Random walks on graphical Sierpinski carpets

Martin T. Barlow[1]
Department of Mathematics
University of British Columbia
Vancouver, V6T 1Z2
Canada

and

Richard F. Bass[2]
Department of Mathematics
University of Connecticut
Storrs, CT 06269
U.S.A.

Abstract. We consider random walks on a class of graphs derived from Sierpinski carpets. We obtain upper and lower bounds (which are non-Gaussian) on the transition probabilities which are, up to constants, the best possible. We also extend some classical Sobolev and Poincaré inequalities to this setting.

[1] Research partially supported by a NSERC (Canada) grant.
[2] Research partially supported by NSF Grant DMS 9322689.

1. Introduction.

We begin with some notation and definitions. Let $G = (V, E)$ be an infinite connected graph. We write $d(x,y)$ for the natural graph distance on V, and for $x \in V$ and $r \geq 0$ set $B(x,r) = \{y \in V : d(x,y) \leq r\}$. Let $a(x,y) = 1$ if $\{x,y\} \in E$ and $a(x,y) = 0$ otherwise, and set

$$\mu(x) = \sum_{y \in V} a(x,y).$$

We assume that G has uniformly bounded vertex degree, that is, there exists $c_1 < \infty$ such that $1 \leq \mu(x) \leq c_1$ for all $x \in V$. We define the measure μ on V by $\mu(A) = \sum_{x \in A} \mu(x)$.

The (discrete time) simple random walk (SRW) X on G is the Markov chain $X = (X_n, n \in \mathbb{Z}_+, \mathbb{P}^x, x \in V)$ with transition probabilities

$$\mathbb{P}^z(X_{n+1} = y \mid X_n = x) = \frac{a(x,y)}{\mu(x)};$$

thus at each time step X moves from a site x to one of the neighbors of x with equal probability for each. We define

$$p(n,x,y) = \mathbb{P}^x(X_n = y)\mu(y)^{-1}. \tag{1.1}$$

(The term $\mu(y)^{-1}$ above looks slightly unnatural, but it has the advantage of making $p(n,x,\cdot)$ the density of $\mathbb{P}^x(X_n \in \cdot)$, with respect to μ, and also makes $p(n,x,y) = p(n,y,x)$).

Associated with X is the μ-symmetric continuous time random walk Y_t on the graph $G = (V,E)$. This is the (continuous time) Markov process $Y = (Y_t, t \geq 0, \mathbb{Q}^x, x \in V)$ with infinitesimal generator

$$\mathcal{L}f(x) = \mu(x)^{-1} \sum_y a(x,y)(f(y) - f(x)). \tag{1.2}$$

If

$$\mathcal{E}(f,f) = \tfrac{1}{2} \sum_{x \in V} \sum_{y \in V} a(x,y)(f(x) - f(y))^2, \qquad f \in L^2(V,\mu), \tag{1.3}$$

then Y is the Markov process with Dirichlet form $(\mathcal{E}, L^2(V,\mu))$ on $L^2(V,\mu)$ – see [FOT]. Note that Y waits at a vertex an independent exponential length of time (with parameter 1) and then jumps to one of the neighboring vertices, with equal probability for each. If T_n is the time of the n-th jump of Y then $\widehat{X}_n = Y_{T_n}$ is a discrete time SRW on G (and so is equal in law to X). Set

$$q(t,x,y) = \mathbb{Q}^x(Y_t = y)\mu(y)^{-1}, \quad t \geq 0,\, x, y \in V. \tag{1.4}$$

There has been much interest in recent years in the general problem of estimating p_n and q_t from geometrical information about the graph G. For a sample of papers see [V1], [Da], [De], [P], [Co], [HSC], and for a survey see [W]. There are, of course, many similarities between this problem and the one of estimating the large time behavior of the heat kernel on a manifold M from geometrical information about M.

Given the similarity between the processes X and Y one would expect that p_n and q_t should satisfy similar bounds. This is usually the case, but the problem for p_n is generally more difficult, since X is less smooth than Y. Indeed, given bounds on p_n, simple integration gives bounds on q_t, since the number of jumps of Y in $[0,t]$ is a Poisson r.v. with mean t which is independent of $\widehat{X}$. So

$$q(t,x,y) = \sum_{n=0}^{\infty} \frac{e^{-t}t^n}{n!} p(n,x,y). \tag{1.5}$$

In obtaining bounds on p_n or q_t one almost always finds that the easiest procedure is to obtain the bounds in the following order:

1. Global upper bounds,
2. Off-diagonal upper bounds,
3. On-diagonal lower bounds,
4. Off-diagonal lower bounds,

where each step builds on information obtained in the previous step.

For the global upper bounds there is a very elegant theory, based on the work in [V1], which gives equivalences between these bounds and certain isoperimetric and Sobolev inequalities.

Notation. For $f : V \to \mathbb{R}$ set

$$||f||_p^p = \sum_x |f(x)|^p \mu(x).$$

For $e = \{x,y\} \in E$ define $a(e) = a(x,y)$ $(= a(y,x))$. We regard a as a measure on E. If $f : V \to \mathbb{R}$ define $|\nabla f| : E \to \mathbb{R}$ by

$$|\nabla f|(x,y) = |f(x) - f(y)|,$$

and set

$$||\nabla f||_p^p = \sum_{e \in E} a(e)(|\nabla f|(e))^p, \quad p \geq 1.$$

Note that $||\nabla f||_2^2 = \mathcal{E}(f,f)$, and also that

$$c_1 \mathcal{E}(f,f) \le ||f||_2^2 \le ||f||_1^2. \tag{1.6}$$

For $A \subset V$ set $\partial A = \{x \in A^c : d(x,A) = 1\}$, where $d(x,A) = \min_{y \in A} d(x,y)$. The cardinality of a set A will be denoted $|A|$.

Definition 1.1. Let $p \ge 1$.

G satisfies a *p-isoperimetric inequality* (I_p) if there exists $c_1 < \infty$ such that

$$\mu(A)^{p-1} \le c_1 \mu(\partial A)^p \text{ for all finite } A \subset V.$$

G satisfies a (α, p)*-Sobolev inequality* (S_p^α) (here $\alpha = 1$ or 2) if there exists $c_2 > 0$ such that

$$||\nabla f||_\alpha \ge c_2 ||f||_{\alpha p/(p-\alpha)}, \qquad \text{for } f \text{ with compact support.}$$

G satisfies a *p-Nash inequality* (N_p) if for some $c_3 > 0$

$$\mathcal{E}(f,f) \ge c_3 ||\nabla f||_2^{2+4/p} ||f||_1^{-4/p} \qquad \text{for } f \text{ with compact support.}$$

G satisfies (P_p) if for some $c_4 > 0$

$$p(n,x,y) \le c_4 n^{-p/2}, \quad n \ge 0,\ x,y \in V.$$

G satisfies (Q_p) if for some $c_5 > 0$

$$q(t,x,y) \le c_5 t^{-p/2}, \quad t \ge 0,\ x,y \in V.$$

Theorem 1.2. *(See [V1], [CKS]).*
(a) If $p > 1$ *then* $(I_p) \iff (S_p^1) \Rightarrow (N_p) \iff (Q_p) \iff (P_p)$.
(b) If $p > 2$ *then* $(I_p) \iff (S_p^1) \Rightarrow (S_p^2) \iff (N_p) \iff (Q_p) \iff (P_p)$.

Thus Theorem 1.2 enables one to obtain global upper bounds on $q(t,x,y)$ starting from an isoperimetric inequality on G. Bounds of this kind are called 'on-diagonal': they follow from the same bound on $q(t,x,x)$ since

$$q(t,x,y) \le q(t,x,x)^{1/2} q(t,y,y)^{1/2}. \tag{1.7}$$

Off-diagonal upper bounds now follow by a variety of techniques: 'Davies method' (see [CKS]), or the wave equation approach (as in [V2], [Ca]), which yields the general bound,

$$p(n,x,y) \le 2(\mu_x/\mu_y)^{1/2} e^{-d(x,y)^2/2n}, \tag{1.8}$$

valid for the SRW X on any infinite graph.

Theorem 1.3. *([V2], [HSC]). Suppose G satisfies (Q_p). Then*

$$p(n,x,y) \le c_1 n^{-p/2} \exp(-c_2 d(x,y)^2/n), \quad x,y \in V,\ n \ge 1, \tag{1.9}$$

$$q(t,x,y) \le c_3 t^{-p/2} \exp(-c_4 d(x,y)^2/t), \quad x,y \in V,\ d(x,y) \le t. \tag{1.10}$$

Remark. Note that $p(n,x,y) = 0$ if $d(x,y) > n$. We cannot expect the Gaussian decay in (1.10) to hold for $d(x,y) \gg t$. For example, for the SRW on $\mathbb{Z}$, if $t = 1$ and $m \gg 1$ then by (1.3)

$$q(1,0,m) \geq (e^{-1}/m!)p(m,0,m) \sim cm^{-1/2}(2e)^{-m}m^{-m},$$

which decays as $e^{-m \log m}$.

For regular graphs we can summarize much of the information above in terms of a number of 'dimensions' of the graph V.

Definition 1.4. We say

G has *uniform volume growth* d_f if

$$c_1 r^{d_f} \leq \mu(B(x,r)) \leq c_2 r^{d_f}, \quad r \geq 1, x \in V. \tag{1.11}$$

(Note that (1.11) implies that G satisfies a volume doubling condition).

The *isoperimetric dimension* d_i of G is defined by

$$d_i = d_i(G) = \sup\{p : G \text{ satisfies } (I_p)\}.$$

The *spectral dimension* d_s of G is defined by

$$d_s = \sup\{p : G \text{ satisfies } (Q_p)\}.$$

Theorem 1.2(a) implies that $d_i \leq d_s$, and it is not hard to deduce from (1.8) that $d_s \leq d_f$.

