The martingale problem for a class of stable-like processes

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May 27, 2008

Abstract

Let $\alpha \in (0,2)$ and consider the operator

$$\mathcal{L}f(x) = \int [f(x+h) - f(x) - 1_{(|h|\le 1)} \nabla f(x) \cdot h] \frac{A(x,h)}{|h|^{d+\alpha}}\, dh$$

for $f \in C^2(\mathbb{R}^d)$, where the $\nabla f(x) \cdot h$ term is omitted if $\alpha < 1$. We consider the martingale problem corresponding to the operator $\mathcal{L}$ and under mild conditions on the function A prove that there exists a unique solution.

Keywords: martingale problem, stable-like processes, symmetric stable process, stochastic differential equation, jump process, Poisson point process, Harnack inequality.

Subject Classification: Primary 60J75; Secondary 60H10, 60G52

1 Introduction

A stable-like process is a pure jump process where the jump intensity kernel is comparable in some sense to that of one or more stable processes. The

*Research partially supported by NSF grant DMS-0601783.

term was introduced in [3] for processes whose associated operators were of the form

$$\int_{\mathbb{R}^d} [f(x+h) - f(x) - 1_{(|h|\le 1)} \nabla f(x) \cdot h] \frac{dh}{|h|^{1+\alpha(x)}}, \qquad f \in C^2(\mathbb{R}^d), \quad x \in \mathbb{R}^d,$$

and the use of the term was extended in [10] to refer to symmetric Markov processes whose jump kernels $J(x, y)$ were comparable to $|x - y|^{-d-\alpha}$ for a fixed α.

In this paper we fix $\alpha \in (0, 2)$. For $\alpha \in [1, 2)$ we consider jump processes associated to the operator

$$\mathcal{L}f(x) = \int [f(x+h) - f(x) - 1_{(|h|\le 1)} \nabla f(x) \cdot h] \frac{A(x,h)}{|h|^{d+\alpha}}\, dh, \tag{1.1}$$

and for $\alpha \in (0, 1)$ associated to the operator

$$\mathcal{L}f(x) = \int [f(x+h) - f(x)] \frac{A(x,h)}{|h|^{d+\alpha}}\, dh, \tag{1.2}$$

where $A(x, h)$ is bounded above and below by positive constants not depending on x or h. For the domain of $\mathcal{L}$ we take the class of C^2 functions such that the function and its first and second partial derivatives are bounded. These jump processes, when at a point x, jump to $x + h$ with intensity given by $A(x, h)|h|^{-d-\alpha}$. These processes stand in the same relationship to symmetric stable processes of index α as uniformly elliptic operators in non-divergence form do to Brownian motion.

For $\alpha \ge 1$ the $\nabla f(x) \cdot h$ term is needed to guarantee convergence of the integral, while for $\alpha < 1$ the $\nabla f(x) \cdot h$ term cannot be present, or else the jumps of the process will be dominated by the drift.

Processes corresponding to $\mathcal{L}$ given by (1.1) or (1.2) were considered in [9] and [16], where Harnack inequalities and regularity of harmonic functions were proved. It is natural to ask whether there exists a process corresponding to $\mathcal{L}$, and if so, is there only one.

We view this question as a martingale problem. Let $\Omega = D([0, \infty))$, the set of paths that are right continuous with left limits, endowed with the Skorokhod topology. Set $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and let $\mathcal{F}_t$ be the right continuous filtration generated by the process X. A probability measure $\mathbb{P}$ is a solution to the martingale problem for $\mathcal{L}$ started at x if $\mathbb{P}(X_0 = x) = 1$ and $f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\,ds$ is a martingale whenever f is a C^2 function such that f and its first and second partial derivatives are bounded. The question to be answered is the existence and uniqueness of a solution to the martingale problem for $\mathcal{L}$.

Our results on existence are merely an application of techniques used in [4] and the novelty in the current paper is a sufficient condition for uniqueness. Let $\eta > 0$ and set

$$\psi_\eta(r) = (1 + \log^+(1/r))^{1+\eta}, \qquad r > 0. \tag{1.3}$$

We require continuity in x of the function $A(x,h)$, with more continuity the smaller h is. More specifically, let

$$\overline{A}(x,h) = A(x,h)\psi_\eta(|h|). \tag{1.4}$$

Our main assumption is that $\overline{A}(x,h)$ be continuous in x, uniformly in h. We assume

Assumption 1.1 *(a) There exist $c_1, c_2 > 0$ such that $c_1 \le A(x,h) \le c_2$ for all x and h.*

(b) There exists $\eta > 0$ such that for every $y \in \mathbb{R}^d$ and every $b > 0$

$$\lim_{x \to y} \sup_{|h| \le b} |\overline{A}(x,h) - \overline{A}(y,h)| = 0.$$

Part (a) of Assumption 1.1 may be regarded as the jump process equivalent of uniform ellipticity.

We then have

Theorem 1.2 *Suppose Assumption 1.1 holds and $x_0 \in \mathbb{R}^d$. Then there is one and only one solution to the martingale problem for $\mathcal{L}$ started at x_0.*

As we alluded to above, existence is already known and can in fact be proved under slightly weaker hypotheses. Some other generalizations are possible; see Remarks 4.7 and 4.10. Our theorem also extends some of the results obtained in [12]; see Remark 4.9.

We do not know if our theorem is still true if $\overline{A}$ is replaced by A in Assumption 1.1. We point out that uniqueness for the martingale problem for jump processes does not always hold; see [2, Section 6]. We suspect, based on the example of [15], that the continuity of A cannot be dispensed with.

In additions to the papers [9] and [16], which consider the processes described above, a similar model to ours is considered by [12]; see Remark 4.9. Results for related models can be found in [2], [6], [7], [8], and [10].

In the next section we establish some estimates. An approximation is given in Section 3 and Theorem 1.2 is proved in Section 4.

2 Estimates

Let $B(x,r) = \{y \in \mathbb{R}^d : |y-x| < r\}$. Let C^k be the functions which are k times continuously differentiable, C_b^k the elements of C^k such that the function and its partial derivatives up to order k are bounded, and C_K^k the functions in C^k that have compact support. We use the probabilist's version of the Fourier transform:

$$\widehat{f}(u) = \int e^{iu\cdot x} f(x)\, dx, \qquad u \in \mathbb{R}^d.$$

For processes whose paths are right continuous with left limits, we set $X_{t-} = \lim_{s<t, s\to t} X_s$ and $\Delta X_t = X_t - X_{t-}$. We use the letter c with or without subscripts to denote constants whose value is unimportant and may change from line to line.

We suppose throughout the remainder of the paper that Assumption 1.1 holds.

Definition 2.1 We say a collection $\{\mathbb{P}^x\}$ of probability measures is a strong Markov family of solutions to the martingale problem for $\mathcal{L}$ if for each $x \in \mathbb{R}^d$, $\mathbb{P}^x$ is a solution to the martingale problem for $\mathcal{L}$ started at x and in addition the strong Markov property holds: for any finite stopping time T, any Y bounded and $\mathcal{F}_\infty$-measurable, and any $x \in \mathbb{R}^d$,

$$\mathbb{E}^x[Y \circ \theta_T \mid \mathcal{F}_T] = \mathbb{E}^{X_T}[Y], \qquad \mathbb{P}^x - \text{a.s.}$$

Remark 2.2 We will sometimes work with strong Markov families of solutions, in which case the notation $\mathbb{P}^x$ is appropriate, and sometimes with arbitrary solutions to a martingale problem. In the latter case the notation used for the probability measure is then $\mathbb{P}$.

Proposition 2.3 *Suppose $r < 1$, $x \in \mathbb{R}^d$, and $\mathbb{P}$ is a solution to the martingale problem for $\mathcal{L}$ started at x. There exists c_1 not depending on x such that*

$$\mathbb{P}(\sup_{s\le t} |X_s - x| \ge r) \le c_1 t/r^\alpha, \qquad t > 0.$$

Proof. Let $f : \mathbb{R}^d \to [0,1]$ be a C^2 function such that $f(0) = 0$ and $f(y) = 1$ if $|y| > 1$. Let $f_{rx}(y) = f((y-x)/r)$. There exists a constant c such that the first derivatives of f_{rx} are bounded by c/r and the second derivatives are bounded by c/r^2. By Taylor's theorem,

$$|f_{rx}(z+h) - f_{rx}(z) - \nabla f_{rx}(z)\cdot h| \le c|h|^2/r^2$$

and

$$|f_{rx}(z+h) - f_{rx}(z)| \le c|h|/r.$$

Suppose $\alpha \ge 1$. Then

$$\begin{aligned} |\mathcal{L} f_{rx}(z)| \le & \int_{|h|\le r} |f_{rx}(z+h) - f_{rx}(z) - \nabla f_{rx}(z)\cdot h| \frac{A(x,h)}{|h|^{d+\alpha}}\, dh \\ & + \int_{1\ge |h|>r} |f_{rx}(z+h) - f_{rx}(z) - \nabla f_{rx}(z)\cdot h| \frac{A(x,h)}{|h|^{d+\alpha}}\, dh \\ & + \int_{|h|>1} |f_{rx}(z+h) - f_{rx}(z)| \frac{A(x,h)}{|h|^{d+\alpha}}\, dh \\ \le & \frac{c}{r^2} \int_{|h|\le r} \frac{|h|^2}{|h|^{d+\alpha}}\, dh + \frac{c}{r}\int_{|h|>r} \frac{|h|}{|h|^{d+\alpha}}\, dh \\ \le & c r^{-\alpha}. \end{aligned}$$

