\
&\leq 2\sigma^i(c_2(\rho/\sigma)^i + c_5(\rho/\sigma) + (1-a)).
\end{aligned}
$$

If we choose ρ small enough so that $(c_2+c_5)(\rho/\sigma) \leq a/2$ and $\sigma \in (1-a/2, 1)$, then the left hand side of (3.13) is bounded by

$$2\sigma^i(1-a/2) \leq 2\sigma^{i+1}.$$

This proves the induction hypothesis, and hence the proposition. □

The main estimate we need is the following. Let

$$S_\varepsilon = \{z : -\varepsilon < v(z) < 0\}.$$

Proposition 3.3. *There exist $c_1, b > 0$ such that if $v(x_0) \in [2,3]$ and $\varepsilon > 0$, then*

$$\mathbb{P}^{x_0}(X_{\tau_D} \in S_\varepsilon) \leq c_1\varepsilon^b.$$

Proof. Let ρ, σ be as in Proposition 3.2. Given ε there exists i such that $\rho^{i+1} \leq \varepsilon < \rho^i$. Let $D^* = \{z : -\rho^i < v(z)\}$. If $X_{\tau_D} \in S_\varepsilon$, then $T_{S_\varepsilon} < \tau_{D^*}$. By the above proposition applied to D^*,

$$\mathbb{P}^{x_0}(T_{S_\varepsilon} < \tau_{D^*}) \leq 2\sigma^i.$$

If we choose $b = \log\sigma/\log\rho$ and $c_1 = 2\rho^{-b}$, then

$$2\sigma^i = \frac{2}{\rho^b}(\rho^{i+1})^b \leq c_1\varepsilon^b,$$

which completes the proof. □

Since for $r > 0$ the process $r^{-\alpha} X_{rt}$ has the same law under $\mathbb{P}^0$ as X_t and rD is again the region above the graph of a Lipschitz function with Lipschitz contant λ, a scaling argument yields

Corollary 3.4. *There exist $c_1, b > 0$ such that if $x_0 \in D$, then*

$$\mathbb{P}^{x_0}(X_{\tau_D} \in S_{\varepsilon v(x_0)}) \leq c_1 \varepsilon^b. \tag{3.14}$$

The constant in Corollary 3.4 is not necessarily the same as the one in Proposition 3.3.

Although dominated convergence and the Harnack inequality give us something like (3.14) for x_0 such that $v(x_0) \in [2, 3]$, the uniformity over $v(x_0)$ requires the extra work above.

In the following, again the uniformity is where the work comes in.

Proposition 3.5. *Let $\varepsilon > 0$. There exists η such that if $v(x_0) < \eta$, then*

$$\mathbb{P}^{x_0}(\tau_{B(x_0,1)} < \tau_D) < \varepsilon.$$

Proof. Let $z \in \partial D$. We will show that there exists $\rho \in (0, \frac{1}{4})$ and $\sigma \in (\frac{1}{2}, 1)$ such that if $x \in D \cap B(z, \rho^i)$, then

$$h(x) = \mathbb{P}^x(\tau_{B(z,1/2)} < \tau_D) \leq 2\sigma^i.$$

This is true for $i = 1$. We suppose it is true for all $j \leq i$ and prove it for $i + 1$. Let $U_i = B(z, \rho^i)$.

If $x \in U_{i+1}$, then since h is regular α-harmonic in $B(z, 1/2) \cap D$ and 0 in D^c,

$$\begin{aligned} h(x) &= \mathbb{E}^x[h(X_{\tau_{U_i} \wedge \tau_D})] \\ &\leq \mathbb{E}^x[h(X_{\tau_{U_i}}); \tau_{U_i} < \tau_D, X_{\tau_{U_i}} \in U_{i-1}] \\ &\quad + \sum_{j=2}^{i-1} \mathbb{E}^x[h(X_{\tau_{U_i}}); X_{\tau_{U_i}} \in U_{i-j} \setminus U_{i-j+1}] \\ &\quad + \mathbb{E}^x[h(X_{\tau_{U_i}}); X_{\tau_{U_i}} \notin U_1]. \end{aligned}$$