The estimates of Theorem 1.3, together with a bound on the volume growth of G gives easily an on-diagonal lower bound for q – see for example Theorem 6.1 below. However, the final step, obtaining off-diagonal bounds on p or q is harder – generally some more information of an analytic kind about G is needed, such as that G satisfies a Poincaré inequality.

For many graphs the upper bound techniques outlined above do provide bounds which are, up to constants, best possible. However, there are graphs which have a quite regular structure but for which the bounds in Theorem 1.3 are far from giving the true behavior of $p(n,x,y)$. These 'fractal graphs' have large holes, so that $d_i < d_f$; this acts to restrict the dispersion of the random walk, and causes p and q to decay in a non-Gaussian fashion.

We will not attempt to define 'fractal graphs' but will simply remark that given any finitely ramified fractal F in one of the families which has been studied in the diffusions on fractals literature (such as nested fractals, post-critically finite self-similar sets, or Sierpinski carpets – see [B1]), there is a natural way to construct an infinite graph G, such that the large-scale structure of G mimics the microscopic structure of F.

The first treatment of heat kernel bounds on fractal graphs was by Jones [J], on the graphical Sierpinski gasket. (See Figure 1.1). This graph has uniform volume growth $d_f = \log 3/\log 2$, but since $|\partial B(0, 2^n - 1)| = 2$ for all n it satisfies $d_i = 1$. (So the technique of obtaining bounds on q_t from an isoperimetric inequality fails completely in this case).

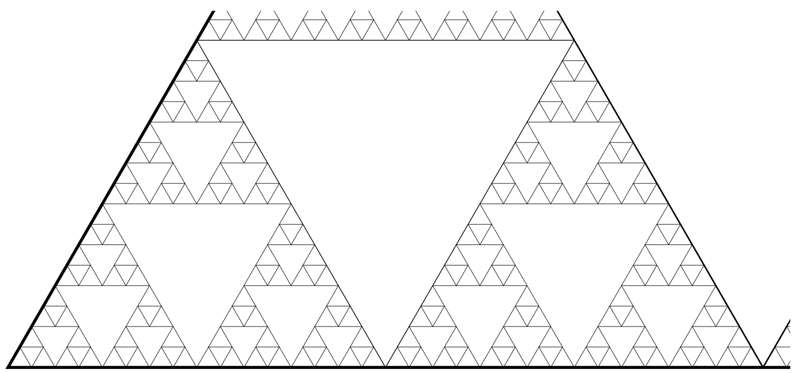

Figure 1: The graphical Sierpinski gasket.

In this paper we study a class of graphs derived from generalized Sierpinski carpets. We defer the full definition of these graphs to Section 2. The simplest examples of the graphs in this class are those derived from the standard d-dimensional Sierpinski carpet.

Let $H_0 = \mathbb{Z}^d_+$. For $x = (n_1, n_2, ..., n_d) \in H_0$ write n_i in ternary: so $n_i = \sum_{r=0}^{\infty} n_{ir} 3^r$, where $n_{ir} \in \{0, 1, 2\}$, and $n_{ir} = 0$ for all but finitely many r. Set

$$J_k = \{(n_1, ..., n_d) : n_{ik} = 1 \text{ for } 1 \le i \le d\},$$

so that J_k consists of a union of disjoint cubes of side 3^k: the cube in J_k closest to the origin is $\{3^k, \dots, 2.3^k - 1\}^d$. Now set

$$H_n = H_0 - \bigcup_{k=1}^{n} J_k, \quad H = \bigcap_{n=0}^{\infty} H_n.$$

To match our notation below we take $V = V^{(d)} = a + H$, where $a = (\frac{1}{2}, ..., \frac{1}{2}) \in \mathbb{Z}^d$. For $x, y \in V$ let $\{x, y\} \in E$ if and only if $|x - y| = 1$.

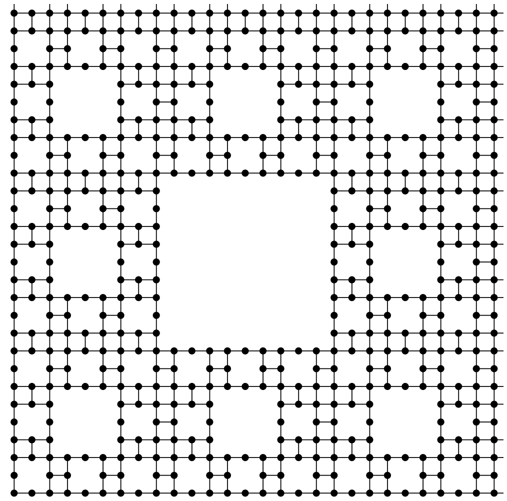

Figure 1.2: The graphical Sierpinski carpet $V^{(2)}$.

It is not hard to see that V has uniform volume growth $d_f = d_f(V^{(d)}) = \log(3^d - 1)/\log 3$. However the presence of large holes means that the process X moves less rapidly on V than on $\mathbb{Z}^d$; the holes act as obstacles which take a significant amount of time for X to find its way around. We shall see below that there exists a parameter d_w (called the *walk dimension of* V) such that writing $\sigma(x,r) = \min\{n \geq 0 : X_n \notin B(x,r)\}$,

$$c_1 r^{d_w} \leq \mathbb{E}^x \sigma(x,r) \leq c_2 r^{d_w}, \; r \geq 0. \qquad (1.12)$$

(Compare this with the classical estimate $\mathbb{E}^x \sigma(x,n) \approx n^2$ for a simple random walk in $\mathbb{Z}^2$). The estimate (1.12) shows that $p(n,x,\cdot)$ puts nearly all its mass on a ball of radius roughly n^{1/d_w}. So if $p(n,x,.)$ were reasonably evenly spread on this set, we would have (since $\mu(B(x,n^{1/d_w}) \approx n^{d_f/d_w})$ that

$$p(n,x,y) \leq c n^{-d_f/d_w}, \; x,y \in V, \; n \geq 1. \qquad (1.13)$$

We prove this estimate below (Theorem 5.2) by deriving it from a Nash inequality which holds in the continuous setting. By Theorem 1.3 we deduce therefore from (1.13) that $p(n,x,y)$ satisfies (1.9) with $p = d_s/2$. However, this upper bound is still not a very good one.

We prove the following sharp (up to constants) upper and lower bounds for $p(n,x,y)$.

Theorem 1.5. *There exists $d_w > 2$, and constants c_1, c_2, c_3, c_4 such that if n and $d(x,y)$ are both odd or both even and $d(x,y) \leq n$, then*

$$\begin{aligned} c_1 n^{-d_f/d_w} \exp(-c_2 (d(x,y)^{d_w}/n)^{1/(d_w-1)}) &\leq p(n,x,y) \qquad (1.14) \\ &\leq c_3 n^{-d_f/d_w} \exp(-c_4 (d(x,y)^{d_w}/n)^{1/(d_w-1)}). \end{aligned}$$

Note that since G is, like $\mathbb{Z}^d$, bipartite, the random walk has period two, and hence the transition probabilities will be 0 if one of $d(x, y)$ and n is even and the other is odd. Note also that when $d(x, y) \geq halfn$ then the bounds (1.14) and (1.9) take the same form. The estimate (1.14) implies immediately that the spectal dimension d_s for G is $d_s = 2d_f/d_w$.

Let $C = \{x : |x_i| \leq \frac{1}{2}\}$ and $P = \cup_{x \in V}(x + C)$: we call P the *pre-carpet*. Our results here rely heavily on the estimates for Brownian motion on P with normal reflection on ∂P developed by us in [BB3]. We refer the reader to that paper for more discussion of the significance of d_w and d_s and for background on diffusions on fractals. See also [Ba].

Our techniques may perhaps be of some interest as well in showing how one might obtain estimates on graphs from information on a related continuous-time continuous-space process.

In Section 2 we introduce the notation we will use and recall the key inequalities we need from [BB3]. Section 3 extends the Sobolev and Poincaré inequalities to the graphical setting. In Section 4 we show how the coupling argument of [BB3] yields a Harnack inequality for the random walks on our graphs. The upper bound in (1.14) is derived in Section 5 (see Theorem 5.6) and the lower bound in Section 6 (see Theorem 6.5). Section 7 contains some further results, such as a Liouville theorem, and estimates for the process Y on the graph G.

We use the letter c with subscripts to denote constants which depend only on the graph G. We renumber the constants for each lemma, proposition, theorem, and corollary.

2. Notation and preliminaries.

Let $d \geq 2$, $F_0 = [0, 1]^d$, and let $l_F \in \mathbb{N}$, $l_F \geq 3$ be fixed. For $n \in \mathbb{Z}$ let $\mathcal{S}_n$ be the collection of closed cubes of side l_F^n with vertices in $l_F^n \mathbb{Z}^d$. For $A \subseteq \mathbb{R}^d$, set

$$\mathcal{S}_n(A) = \{S : S \subset A, S \in \mathcal{S}_n\}.$$

For $S \in \mathcal{S}_n$, let Ψ_S be the orientation preserving affine map which maps F_0 onto S.

We define a decreasing sequence (F_n) of closed subsets of F_0. Let $1 \leq m_F \leq l_F^d$ be an integer, and let F_1 be the union of m_F distinct elements of $\mathcal{S}_{-1}(F_0)$. We impose the following conditions on F_1:

Hypotheses 2.1.