Therefore by Doob's optional stopping theorem, if $\tau_r = \inf\{t : |X_t - X_0| \ge r\}$, then

$$\begin{aligned} \mathbb{P}(\tau_r \le t) &\le \mathbb{E}\, f_{rx}(X_{\tau_r\wedge t}) - f_{rx}(x) \\ &= \mathbb{E} \int_0^{\tau_r \wedge t} \mathcal{L} f_{rx}(X_s)\, ds \\ &\le ct/r^\alpha. \end{aligned}$$

The case $\alpha < 1$ is similar; in this case we write

$$
\begin{aligned}
|\mathcal{L} f_{rx}(z)| &\le \int_{|h|\le r} |f_{rx}(z+h) - f_{rx}(z)| \frac{A(x,h)}{|h|^{d+\alpha}}\, dh \\
&\qquad + \int_{|h|>r} |f_{rx}(z+h) - f_{rx}(z)| \frac{A(x,h)}{|h|^{d+\alpha}}\, dh \\
&\le c \int_{|h|\le r} \frac{|h|}{r} \frac{dh}{|h|^{d+\alpha}} + c \int_{|h|>r} \frac{dh}{|h|^{d+\alpha}} \\
&\le c r^{-\alpha},
\end{aligned}
$$

and the remainder of the proof is as in the $\alpha \ge 1$ case. □

Proposition 2.4 *If $f \in C_b^2$, then $\mathcal{L} f$ is continuous.*

Proof. Let $\varepsilon > 0$ and suppose that $\alpha \ge 1$, the case when $\alpha < 1$ being very similar. Let $\delta \in (0,1)$ and write

$$
\begin{aligned}
\mathcal{L} f(x) = &\int_{|h|\le \delta} [f(x+h) - f(x) - \nabla f(x) \cdot h] \frac{A(x,h)}{|h|^{d+\alpha}}\, dh \\
&+ \int_{\delta<|h|\le 1} [f(x+h) - f(x) - \nabla f(x) \cdot h] \frac{A(x,h)}{|h|^{d+\alpha}}\, dh \\
&+ \int_{1<|h|\le \delta^{-1}} [f(x+h) - f(x)] \frac{A(x,h)}{|h|^{d+\alpha}}\, dh \\
&+ \int_{\delta^{-1}<|h|} [f(x+h) - f(x)] \frac{A(x,h)}{|h|^{d+\alpha}}\, dh.
\end{aligned}
$$

The first term is bounded by

$$
c \int_{|h|\le \delta} \frac{|h|^2}{|h|^{d+\alpha}}\, dh,
$$

where c depends on f. This is less than ε if δ is sufficiently small. The fourth term is bounded by

$$
c \int_{|h|>\delta^{-1}} \frac{dh}{|h|^{d+\alpha}},
$$

where again c depends on f. This will also be less than ε if δ is sufficiently small. The second and third terms are continuous in x by dominated convergence and the continuity of $A(x,h)$ in x. $\square$

Proposition 2.5 *Suppose $\{\mathbb{P}^x\}$ is a strong Markov family of solutions to the martingale problem for $\mathcal{L}$. Let $x_0 \in \mathbb{R}^d$, suppose $r < 1$, and $\tau_r = \inf\{t : |X_t - x_0| > r\}$.*

(a) If $\varepsilon \in (0,1)$, there exists c_1 (depending on ε) such that

$$\inf_{z\in B(x_0,(1-\varepsilon)r)} \mathbb{E}^z \tau_r \geq c_1 r^\alpha.$$

(b) There exists c_2 such that

$$\sup_z \mathbb{E}^z \tau_r \leq c_2 r^\alpha.$$

Proof. The proof consists of minor modifications to the proofs of [4, Lemmas 3.2 and 3.3]. $\square$

Proposition 2.6 *Let $\mathbb{P}$ be a solution to the martingale problem for $\mathcal{L}$ started at some point x_0. If B and C are Borel sets whose closures are disjoint, then*

$$\sum_{s\leq t} 1_B(X_{s-})1_C(X_s) - \int_0^t 1_B(X_s) \int_C \frac{A(X_s, u-X_s)}{|u-X_s|^{d+\alpha}}\, du\, ds$$

is a martingale with respect to $\mathbb{P}$.

Proof. Suppose B and C are disjoint compact sets, $f \in C_b^2$ is 0 on B and 1 on C, and ∇f is 0 on B. Then

$$f(X_t) - f(X_0) = M_t + \int_0^t \mathcal{L}f(X_s)\, ds,$$

where M_t is a martingale. Since stochastic integrals of predictable processes with respect to martingales are martingales, it follows that $\int_0^t 1_B(X_{s-})\,dM_s$ is also a martingale. By Ito's formula

$$f(X_t)-f(X_0)=\int_0^t \nabla f(X_{s-})\cdot dX_s+\sum_{s\le t}[f(X_s)-f(X_{s-})-\nabla f(X_{s-})\cdot\Delta X_s].$$

Hence

$$\begin{aligned}\int_0^t 1_B(X_{s-})\nabla f(X_{s-})\cdot dX_s& \\ &+\sum_{s\le t}1_B(X_{s-})[f(X_s)-f(X_{s-})-\nabla f(X_{s-})\cdot\Delta X_s] \\ &-\int_0^t 1_B(X_{s-})\mathcal{L}f(X_s)\,ds\end{aligned} \tag{2.1}$$

is a martingale. Since $f\in C^2$ and both f and ∇f are 0 on B, the first term of (2.1) is equal to 0 and the second term of (2.1) is

$$\sum_{s\le t}1_B(X_{s-})f(X_s).$$

We have

$$\begin{aligned}1_B(x)\mathcal{L}f(x)&=1_B(x)\int[f(x+h)-f(x)-1_{(|h|\le 1)}\nabla f(x)\cdot h]\frac{A(x,h)}{|h|^{d+\alpha}}\,dh\\ &=1_B(x)\int f(x+h)\frac{A(x,h)}{|h|^{d+\alpha}}\,dh\\ &=1_B(x)\int f(u)\frac{A(x,u-x)}{|u-x|^{d+\alpha}}\,du.\end{aligned}$$

Putting this in (2.1), and using the fact that X_s differs from X_{s-} on a set of times having Lebesgue measure 0, the last term in (2.1) is

$$\int_0^t 1_B(X_s)\int f(u)\frac{A(X_s,u-X_s)}{|u-X_s|^{d+\alpha}}\,du\,ds.$$

Our result follows by using a limit argument. $\square$

Proposition 2.7 *Suppose $\{\mathbb{P}^x\}$ is a strong Markov family of solutions to the martingale problem for $\mathcal{L}$. Suppose g is bounded and measurable and $\lambda > 0$. Let*

$$S_\lambda g(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} g(X_t)\, dt.$$

Then $S_\lambda g$ is Hölder continuous in x.

Proof. The proof follows by [9, Theorem 4.3] and the arguments leading up to it. See also [16]. □

Let $z \in \mathbb{R}^d$ and let $\mathcal{M}^z$ be the operator on C_b^2 functions defined by

$$\mathcal{M}^z f(x) = \int [f(x+h) - f(x) - \nabla f(x) \cdot h 1_{(|h|\le 1)}] \frac{A(z,h)}{|h|^{d+\alpha}}\, dh, \qquad (2.2)$$

where the $\nabla f(x) \cdot h$ term is missing if $\alpha < 1$. Let R_λ^z be the resolvent for the Lévy process whose infinitesimal generator is $\mathcal{M}^z$ and let P_t^z be the corresponding transition operator. We define the Fourier transform of $\mathcal{M}^z$ by setting $C_u(x) = e^{iu\cdot x}$ and $\widehat{\mathcal{M}}^z(u) = e^{-iu\cdot x}\mathcal{M}^z C_u(x)$.

Proposition 2.8 *With $\mathcal{M}^z$ as above,*

$$\mathrm{Re}\,(\widehat{\mathcal{M}}^z(u)) \le \begin{cases} 0, & |u| \le 1; \\ -c|u|^\alpha, & |u| > 1. \end{cases}$$

Proof.

$$\widehat{\mathcal{M}}^z(u) = \int [e^{iu\cdot h} - 1 - iu \cdot h 1_{(|h|\le 1)}] \frac{A(z,h)}{|h|^{d+\alpha}}\, dh,$$

with the $iu \cdot h1_{(|h|\le 1)}$ term missing if $\alpha < 1$. So

$$-\mathrm{Re}\,(\widehat{\mathcal{M}}^z(u)) = \int [1 - \cos(u \cdot h)] \frac{A(z,h)}{|h|^{d+\alpha}}\, dh,$$

and the assertion in the case $|u| \le 1$ is immediate.