Using (3.1) and the induction hypothesis, there exists $a > 0$ not depending on i such that the first term on the right is bounded by $2(1 - a)\sigma^{i-1}$. Using the induction hypothesis and (2.3), the second term is bounded by

$$2c_1 \sum_{j=2}^{i-1} \sigma^{i-j} \rho^{j\alpha}.$$

Finally, using (2.3), the third term is bounded by $c_2\rho^{i\alpha}$. We will take ρ small enough so that $\rho^\alpha/\sigma < \frac{1}{2}$. Adding the three terms together, we have the bound

$$\begin{aligned}2(1-a)\sigma^{i-1} &+ 2c_1\sum_{j=2}^{i-1}\sigma^{i-j}\rho^{j\alpha} + c_2\rho^{i\alpha} \\ &\le 2(1-a)\sigma^{i-1} + c_3\sigma^i(\rho^\alpha/\sigma) + c_2\rho^{i\alpha} \\ &\le 2\sigma^{i-1}[(1-a) + c_3(\rho^\alpha/\sigma) + c_4(\rho^\alpha/\sigma)].\end{aligned}$$

We choose ρ small so that $\rho^\alpha/\sigma < \frac{1}{2}$ and $2(c_3+c_4)\rho < a/2$ and then choose $\sigma \in (\sqrt{1-a/2}, 1)$. So the above is bounded by

$$2\sigma^{i-1}[(1-a) + (a/2)] \le 2\sigma^{i-1}(1-a/2) \le 2\sigma^{i+1},$$

and the induction step is established. The result follows easily from this. □

Remark. The above proposition actually shows that $\mathbb{P}^{x_0}(\tau_{B(z,1/2)} < \tau_D)$ goes to 0 as some power of $v(x_0)$. An argument using the boundary Harnack principle shows that the same is true for the Green function $G_D(x_0, y)$ if y is fixed. This complements the estimates in [18] on the behavior of the Green function.

Proposition 3.6. *Let $\varepsilon > 0$. There exist $\eta, K > 0$ such that if $v(x_0) \le \eta$, then*

$$\mathbb{P}^{x_0}(|X_{\tau_D} - x_0| > K) < \varepsilon.$$

Proof. By Proposition 3.5, there exists η such that

$$\mathbb{P}^{x_0}(\tau_{B(x_0,1)} < \tau_D) < \varepsilon/2 \tag{3.15}$$

if $v(x_0) \le \eta$. Provided $K > 1$, notice that it is not possible that $|X_{\tau_D} - x_0| > K$ on the event where $\tau_{B(x_0,1)} > \tau_D$. So we have

$$\begin{aligned}\mathbb{P}^{x_0}(|X_{\tau_D} - x_0| > K) &= \mathbb{P}^{x_0}(|X_{\tau_D} - x_0| > K, \tau_{B(x_0,1)} < \tau_D) \\ &\quad + \mathbb{P}^{x_0}(|X_{\tau_D} - x_0| > K, \tau_{B(x_0,1)} = \tau_D).\end{aligned}$$

The first term on the right is less than $\varepsilon/2$ by (3.15). Using (2.3) let us choose $K > 1$ large so that

$$\mathbb{P}^{x_0}(|X_{\tau_{B(x_0,1)}} - x_0| > K) < \varepsilon/2.$$

So the second term on the right is also less than $\varepsilon/2$. □

A scaling argument shows that if $v(x_0) \leq 1$, then $\mathbb{P}^{x_0}(|X_{\tau_D} - x_0| > K/\eta) < \varepsilon$. Let $M = K/\eta$. Using scaling again, we have

Corollary 3.7. *Let $\varepsilon > 0$. There exists $M > 0$ such that if $x_0 \in D$, then*

$$\mathbb{P}^{x_0}(|X_{\tau_D} - x_0| > Mv(x_0)) < \varepsilon.$$

4 Maximal functions

Let D be the region above a bounded Lipschitz function $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ with Lipschitz constant λ. Set $\lambda_0 = 6(\lambda + 1)$. For i an integer and $\widetilde{x} \in \mathbb{R}^{d-1}$, let

$$\begin{aligned} A_i(\widetilde{x}) \quad = \quad & \{(\widetilde{t}, t_d) \in \mathbb{R}^d : \ |t_k - x_k| \leq 2^{-i-1}, k = 1, \dots, d-1, \\ & |t_d - (\Gamma(\widetilde{x}) - 3\lambda_0 2^{-i-1})| \leq 2^{-i-1}\lambda_0\}, \end{aligned} \tag{4.1}$$

and let $\widetilde{A}_i(\widetilde{x})$ be the $(d-1)$-dimensional cube with center $\widetilde{x}$ and side length 2^{-i}. Note that $|A_i(\widetilde{x})| = 2^{-id}\lambda_0$. Set

$$\mathcal{B}_i = \{\widetilde{x} \in \mathbb{R}^{d-1} : \text{ each coordinate of } \widetilde{x} \text{ is a multiple of } 2^{-i}\}.$$

Let us define for $i \geq 0$ and $L \geq 0$

$$F_i(\widetilde{x}) = \frac{1}{|A_i(\widetilde{x})|} \int_{A_i(\widetilde{x})} |f(y)| dy, \quad G_{iL}(\widetilde{x}) = \sup_{\widetilde{y} \in \mathcal{B}_i, |\widetilde{y}| \leq L2^{-i}} F_i(\widetilde{x} + \widetilde{y}).$$

$$M_L f(\widetilde{x}) = \sup_{i \geq 0} G_{iL}(\widetilde{x}).$$

We use $\| \cdot \|_1$ to denote the L^1 norm of a function with respect to $(d-1)$-dimensional Lebesgue measure.