(H1) *(Symmetry)* *F_1 is preserved by all the isometries of the unit cube F_0.*

(H2) *(Connectedness)* $\mathrm{Int}(F_1)$ *is connected, and contains a path connecting the hyperplanes $\{x_1 = 0\}$ and $\{x_1 = 1\}$.*

(H3) *(Non-diagonality) Let B be a cube in F_0 which is the union of 2^d distinct elements of $\mathcal{S}_{-1}$. (So B has side length $2l_F^{-1}$). Then if* $\operatorname{int}(F_1 \cap B)$ *is non-empty, it is connected.*

(H4) *(Borders included) F_1 contains the line segment*

$$\{x : 0 \le x_1 \le 1, x_2 = \ldots = x_d = 0\}.$$

We may think of F_1 as being derived from F_0 by removing the interiors of $l_F^d - m_F$ squares in $\mathcal{S}_{-1}(F_0)$. Given F_1, F_2 is obtained by removing the same pattern from each of the squares in $\mathcal{S}_{-1}(F_1)$. Iterating, we obtain a sequence (F_n), where F_n is the union of m_F^n squares in $\mathcal{S}_{-n}(F_0)$. Formally, we define

$$F_{n+1} = \bigcup_{S \in \mathcal{S}_{-n}(F_n)} \Psi_S(F_1) = \bigcup_{S \in \mathcal{S}_{-1}(F_1)} \Psi_S(F_n), \quad n \ge 1.$$

We call the set $F = \cap_{n=0}^{\infty} F_n$ a *generalized Sierpinski carpet.* Let d_f denote the Hausdorff dimension of F; by [Hu] $d_f = \log m_F / \log l_F$.

Let

$$P = \bigcup_{r=0}^{\infty} l_F^r F_r; \tag{2.1}$$

we call P the *pre-carpet* (see [O]). Let μ_P be Lebesgue measure restricted to P.

We now define our graph $G = (V, E)$. Let V be the collection of centers of cubes in $\mathcal{S}_0(P)$, and for x, $y \in V$ let $\{x, y\} \in E$ if and only if $|x - y| = 1$. Let a, μ, X, $p(n, x, y)$, Y, $q(t, x, y)$, and $\mathcal{E}$ be as in Section 1. If we restrict the sum in (1.3) to $x, y \in D \subset V$, we will write $\mathcal{E}_D(f, f)$. For $S \in \mathcal{S}_0(P)$ write z_S for the center of S.

Recall that $B(x, r) = \{y \in V : d(x, y) \le r\}$. We will define $S(x, r) \subset P$ by

$$S(x, r) = \bigcup \{S \in \mathcal{S}_0(P) : d(z_S, x) \le r\}.$$

We let

$$\sigma(x, r) = \sigma^X(x, r) = \sigma(x, r, X) = \inf\{n \ge 0 : X_n \notin B(x, r)\}. \tag{2.2}$$

Note the following simple result, which holds for any graph for which the vertex degree is bounded.

Lemma 2.2. *Let $G = (V, E)$, let d be the usual graph distance on G, and suppose $|B(x, 1)| \le M$ for all $x \in V$. Let*

$$\mathcal{E}_G^{(n)}(g, g) = \tfrac{1}{2} \sum_x \sum_y 1_{(d(x,y) \le n)} (g(x) - g(y))^2.$$

Then there exists $c_1(n, M)$, depending only on n, M, such that

$$\mathcal{E}_G^{(n)}(g, g) \le c_1(n, M) \mathcal{E}_G^{(1)}(g, g).$$

Finally, we need to summarize some inequalities obtained for P in [BB3]. Define

$$\mathcal{E}^0(f, f) = \tfrac{1}{2} \int_P |\nabla f(z)|^2 \, dz.$$

This is the Dirichlet form associated with W, reflecting Brownian motion on P, with with normal reflection on ∂P. Associated with W and P is a constant d_w, called the walk dimension: see [BB3] for the definition of d_w in terms of electrical resistance. Set

$$d_s = 2 d_f / d_w.$$

The process W on P is transient if and only if $d_s > 2$ – see [SCH, Theorem 8.1].

We now suppose $d_s > 2$, and define the capacity of A with respect to W. Let $u(x, y)$ denote the Green function of W, and let $\partial_P A$ denote the relative boundary of A in P. We will need to consider only sets A for which all points of $\partial_P A$ are regular for $A^c \subset P$. For $A \subset P$ the capacity of A is defined by

$$C_0(A) = \sup\{\nu(A) : \sup_x \int_P u(x, y) \nu(dy) \le 1\}.$$

It is standard (see [FOT]) that $C_0(A)$ also satisfies

$$C_0(A) = \inf\{\mathcal{E}^0(f, f) : f = 1 \text{ on } A, \quad f(x) \to 0 \text{ as } x \to \infty\}. \tag{2.3}$$

We start with the mass-capacity inequality.

Theorem 2.3. *([BB3, Sect. 7]). Suppose $d_s > 2$. There exists c_1 such that if A is a subset of P and $\mu_P(A) \ge 1$, then*

$$C_0(A) \ge c_1 \mu_P(A)^{(d_s - 2)/d_s}.$$

We have the following conditional Nash inequality for $\mathcal{E}^0$.

Theorem 2.4. *([BB3, Sect. 7]). For each $c_1 > 0$ there exists c_2 such that*

$$\|f\|_2^{2+4/d_s} \le c_2 \mathcal{E}^0(f,f) \|f\|_1^{4/d_s} \qquad \textit{whenever } \mathcal{E}^0(f,f) \le c_1 \|f\|_1^2$$

Finally we give a Poincaré inequality for P. Write

$$\mathcal{E}^0_{S(x_0,r)}(f,f) = \int_{S(x_0,r)} |\nabla f(x)|^2 dx$$

and

$$f_{S(x_0,r)} = \int_{S(x_0,r)} f(z)\, dz.$$

Theorem 2.5. *([BB3, Sect. 7]). There exists c_1 such that if $x_0 \in P$ and $r \ge 0$, then*

$$\int_{S(x_0,r)} |f - f_{S(x_0,r)}|^2 \le c_1 r^{d_w} \mathcal{E}^0_{S(x_0,r)}(f,f).$$

In [BB3] we proved the Poincaré inequality for cubical domains. The same proof works, however, for $S(n,x)$. In any case we do not use Theorem 2.5 below.

3. Analytic Inequalities for G.

In this section we will derive Sobolev and Poincaré inequalities for $\mathcal{E}$ from the corresponding inequalities for the Dirichlet form $\mathcal{E}^0$ on the pre-carpet P.

To approximate functions on G by functions on P it will be useful to introduce another graph $H = (U \cup W, E_H)$. Let D be a union of cubes in $\mathcal{S}_0(P)$. Let $U = \mathbb{Z}^d \cap D$, $W = V \cap D$, and let $\{x,y\} \in E_H$ if and only if there exists $S \in \mathcal{S}_0(D)$ such that $x, y \in S$, and exactly one of x, y is in each of U, W. (So H consists of the centers and corners of the cubes in $\mathcal{S}_0(D)$, with edges joining the center of each cube $S \in \mathcal{S}_0(D)$ to its corners). Let d_H be graph distance on H, let $b(x,y) = 1_{(d_H(x,y)=1)}$, and let

$$\begin{aligned} \mathcal{E}^H_D(h,h) &= \tfrac{1}{2} \sum_{x \in U \cup W} \sum_{y \in U \cup W} b(x,y)(h(x) - h(y))^2 \\ &= \sum_{x \in U} \sum_{y \in W} b(x,y)(h(x)-h(y))^2. \end{aligned}$$

Set $B(x) = \sum_{y \in U \cup W} b(x,y)$, and let $\alpha(x) = 2^{-d} B(x)$. We regard α as a measure on $U \cup W$; note that $\alpha(x) = 1$ for $x \in W$. For $h : U \cup W \to \mathbb{R}$ write

$$||h||_p^p = \sum_{x \in U \cap W} |h(x)|^p \alpha(x), \qquad p \ge 1.$$

Note that

$$||h||_p^p = \sum_{S \in \mathcal{S}_0(D)} \Big(h(z_S) + 2^{-d} \sum_{y \in U \cap S} |h(y)|^p \Big). \tag{3.1}$$

If $g \in L^2(W, \alpha)$ define Γg on $U \cup W$ by:

$$\Gamma g = \begin{cases} g(x), & x \in W, \\ B(x)^{-1} \sum_y b(x,y) g(y) & x \in U. \end{cases}$$

Lemma 3.1. *(a) If $p \geq 1$ and $g \in L^p(W, \alpha)$ then there exists $c_1(p)$ such that*

$$\sum_{x \in W} |g(x)|^p \leq \sum_{x \in U \cup W} |\Gamma g(x)|^p \alpha(x) \leq c_1(p) \sum_{x \in W} |g(x)|^p.$$

(b) If $g \in L^1(W, \alpha)$ then

$$\sum_{x \in U} \Gamma g(x) \alpha(x) = \sum_{x \in W} g(x) \alpha(x).$$

(c) There exists c_2 such that

$$\mathcal{E}_D^H(\Gamma g, \Gamma g) \leq c_2 \mathcal{E}_D(g, g).$$

Proof. (a) As $\Gamma g(x) = g(x)$ and $\alpha(x) = 1$ for $x \in W$ the left hand side is obvious. For $x \in U$

$$|\Gamma g(x)|^p \leq c_3 \Big| \sum_{y \in W} b(x,y) g(y) \Big|^p \leq c_4 \sum_{y \in W} b(x,y) |g(y)|^p,$$

and summing over $x \in U$ gives the right hand side of (a).
(b) This follows immediately from the calculation

$$\sum_{x \in U} \Gamma g(x) \alpha(x) = \sum_{x \in U} 2^{-d} \sum_{y \in W} b(x,y) g(y) = \sum_{y \in U} g(y) \alpha(y).$$