If $|u| > 1$, setting $u = rv$, where $|v| = 1$ and $r \in (1, \infty)$,

$$\begin{aligned}
-\mathrm{Re}\,(\widehat{\mathcal{M}}^z(u)) &\geq c \int_{|h|\leq 1} [1 - \cos(u \cdot h)] \frac{1}{|h|^{d+\alpha}}\, dh \\
&= c \int_{|h|\leq 1} [1 - \cos(r(v \cdot h))] \frac{1}{|h|^{d+\alpha}}\, dh \\
&= c r^\alpha \int_{|h|\leq r} [1 - \cos(v \cdot h)] \frac{1}{|h|^{d+\alpha}}\, dh \\
&\geq c r^\alpha \int_{|h|\leq 1} [1 - \cos(v \cdot h)] \frac{1}{|h|^{d+\alpha}}\, dh,
\end{aligned}$$

using a change of variables. The integral in the last line is bounded below by a constant, using rotational invariance, and the result follows on noting $r = |u|$. □

Corollary 2.9 *If $p^z(t, x, y) = \overline{p}^z_t(x-y)$ is the transition density for the Lévy process with generator $\mathcal{M}^z$, then for each t, $\sup_z \|\overline{p}^z_t\|_2 \leq c(t) < \infty$.*

Moreover, if $r^z_\lambda(x) = \int_0^\infty e^{-\lambda t}\overline{p}^z_t(x)\, dt$ and $\lambda \geq 1$, then

$$|\widehat{r^z}_\lambda(u)| \leq \frac{c}{\lambda + |u|^\alpha}.$$

Proof. The Fourier transform of $\overline{p}^z_t$ is $e^{t\widehat{\mathcal{M}}^z(u)}$, and so

$$|e^{t\widehat{\mathcal{M}}^z(u)}| = e^{t\mathrm{Re}\,(\widehat{\mathcal{M}}^z(u))}.$$

With the estimates from Proposition 2.8, this is less than or equal to 1 if $|u| \leq 1$ and less than or equal to $e^{-ct|u|^\alpha}$ if $|u| > 1$, where c does not depend on z. So the Fourier transform of $\overline{p}^z_t$ is in L^2, hence $\overline{p}^z_t$ is in L^2 by Plancherel's theorem, with a bound not depending on z.

Now

$$|\widehat{r^z}_\lambda(u)| = \Big|\frac{1}{\lambda - \widehat{\mathcal{M}}^z(u)}\Big| \leq \frac{1}{\mathrm{Re}\,(\lambda - \widehat{\mathcal{M}}^z(u))}.$$

This is less than or equal to c/λ if $|u| \leq 1$ and less than $1/(\lambda + c|u|^\alpha)$ if $|u| > 1$. Since $\lambda \geq 1$, this proves the corollary. □

Recall that R^z_λ is the resolvent operator.

Proposition 2.10 *If $f \in L^2$, $\|R^z_\lambda f\|_2 \le \frac{1}{\lambda}\|f\|_2$.*

Proof. By Jensen's inequality, $\|P^z_t f\|_2 \le \|f\|_2$. We now apply Minkowski's inequality for integrals. $\square$

Proposition 2.11 *Let $\lambda \ge 1$, R^z_λ be as above, $h \in \mathbb{R}^d$, and $f \in L^2 \cap C^2_K$. Set*

$$g(x) = R^z_\lambda f(x+h) - R^z_\lambda f(x)$$

and

$$G(x) = R^z_\lambda f(x+h) - R^z_\lambda f(x) - \nabla R^z_\lambda f(x) \cdot h.$$

(a) If $\alpha < 1$, then

$$\|g\|_2 \le c|h|^\alpha \|f\|_2.$$

(b) If $\alpha \in (0,2)$, then

$$\|g\|_2 \le \frac{c}{\lambda}\|f\|_2.$$

(c) If $\alpha \in [1,2)$, then

$$\|G\|_2 \le c|h|^\alpha \|f\|_2.$$

(d) If $\alpha \in [1,2)$, then

$$\|G\|_2 \le c\Big(\frac{1}{\lambda} + |h|\Big)\|f\|_2.$$

Proof. First of all, if $f \in L^2 \cap C^2_K$, then $R^z_\lambda f \in L^2 \cap C^2_b$ by Proposition 2.10 and translation invariance. So $\nabla R^z_\lambda f$ is well defined. By translation invariance, $\frac{\partial R^z_\lambda f}{\partial x_i} = R^z_\lambda(\frac{\partial f}{\partial x_i})$, and $\frac{\partial f}{\partial x_i} \in C^1_K \subset L^2$, so $R^z_\lambda(\frac{\partial f}{\partial x_i}) \in C^1_b$, and is in L^2. Therefore to prove the proposition it suffices to look at Fourier transforms and to use Plancherel's theorem.

(a) We have

$$\widehat{g}(u) = \widehat{f}(u)\widehat{r^z_\lambda}(u)[e^{iu\cdot h} - 1],$$

so using Corollary 2.9

$$|\widehat{g}(u)| \le \frac{c|\widehat{f}(u)|}{\lambda + |u|^\alpha}|h|^\alpha |u|^\alpha \le c|\widehat{f}(u)|\,|h|^\alpha.$$

Therefore

$$\|\widehat{g}\|_2 \le c|h|^\alpha \|\widehat{f}\|_2,$$

and the result follows by Plancherel's theorem.

(b) As in (a), but using $|e^{iu\cdot h} - 1| \le 2$, we have

$$|\widehat{g}(u)| \le \frac{2c|\widehat{f}(u)|}{\lambda},$$

and we use Plancherel's theorem as in (a).

(c)

$$\widehat{G}(u) = \widehat{f}(u)\widehat{r^z}_\lambda(u)[e^{iu\cdot h} - 1 - iu\cdot h].$$

Now

$$\begin{aligned} |e^{iu\cdot h} - 1 - iu\cdot h| &= \Big|\int_0^{u\cdot h} [ie^{is} - i]\, ds\Big| \\ &\le c\int_0^{|u\cdot h|} |s|^{\alpha-1}\, ds \\ &\le c|u\cdot h|^\alpha. \end{aligned}$$

Hence

$$|\widehat{G}(u)| \le c\frac{|\widehat{f}(u)|}{\lambda + |u|^\alpha}|u|^\alpha |h|^\alpha \le c|h|^\alpha |\widehat{f}(u)|.$$

(d) Similarly to the proofs of (b) and (c),

$$|\widehat{G}(u)| \le c\frac{|\widehat{f}(u)|}{\lambda + |u|^\alpha}(2 + |u\cdot h|).$$

If $|u| \le 1$, then

$$|\widehat{G}(u)| \le \frac{c}{\lambda}|\widehat{f}(u)|(2 + |h|).$$

On the other hand, if $|u| > 1$, then since $\alpha \ge 1$ and

$$\frac{|u\cdot h|}{\lambda + |u|^\alpha} \le \frac{|u|\,|h|}{|u|^\alpha} \le |h|,$$

we have

$$|\widehat{G}(u)| \le c|h|\,|\widehat{f}(u)|.$$

□

Using this proposition we can extend the definition of the functions g, G and extend the above estimates to every $f \in L^2$.

3 Approximation

A key step in the uniqueness proof is to get a bound on the resolvent for an arbitrary solution to the martingale problem for $\mathcal{L}$. We do that by an approximation procedure.

We begin with

Definition 3.1 Let $(S, \mathcal{S}, \lambda)$ be a measure space, where λ is a σ-finite measure. A random measure $\mu([0,t] \times A)(\omega)$ is a Poisson point process with intensity measure λ if

(a) whenever $A \in \mathcal{S}$ and $\lambda(A) < \infty$, $N_t(A) = \mu([0,t] \times A)$ is a Poisson process with intensity $\lambda(A)$ and

(b) If $n \geq 1$ and $A_1, \dots, A_n \in \mathcal{S}$ are disjoint with $\lambda(A_1), \dots, \lambda(A_n) < \infty$, then the processes $N_t(A_i)$, $i = 1, \dots, n$, are independent.

Proposition 3.2 *Suppose the following hold.*

(1) If $A \in \mathcal{S}$ and $\lambda(A) < \infty$, then $N_t(A)$ is a process starting at 0 with paths that are constant except for jumps that are of size one.

(2) If $A \in \mathcal{S}$ and $\lambda(A) < \infty$, then $N_t(A)$ has paths that are right continuous with left limits.

(3) If $A \in \mathcal{S}$ and $\lambda(A) < \infty$, then $N_t(A) - \lambda(A)t$ is a martingale.

(4) If $A, B \in \mathcal{S}$, $\lambda(A), \lambda(B) < \infty$ and A and B are disjoint, then $N_t(A)$ and $N_t(B)$ have no jumps in common.

Then $\mu([0,t] \times A) = N_t(A)$ is a Poisson point process.

Proof. Property (a) of the definition of Poisson point process follows from a very slight modification of [14, III.T12]. In addition, that theorem shows that if $t > s$, then $\sigma(N_t(A) - N_s(A) : A \in \mathcal{S}\})$ is independent of $\mathcal{F}_s$.

We next prove that if A_i, $i = 1, \dots, n$, are disjoint sets of finite λ-measure and $t_0 > 0$, then

$$N_{t_0}(A_1), \dots, N_{t_0}(A_n) \text{ are independent random variables.} \tag{3.1}$$

To prove (3.1), we do the case when $n = 2$, the general case being very similar. Let u_1, u_2 be two reals and define

$$M_t^j = \exp\Big(iu_j N_{t\wedge t_0}(A_j) - \lambda(A_j)(t\wedge t_0)(e^{iu_j} - 1)\Big), \qquad j = 1, 2.$$

Because $N_t(A_j)$ is a Poisson process with intensity $\lambda(A_j)$, each M_t^j is a martingale with $M_0^j = 1$.