The following proposition's proof is similar to the analogous one in [3], but we take this opportunity to correct some errors in that paper. Note also that G_{iL} is defined differently here.

Proposition 4.1. *Suppose $f \in \Lambda_\beta^{p,\infty}(\mathbb{R}^d)$, $\beta p > 1$, $p > 1$, and the support of f is contained in $B(0, J)$ for some $J \geq 1$. Then*

$$\|M_L f(\widetilde{x})\|_1 < \kappa \|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)}. \tag{4.2}$$

The constant κ depends on d, p, L, J, λ, and β. In particular, $M_L f(\widetilde{x})$ is finite a.e.

Proof. Since

$$
\begin{aligned}
|G_{iL}(\widetilde{x}) - G_{i+1,L}(\widetilde{x})| &\leq \sup_{\widetilde{y},\widetilde{z}\in\mathcal{B}_{i+1}, |\widetilde{y}|\vee|\widetilde{z}|\leq 2L2^{-i}} |F_i(\widetilde{x}+\widetilde{y}) - F_{i+1}(\widetilde{x}+\widetilde{z})| \\
&\leq \sum_{\widetilde{y},\widetilde{z}\in\mathcal{B}_{i+1}, |\widetilde{y}|\vee|\widetilde{z}|\leq 2L2^{-i}} |F_i(\widetilde{x}+\widetilde{y}) - F_{i+1}(\widetilde{x}+\widetilde{z})|,
\end{aligned}
$$

we have

$$
\begin{aligned}
\|G_{iL}(\widetilde{x}) - G_{i+1,L}(\widetilde{x})\|_1 &\leq \sum_{\widetilde{y},\widetilde{z}\in\mathcal{B}_{i+1}, |\widetilde{y}|\vee|\widetilde{z}|\leq 2L2^{-i}} \|F_i(\widetilde{x}+\widetilde{y}) - F_{i+1}(\widetilde{x}+\widetilde{z})\|_1 \\
&\leq c_1 L^{d-1} \sup_{\widetilde{w}\in\mathcal{B}_{i+1}, |\widetilde{w}|\leq 4L2^{-i}} \|F_i(\widetilde{x}) - F_{i+1}(\widetilde{x}+\widetilde{w})\|_1. \qquad (4.3)
\end{aligned}
$$

Let $A_{ij}(\widetilde{x}),\ j = 1, \cdots, 2^d$, be a subdivision of $A_i(\widetilde{x})$ into 2^d equal rectangular solids, each of which is congruent to $A_{i+1}(\widetilde{x})$; to be more precise, if $A_i(\widetilde{x}) = \prod_{i=1}^d [a_i, b_i]$, then each $A_{ij}(\widetilde{x})$ is of the form $\prod_{i=1}^d [d_i, e_i]$, where for each i either $d_i = a_i$ and $e_i = (a_i + b_i)/2$ or else $d_i = (a_i + b_i)/2$ and $e_i = b_i$.

Fix a $\widetilde{w} \in \mathcal{B}_{i+1}$ with $|\widetilde{w}| \leq 4L2^{-i}$. Let $T_{ij} : \mathbb{R}^d \to \mathbb{R}^d$ be the linear map defined by

$$
T_{ij}(\widetilde{x}, x_d) = (\widetilde{x} + \widetilde{t}_{ij}, x_d + t_{ij}^d),
$$

where $t_{ij} = (\widetilde{t}_{ij}, t_{ij}^d)$ are points such that $A_{ij}(\widetilde{x}) = T_{ij}(A_{i+1}(\widetilde{x}+\widetilde{w}))$; we can find a constant c_2 not depending on i such that $|t_{ij}| \leq c_2 2^{-i}$ for $j = 1, \ldots, 2^d$.