(c) Write $h = \Gamma g$. If $x \in U$, $y \in W$, and $\{x, y\}$ is an edge, let $z = z(x,y) \in W$ be such that $\{x, z\} \in E_H$, and $|g(z) - g(y)| \geq |h(x) - h(y)|$; such a choice is possible from the definition of $h(x)$. Then $d(y,z) \leq 2^d$, and so

$$\begin{aligned} \mathcal{E}_D^H(h,h) &= \sum_{y \in W} \sum_{x \in U} b(x,y) (h(y) - h(x))^2 \\ &\leq \sum_{y \in W} \sum_{x \in U} (g(y) - g(z(x,y)))^2 \\ &\leq \sum_{y \in W} \sum_{z \in W} 1_{(d(y,z) \leq 2^d)} (g(y) - g(z))^2 \\ &= \mathcal{E}_D^{(2^d)}(g,g) \leq c_5 \mathcal{E}_D(g,g), \end{aligned}$$

where we used Lemma 2.2 in the last line. □

We now perform a similar procedure to relate $\mathcal{E}_D^H$ and the Dirichlet form $\mathcal{E}_D^0$ for reflecting Brownian motion in D, defined by

$$\mathcal{E}_D^0(f,f) = \tfrac{1}{2}\int_D |\nabla f|^2.$$

If $h : U \cup W \to \mathbb{R}$, let

$$\widetilde{h}(x) = 2^{-d} \sum_{y \in U} b(x,y)h(y), \quad x \in W.$$

Define for $y \in U$, $z \in W$,

$$\varphi_y(x) = 0 \vee \prod_{i=1}^{d} (1 - |x_i - y_i|),$$
$$\psi_z(x) = 0 \vee \prod_{i=1}^{d} (1 - 2|x_i - z_i|).$$

Note that if $S \in \mathcal{S}_0(D)$, $y \in S \cap U$ then

$$\int_S \varphi_y(x)dx = \int_S \psi_{z_S}(x)dx = 2^{-d} \tag{3.2}$$

and

$$\int_S |\nabla \varphi_y(x)|^2 dx \le 1, \qquad \int_S |\nabla \psi_y(x)|^2 dx \le 2. \tag{3.3}$$

Given $h : U \cup W \to \mathbb{R}$ define $\Lambda h : D \to \mathbb{R}_+$ as follows. For $S \in \mathcal{S}_0(D)$, Λh is defined on S by

$$\Lambda h(x) = \sum_{y \in U \cap S} h(y)\varphi_y(x) + (h(z_S) - \widetilde{h}(z_S))\psi_{z_S}(x), \qquad x = (x_1, \dots, x_d) \in S.$$

Note that the definition of Λh agrees on the intersection of two cubes in $\mathcal{S}_0(D)$, and that $\Lambda h(x) = h(x)$ for $x \in U \cup W$. So Λh is a continuous extension of h from $D \cap (U \cup W)$ to D.

Lemma 3.2. *(a) For $p \geq 1$, if $h \in L^p(U \cup W, \alpha)$, then there exist $c_1(p)$ and $c_2(p)$ such that*

$$c_1(p)\|h\|_p^p \leq \int_D |\Lambda h(x)|^p \, dx \leq c_2(p)\|h\|_p^p.$$

$$\text{(b)} \qquad \int_D \Lambda h(x)dx = 2^{-d} \sum_{x \in W} \alpha(x)h(x) + (1 - 2^{-d}) \sum_{y \in U} \alpha(y)h(y).$$

(c) There exists c_3 such that

$$\mathcal{E}_D^0(\Lambda h, \Lambda h) \leq c_3 \mathcal{E}_D^H(h, h).$$

Proof. (a) Let $S \in \mathcal{S}_0(D)$, and let $y' \in S \cap (U \cup W)$ be a point at which $|h|$ attains its maximum. Then

$$|h(z_S)|^p + 2^{-d} \sum_{y \in U \cap S} |h(y)|^p \leq 2|h(y')|^p. \tag{3.4}$$

On the other hand, as $\Lambda h(y') = h(y')$, and the modulus of continuity of Λh in S is bounded by a constant times $h(y')$, there exists $c_4(p) > 0$ such that

$$\int_S \Lambda h(x)dx \geq c_4(p)|h(y')|^p. \tag{3.5}$$

The left hand side of (a) now follows on summing over $S \in \mathcal{S}_0(D)$, and using (3.1), (3.4) and (3.5). As

$$\int_S |\varphi_y(x)|^p dx = c_5(p), \qquad \int_S |\psi_{z_S}(x)|^p dx = c_6(p),$$

the right hand side of (a) follows easily from the definition of Λh.

(b) This follows from (3.2) and the definition of Λh by an straightforward calculation.

(c) Write $f = \Lambda h$, and fix $S \in \mathcal{S}_0(D)$. Then for $x \in S$, as $\sum_{y \in U \cap S} \varphi_y = 1$ on S,

$$\begin{aligned} f(x) &= \sum_{y \in U \cap S} h(y)\varphi_y(x) + (h(z_S) - \widetilde{h}(z_S))\psi_{z_S}(x) \\ &= \sum_{y \in U \cap S} (h(y) - h(z_S))(\varphi_y(x) - 2^{-d}\psi_{z_S}(x)) + h(z_S). \end{aligned}$$

Differentiating and using Cauchy-Schwarz it follows that

$$|\nabla f(x)|^2 \le 2 \sum_{y\in U\cap S} (h(y)-h(z_S))^2 \Big(\sum_{y\in U\cap S} |\nabla \varphi_y(x)|^2 + 4^{-d}|\nabla \psi_S(x)|^2 \Big).$$

Using (3.3) we therefore have

$$\int_S |\nabla f(x)|^2 \le c_7 \sum_{y\in U\cap S} (h(y)-h(z_S))^2,$$

and summing over $S\in \mathcal{S}_0(D)$ gives (c). □

We can now deduce Nash and Sobolev inequalities for $\mathcal{E}$ from those for $\mathcal{E}^0$. For the next few results, we take $D=P$, so that $W=V$.

Theorem 3.3. *Let $g\in L^2(W,|\cdot|)$. Then*

$$||g||_2^{1+4/d_s} \le c_1 \mathcal{E}(g,g)||g||_1^{4/d_s}. \qquad (3.6)$$

Further, if $d_s>2$ then

$$||g||_p^2 \le c_2\mathcal{E}(g,g), \qquad p=2d_s/(d_s-2). \qquad (3.7)$$

Proof. Since $\mathcal{E}(g^+,g^+)\le \mathcal{E}(g,g)$, it is sufficient to prove the theorem for non-negative g. Let $g\ge 0$, $h=\Gamma g$, $f=\Lambda h$. By Lemmas 3.1 and 3.2 we have, for $p=1,2$,

$$c_3(p)||g||_p \le ||f||_p \le c_4(p)||g||_p, \qquad (3.8)$$

and

$$\mathcal{E}^0(f,f)\le c_5\mathcal{E}_H(h,h)\le c_6\mathcal{E}(g,g). \qquad (3.9)$$

Therefore, using (2.4), $\mathcal{E}^0(f,f)\le c_7||f||_1^2$. So, by Theorem 2.4 we have

$$||f||_2^{1+4/d_s} \le c_1\mathcal{E}^0(f,f)||f||_1^{4/d_s},$$

and (3.6) then follows from (3.8) and (3.9).

If $d_s>2$ then (3.7) is immediate from (3.6) and Theorem 2.16 of [CKS]. □

The Sobolev inequality (3.7) implies a mass-capacity inequality. Write $C_G(A)$ for the capacity of $A\subset V$. This is defined when $d_s>2$ by

$$C_G(A)=\sup\{\nu(A):\sum_y g(x,y)\nu(\{y\})\le 1 \text{ for all } x\in V\},$$

where $g(x,y) = \sum_n p(n,x,y)$ is the Green function on V associated with X. It is standard ([FOT]) that $C_G(A)$ can also be defined by

$$C_G(A) = \inf\{\mathcal{E}(f,f) : f = 1 \text{ on } A, \quad f(x) \to 0 \text{ as } x \to \infty\}. \tag{3.10}$$

Proposition 3.4. *If $d_s > 2$ then for $A \subset V$*

$$C_G(A) \geq c_1 |A|^{(d_s-2)/d_s}.$$

Proof. We apply (3.7) to functions g that are one on A and tend to 0 as $|x| \to \infty$ to obtain

$$|A|^{2/p} \leq c_2 \mathcal{E}(g,g).$$

Taking the infimum over all such g and using (3.10) proves the proposition. □

We prove a Poincaré inequality for $\mathcal{E}$.

Theorem 3.5. *Let $n \geq 0$, $x_0 \in P$, and let $A = B(x_0, n)$. There exists c_1 such that if $g : V \cap D \to \mathbb{R}$, then*

$$\sum_{x \in D \cap V} |g(x) - g_D|^2 \leq c_1 n^{d_w} \mathcal{E}_D(g,g),$$

where $g_D = |D \cap V|^{-1} \sum_{x \in D \cap V} g(x)$.