Since $N_t(A_1)$ and $N_t(A_2)$ have no jumps in common and are non-decreasing, the quadratic variation process $[N_\cdot(A_1), N_\cdot(A_2)]_t$ is zero. So by Ito's product formula,

$$M_\infty^1 M_\infty^2 = M_0^1 M_0^2 + \int_0^\infty M_{s-}^1 \, dM_s^2 + \int_0^\infty M_{s-}^2 \, dM_s^1,$$

or $\mathbb{E}\,[M_\infty^1 M_\infty^2] = 1$. It follows that

$$\begin{aligned} \mathbb{E}\left[e^{iu_1 N_{t_0}(A_1)} e^{iu_2 N_{t_0}(A_2)}\right] &= e^{\lambda(A_1)t_0(e^{iu_1}-1)} e^{\lambda(A_2)t_0(e^{iu_2}-1)} \\ &= \mathbb{E}\left[e^{iu_1 N_{t_0}(A_1)}\right] \mathbb{E}\left[e^{iu_2 N_{t_0}(A_2)}\right]. \end{aligned}$$

This holds for every u_1, u_2, so $N_{t_0}(A_1)$ and $N_{t_0}(A_2)$ are independent.

A very similar argument shows that if $0 < s_0 < t_0$, then $N_{t_0}(A_1) - N_{s_0}(A_1), \ldots, N_{t_0}(A_n) - N_{s_0}(A_n)$ are independent random variables. This and the independence of $\sigma(N_t(A) - N_s(A) : A \in \mathcal{S})$ from $\mathcal{F}_s$ when $t > s$ implies part (b) of Definition 3.1. □

We next construct a function $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that for every Borel set B and every $x \in \mathbb{R}^d$

$$\int 1_B(F(x,u)) \frac{du}{|u|^{d+\alpha}} = \int_B \frac{A(x,h)}{|h|^{d+\alpha}} \, dh. \tag{3.2}$$

Such constructions are known (see [13]), but we want our F to be continuous in x as well, and the existing constructions do not necessarily possess this property. (When $d > 1$, F satisfying (3.2) are by no means unique.) We will do the case $d = 2$ for simplicity of notation, but the idea for higher dimensions is essentially the same.

Fix x. We define F for u in the first quadrant, and the other quadrants are done similarly. Set $r_0 = \infty$ and choose $r_1 > r_2 > \cdots > 0$ such that

$$\int_{[r_{i+1}, r_i) \times [0,\infty)} \frac{A(x,h)}{|h|^{2+\alpha}}\, dh = 2^{-1}, \qquad i = 0, 1, \ldots$$

For each strip $[r_{i+1}, r_i) \times [0, \infty)$, let $s_0 = \infty$, $s_2 = 0$, and choose $s_1 > 0$ such that

$$\int_{[r_{i+1}, r_i) \times [s_{j+1}, s_j)} \frac{A(x,h)}{|h|^{2+\alpha}}\, dh = 2^{-2}, \qquad j = 0, 1.$$

Let $\mathcal{R}_1 = \mathcal{R}_1(x)$ be the collection of such rectangles. (Some of our rectangles will be semi-infinite.) Note that each rectangle in $\mathcal{R}_1$ has the same mass with respect to the measure $A(x,h)/|h|^{2+\alpha}\, dh$, but the rectangles are not congruent in shape.

Set $v_0 = \infty$ and choose $v_1 > v_2 > \cdots > 0$ such that

$$\int_{[v_{i+1}, v_i) \times [0,\infty)} \frac{du}{|u|^{2+\alpha}} = 2^{-1}, \qquad i = 0, 1, \ldots$$

For each strip $[v_{i+1}, v_i) \times [0, \infty)$, let $w_0 = \infty$, $w_2 = 0$, and choose $w_1 > 0$ such that

$$\int_{[v_{i+1}, v_i) \times [w_{j+1}, w_j)} \frac{du}{|u|^{2+\alpha}} = 2^{-2}, \qquad j = 0, 1.$$

Let $\mathcal{V}_1$ be the collection of such rectangles. Note $\mathcal{V}_1$ does not depend on x.

Let Γ_1 be the map from $\mathcal{V}_1$ to $\mathcal{R}_1$ taking the element $[v_{i+1}, v_i) \times [w_{j+1}, w_j)$ of $\mathcal{V}_1$ to the element $[r_{i+1}, r_i) \times [s_{j+1}, s_j)$ of $\mathcal{R}_1$.

If $[r, r') \times [s, s')$ is an element of $\mathcal{R}_1$, choose r'' such that

$$\int_{[r, r'') \times [s, s')} \frac{A(x,h)}{|h|^{2+\alpha}}\, dh = 2^{-3},$$

and then s'', s''' such that the integrals of $A(x,h)/|h|^{2+\alpha}$ over $[r, r'') \times [s, s'')$ and over $[r'', r') \times [s, s''')$ are both equal to 2^{-4}. We put the 4 rectangles $[r, r'') \times [s, s'')$, $[r, r'') \times [s'', s')$, $[r'', r') \times [s, s''')$, and $[r'', r') \times [s''', s')$ into $\mathcal{R}_2 = \mathcal{R}_2(x)$ and do this for each rectangle in $\mathcal{R}_1$. We divide each rectangle of $\mathcal{V}_1$ similarly into 4 rectangles of mass 2^{-4} with respect to the measure $du/|u|^{2+\alpha}$ and let $\mathcal{V}_2$ be the collection of such subrectangles. We define the map Γ_2 from $\mathcal{V}_2$ into $\mathcal{R}_2$ that takes a rectangle of $\mathcal{V}_2$ into the corresponding rectangle of $\mathcal{R}_2$.

We define $\mathcal{R}_m$ and $\mathcal{V}_m$ for all m by induction. For each rectangle in $\mathcal{R}_m$, we divide it in two by introducing a vertical line segment (or ray) so that each of the two resulting subrectangles has equal mass with respect to the measure $A(x,h)|h|^{-2-\alpha}\,dh$. Each of these two subrectangles is divided into two by introducing two horizontal line segments so that all of the four resulting subrectangles have equal mass with respect to $A(x,h)|h|^{-2-\alpha}\,dh$. We do this for each rectangle in $\mathcal{R}_m$, and let $\mathcal{R}_{m+1}$ be the collection of all the subrectangles formed in this way. We construct $\mathcal{V}_{m+1}$ from $\mathcal{V}_m$ in a similar way, but we use the measure $|u|^{-2-\alpha}\,du$ instead. Define the maps $\Gamma_m = \Gamma_{m,x}$ that take the rectangles of $\mathcal{V}_m$ into the corresponding rectangles of $\mathcal{R}_m$.

By virtue of Assumption 1.1(a), any compact subset of $(0,\infty)^2$ will, provided m is sufficiently large, be covered by finitely many rectangles such that each of these rectangles belongs to $\mathcal{R}_m$ and each rectangle has finite side lengths. Also, if R is rectangle in $\mathcal{R}_m$ which has finite side lengths, then any rectangle in $\mathcal{R}_{m+1}$ contained in R will have side lengths at most a fixed fraction (strictly less than 1) of the corresponding side lengths of R. Similar comments apply to $\mathcal{V}_m$.

Now define $F_m(x,u) : (0,\infty)^2 \to (0,\infty)^2$ by setting $F_m(x,u)$ to be the lower left hand point of $\Gamma_m(U)$ if $u \in U \in \mathcal{V}_m$. Recall x is fixed. We check that $F_m(x,u)$ converges uniformly over u in compact subsets of $(0,\infty)^2$. Consider a compact subset of $(0,\infty)^2$, and take m large enough so that this subset is covered by finitely many rectangles $\{U_m^i\}$, each in $\mathcal{V}_m$ and each with finite side lengths. Let δ_m be the maximum of the side lengths of the rectangles $\{\Gamma_m(U_m^i)\}$, and observe that by the paragraph above, $\delta_m \to 0$ as $m \to \infty$. If $u \in U_n^j \subset U_m^i$, where $n \geq m$, then both $F_m(x,u)$ and $F_n(x,u)$ lie in $\Gamma_m(U_m^i)$, so $|F_n(x,u) - F_m(x,u)| \leq \sqrt{2}\delta_m$. This implies the uniform convergence of $F_m(x,u)$ on compact subsets of $(0,\infty)^2$.

Define $F(x,u)$ to be the limit of $F_m(x,u)$. If u_1 and u_2 are distinct points of $(0,\infty)^2$, then for some m large enough, u_1 and u_2 will lie in distinct rectangles U_1 and U_2 of $\mathcal{V}_m$. Moreover, if m is large enough, the distance between U_1 and U_2 will be positive. For all $n \geq m$, $F_n(x,u_i)$ will lie in $\Gamma_m(U_i)$, $i = 1,2$. We conclude that the map $u \to F(x,u)$ is one-to-one.

The construction shows that the equality (3.2) holds if $m \geq 1$ and $B \in \mathcal{V}_m$. Linearity and a limit argument shows that it holds for every Borel set contained in $(0,\infty)^2$.