Then

$$
\begin{aligned}
&F_i(\widetilde{x}) - F_{i+1}(\widetilde{x}+\widetilde{w}) \qquad (4.4) \\
&= \frac{1}{|A_i(\widetilde{x})|} \int_{A_i(\widetilde{x})} |f(y)| dy - \frac{1}{|A_{i+1}(\widetilde{x}+\widetilde{w})|} \int_{A_{i+1}(\widetilde{x}+\widetilde{w})} |f(y)| dy \\
&= \sum_{j=1}^{2^d} \frac{1}{|A_i(\widetilde{x})|} \Big[\int_{A_{ij}(\widetilde{x})} |f(y)| dy - \int_{A_{i+1}(\widetilde{x}+\widetilde{w})} |f(y)| dy \Big] \\
&= \sum_{j=1}^{2^d} 2^{id} \lambda_0^{-1} \Big[\int_{T_{ij}(A_{ij}(\widetilde{x}+\widetilde{w}))} |f(y)| dy - \int_{A_{i+1}(\widetilde{x}+\widetilde{w})} |f(y)| dy \Big] \\
&= \sum_{j=1}^{2^d} 2^{id} \lambda_0^{-1} \int_{A_{i+1}(\widetilde{x}+\widetilde{w})} \left(|f(T_{ij}(y))| - |f(y)| \right) dy.
\end{aligned}
$$

Integrating (4.4) gives

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} |F_i(\widetilde{x}+\widetilde{w}) - F_{i+1}(\widetilde{x})| d\widetilde{x} \\
&\quad \le \sum_{j=1}^{2^d} 2^{id}\lambda_0^{-1} \int_{\mathbb{R}^{d-1}} \int_{A_{i+1}(\widetilde{x}+\widetilde{w})} |f(y) - f(T_{ij}(y))| dy\, d\widetilde{x} \\
&\quad = \sum_{j=1}^{2^d} 2^{id}\lambda_0^{-1} \int_{\mathbb{R}^{d-1}} \int_{\widetilde{A}_{i+1}(\widetilde{x}+\widetilde{w})} \int_{\Gamma(\widetilde{x}+\widetilde{w})-2^{-i}\lambda_0}^{\Gamma(\widetilde{x}+\widetilde{w})-2^{-i-1}\lambda_0} |f(y) - f(T_{ij}(y))| dy_d\, d\widetilde{y}\, d\widetilde{x}.
\end{aligned}$$

Let $\widetilde{z} = \widetilde{y} - (\widetilde{x}+\widetilde{w}),\ z_d = y_d$. Then by Fubini's theorem the last expression can be written as

$$\begin{aligned}
&\sum_{j=1}^{2^d} 2^{id}\lambda_0^{-1} \int_{\widetilde{A}_{i+1}(\widetilde{0})} \int_{\mathbb{R}^{d-1}} \int_{\Gamma(\widetilde{x}+\widetilde{w})-2^{-i}\lambda_0}^{\Gamma(\widetilde{x}+\widetilde{w})-2^{-i-1}\lambda_0} |f(\widetilde{z}+\widetilde{x}+\widetilde{w}, z_d) \\
&\qquad - f(\widetilde{z}+\widetilde{x}+\widetilde{w}+\widetilde{t}_{ij}, z_d + t_{ij}^d)| dz_d\, d\widetilde{x}\, d\widetilde{z} \\
&\le \sum_{j=1}^{2^d} 2^{id}\lambda_0^{-1} \int_{\widetilde{A}_{i+1}(\widetilde{0})} \Big(\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |f(\widetilde{z}+\widetilde{x}+\widetilde{w}, z_d) \\
&\qquad - f(\widetilde{z}+\widetilde{x}+\widetilde{w}+\widetilde{t}_{ij}, z_d + t_{ij}^d)|^p dz_d\, d\widetilde{x} \Big)^{1/p} \\
&\qquad \times \left(\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \chi_{B(0,J)\cap(\mathbb{R}^{d-1}\times[-2^{-i}, -2^{-i-1}])} dz_d\, d\widetilde{x} \right)^{1/q} d\widetilde{z} \\
&\le \sum_{j=1}^{2^d} 2^{id}\lambda_0^{-1} \int_{\widetilde{A}_{i+1}(\widetilde{0})} |(\widetilde{t}_{ij}, t_{ij}^d)|^\beta \\
&\qquad \times \|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)} J^{(d-1)/q} 2^{-i/q}\, d\widetilde{z},
\end{aligned} \tag{4.5}$$

where we use Hölder's inequality with $p^{-1} + q^{-1} = 1$. From (4.5) we have

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} |F_i(\widetilde{x}) - F_{i+1}(\widetilde{x}+\widetilde{w})| d\widetilde{x} \\
&\qquad \le c_3 \sum_{j=1}^{2^d} 2^i |t_{ij}|^\beta \|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)} 2^{-i/q}.
\end{aligned} \tag{4.6}$$