Proof. Let $D = S(x_0, n)$. By subtracting a constant from g we can assume that $g_D = 0$. Let $h = \Gamma g$, $f = \Lambda h$. By part (b) of Lemmas 3.1 and 3.2, we have

$$0 = \sum_{x \in V} h(x) = \sum_{x \in U} \alpha(x) h(x) = \int_D f(x) dx,$$

so, by Theorem 2.5,

$$\int_D f^2 \leq c_2 n^{d_w} \mathcal{E}_0^D(f,f).$$

On the other hand, by Lemmas 3.1 and 3.2,

$$\sum_{x \in D \cap V} g(x)^2 \leq c_3 \int_D f^2, \qquad \mathcal{E}_0^D(f,f) \leq c_4 \mathcal{E}^D(g,g),$$

and combining these inequalities proves the theorem. □

Proposition 3.6. *Suppose $d_s > 2$. There exist c_1 and c_2 such that for all x and for all $r \geq 1$*

$$c_1 r^{d_f - d_w} \leq C_G(B(x,r)) \leq c_2 r^{d_f - d_w}.$$

Proof. The lower bound follows from the mass-capacity inequality for G:

$$C_G(B(x,r))^{d_s/(d_s-2)} \geq c_3|B(x,r)| \geq c_4 r^{d_f}.$$

So

$$C_G(B(x,r)) \geq c_5 r^{d_f(d_s-2)/d_s} = c_5 r^{d_f-d_w}.$$

We obtain the upper bound as follows. Let π be the capacitary measure for $S(x,r)$ with respect to Brownian motion on the pre-carpet. Let $z \in \partial S(x,2r)$. Then as $u(\cdot,z)$, the Green function for reflecting Brownian motion in P, is harmonic in $S(x,r)$, by the Harnack inequality ([BB3, Sect. 4]),

$$1 \geq u\pi(z) = \int_{S(x,r)} u(y,z)\,\pi(dy) \geq c_6 u(x,z)\pi(S(x,r)).$$

Using this,

$$C_0(S(x,r)) = \pi(S(x,r)) \leq c_6^{-1} u(x,z)^{-1}.$$

By [BB3], $u(x,z) \geq c_7|x-z|^{d_f-d_w}$, so

$$C_0(S(x,r)) \leq c_8|x-z|^{d_f-d_w} \leq c_9 r^{d_f-d_w}.$$

Using (2.3) it follows that we can find g such that $\int |\nabla g|^2 \leq 2c_9 r^{d_f-d_w}$, $g=1$ on $S(x,r)$, and $g(y)\to 0$ as $|y|\to\infty$.

For each $x\in V$, let S_x be the unit cube whose center is x. Define h on V by

$$h(x) = \int_{S_x} g(y)\,dy.$$

Suppose $x,y\in V$ with $|x-y|=1$. Let $e=y-x$. Then

$$\begin{aligned}(h(y)-h(x))^2 &= \Big|\int_{S_y} g - \int_{S_x} g\Big|^2 = \Big|\int_{S_x} (g(z+e)-g(z))\,dz\Big|^2 \\ &= \Big|\int_{S_x}\int_0^1 \nabla g(z+te)\,dt\,dz\Big|^2 \leq \int_0^1\int_{S_x} |\nabla g(z+te)|^2\,dz\,dt \\ &\leq \int_{S_x\cup S_y} |\nabla g(z)|^2\,dz.\end{aligned}$$

Summing over all pairs (x,y) with $|y-x|=1$, we obtain (with $D=P$)

$$\mathcal{E}_D^H(h,h) \leq 2^d \mathcal{E}_0^D(g,g).$$

Our result follows by (3.10), and the inequalities $\mathcal{E}_G^D(h,h) \leq c_{11}\mathcal{E}_H^D(h,h)$, and $C_G(B(x,r)) \leq C_G(S(x,2r))$. □

4. Coupling and Harnack inequalities.

The most difficult step in [BB3] was the proof of a coupling result for reflecting Brownian motion on the pre-carpet P. We now translate this theorem into the graph setting.

Call $A \subset \mathbb{R}^d$ a *half-face* if there exists $i \in \{1, \dots, d\}$, $a = (a_1, \dots, a_d) \in \frac{1}{2}\mathbb{Z}^d$ with $a_i \in \mathbb{Z}$ such that

$$A = \{x : x_i = a_i, \quad a_j \le x_j \le a_j + 1/2 \quad \text{for } j \ne i\}.$$

Let $\mathcal{A}$ be the collection of half-faces, and for $A \in \mathcal{A}$ let w_A be the center of A. For $x, y \in \mathbb{R}^d$ write $[x, y]$ for the line segment with endpoints x, y. For $S \in \mathcal{S}_0$ set

$$\Gamma_S = \big\{ [z_S, w_A] : A \subset S \big\}.$$

(So, if $d = 3$, Γ_S consists of the 24 line segments, each joining the center of S to the center of one of the half-faces bordering S). Set

$$\Gamma = \bigcup \big\{ \Gamma_S : S \in \mathcal{S}(P) \big\}.$$

Let $Z = (Z_t, t \ge 0, \mathbb{Q}^x, x \in \Gamma)$ be the 'cable process' of Varopoulos [V2] on the set Γ. Z is a diffusion on Γ: on the cables $[z_S, w_A]$ Z behaves like a standard 1-dimensional Brownian motion, while at the vertices z_S (and the 'trivial' vertices w_A) Z behaves like a Walsh Brownian motion (see [BPY]), taking excursions along each of the cables meeting z_S with equal probability.

Let $\gamma(dx)$ be 1-dimensional Lebesgue measure on Γ. Z is the diffusion associated with the Dirichlet form

$$\tfrac{1}{2} \int_\Gamma |\nabla g(x)|^2 \, dx,$$

on $\mathcal{D}^Z \subset L^2(\Gamma, \gamma)$. Here, $\mathcal{D}^Z$ is the set of all $g \in L^2(\Gamma, \gamma)$ such that the integral above is finite, and for $x \in \Gamma - V$, $\nabla g(x)$ is the derivative of g in the direction of the line segment containing z. (Since $\gamma(V) = 0$ the definition of $\nabla g(x)$ for $x \in V$ is not relevant).

We can now use the argument of [BB3, Sect. 3] to prove a coupling theorem for Z. While the arguments of [BB3] are written for reflecting Brownian motion W on P, all they use is that W satisfies certain symmetry conditions, and the fact that the set of boundaries between half faces

$$A^* = \bigcup \{ A \cap B; A, B \in \mathcal{A}, A \ne B \}$$

is polar for W – see [SCH, Remark 3.27]. Z satisfies the same symmetry conditions, and since $A^* \cap \Gamma = \emptyset$, A^* is polar for Z. Hence we deduce the following coupling result.

Given two processes, X, Y, define

$$T_C(X,Y) = \inf\{t \geq 0 : X_t = Y_t\}.$$

Theorem 4.1. *There exists $r_0 \geq 1$, and constants $p_4 = p_4(d, l_F) > 0$, $c_1 > 0$ such that if $x_0 \in V$, $r \geq r_0$, $x, y \in \Gamma \cap S(x, c_1 r)$ then there exists a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ carrying processes $(Z_t^x, t \geq 0)$, $(Z_t^y, t \geq 0)$ with the following properties.*
(a) For $z = x, y$ the process $(Z_t^z, t \geq 0)$ is equal in law to $(Z_t, t \geq 0)$ under $\mathbb{Q}^z$. In particular $Z_0^x = x$, $Z_0^y = y$.
(b)

$$\mathbb{P}\big(T_C(Z^x, Z^y) < \sigma(x_0, r, Z^x) \wedge \sigma(x_0, r, Z^y)\big) > p_4 > 0.$$

Definition 4.2. Let $\mathcal{L}_\Gamma$ be the infinitesimal generator of Z. If D is an open subset of Γ we say h is *Z-harmonic on D* if $\mathcal{L}_\Gamma h = 0$ on D.

If A is a subset of V we say h is *X-harmonic* (or just harmonic) on A if $\mathcal{L}h = 0$ on D, where $\mathcal{L}$ is given by (1.2).

Note that if D is an open subset of Γ, and $J = \{x \in V \cap D : B(x,1) \subset D\}$ then if h is Z-harmonic on D, $h|_V$ is X-harmonic on J.

Once we have coupling, we deduce a uniform Harnack inequality for Z_t. The proof is the same word for word as Section 4 of [BB3].

Theorem 4.3. *Let $x_0 \in V$. There exists c_1 such that if u is nonnegative and Z-harmonic in $B(x_0, 2r) \subset \Gamma$ then*

$$u(y) \leq c_1 u(x), \qquad x, y \in S(x_0, r) \cap \Gamma.$$

Using the fact that $u|_V$ is (X)-harmonic we therefore obtain immediately a Harnack inequality for the discrete random walk X on G.

Theorem 4.4. *Let $x_0 \in V$. There exists c_1 such that if u is nonnegative and harmonic in $B(x_0, 2r)$, then*

$$u(y) \leq c_1 u(x), \qquad x, y \in B(x_0, r).$$

Proof. Let u be harmonic for $B(x_0, 2r)$. By linear interpolation extend u to a function v on $S(x_0, 2r) \cap \Gamma$. Since v is harmonic with respect to the cable process Z_t, the result follows by Theorem 4.3. □

Let $\mathrm{Osc}_A\, h$ denote the difference of the supremum and infimum of h on A. A consequence of the Harnack inequality is the following.

Proposition 4.5. *If h is bounded and harmonic on $B(x,2r)$, then there exists $\theta<1$ independent of x, r, and h such that*

$$\operatorname*{Osc}_{B(x,r)} h \le \theta \operatorname*{Osc}_{B(x,2r)} h.$$

Proof. By looking at $Ah+B$ for suitable constants A and B, we may assume $\sup_{B(x,2r)} h = 1$ and $\inf_{B(x,2r)} h = 0$. By looking at $1-h$ if necessary, we may assume $h(x) \ge 1/2$. By the Harnack inequality, $h \ge c_1/2$ on $B(x,r)$. Therefore $\operatorname{Osc}_{B(x,r)} h \le 1 - c_1/2$, and our result follows by setting $\rho = 1 - c_1/2$. □

5. Upper bounds.

Remark 5.1. It is easy to see that the random walk X_n on G has period 2, and so $p(n,x,y)$ is equal to 0 unless n and $d(x,y)$ have the same parity (i.e., both are even or both are odd).

Upper bounds for $p(n,x,x)$ are quick.