Suppose $x_0 \in \mathbb{R}^2$ is fixed and $m \geq 1$. Given a compact subset H of $(0,\infty)^2$ and $\varepsilon > 0$, choose m large enough so that H can be covered by finitely many rectangles $\{U_m^i\}$ with the properties that each $U_m^i \in \mathcal{V}_m$, each rectangle U_m^i has finite side lengths, and in addition each rectangle of $\mathcal{V}_m$ whose closure intersects the closure of one of the U_m^i also has finite side lengths. Taking m even larger if necessary, we can arrange that the maximum side length of any $\Gamma_{m,x_0}(U_m^i)$ is at most ε. Recall that the construction of the rectangles in $\mathcal{V}_m$ is independent of the point x (this is not the case for $\mathcal{R}_m$). By taking x sufficiently close to x_0 the boundaries of each rectangle in $\mathcal{R}_i(x)$, $i \leq m$, can be made as close as we please to the boundaries of the corresponding rectangle of $\mathcal{R}_i(x_0)$; we are using the continuity of $A(x,h)$ here. Take x sufficiently close to x_0 so that the lower left hand corner of each $\Gamma_{m,x}(U_m^i)$ is within ε of the lower left hand corner of $\Gamma_{m,x_0}(U_m^i)$ and the side lengths of each $\Gamma_{m,x}(U_m^i)$ are all less than 2ε. Then for each $u \in H$, $|F_m(x,u) - F_m(x_0,u)| < \varepsilon$, and if $n \geq m$, $|F_n(x_0,u) - F_m(x_0,u)| < \sqrt{2}\varepsilon$ and $|F_n(x,u) - F_m(x,u)| < 2\sqrt{2}\varepsilon$. We conclude that for $n \geq m$,

$$|F_n(x,u) - F_n(x_0,u)| \leq (3\sqrt{2}+1)\varepsilon < 6\varepsilon.$$

Letting $n \to \infty$, we have $|F(x,u) - F(x_0,u)| \leq 6\varepsilon$ for $u \in H$, and we deduce that $F(x,u)$ is continuous in x, uniformly over $u \in H$.

Using Assumption 1.1(a), the construction also tells us that there exists β such that

$$\beta|u| \leq |F(x,u)| \leq \beta^{-1}|u|, \qquad x \in \mathbb{R}^2, \quad u \neq 0. \tag{3.3}$$

From now on, we no longer assume $d = 2$. For each x, let $G(x,\cdot)$ be the inverse of $F(x,\cdot)$. Define

$$N_t(C) = \sum_{s \leq t} 1_{(G(X_{s-},\Delta X_s) \in C)}$$

and

$$\lambda(C) = \int_C \frac{du}{|u|^{d+\alpha}}.$$

Proposition 3.3 *Let $x_0 \in \mathbb{R}^d$ and let $\mathbb{P}$ be a solution to the martingale problem for $\mathcal{L}$ started at x_0. Then with respect to $\mathbb{P}$, $N_t(\cdot)$ is a Poisson point process with intensity measure λ.*

Proof. Let $F(x,C)=\{F(x,z):z\in C\}$. Then

$$N_t(C)=\sum_{s\le t} 1_{(\Delta X_s\in F(X_{s-},C))}.$$

By Proposition 2.6 and a limit argument, the right hand side is equal to a martingale plus

$$\int_0^t \int_{F(X_{s-},C)} \frac{A(X_s,h)}{|h|^{d+\alpha}}\,dh\,ds.$$

By (3.2) and the fact that X has only countably many jumps, this in turn is equal to

$$\begin{aligned}\int_0^t \int 1_{(F(X_{s-},u)\in F(X_{s-},C))}\frac{du}{|u|^{d+\alpha}}\,ds &= \int_0^t \int_C \frac{du}{|u|^{d+\alpha}}\,ds\\ &= \lambda(C)t.\end{aligned}$$

Therefore by Proposition 3.2 we see that $N_t(\cdot)$ is a Poisson point process. □

Set $\mu([0,t]\times C)=N_t(C)$. Note the definition of μ does not depend on $\mathbb{P}$.

Proposition 3.4 *X_t solves the stochastic differential equation*

$$\begin{aligned}X_t = X_0 &+ \int_0^t \int_{|F(X_{s-},z)|\le 1} F(X_{s-},z)\,(\mu(dz\,ds)-\lambda(dz)\,ds)\\ &+ \int_0^t \int_{|F(X_{s-},z)|>1} F(X_{s-},z)\,\mu(dz\,ds). \qquad (3.4)\end{aligned}$$

Proof. Let $\delta>0$. Set $G(x,D)=\{G(x,w):w\in D\}$, $D^\delta=\{y:|y|>\delta\}$,

$$\begin{aligned}H^\delta_t &= \sum_{s\le t}\Delta X_s 1_{(1\ge|\Delta X_s|>\delta)},\\ \widetilde{H^\delta}_t &= \int_0^t \int_{G(X_{s-},D^\delta\setminus D^1)} F(X_{s-},z)\,\lambda(dz)\,ds,\\ K_t &= \sum_{s\le t}\Delta X_s 1_{(|\Delta X_s|>1)}.\end{aligned}$$

If $\Delta X_s \neq 0$, the definition of μ via N_t shows that μ assigns unit mass to some point (z,s) satisfying $z = G(X_{s-}, \Delta X_s)$, or with $\Delta X_s = F(X_{s-}, z)$. Hence

$$H_t^\delta = \int_0^t \int_{G(X_{s-}, D^\delta \setminus D^1)} F(X_{s-}, z)\, \mu(dz\, ds) \tag{3.5}$$

and

$$K_t = \int_0^t \int_{G(X_{s-}, D^1)} F(X_{s-}, z)\, \mu(dz\, ds). \tag{3.6}$$

By [13, Theorem II.10], there exists a function $\overline{F}(x,z)$ satisfying

$$\int 1_B(\overline{F}(x,u)) \frac{du}{|u|^{d+\alpha}} = \int_B \frac{A(x,h)}{|h|^{d+\alpha}}\, dh, \tag{3.7}$$

for B Borel and and a Poisson point process $\overline{\mu}$ such that X_t solves

$$\begin{aligned} X_t = X_0 &+ \int_0^t \int_{|\overline{F}(X_{s-},z)| \le 1} \overline{F}(X_{s-}, z)\, (\overline{\mu}(dz\, ds) - \lambda(dz)\, ds) \\ &+ \int_0^t \int_{|\overline{F}(X_{s-},z)| > 1} \overline{F}(X_{s-}, z)\, \overline{\mu}(dz\, ds). \end{aligned}$$

From this equation we see that $\overline{\mu}$ gives unit mass to a point (z,s) if and only if $\Delta X_s = \overline{F}(X_{s-}, z)$. Set

$$V_t^\delta = X_t - X_0 - K_t - (H_t^\delta - \widetilde{H^\delta}_t).$$

We then have

$$V_t^\delta = \int_0^t \int_{|\overline{F}(X_{s-},z)| \le \delta} \overline{F}(X_{s-}, z)\, (\overline{\mu}(dz\, ds) - \lambda(dz)\, ds).$$

A limit argument and (3.7) show that

$$\int |\overline{F}(x,u)|^2 \frac{du}{|u|^{d+\alpha}} = \int |h|^2 \frac{A(x,h)}{|h|^{d+\alpha}}\, dh,$$

which is bounded uniformly in x. Consequently each component of V^δ is a pure jump martingale and $\mathbb{E} \sup_{s \le t} |V_t^\delta|^2 \to 0$ as $\delta \to 0$.

On the other hand, using (3.5) and (3.6),

$$V_t^\delta = X_t - \Big(X_0 + \int_0^t \int_{\delta<|F(X_{s-},z)|\le 1} F(X_{s-},z)\,(\mu(dz\,ds) - \lambda(dz)\,ds)$$
$$+ \int_0^t \int_{|F(X_{s-},z)|>1} F(X_{s-},z)\,\mu(dz\,ds)\Big).$$

Our conclusion follows. □

Fix $x_0 \in \mathbb{R}^d$ and define Y_s^n to be equal to x_0 if $s < 1/n$ and equal to $X_{(k-1)/n}$ if $k/n \le s < (k+1)/n$. The reason for the $1/n$ delay will appear in (4.12). Let

$$X_t^n = X_0 + \int_0^t \int_{|F(Y_s^n,z)|\le 1} F(Y_s^n,z)(\mu(dz\,ds) - \lambda(dz)\,ds)$$
$$+ \int_0^t \int_{|F(Y_s^n,z)|>1} F(Y_s^n,z)\,\mu(dz\,ds). \tag{3.8}$$

Proposition 3.5 *Let $x \in \mathbb{R}^d$ and let $\mathbb{P}$ be a solution to the martingale problem for $\mathcal{L}$ started at x. For each t_0*

$$\sup_{t\le t_0} |X_t - X_t^n| \to 0$$

in probability as $n \to \infty$.

Proof. Except for $s = 0$, notice $Y_s^n \to X_{s-}$ a.s. under $\mathbb{P}$, using the fact that the paths of X_t have left limits. Except for z in the boundary of any of the 2^d orthants, $F(Y_s^n, z) \to F(X_{s-}, z)$ a.s. if $s > 0$.