Since $|t_{ij}| \le c_4 L 2^{-i}$, for $j = 1, \dots, 2^d$, $\widetilde{w} \in \mathcal{B}_{i+1}$, and $|\widetilde{w}| \le 4L2^{-i}$, from (4.3) and (4.6) we have

$$\begin{aligned} \|G_{iL}(\widetilde{x}) - G_{i+1,L}(\widetilde{x})\|_1 &\le c_5 2^{i(1-1/q)} (2^{-i})^{\beta} \|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)} \\ &\le c_5 2^{((1/p)-\beta)i} \|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)}. \end{aligned} \tag{4.7}$$

To prove $\|M_L f(\widetilde{x})\|_1 < \kappa \|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)}$, note that

$$\sup_i |G_{iL}(\widetilde{x})| \le |G_{0L}(\widetilde{x})| + \sum_{i=1}^{\infty} |G_{iL}(\widetilde{x}) - G_{i-1,L}(\widetilde{x})|.$$

Clearly $\|G_{0L}\|_1 < c_6 \|f\|_p$ because f is in L^p and has compact support. Hence by (4.7)

$$\begin{aligned} \|M_L f(\widetilde{x})\|_1 = \|\sup_i G_{iL}(\widetilde{x})\|_1 &\le \|G_{0L}(\widetilde{x})\|_1 + \sum_{i=1}^{\infty} \|G_{iL}(\widetilde{x}) - G_{i-1,L}(\widetilde{x})\|_1 \\ &\le c_6 \|f\|_p + c_7 \|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)} \sum_{i=1}^{\infty} 2^{((1/p)-\beta)i} \\ &< c_8 \|f\|_{\Lambda_\beta^{p,\infty}(\mathbb{R}^d)}, \end{aligned} \tag{4.8}$$

if $\beta p > 1$. □

5 The Fatou theorem

In this section we prove Theorem 1.1.

Proposition 5.1. *Suppose D is the region above the graph of a Lipschitz function with Lipschitz constant λ and f is bounded, has compact support, and is in $\Lambda_\beta^{p,\infty}(\mathbb{R}^d)$ with $\beta p > 1$. Then u_f converges nontangentially at almost every point on the boundary.*

Proof. Let $\varepsilon > 0$. By Corollary 3.4 there exists a_1 such that for $x \in D$

$$\mathbb{P}^x(X_{\tau_D} \in S_{a_1 v(x)}) < \varepsilon.$$

By Corollary 3.7 there exists a_2 such that for $x \in D$

$$\mathbb{P}^x(X_{\tau_D} \in B(x, a_2 \delta_D(x))^c) < \varepsilon.$$

Recall the definition of $A_i(\widetilde{x})$ given in (4.1). We claim that if $v(x)$ is sufficiently small, there exist positive integers b_1, b_2, and L independent of $v(x)$ and an integer $i_0 \geq b_1 + 1$ depending on x such that

$$\Big(D^c \cap B(x, a_2\delta_D(x))\Big) \setminus S_{a_1 v(x)} \subset \bigcup_{i=i_0-b_1}^{i_0+b_2} \Big(\bigcup_{\{\widetilde{t}_j \in \mathcal{B}_i, |\widetilde{t}_j| \leq L2^{-i}\}} A_i(\widetilde{x} + \widetilde{t}_j)\Big). \quad (5.1)$$

To see this, first choose i_0 such that $\lambda_0 2^{-i_0} \leq \delta_D(x) < \lambda_0 2^{-i_0+1}$. We can then choose b_1 and b_2 independently of i_0 such that if y is an element of the left hand side of (5.1), then $y \in \cup_{i=i_0-b_1}^{i_0+b_2} A_i(\widetilde{y})$. Now if y is an element of the left hand side of (5.1), then the distance between $\widetilde{x}$ and $\widetilde{y}$ is less than $a_2\delta_D(x) \leq c_1 2^{-i_0}$; so if we take L larger than $c_1 2^{b_2+1}$, then y will be an element of the right hand side of (5.1). This proves (5.1).