Theorem 5.2. *There exists c_1 such that if $x \in V$, then*

$$p(n,x,x) \le c_1 n^{-d_s/2}.$$

Proof. This follows from the Nash inequality (3.6) and Theorem 2.1 of [CKS]. □

The main work is in obtaining good upper bounds for $p(n,x,y)$ when $|x-y|$ is relatively large. For simplicity we consider only the case $d_s > 2$ in this section, and consider the case $d_s \le 2$ in Section 7. First we obtain estimates on the Green function $g(x,y)$ for the random walk on G, defined by

$$g(x,y) = \sum_{m=0}^{\infty} p(m,x,y).$$

Theorem 5.3. *There exist c_1 and c_2 such that*

$$c_1 d(x,y)^{d_w - d_f} \le g(x,y) \le c_2 d(x,y)^{d_w - d_f}. \tag{5.1}$$

Proof. Fix $x \in V$. Write $N(x,r)$ for $\partial(B(x,r)^c)$; note that $N(x,r)$ consists of those points of $B(x,r)$ which have a neighbor in $B(x,r)^c$. It is clear from the geometry of V that there exist r_0 and M, both independent of x, with the following property: if $r \ge r_0$, there exist M points $x_1, \ldots, x_M$ in $N(x,4r)$ such that $x_{i+1} \in B(x_i,r)$, $i = 1, \ldots, M-1$, and $N(x,4r) \subseteq \cup_{i=1}^M B(x_i,r)$; we emphasize that M does not depend on r.

Suppose $r \geq r_0$. By the Harnack inequality repeated M times on the sets $B(x_i, 2r)$, there exists c_3 such that if u is nonnegative and harmonic on $V - \{x\}$, then

$$\sup_{N(x,4r)} u \leq c_3 \inf_{N(x,4r)} u. \tag{5.2}$$

We apply this to $g(x, \cdot)$. Fix $x_0 \in N(x, 4r)$. Let π be the capacitary measure for $B(x, 4r)$. It is well-known that $g\pi$ is equal to 1 on $B(x, 4r)$, and moreover π is supported on $N(x, 4r)$. We then have

$$1 = g\pi(x) = \sum_{z \in N(x,4r)} g(x, z)\pi(\{z\}).$$

By the Harnack inequality (5.2) there exist c_4 and c_5 such that

$$c_4 g(x, x_0) \leq g(x, z) \leq c_5 g(x, x_0), \qquad z \in N(x, 4r).$$

Thus

$$1 \leq c_5 g(x, x_0)\pi(N(x, 4r)) = c_5 g(x, x_0) C_G(B(x, 4r))$$

and

$$1 \geq c_4 g(x, x_0)\pi(N(x, 4r)) = c_4 g(x, x_0) C_G(B(x, 4r)).$$

Using Proposition 3.6 we have the inequality (5.1) for $g(x, x_0)$. By the Harnack inequality we have our result for $g(x, y)$ when $r \leq d(x, y) \leq 8r$.

It remains to consider the case when $d(x, y) < r_0$. As

$$g(x, y) = \sum_{m=0}^{\infty} p(m, x, y),$$

and there is positive probability that X_n will hit y before $2r$ steps, the lower bound is clear.

We examine the upper bound. Since $g(x, y) \leq g(x, x)$ by the maximum principle, we need only show $g(x, x) \leq c_6$. There exists $c_7 > 0$ independent of y such that starting at y, there is probability at least c_7 that X_n will leave $B(x, r_0)$ within $2r_0$ steps. The probability that X_n remains within $B(x, r_0)$ after $2jr_0$ steps is then, by a standard renewal argument, bounded by $(1 - c_7)^j$. Thus

$$\mathbb{E}^x \sum_{m=0}^{\sigma(x,r_0)} 1_{\{x\}}(X_m) \leq \mathbb{E}^x \sigma(x, r_0) \leq c_8.$$

By the strong Markov property,

$$g(x, x) \leq \mu(x)^{-1} \mathbb{E}^x \sum_{m=0}^{\sigma(x,r_0)} 1_{\{x\}}(X_m) + \mathbb{E}^x g(X(\sigma(x, r_0)), x),$$

and this is bounded by $c_8 + c_2 r_0^{d_w - d_f} < \infty$. □

The next step is to obtain upper and lower bounds on $\mathbb{E}^y \sigma(x, r)$.

Theorem 5.4. *There exist c_1 and c_2 such that if $x \in V$,*

$$c_1 r^{d_w} \le \mathbb{E}^x \sigma(x,r),$$

and

$$\mathbb{E}^y \sigma(x,2r) \le c_2 r^{d_w}, \qquad y \in B(x,r).$$

Proof. The upper bound comes from Theorem 5.3 and

$$\begin{aligned}\mathbb{E}^y \sigma(x,2r) &\le \mathbb{E}^y \sum_{m=0}^{\infty} 1_{B(x,2r)}(X_m)\\ &\le \sum_{B(x,2r)} g(y,z)\mu(z) \le c_3 \sum_{B(x,2r)} d(y,z)^{d_w-d_f} \le c_4 r^{d_w}.\end{aligned}$$

For the lower bound we will choose $\theta \in (0,1)$ and r_0 in a moment and establish the lower bound first for $r \ge r_0$. We write

$$\begin{aligned}&\mathbb{E}^y \sum_{m\le\sigma(x,r)} 1_{B(x,\theta r)}(X_m) \qquad (5.3)\\ &\qquad = \mathbb{E}^y \sum_{m=0}^{\infty} 1_{B(x,\theta r)}(X_m) - \mathbb{E}^y \mathbb{E}^{X(\sigma(x,r))} \sum_{m=0}^{\infty} 1_{B(x,\theta r)}(X_m).\end{aligned}$$

The first term on the right hand side of (5.2) is equal to

$$\sum_{z\in B(x,\theta r)} g(x,z)\mu(z) \ge \sum_{z\in B(x,\theta r)} (c_1 d(x,z))^{d_w-d_f} \ge c_5(\theta r)^{d_w},$$

provided $\theta r \ge 1$. The second term on the right hand side of (5.3) is equal to

$$\mathbb{E}^x \sum_{z\in B(x,\theta r)} g(X(\sigma(x,2r)),z)\mu(z). \qquad (5.4)$$

For $z \in B(x,\theta r)$ and $y \in \partial B(x,2r)$, we have $g(y,z) \le c_6 r^{d_w-d_f}$, so (5.4) is bounded above by

$$c_7(\theta r)^{d_f} c_6 r^{d_w-d_f} = c_8 \theta^{d_f} r^{d_w}.$$

Therefore

$$\begin{aligned}\mathbb{E}^x \sigma(x,r) &\ge \mathbb{E}^x \sum_{m\le\sigma(x,r)} 1_{B(x,\theta r)}(X_m)\\ &\ge c_5 \theta^{d_w} r^{d_w} - c_8 \theta^{d_f} r^{d_w},\end{aligned}$$

provided $\theta r \ge 1$.

As $d_s > 2$, then $d_f > d_w$. Choose θ small so that $c_5\theta^{d_w} \ge 2c_8\theta^{d_f}$, and then r_0 large so that $\theta r_0 \ge 1$. For $r \ge r_0$ we then have

$$\mathbb{E}^x \sigma(x,r) \ge c_8 \theta^{d_f} r^{d_w}.$$

The case $r \le r_0$ is trivial if we adjust c_1, since $\mathbb{E}^x \sigma(x,r) \ge 1$. □

We now follow [BB1] and [BB2] closely to get a tightness estimate.

Theorem 5.5. *There exist c_1 and c_2 such that*

$$\mathbb{P}^x(\sup_{m\le n}|X_m - X_0| > \lambda) \le c_1 \exp\Big(-c_2\Big(\frac{\lambda^{d_w}}{n}\Big)^{1/(d_w-1)}\Big).$$

Proof. From Theorem 5.4 we have

$$\begin{aligned} c_3 r^{d_w} &\le \mathbb{E}^x\sigma(x,r) \le n + \mathbb{E}^x\Big[\mathbb{E}^{X_n}\sigma(x,r);\sigma(x,r)>n\Big] \\ &\le n + c_4 r^{d_w}\mathbb{P}^x(\sigma(x,r)>n) \\ &= n + c_4 r^{d_w} - c_4 r^{d_w}\mathbb{P}^x(\sigma(x,r)\le n). \end{aligned}$$

We thus obtain

$$\mathbb{P}^x(\sigma(x,r)\le n) \le c_5 + c_6 n r^{-d_w},$$

or

$$\mathbb{P}^x(\sigma(x,r)r^{-d_w}\le x) \le c_5 + c_6 x, \qquad x>0. \tag{5.5}$$

In order for $|X_m - X_0|$ to exceed λ for some $m\le n$, X_m must exit disjoint balls of the form $B(x_i,r)$ at least $M=\lambda/2r$ times. Let Y_k, $k=1,\ldots,M$, be r^{-d_w} times the time to exit the kth ball. By the strong Markov property,

$$\mathbb{P}(Y_i\le x \mid Y_1,\ldots,Y_{i-1}) \le c_5 + c_6 x.$$

So

$$\mathbb{E}\Big(e^{-uY_i}\mid Y_1,\ldots,Y_{i-1}\Big) \le c_5 + \int_0^{(1-c_5)/c_6} e^{-ux}c_6\,dx \le c_5 + c_6 u^{-1}.$$

Then

$$\begin{aligned} \mathbb{P}\Big(\sum_{i=1}^M Y_i \le x\Big) &= \mathbb{P}\Big(e^{-u\sum Y_i} \ge e^{-ux}\Big) \le e^{ux}\mathbb{E}e^{-u\sum Y_i} \\ &\le e^{ux}(c_5 + c_6u^{-1})^M \\ &\le c_5^M \exp\Big(ux + \frac{c_6 M}{c_5 u}\Big). \end{aligned}$$

Taking $u = (c_6M/c_5x)^{\frac12}$,

$$\mathbb{P}(\sum Y_i \le x) \le \exp\Big(2(c_6Mx/c_5)^{\frac12} - M\log(1/c_5)\Big).$$

We now choose r so that $M\in[c_7\log(c_8/n), 2c_7\log(c_8/n)]$ for appropriate c_7 and c_8. □

We now get, following [BB2], the off-diagonal estimate.