Let X_t^I be the first double integral on the right in (3.4) and X_t^{II} the second. Let

$$Z_t^n = X_0 + \int_0^t \int_{|F(X_{s-},z)|\le 1} F(Y_s^n,z)(\mu(dz\,ds) - \lambda(dz)\,ds)$$
$$+ \int_0^t \int_{|F(X_{s-},z)|>1} F(Y_s^n,z)\mu(dz\,ds)$$
$$= X_0 + Z_t^{n,I} + Z_t^{n,II}.$$

Using Doob's inequality on each component and basic properties of stochastic integrals with respect to Poisson point processes (see, e.g., [11]),

$$\begin{aligned}\mathbb{E}\sup_{t\le t_0}|X^I_t-Z^{n,I}_t|^2&\le c\mathbb{E}\,|X^I_{t_0}-Z^{n,I}_{t_0}|^2\\&=c\mathbb{E}\int_0^{t_0}\int_{|F(X_{s-},z)|\le 1}|F(X_{s-},z)-F(Y^n_s,z)|^2\,\lambda(dz)\,ds.\end{aligned}$$

Using (3.3), the integrand is bounded by

$$c|z|^2 1_{(|z|\le\beta^{-1})},$$

which is integrable with respect to $\lambda(dz)\,ds$. For $s>0$ and all z not on the boundary of any of the orthants, and hence for almost every z with respect to λ, the integrand tends to 0, a.s. Therefore by dominated convergence

$$\mathbb{E}\sup_{t\le t_0}|X^I_t-Z^{n,I}_t|^2\to 0.$$

Since $|F(X_{s-},z)|>1$ implies $|z|\ge\beta$ by (3.3), with probability one, there are only finitely many points (z,s) with $|z|\ge\beta$ charged by μ before time t_0. Also with probability one, none of the z values will lie on the boundary of any of the orthants. It follows then that $\sup_{t\le t_0}|X^{II}_t-Z^{n,II}_t|\to 0$ as $n\to\infty$.

Notice

$$X^n_t-Z^n_t=\int_0^t\int_{C(n,s)}F(Y^n_s,z)\,\lambda(dz)\,ds,$$

where

$$\begin{aligned}C(n,s)=\{&|F(Y^n_s,z)|\le 1,|F(X_{s-},z)|>1\}\\&\cup\{|F(X_{s-},z)|\le 1,|F(Y^n_s,z)|>1\}.\end{aligned}$$

Therefore

$$\mathbb{E}\sup_{t\le t_0}|X^n_t-Z^n_t|\le\sum_{i=1}^5\mathbb{E}\int_0^{t_0}\int_{D^i(n,s)}|F(Y^n_s,z)|\,\lambda(dz)\,ds,$$

where

$$\begin{aligned}D^1(n,s)&=\{|F(Y^n_s,z)|\le 1-\gamma,|F(X_{s-},z)|>1\},\\D^2(n,s)&=\{|F(Y^n_s,z)|\le 1,|F(X_{s-},z)|\ge 1+\gamma\},\\D^3(n,s)&=\{|F(Y^n_s,z)|\ge 1+\gamma,|F(X_{s-},z)|\le 1\},\\D^4(n,s)&=\{|F(Y^n_s,z)|>1,|F(X_{s-},z)|\le 1-\gamma\},\\D^5(n,s)&=\{1-\gamma\le|F(Y^n_s,z)|,|F(X_{s-},z)|<1+\gamma\},\end{aligned}$$

and $\gamma > 0$ will be chosen in a moment. By (3.3), if $|F(X_{s-},z)| \geq 1-\gamma$, then $|z| \geq c$. So using (3.2) and Assumption 1.1

$$\begin{aligned}
&\mathbb{E}\int_0^{t_0}\int_{D^5(n,s)} |F(Y_s^n,z)|\,\lambda(dz)\,ds \\
&\qquad \leq (1+\gamma)\mathbb{E}\int_0^{t_0}\int 1_{B(0,1+\gamma)\setminus B(0,1-\gamma)}(F(X_{s-},z))\,\lambda(dz)\,ds \\
&\qquad = (1+\gamma)\mathbb{E}\int_0^{t_0}\int_{B(0,1+\gamma)\setminus B(0,1-\gamma)} \frac{A(X_{s-},h)}{|h|^{d+\alpha}}\,dh\,ds \\
&\qquad \leq c\gamma t_0.
\end{aligned}$$

So the integral over $D^5(n,s)$ can be made as small as we like by taking γ sufficiently small. Once γ is chosen, observe that $1_{D^1(n,s)} \to 0$ a.s. for every $s > 0$ because $Y_s^n \to X_{s-}$. Also, on $D^1(n,s)$, we have $|F(X_{s-},z)| > 1$, and as above $|z| > c$, so $|F(Y_s^n,z)|1_{D^1(n,s)}$ is dominated by $(1+\gamma)1_{(|z|\geq c)}$, which is integrable with respect to $\lambda(dz)\,ds$. So by dominated convergence,

$$\mathbb{E}\int_0^{t_0}\int_{D^1(n,s)} |F(Y_s^n,z)|\,\lambda(dz)\,ds \to 0.$$

The argument for $D^2(n,s), D^3(n,s)$, and $D^4(n,s)$ is the same. Hence

$$\mathbb{E}\sup_{t\leq t_0}|X_t^n - Z_t^n| \to 0$$

. □

Proposition 3.6 *Let $x_0 \in \mathbb{R}^d$ and let $\mathbb{P}$ be a solution to the martingale problem for $\mathcal{L}$ started at x_0. If $f \in C_b^2$, then*

$$f(X_t^n) - f(X_0^n) - \int_0^t \mathcal{M}^{Y_s^n} f(X_s^n)\,ds$$

is a martingale under $\mathbb{P}$, where $\mathcal{M}^y$ is defined in (2.2).

Proof. If μ assigns unit mass to (z,s), then $\Delta X_s^n = F(Y_s^n,z)$. By Ito's formula

$$
\begin{aligned}
f(X_t^n) &- f(X_0^n)\\
&= \text{martingale } + \int_0^t \int_{|F(X_{s-},z)|>1} \nabla f(X_{s-}^n)\cdot F(Y_s^n,z)\,\mu(dz\,ds)\\
&\quad + \sum_{s\le t}[f(X_s^n)-f(X_{s-}^n)-\nabla f(X_{s-}^n)\cdot \Delta X_s^n]\\
&= \text{martingale } + \int_0^t \int [f(X_{s-}^n+F(Y_s^n,z))-f(X_{s-}^n)\\
&\quad - \nabla f(X_{s-}^n)\cdot F(Y_s^n,z)1_{(|F(Y_s^n,z)|\le 1)}]\,\mu(dz\,ds)\\
&= \text{martingale } + \int_0^t \int [f(X_{s-}^n+F(Y_s^n,z))-f(X_{s-}^n)\\
&\quad - \nabla f(X_{s-}^n)\cdot F(Y_s^n,z)1_{(|F(Y_s^n,z)|\le 1)}]\,|z|^{-(d+\alpha)}\,dz\,ds.
\end{aligned}
$$

Fix y and if $\alpha \ge 1$, let

$$g(v) = f(y+v) - f(y) - \nabla f(y)\cdot v1_{(|v|\le 1)}.$$

A limit argument using (3.2) shows

$$\int g(F(x,z))\frac{1}{|z|^{d+\alpha}}\,dz = \int g(h)\frac{A(x,h)}{|h|^{d+\alpha}}\,dh.$$

Now taking $y = X_{s-}^n$ and $x = Y_s^n$ shows that $f(X_t^n) - f(X_0^n)$ is equal to a martingale plus

$$\int_0^t \int [f(X_{s-}^n + h) - f(X_{s-}^n) - \nabla f(X_{s-}^n)\cdot h1_{(|h|\le 1)}]\frac{A(Y_s^n,h)}{|h|^{d+\alpha}}\,dh\,ds,$$

which proves the proposition when $\alpha \ge 1$. The case $\alpha < 1$ is similar. □

4 Existence and uniqueness

Theorem 4.1 *Suppose Assumption 1.1 holds. Then for each x there exists a solution to the martingale problem for $\mathcal{L}$ started at x.*

Proof. In view of Propositions 2.3 and 2.4, existence of a solution follows by the proof in [4, Section 3], with minor modifications to handle the case of d dimensions. □

Remark 4.2 It is easy to see by the same arguments that existence holds if $A(x,h)$ is bounded above and below by positive constants and for each h, $A(x,h)$ is continuous in x.

We now turn to the proof of uniqueness. Fix $x_0 \in \mathbb{R}^d$. If $\mathcal{G}$ is the set of solutions to the martingale problem for $\mathcal{L}$ started at x_0, then $\mathcal{G}$ is a tight family by the proof in [4, Section 3]. Any subsequential limit point of $\mathcal{G}$ is in $\mathcal{G}$ by the arguments in that same section, and therefore $\mathcal{G}$ is compact. Hence by the proofs in [17, Chapter 12], it suffices to consider uniqueness of strong Markov families of solutions $\{\mathbb{P}^x\}$ to the martingale problem for $\mathcal{L}$.

Let $\eta > 0$ and let ψ_η be as in (1.3). We will sometimes make the following temporary assumption, where we will choose ζ later:

Assumption 4.3 *There exists ζ such that*

$$|A(x,h) - A(x_0,h)| \le \frac{\zeta}{\psi_\eta(|h|)}, \qquad x \in \mathbb{R}^d, \quad |h| \le 1.$$

For the rest of this section we take $\alpha \ge 1$, the case $\alpha < 1$ being similar.