Choose $\beta' \in (1/p, \beta)$. Let κ be the constant in (4.2) when we replace β by β'; recall that κ depends on L. By [3], Lemma 4.1, we can choose $h \in \Lambda^{p,\infty}_{\beta'}(\mathbb{R}^d)$ such that $\|h\|_\infty \leq 2\|f\|_\infty$,

$$\|h\|_{\Lambda^{p,\infty}_{\beta'}(\mathbb{R}^d)} < \varepsilon^2/(\kappa(b_1 + b_2 + 1)L^{d-1}),$$

and $g = f - h$ is continuous with compact support. For any function k on D, define

$$\Omega k(x) = \limsup_{z \in C(x), z \to x} u_k(z) - \liminf_{z \in C(x) z \to x} u_k(z)$$

for $x \in \partial D$. We want to show that if $x_0 \in \partial D$, then

$$\Omega f(x_0) \leq c_2 \varepsilon \|f\|_\infty + c_3 M h(x_0). \quad (5.2)$$

If $z \in A_i(\widetilde{x} + t_j)$, then $P_D(x, z) \leq c_4 \delta_D(z)^{-d}$ by Proposition 2.1. Suppose $x \in D$ with $\widetilde{x} = \widetilde{x}_0$. Writing h^+ for the positive part of h and defining u_{h^+}

analogously to (1.1), we have

$$\begin{aligned}
u_{h^+}(x) &\leq \mathbb{E}^x[h^+(X_{\tau_D}); X_{\tau_D} \in S_{a_1 v(x)}] + \mathbb{E}^x[h^+(X_{\tau_D}); X_{\tau_D} \notin B(x, a_2 \delta_D(x))] \\
&\quad + \sum_{i=i_0-b_1}^{i_0+b_2} \sum_{\{\widetilde{t}_j \in \mathcal{B}_i, |\widetilde{t}_j| \leq L2^{-i}\}} \mathbb{E}^x[h^+(X_{\tau_D}); X_{\tau_D} \in A_i(\widetilde{x} + \widetilde{t}_j)] \\
&\leq 2\|h\|_\infty \varepsilon + \sum_{i=i_0-b_1}^{i_0+b_2} \sum_{\{\widetilde{t}_j \in \mathcal{B}_i, |\widetilde{t}_j| \leq L2^{-i}\}} \int_{A_i(\widetilde{x}+\widetilde{t}_j)} h^+(z) P_D(x,z)\, dz \\
&\leq 2\|h^+\|_\infty \varepsilon + c_5 \sum_{i=i_0-b_1}^{i_0+b_2} \sum_{\{\widetilde{t}_j \in \mathcal{B}_i, |\widetilde{t}_j| \leq L2^{-i}\}} F_i(\widetilde{x} + \widetilde{t}_j) \\
&\leq 2\|h^+\|_\infty \varepsilon + c_6 \sum_{i=i_0-b_1}^{i_0+b_2} L^{d-1} G_{iL}(\widetilde{x}) \\
&\leq 2\|h^+\|_\infty \varepsilon + c_6 (b_1 + b_2 + 1) L^{d-1} M_L h^+(\widetilde{x}),
\end{aligned}$$

where we define F_i and G_{iL} in terms of h^+ instead of f. By the Harnack inequality for nonnegative α-harmonic functions,

$$\begin{aligned}
\limsup_{y \in C(x_0), y \to x_0} u_{h^+}(y) &\leq c_7 \|h^+\|_\infty \varepsilon + c_7 (b_1 + b_2 + 1) L^{d-1} M_L h^+(\widetilde{x}_0) \\
&\leq c_8 \|f\|_\infty \varepsilon + c_8 (b_1 + b_2 + 1) L^{d-1} M_L h(\widetilde{x}_0).
\end{aligned}$$

We have a similar estimate when h^+ is replaced by h^-. Since g is continuous with compact support,

$$\begin{aligned}
\Omega f(x_0) &\leq \Omega g(x_0) + \Omega h(x_0) = \Omega h(x_0) \\
&\leq 2c_8 \|f\|_\infty \varepsilon + 2c_8 (b_1 + b_2 + 1) L^{d-1} M_L h(\widetilde{x}_0)
\end{aligned}$$

as desired.

By Proposition 4.1,

$$\begin{aligned}
&|\{x_0 \in \partial D : (b_1 + b_2 + 1) L^{d-1} M_L h(\widetilde{x}_0) > \varepsilon\}| \leq \frac{(b_1 + b_2 + 1) L^{d-1}}{\varepsilon} \|M_L h\|_1 \\
&\leq \frac{\kappa (b_1 + b_2 + 1) L^{d-1}}{\varepsilon} \|h\|_{\Lambda^{p,\infty}_{\beta'}(\mathbb{R}^d)} \leq c_9 \varepsilon.
\end{aligned}$$

Recall that $(d-1)$-dimensional Lebesgue measure is comparable to surface measure. Therefore, except for a set of surface measure at most $c_{10}\varepsilon$ we have

$$\Omega f(x_0) \leq \Omega h(x_0) \leq (2c_8 \|f\|_\infty + 2c_8)\varepsilon. \tag{5.3}$$

Since ε is arbitrary, this proves the proposition. $\square$

To handle bounded Lipschitz domains we first need some lemmas.