Theorem 5.6. *There exist c_1 and c_2 such that*

$$p(n,x,y) \le c_1 n^{-d_s/2} \exp\Big(-c_2 \Big(\frac{d(x,y)^{d_w}}{n} \Big)^{1/(d_w-1)} \Big). \qquad (5.6)$$

Proof. Fix x,y,n, let $A_x = \{z \in V : d(x,z) \le d(x,y)\}$, $A_y = \{z \in V : d(y,z) \le d(x,y)\}$, and $S = \inf\{n \ge 0 : |X_n - X_0| \ge d(x,y)/3\}$. Let us first suppose n is even. Then

$$\mu(y)p(n,x,y) = \mathbb{P}^x(X_n = y, X_{n/2} \in A_x) + \mathbb{P}^x(X_n = y, X_{n/2} \in A_y). \quad (5.7)$$

For the second term,

$$\begin{aligned} \mathbb{P}^x(X_n = y, X_{n/2} \in A_y) &= \mathbb{P}^x(S < n/2, X_n = y, X_{n/2} \in A_y) \\ &\le \mathbb{P}^x(S < n/2) \sup_{y' \in A_y} \mathbb{P}^{y'}(X_{n/2} = y) \\ &\le \mathbb{P}^x(S < n/2) \sup_{y' \in A_y} p(n/2, y', y)\mu(y). \end{aligned}$$

For the first term in (5.7), by symmetry,

$$\mu(x)\mathbb{P}^x(X_n = y, X_{n/2} \in A_x) = \mu(y)\mathbb{P}^y(X_n = x, X_{n/2} \in A_x),$$

which is bounded the same way.

Using Theorems 5.2 and 5.5, we have our result when n is even. If n is odd, we have by the Markov property that

$$p(n,x,y) = \sum_{\{z:(y,z)\in E\}} p(n-1,x,z)p(1,z,y)\mu(z);$$

applying (5.5) to $p(n-1,x,z)$ establishes (5.6) when n is odd. □

6. Lower bounds.

Theorem 6.1. *There exists c_1 such that if n is even,*

$$p(n,x,x) \ge c_1 n^{-d_s/2}.$$

Proof. Let $m = n/2$. Recall

$$\mathbb{P}^x(\sigma(x,r) \le n) \le c_2 \exp(-c_3(r^{d_w}/n)^{1/(d_w-1)}). \qquad (6.1)$$

Pick r to be the first integer such that $c_2 \exp(-c_3(r^{d_w}/n)^{1/(d_w-1)}) \le 1/2$. Then

$$\mathbb{P}^x(X_m \in B(x,r)) \ge \mathbb{P}^x(\sigma(x,r) > n/2) \ge \tfrac{1}{2},$$

while

$$\mu(B(x,r)) \le c_4 r^{d_f} \le c_5 m^{d_s/2}.$$

By Cauchy-Schwarz,

$$\begin{aligned}\tfrac{1}{4} \le [\mathbb{P}^x(X_{n/2} \in B(x,r))]^2 &= \Big[\sum_{B(x,r)} p(n/2,x,y)\mu(y) \Big]^2 \\ &\le \mu(B(x,r)) \sum_{B(x,r)} p(n/2,x,y)^2 \mu(y) \\ &\le \mu(B(x,r)) p(n,x,x).\end{aligned}$$

Using the estimate for $\mu(B(x,r))$ yields the theorem. □

Fix $n \ge 1$, such that n is even, let A be a large constant such that $c_2 \exp(-c_3 A^{d_w/(d_w-1)}) \le 1/2$, where c_2 and c_3 are as in (6.1), and let $D_n = S(x, An^{1/d_w})$. Let $\overline{p}(m,y,z)$ be the transition probabilities for the random walk X_m killed on exiting D_n and let $\overline{g}(y,z) = \sum_{m=0}^{\infty} \overline{p}(m,y,z)$ be the corresponding Green function.

The same as argument as that in Theorem 6.1 proves

Corollary 6.2. *There exists c_1 such that if n is even, $\overline{p}(n,x,x) \ge c_1 n^{-d_s/2}$.*

Fix x and let $F_x = \{y \in D_n : d(x,y) \text{ is even}\}$. To get a lower bound for the off-diagonal terms, we start with

Proposition 6.3. *There exist c_1 and $h : D_n \to \mathbb{R}$ such that $|h|$ is bounded by $c_1 n^{-d_s/2-1}$ and if n is even $\overline{g}h(y) = \overline{p}(n,x,y)$ if $y \in F_x$.*

Proof. By the semigroup property, $\overline{p}(2,\cdot,\cdot)$ is a symmetric nonnegative definite matrix. Thus if we consider $\overline{p}(2,\cdot,\cdot)$ restricted to $F_x \times F_x$, we still have a symmetric nonnegative definite matrix, and by the spectral theorem there exist a collection of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_M$ and a collection of corresponding orthonormal eigenvectors $\varphi_i : F_x \to \mathbb{R}$ (orthonormal with respect $\mu|_{F_x}$) such that

$$\overline{p}(2,y,z) = \sum_{i=1}^{M} \lambda_i \varphi_i(y) \varphi_i(z).$$

We extend φ_i to all of D_n by setting it equal to zero on $D_n - F_x$.

By the semigroup property,

$$\overline{p}(2m+2,y,z) = \sum_{w \in D_n} \overline{p}(2m,y,w)\overline{p}(2,z,w)\mu(w).$$

For $y, z \in F_x$ we know $\overline{p}(2, z, w)$ is 0 unless w is also in F_x; hence we may replace the sum in the above equation by a sum over $w \in F_x$. Using the eigenvalue expansion and induction, we then obtain

$$\overline{p}(2m, y, z) = \sum_{i=1}^{M} \lambda_i^m \varphi_i(y)\varphi_i(z), \qquad y, z \in F_x.$$

Since $\overline{p}(2, y, z)$ is nonegative definite, it follows that $\lambda_i \geq 0$. Since $\overline{p}$ is a submarkovian kernel, $\lambda_i \leq 1$ for all i.

Let

$$h(z) = \sum_{i=1}^{M} (1 - \lambda_i)\lambda_i^n \varphi_i(x)\varphi_i(z)$$

if $z \in F_x$ and zero otherwise. If $y \in F_x$, then

$$\overline{g}h(y) = \sum_{z \in D_n} \overline{g}(y, z)h(z) = \sum_{m=0}^{\infty} \sum_{z \in D_n} \overline{p}(m, y, z)h(z).$$

Since h is zero on $D_n - F_x$ and $y \in F_x$, then $\overline{p}(m, y, z)h(z)$ is nonzero only if m is even and $z \in F_x$. Therefore

$$\begin{aligned}
\overline{g}h(y) &= \sum_{m=0}^{\infty} \sum_{z \in F_x} \overline{p}(2m, y, z)h(z) \\
&= \sum_{m=0}^{\infty} \sum_{i=1}^{M} (1 - \lambda_i)\lambda_i^m \lambda_i^n \varphi_i(x)\varphi_i(y) = \sum_{i=1}^{M} \lambda_i^n \varphi_i(x)\varphi_i(y) \\
&= \overline{p}(n, x, y).
\end{aligned}$$

It remains to bound $|h(z)|$. By the Cauchy-Schwarz inequality,

$$|h(z)| \leq \Big(\sum_{i=1}^{M} (1 - \lambda_i)\lambda_i^n \varphi_i(x)^2 \Big)^{1/2} \Big(\sum_{i=1}^{M} (1 - \lambda_i)\lambda_i^n \varphi_i(z)^2 \Big)^{1/2}.$$

Note the function $\lambda \to (1 - \lambda)\lambda^{n/2}$ is bounded by c_2/n for $\lambda \geq 0$. We then observe that

$$\begin{aligned}
\sum_{i=1}^{M} (1 - \lambda_i)\lambda_i^n \varphi_i(z)^2 &\leq \frac{c_2}{n} \sum_{i=1}^{M} \lambda_i^{n/2} \varphi_i(z)^2 \\
&\leq \frac{c_2}{n} \overline{p}(n/2, z, z) \leq \frac{c_2}{n} p(n/2, z, z) \\
&\leq c_3 n^{-d_s/2 - 1}.
\end{aligned}$$

This completes the proof. □

Proposition 6.4. *There exist c_1 and c_2 such that if n and $d(x,y)$ are even and $d(x,y) \le c_1 n^{1/d_w}$, then $p(n,x,y) \ge c_2 n^{-d_s/2}$.*

Proof. We will choose a constant c_3 in a moment. Let $\rho = \min\{n \ge 0; n \in 2\mathbb{Z},\ n \ge \sigma(x, c_3 n^{1/d_w})\}$. For any $y \in F_x$,

$$\overline{p}(n,x,y) = \overline{g}h(y) = \mathbb{E}^y \sum_{m=0}^{\rho} h(X_m) + \mathbb{E}^y \overline{g}h(X_\rho). \tag{6.2}$$

The first term on the right is bounded by

$$\|h\|_\infty \mathbb{E}^y \rho \le c_4 c_3^{d_w} n \|h\|_\infty,$$

using Theorem 5.4. So by Proposition 6.3, the first term on the right is bounded by $c_4 c_3^{d_w} n^{-d_s/2}$. The second term on the right of (6.2) is harmonic in $S(x, c_3 n^{1/d_w})$. By Proposition 4.5, there exists $c_5 < 1$ such that if $y \in S(x, 2^{-k} c_3 n^{1/d_w})$, then

$$|\mathbb{E}^y \overline{g}h(X_\rho) - \mathbb{E}^x \overline{g}h(X_\rho)| \le c_6 c_5^k \|\overline{g}h\|_\infty \le c_7 c_5^k n^{-d_s/2}.$$

Therefore,

$$\overline{p}(n,x,x) - \overline{p}(n,x,y) \le 2c_4 c_3^{d_w} n^{-d_s/2} + c_7 c_5^k n^{-d_s/2}$$

for $y \in S(x, 2^{-k} c_3 n^{1/d_w}) \cap F_x$.