Let $\mathcal{M}^z f$ be defined by (2.2) and let R^z_λ be the corresponding resolvent. Define an operator $\mathcal{H}$ by

$$\mathcal{H}f(x) = \int_{|h|\le 1} |f(x+h) - f(x) - \nabla f(x)\cdot h| \frac{\zeta}{\psi_\eta(|h|)|h|^{d+\alpha}}\, dh + \int_{|h|>1} |f(x+h) - f(x)|\, \frac{dh}{|h|^{d+\alpha}}, \qquad f \in C^2_b.$$

Proposition 4.4 *There exists a constant c_1 not depending on x_0 such that if $\lambda \ge 1$, then*

$$\|\mathcal{H}R^{x_0}_\lambda f\|_2 \le c_1(\zeta + \lambda^{-1})\|f\|_2, \qquad f \in L^2 \cap C^2_b. \tag{4.1}$$

Proof. By Minkowski's inequality for integrals,

$$\begin{aligned}\|\mathcal{H}R_\lambda^{x_0}f\|_2 \le \int_{|h|\le 1} &\|R_\lambda^{x_0}f(x+h)-R_\lambda^{x_0}f(x)\\ &-\nabla R_\lambda^{x_0}f(x)\cdot h\|_2\frac{\zeta}{\psi_\eta(h)|h|^{d+\alpha}}\,dh\\ &+\int_{|h|>1}\|R_\lambda^{x_0}f(x+h)-R_\lambda^{x_0}f(x)\|_2\frac{c}{|h|^{d+\alpha}}\,dh. \end{aligned}\tag{4.2}$$

By Proposition 2.11(c) and the definition of ψ_η given in (1.3), the first term on the right of (4.2) is bounded by

$$c\int_{|h|\le 1}|h|^\alpha\frac{\zeta}{\psi_\eta(h)|h|^{d+\alpha}}\,dh\|f\|_2\le c\zeta\|f\|_2. \tag{4.3}$$

By Proposition 2.11(b) the second term on the right of (4.2) is bounded by

$$\frac{c}{\lambda}\int_{|h|>1}\frac{dh}{|h|^{d+\alpha}}\,\|f\|_2. \tag{4.4}$$

□

Corollary 4.5 *Suppose Assumption 4.3 holds and $\lambda\ge 1$. There exists κ such that*

$$\|(\mathcal{L}-\mathcal{M}^{x_0})R_\lambda^{x_0}f\|_2\le\kappa(\zeta+\lambda^{-1})\|f\|_2,\qquad f\in L^2\cap C_b^2, \tag{4.5}$$

and

$$\|\sup_{w\in\mathbb{R}^d}|\mathcal{M}^wR_\lambda^{x_0}f(\cdot)-\mathcal{M}^{x_0}R_\lambda^{x_0}f(\cdot)|\,\|_2\le\kappa(\zeta+\lambda^{-1})\|f\|_2,\qquad f\in L^2\cap C_b^2. \tag{4.6}$$

Proof. If Assumption 4.3 holds, then

$$\begin{aligned}|(\mathcal{L}-\mathcal{M}^{x_0})&R_\lambda^{x_0}f(x)|\\ &=\Big|\int[R_\lambda^{x_0}f(x+h)-R_\lambda^{x_0}f(x) \qquad (4.7)\\ &\qquad-\nabla R_\lambda^{x_0}f(x)\cdot h1_{(|h|\le 1)}]\frac{A(x,h)-A(x_0,h)}{|h|^{d+\alpha}}\,dh\Big|\\ &\le c\mathcal{H}R_\lambda^{x_0}f(x)\end{aligned}$$

and for each w

$$
\begin{aligned}
|(\mathcal{M}^w R_\lambda^{x_0} f(x,y) - \mathcal{M}^{x_0} R_\lambda^{x_0} f(x)| \\
\le \Big| \int_{|h|\le 1} [R_\lambda^{x_0} f(x+h) - R_\lambda^{x_0} f(x) \qquad\qquad (4.8) \\
- \nabla R_\lambda^{x_0} f(x) \cdot h 1_{|h|\le 1)}] \frac{A(w,h) - A(x_0,h)}{|h|^{d+\alpha}} \, dh \Big| \\
+ \Big| \int_{|h|\ge 1} [R_\lambda^{x_0} f(x+h) - R_\lambda^{x_0} f(x)] \frac{A(w,h) + A(x_0,h)}{|h|^{d+\alpha}} \, dh \Big| \\
\le c\mathcal{H} R_\lambda^{x_0} f(x).
\end{aligned}
$$

Now combine Proposition 4.4, (4.7), and (4.8). □

Proposition 4.6 *Let $\{\mathbb{P}^x\}$ be a strong Markov family of solutions to the martingale problem for $\mathcal{L}$. For f bounded and measurable set*

$$S_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t) \, dt.$$

Suppose Assumption 4.3 holds with ζ and λ chosen so that $\lambda \ge 1$ and $\kappa(\zeta + \lambda^{-1}) \le 1/2$, where κ is as in Corollary 4.5. Let $\rho \in L^2$ be non-negative with compact support. Then

$$\sup_{\|g\|_2 \le 1} \Big| \int S_\lambda g(x) \rho(x) \, dx \Big| < \infty.$$

Proof. Fix $x_0 \in \mathbb{R}^d$, define X_t^n as in Section 3, and define

$$S_\lambda^n g(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} g(X_t^n) \, dt, \qquad g \in C_b^2.$$

Step 1: Our first goal is to show that if

$$\Lambda_n = \sup_{\|g\|_2 \le 1} \Big| \int S_\lambda^n g(x) \rho(x) \, dx \Big|, \tag{4.9}$$

then $\Lambda_n < \infty$. The value of Λ_n will depend on ρ.

To prove (4.9) it suffices to suppose $g \geq 0$ since we can write an arbitrary g as the difference of its positive and negative parts. Suppose $g \in C_K^2$ and write

$$S_\lambda^n g(x) = \mathbb{E}^x \int_0^{1/n} e^{-\lambda t} g(X_t^n)\, dt + \sum_{k=1}^{\infty} \mathbb{E}^x \int_{k/n}^{(k+1)/n} e^{-\lambda t} g(X_t^n)\, dt. \tag{4.10}$$

Over the time interval $[0, 1/n)$, the process X_t^n behaves like the Lévy process corresponding to $\mathcal{M}^{x_0}$ started at x. So the first term on the right hand side of (4.10) is bounded by $R_\lambda^{x_0} g(x)$. By the Cauchy-Schwarz inequality and Proposition 2.10,

$$\Big| \int R_\lambda^{x_0} g(x) \rho(x)\, dx \Big| \leq \|R_\lambda^{x_0} g\|_2 \|\rho\|_2 \leq \frac{c}{\lambda} \|g\|_2. \tag{4.11}$$

The k^{th} term on the right hand side of (4.10) is

$$\begin{aligned} &e^{-\lambda(k-1)/n} \mathbb{E}^x \int_{k/n}^{(k+1)/n} e^{-\lambda(t-(k-1)/n)} g(X_t^n)\, dt \\ &\qquad \leq c e^{-\lambda k/n} \mathbb{E}^x \Big[\mathbb{E}^x \Big[\int_{1/n}^{2/n} g(X^n_{t+\frac{k-1}{n}})\, dt \mid \mathcal{F}_{(k-1)/n} \Big] \Big]. \end{aligned}$$

Let us temporarily write $\overline{Y}$ for $Y^n_{(k-1)/n}$. Conditional on $\mathcal{F}_{(k-1)/n}$, the process X_t^n over the time interval $[k/n, (k+1)/n)$ behaves like the Lévy process corresponding to $\mathcal{M}^{\overline{Y}}$ started at $X^n_{(k-1)/n}$ and run over the time interval $[1/n, 2/n]$. Therefore

$$\begin{aligned} \mathbb{E}^x \Big[\int_{1/n}^{2/n} g(X^n_{t+\frac{k-1}{n}})\, dt \mid \mathcal{F}_{(k-1)/n} \Big] & \qquad\qquad (4.12) \\ \leq \int_{1/n}^{2/n} P_t^{\overline{Y}} g(X^n_{(k-1)/n})\, dt & \\ \leq e^{\lambda/n} P^{\overline{Y}}_{1/n} R_\lambda^{\overline{Y}} g(X^n_{(k-1)/n}). & \end{aligned}$$

Using Corollary 2.9 and Proposition 2.10 we have for $w \in \mathbb{R}^d$

$$\begin{aligned} P^w_{1/n} R^w_\lambda g(v) &= \int \overline{p}^w(1/n, v-z) R_\lambda^w g(z)\, dz \\ &\leq \|\overline{p}^w(1/n, \cdot)\|_2 \|R_\lambda^w g\|_2 \\ &\leq c_n \frac{1}{\lambda} \|g\|_2, \end{aligned}$$

where c_n depends on n. Hence the k^{th} term on the right hand side of (4.10) is bounded by $ce^{-\lambda k/n}\|g\|_2$. Because ρ is in L^2 with compact support,

$$\int \mathbb{E}^x \int_{k/n}^{(k+1)/n} g(X_t^n)\, dt\, \rho(x)\, dx \leq c\|g\|_2 \int \rho(x)\, dx \leq c\|g\|_2.$$

Combining this with (4.11) and taking the supremum over $g \in C_K^2$ with $\|g\|_2 \leq 1$ proves (4.9).