Lemma 5.2. *Let D be a bounded Lipschitz domain, $z \in \partial D$, and $M > 0$. If $x_n \in D$, $n = 1, 2, \ldots$, and $x_n \to z$, then $\mathbb{P}^{x_n}(X_{\tau_D} \in B(z, M) \cap D^c) \to 1$.*

Proof. $z \in \partial D$ is regular for D^c (the proof that every point in ∂D is regular for the complement of the domain is the same as that for Brownian motion; see [1], Proposition II.1.13), and so $\mathbb{P}^{x_n}(\tau_D \le t) \to 1$ for all $t > 0$; see [1], Corollary II.1.11. Let $\varepsilon > 0$ and choose t such that $\mathbb{P}^0(\sup_{s \le t} |X_s| \ge M/2) < \varepsilon$; this is possible by the right continuity of the paths of X_t. We have $|x_n - z| < M/2$ and $\mathbb{P}^{x_n}(\tau_D \le t) \ge 1 - \varepsilon$ for sufficiently large n. So we have

$$\begin{aligned} \mathbb{P}^{x_n}(X_{\tau_D} \in B(z, M) \cap D^c) &\ge \mathbb{P}^{x_n}(\tau_D \le t, \sup_{s \le t} |X_s - x_n| < M/2) \\ &\ge \mathbb{P}^{x_n}(\tau_D \le t) - \mathbb{P}^0(\sup_{s \le t} |X_s| \ge M/2) \\ &\ge 1 - \varepsilon - \varepsilon. \end{aligned}$$

$\square$

Lemma 5.3. *Let D be the region above the graph of a Lipschitz function with Lipschitz constant λ, $z \in \partial D$, and $M > 0$, and let f be a bounded function. If $x_n \in D$, $x_n \to z$, and either $\lim_{x_n \to z} \mathbb{E}^{x_n} f(X_{\tau_D})$ or $\lim_{x_n \to z} \mathbb{E}^{x_n} f(X_{\tau_{D \cap B(z,M)}})$ exists, then they both exist and coincide.*

Proof. Letting $B_M = D \cap B(z, M)$, we have

$$\begin{aligned} \mathbb{E}^{x_n} f(X_{\tau_D}) &= \mathbb{E}^{x_n}[f(X_{\tau_D}); \tau_{B_M} < \tau_D] + \mathbb{E}^{x_n}[f(X_{\tau_D}); \tau_{B_M} = \tau_D] \\ &= \mathbb{E}^{x_n}[f(X_{\tau_D}); \tau_{B_M} < \tau_D] + \mathbb{E}^{x_n}[f(X_{\tau_{B_M}}); \tau_{B_M} = \tau_D] \\ &= \mathbb{E}^{x_n}[f(X_{\tau_D}); \tau_{B_M} < \tau_D] + \mathbb{E}^{x_n} f(X_{\tau_{B_M}}) - \mathbb{E}^{x_n}[f(X_{\tau_{B_M}}); \tau_{B_M} < \tau_D]. \end{aligned} \tag{5.4}$$

Note that the first and third terms of the last line are both bounded by

$$\|f\|_\infty \mathbb{P}^{x_n}(\tau_{B_M} < \tau_D).$$

Since

$$\mathbb{P}^{x_n}(\tau_{B_M} < \tau_D) = 1 - \mathbb{P}^{x_n}(\tau_{B_M} = \tau_D) \le 1 - \mathbb{P}^{x_n}(X_{\tau_D} \in B(z, M) \cap D^c) \to 0$$

as $x_n \to z$ by Lemma 5.2, the first and the last terms in the last line of (5.4) go to 0 as $x_n \to z$, which completes the proof. $\square$

We now prove Theorem 1.1.

Proof. First we will consider the case when D is the region above the graph of a Lipschitz function Γ with Lipschitz constant λ. Suppose that f is bounded, $f \in \Lambda_\beta^{p,\infty}(\mathbb{R}^d)$, and $\beta p > 1$. Let $M > 0$. We will show nontangential convergence for $\widetilde{x}$ in $B(0, M/2) \cap \partial D$. Since M is arbitrary, the theorem in this case will follow. Let $\varphi \in C^\infty$ be a cut-off function such that $\varphi = 1$ on $B(0, 2M)$ and 0 on $B(0, 3M)^c$. Since f is bounded, $f\varphi$ is bounded and supported on $B(0, 3M)$. Since φ is smooth, then $f\varphi \in \Lambda_\beta^{p,\infty}(\mathbb{R}^d)$, so by Proposition 5.1 the Fatou theorem holds for this function. Note that $f = f\varphi + f(1-\varphi)$ and so $u_f(x) = u_{f\varphi}(x) + u_{f(1-\varphi)}(x)$. Thus it is enough to show for $z \in B(0, M/2) \cap \partial D$ that $\lim_{x_n \to z} u_{f(1-\varphi)}(x_n) = 0$ if $x_n \in D$. Since the support of $f(1-\varphi)$ is contained in $B(0, 2M)^c \subset B(z, 3M/2)^c$, we have