Recall there exists c_8 such that $\overline{p}(n,x,x) \ge c_8 n^{-d_s/2}$. Now take c_3 small so that $2c_4 c_3^{d_w} < c_8/3$ and then k large so that $c_7 c_5^k < c_8/3$. Let $c_1 = 2^{-k} c_3$ and let n_0 be chosen so that $c_1 n_0^{1/d_w}$ is larger than 4. Provided $n \ge n_0$ and $y \in S(x, c_1 n^{1/d_w}) \cap F_x$, we have

$$p(n,x,y) \ge \overline{p}(n,x,y) \ge c_8 n^{-d_s/2}/3.$$

We thus have our result if $n \ge n_0$. It is easy to see that the proposition also holds if $n \le n_0$ provided we adjust c_2. □

Finally we use chaining to obtain our result.

Theorem 6.5. *If n and $d(x,y)$ have the same parity and $d(x,y) \le n$, then*

$$p(n,x,y) \ge c_1 n^{-d_s/2} \exp(-c_2 (d(x,y)^{d_w}/n)^{1/(d_w-1)}).$$

Proof. Write D for $d(x,y)$. If z is a neighbor of y,

$$p(n,x,y) \geq p(n-1,x,z)p(1,z,y)\mu(z) \geq c_3 p(n-1,x,z),$$

with c_3 independent of x, y, z, and n. So it suffices to consider the case when n is even.

The case $n \leq 4$ is obvious, so we assume $n > 4$. If $D \leq c_4 n^{1/d_w}$, where c_4 is the constant c_2 in Theorem 6.1, the result is immediate from Theorem 6.1 or Corollary 6.2. So we need to look at the case $n \geq D > c_4 n^{1/d_w}$.

Let k be the smallest integer greater than or equal to $(D^{d_w}/An)^{1/(d_w-1)}$, where A will be chosen in a moment. Let $m_1, \ldots, m_k$ be even integers, each between $[n/k]/4$ and $4[n/k]+2$, so that $m_1 + \ldots + m_k = n$. Note

$$\frac{D}{k} \leq \frac{c_4}{4}\Big(\frac{n}{k}\Big)^{1/d_w}$$

if we take A sufficiently large.

We can find points $z_0, z_1, \ldots, z_k$ such that $z_0 = x, z_k = y$, and

$$d(z_i, z_{i+1}) \leq c_4 [n/k]^{1/d_w}, \qquad i = 0, 1, 2, \ldots, k-1.$$

We now let $C_i = \{w \in G : d(w, z_i) \leq c_4[n/k]^{1/d_w}, d(w, z_i) \text{ even}\}$. So $|C_i| \geq c_5 (n/k)^{d_f/d_w} = c_5 [n/k]^{d_s/2}$. If $w_i \in C_i$ and $w_{i+1} \in C_{i+1}$, then by the triangle inequality $d(w_i, w_{i+1}) \leq 2c_4 [n/k]^{1/d_w}$. We can then write

$$\begin{aligned}
&p(n,x,y) \\
&\quad \geq \sum_{w_1 \in C_1} \sum_{w_2 \in C_2} \cdots \sum_{w_{k-1} \in C_{k-1}} p(m_1, z_0, w_1)\mu(w_1) \\
&\qquad\qquad \times p(m_2, w_1, w_2)\mu(w_2) \ldots p(m_k, w_{k-1}, z_k)\mu(w_{k-1}) \\
&\quad \geq (\Pi_{i=1}^{k-1} \mu(C_i))(c_7 [n/k]^{-d_s/2})^k \\
&\quad \geq (c_5 (n/k)^{d_s/2})^{k-1} (2c_7 n/k)^{-kd_s/2} \\
&\quad \geq c_8^k n^{-d_s/2} k^{d_s/2} \\
&\quad = n^{-d_s/2} \exp(-k \log(1/c_8) + d_s (\log k)/2) \\
&\quad \geq n^{-d_s/2} \exp(-c_9 k).
\end{aligned}$$

Recalling the definition of k, this proves the theorem. □

7. Further results.

(a) $d_s \leq 2$. The place in the argument where $d_s > 2$ was used was in obtaining upper and lower bounds on the expectation of $\sigma(x,r)$. If $d_s \leq 2$,

we modify the argument as follows. Instead of C_G and $g(\cdot,\cdot)$, we use the capacity and Green function for X_n killed on exiting $B(x, c_1 r)$ for suitably large c_1. With this change, the argument goes through as before.

(b) Transience and recurrence. From the estimates on $p(n,x,x)$ we see $\sum_{n=0}^{\infty} p(n,x,x)$ is finite for all x if $d_s > 2$ and identically infinite if $d_s \le 2$. So X_n is transient if and only if $d_s < 2$.

(c) Bounded harmonic functions. We show that the only bounded harmonic functions on G are the constant ones.

Theorem 7.1. *If h is bounded and harmonic on V, then h is identically constant.*

Proof. Let $r > 1$. By iterating Proposition 4.5

$$\operatorname*{Osc}_{B(x,r)} h \le \rho^m \operatorname*{Osc}_{B(x,2^m r)} h \le 2\rho^m \|h\|_\infty .$$

Letting $m \to \infty$, $\operatorname{Osc}_{B(x,r)} h = 0$. Since r is arbitrary, h is constant. □

(d) Continuous time random walk. Much as in Sections 5 and 6 we see there exist c_1, c_2, c_3, c_4 such that for $t \ge d(x,y)$

$$c_1 t^{-d_s/2} \exp(-c_2 (d(x,y)^{d_w}/t)^{1/(d_w-1)}) \le q(t,x,y) \le c_3 t^{-d_s/2} \exp(-c_4 (d(x,y)^{d_w}/t)^{1/(d_w-1)})$$

References

[B1] M. T. Barlow, Diffusions on fractals. To appear in *École d'Été de Probabilités de Saint-Flour XXV – 1995.*

[BB1] M. T. Barlow and R. F. Bass, The construction of Brownian motion on the Sierpinski carpet. *Ann. Inst. H. Poincaré* **25** (1989) 225–257.

[BB2] M.T. Barlow and R.F. Bass, Transition densities for Brownian motion on the Sierpinski carpet. *Probab. Th. Rel. Fields* **91** (1992) 307-330.

[BB3] M.T. Barlow and R.F. Bass, Brownian motion and harmonic analysis on Sierpinski carpets. Preprint.

[BPY] M.T. Barlow, J. Pitman and M. Yor: On Walsh's Brownian Motions. *Sém. Prob. XXIII*, 275-293, Lect. Notes Math. **1372** (1989).

[Ba] R.F. Bass, Diffusions on the Sierpinski carpet. *Trends in Probability and related Analysis*, 1-34, World Scientific, Singapore, 1997.

[CKS] E.A. Carlen, S. Kusuoka, and D.W. Stroock, Upper bounds for symmetric Markov transition functions. *Ann. Inst. H. Poincaré* **23** (1987) 245–287.

[Ca] T.K. Carne, A transmutation formula for Markov Chains. *Bull. Sci. Math.* **109** (1985), 399-405.

[Co] T. Coulhon, Dimension á l'infini d'un semi-groupe analytique. *Bull. Sci. Math.* **114** (1990) 485-500.

[Da] E.B. Davies: Large deviations for heat kernels on graphs. *J. London Math. Soc. (2)* **47** (1993), 65-72.

[De] T. Delmotte: Parabolic Harnack inequality and estimates for Markov chains on graphs. Preprint 1997.

[FOT] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes.* de Gruyter, Berlin, 1994.

[HSC] W. Hebisch, L. Saloff-Coste: Gaussian estimates for Markov Chains and random walks on groups. *Ann. Prob.* **21** (1993), 673-709.

[Hu] J.E. Hutchinson, Fractals and self-similarity. *Indiana J. Math.* **30** (1981) 713-747.

[J] O.D. Jones, Transition probabilities for the simple random walk on the Sierpinski graph. *Stoch. Proc. Appl.* **61** (1996), 45-69.

[O] H. Osada, Isoperimetric dimension and estimates of heat kernels of pre–Sierpinski carpets. *Probab. Th. Rel. Fields* **86** (1990) 469–490.

[P] M.M.H. Pang, Heat kernels of graphs. *J. London Math. Soc. (2)* **47** (1993), 50-64.

[SC] L. Saloff-Coste, A note on Poincaré, Sobolev, and Harnack inequalities. *Inter. Math. Res. Notices* (1992) 27–38.

[V1] N. Th. Varopoulos, Isoperimetric inequalities and Markov chains. *J. Funct. Anal.* **63** (1985) 215-239.

[V2] N. Th. Varopoulos, Long range estimates for Markov chains. *Bull. Sc. Math.* **109** (1985), 225-252.

[W] W. Woess, Random walks on infinite graphs and groups – a survey on selected topics. *Bull. London Math. Soc.* **26** (1994), 1-60.