$$u_{f(1-\varphi)}(x_n) = \mathbb{E}^{x_n} f(1-\varphi)(X_{\tau_D}) \le \|f(1-\varphi)\|_\infty \mathbb{P}^{x_n}(X_{\tau_D} \in B(z, M)^c \cap D^c).$$

By Lemma 5.2 the right hand side goes to 0; this completes the proof in the first case.

Next we consider a bounded Lipschitz domain D. For every $z \in \partial D$ there exist $M > 0$ and a Lipschitz function Γ with Lipschitz constant λ (which can depend on x) such that in some coordinate system $D \cap B(z, M) = D_\Gamma \cap B(z, M)$, where D_Γ is the region above the graph of Γ. Suppose f is bounded and in $\Lambda_\beta^{p,\infty}(\mathbb{R}^d)$ with $\beta p > 1$. By the paragraph above, nontangential limits exist a.e. in D_Γ. By Lemma 5.3 the nontangential limits for D_Γ and for $D_\Gamma \cap B(z, M)$ in $B(z, M)$ are the same. By the same reasoning as in the proof of Lemma 5.3 the nontangential limits for D and for $D \cap B(z, M)$ in $B(z, M)$ are the same. Hence the nontangential limits for D are the same as those for D_Γ in $B(z, M)$.

Finally, suppose D is a bounded Lipschitz domain and $f \in \Lambda_{loc}^b(\beta, p, D^c)$. If $z \in \partial D$, there exists ζ (depending on z) such that f agrees on $B(z, \zeta) \cap D^c$ with a bounded function, $\bar{f}$, say, that is in $\Lambda_\beta^{p,\infty}(\mathbb{R}^d)$. Let φ be a C^∞ cutoff function that is 1 on $B(z, \zeta/2)$ and 0 on $B(z, \zeta)^c$. By the preceding paragraph, nontangential limits exists a.e. for $u_{f\varphi} = u_{\bar{f}\varphi}$ in $B(z, \zeta/4) \cap D$. By the argument in the first paragraph of this proof, nontangential limits also exist there for $u_{f(1-\varphi)}$. Therefore every point of ∂D has a neighborhood in which nontangential convergence in D holds; this proves the theorem. □

Remark. A function f that is in $\Lambda_\beta^{p,\infty}(\mathbb{R}^d)$ need not be continuous unless $\beta p > d$, and so in general its value on ∂D is not defined in a pointwise manner.

However if $\beta' p > 1$, one can define a bounded trace operator $\mathcal{T} : \Lambda^{p,\infty}_{\beta'} \to L^p(\partial D)$ which does give a definition of the restriction of f to ∂D; moreover the definition of $\mathcal{T}(f)$ is independent of the precise value of β'. A proof of this may be derived from the results in [22], Chapter VI. If $\beta p > 1$, choose $\beta' \in (1/p, \beta)$. As in the proof of Proposition 5.1, given ε, a function $f \in \Lambda^{p,\infty}_{\beta}(\mathbb{R}^d)$ can be written as the sum of a continuous function with compact support g plus a function h, where

$$\|h\|_{\Lambda^{p,\infty}_{\beta'}(\mathbb{R}^d)} < \varepsilon.$$

Since g is continuous, it is easy to see that the nontangential limit of u_g agrees with $\mathcal{T}(g)$ at every point of the boundary. Let $\delta > 0$. The proof of Proposition 5.1 (see (5.3)) shows that the nontangential limit of u_h is less than δ except for a set of surface measure less than δ provided we take ε small enough. Since $\mathcal{T}$ is a bounded operator, then $\mathcal{T}(h)$ will be less than δ except for a set of surface measure less than δ provided we take ε small enough. Therefore the nontangential limit of u_f differs from $\mathcal{T}$ by at most 2δ, except for a set of surface measure at most 2δ. Since δ is arbitrary, this shows that the nontangential limit of u_f agrees with $\mathcal{T}(f)$ almost everywhere.

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