Step 2: Next we show

$$\Lambda_n \leq \frac{2\|\rho\|_2}{\lambda}, \qquad n \geq 1. \tag{4.13}$$

Let $f \in C_b^2$. By Proposition 3.6

$$\mathbb{E}^x f(X_t^n) - f(x) = \mathbb{E}^x \int_0^t \mathcal{M}^{Y_s^n} f(X_s^n)\, ds.$$

Multiplying by $e^{-\lambda t}$ and integrating over t from 0 to ∞

$$\begin{aligned} S_\lambda^n f(x) - \frac{1}{\lambda} f(x) &= \mathbb{E}^x \int_0^\infty e^{-\lambda t} \int_0^t \mathcal{M}^{Y_s^n} f(X_s^n)\, ds\, dt \qquad (4.14)\\ &= \mathbb{E}^x \int_0^\infty \mathcal{M}^{Y_s^n} f(X_s^n) \int_s^\infty e^{-\lambda t}\, dt\, ds\\ &= \frac{1}{\lambda}\mathbb{E}^x \int_0^\infty e^{-\lambda s} \mathcal{M}^{Y_s^n} f(X_s^n)\, ds\\ &= \frac{1}{\lambda} S_\lambda^n \mathcal{M}^{x_0} f(x) + \frac{1}{\lambda}\mathbb{E}^x \int_0^\infty e^{-\lambda s}(\mathcal{M}^{Y_s^n} - \mathcal{M}^{x_0}) f(X_s^n)\, ds. \end{aligned}$$

If $g \in C_K^2$, set $f = R_\lambda^{x_0} g$. By translation invariance, $f \in C_b^2$. Standard semigroup manipulations show

$$\mathcal{M}^{x_0} f = \mathcal{M}^{x_0} R_\lambda^{x_0} g = \lambda R_\lambda^{x_0} g - g.$$

Therefore

$$S_\lambda^n R_\lambda^{x_0} g(x) - \frac{1}{\lambda} R_\lambda^{x_0} g(x) \leq S_\lambda^n R_\lambda^{x_0} g(x) - \frac{1}{\lambda} S_\lambda^n g(x) + \frac{1}{\lambda} S_\lambda^n H(x),$$

where

$$H(y) = \sup_{w \in \mathbb{R}^d} |\mathcal{M}^w R_\lambda^{x_0} g(y) - \mathcal{M}^{x_0} R_\lambda^{x_0} g(y)|.$$

We thus have

$$S_\lambda^n g(x) \le R_\lambda^{x_0} g(x) + S_\lambda^n H(x). \tag{4.15}$$

By Corollary 4.5 and our choice of ζ and λ,

$$\|H\|_2 \le \kappa(\zeta + \lambda^{-1})\|g\|_2 \le \frac{1}{2}\|g\|_2.$$

Multiplying (4.15) by $\rho(x)$ and integrating,

$$\begin{aligned} \Big| \int S_\lambda^n g(x)\rho(x)\,dx\Big| &\le \Big| \int R_\lambda^{x_0} g(x)\rho(x)\,dx\Big| + \Big| \int S_\lambda^n H(x)\rho(x)\,dx\Big| \\ &\le \|R_\lambda^{x_0} g\|_2\|\rho\|_2 + \Lambda_n \|H\|_2 \\ &\le \frac{1}{\lambda}\|\rho\|_2\|g\|_2 + \frac{1}{2}\Lambda_n\|g\|_2, \end{aligned}$$

where Λ_n is defined in Step 1. Taking the supremum over $g \in C_K^2$ with $\|g\|_2 \le 1$, we thus have

$$\Lambda_n \le \frac{\|\rho\|_2}{\lambda} + \frac{1}{2}\Lambda_n.$$

In Step 1 we proved $\Lambda_n < \infty$, and we conclude

$$\Lambda_n \le \frac{2}{\lambda}\|\rho\|_2.$$

Step 3: We now pass to the limit in n. By Step 1 and Step 2, if $g \in C_K^2$ with $\|g\|_2 \le 1$, then

$$\Big| \int S_\lambda^n g(x)\rho(x)\,dx\Big| \le \frac{2\|\rho\|_2}{\lambda}.$$

On the other hand,

$$S_\lambda^n g(x) = \mathbb{E}^{\,x} \int_0^\infty e^{-\lambda t} g(X_t^n)\,dt \to \mathbb{E}^{\,x} \int_0^\infty e^{-\lambda t} g(X_t)\,dt = S_\lambda g(x)$$

by dominated convergence. We thus see that

$$\Big| \int S_\lambda g(x)\rho(x)\,dx\Big| \le \frac{2\|\rho\|_2}{\lambda}.$$

Our result follows by taking the supremum over $g \in C_K^2$ with $\|g\|_2 \le 1$. □

Proof of Theorem 1.2: Let $x_0 \in \mathbb{R}^d$. Let $\rho \in L^2$ with compact support. We have seen that it suffices to prove uniqueness when we have a strong Markov family of solutions to the martingale problem for $\mathcal{L}$, so suppose we have two such families $\{\mathbb{P}_i^x\}$, $i = 1, 2$. Define

$$S_\lambda^i f(x) = \mathbb{E}_i^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt, \qquad i = 1, 2,$$

and let

$$S_\lambda^\Delta = S_\lambda^1 - S_\lambda^2.$$

Suppose $\lambda_0 \geq 1$ and ζ are chosen so that $\kappa(\zeta + \lambda_0^{-1}) \leq \frac{1}{2}$ and $\lambda > \lambda_0$, and suppose Assumption 4.3 holds with this choice of ζ.

Since $\mathbb{P}_i^x$ is a solution to the martingale problem for $\mathcal{L}$ started at x, for $f \in C_b^2$

$$\mathbb{E}_i^x f(X_t) - f(x) = \mathbb{E}_i^x \int_0^t \mathcal{L} f(X_s)\, ds.$$

Multiplying by $e^{-\lambda t}$ and integrating over t from 0 to ∞,

$$\begin{aligned} S_\lambda^i f(x) - \frac{1}{\lambda} f(x) &= \mathbb{E}_i^x \int_0^\infty e^{-\lambda t} \int_0^t \mathcal{L} f(X_s)\, ds\, dt \\ &= \mathbb{E}_i^x \int_0^\infty \mathcal{L} f(X_s) \int_s^\infty e^{-\lambda t}\, dt\, ds \\ &= \frac{1}{\lambda} S_\lambda^i \mathcal{L} f(x) \\ &= \frac{1}{\lambda} S_\lambda^i \mathcal{M}^{x_0} f(x) + \frac{1}{\lambda} S_\lambda^i (\mathcal{L} - \mathcal{M}^{x_0}) f(x). \end{aligned}$$

Now take $g \in C_K^2$ and set $f = R_\lambda^{x_0} g$. Then $f \in C_b^2$ and $\mathcal{M}^{x_0} f = \lambda R_\lambda^{x_0} g - g$. Hence

$$S_\lambda^i R_\lambda^{x_0} g(x) - \frac{1}{\lambda} R_\lambda^{x_0} g(x) = S_\lambda^i R_\lambda^{x_0} g(x) - \frac{1}{\lambda} S_\lambda^g(x) + \frac{1}{\lambda} S_\lambda^i (\mathcal{L} - \mathcal{M}^{x_0}) R_\lambda^{x_0} g(x),$$

or

$$S_\lambda^i g(x) = R_\lambda^{x_0} g(x) + S_\lambda^i (\mathcal{L} - \mathcal{M}^{x_0}) R_\lambda^{x_0} g(x). \tag{4.16}$$

Let

$$\Theta = \sup_{\|g\|_2 \leq 1} \Big| \int S_\lambda^\Delta g(x) \rho(x)\, dx \Big|.$$

By Proposition 4.6, we know that $\Theta < \infty$. From (4.16)

$$S_\lambda^\Delta g(x) = S_\lambda^\Delta (\mathcal{L} - \mathcal{M}^{x_0}) R_\lambda^{x_0} g(x).$$

Multiplying by $\rho(x)$ and integrating,

$$\begin{aligned}\Big| \int S_\lambda^\Delta g(x)\rho(x)\,dx \Big| &= \Big| \int S_\lambda^\Delta (\mathcal{L} - \mathcal{M}^{x_0}) R_\lambda^{x_0} g(x)\rho(x)\,dx \Big| \\ &\le \Theta \| (\mathcal{L} - \mathcal{M}^{x_0}) R_\lambda^{x_0} g\|_2.\end{aligned}$$

By Corollary 4.5 this is bounded by $\frac{1}{2}\|g\|_2$. Taking the supremum over $g \in C_K^2$ with $\|g\|_2 \le 1$, we then obtain $\Theta \le \frac{1}{2}\Theta$. Since $\Theta < \infty$, this implies $\Theta = 0$. This can be rewritten as

$$\int S_\lambda^1 g(x)\rho(x)\,dx = \int S_\lambda^2 g(x)\rho(x)\,dx.$$

This is true for each $\rho \in L^2$ with compact support, and we conclude that if $\lambda \ge 1$, then $S_\lambda^1 g(x) = S_\lambda^2 g(x)$ for almost every x. By Proposition 2.7, $S_\lambda^i g(x)$ is continuous in x, so we have equality for all x. By the uniqueness of the Laplace transform and the right continuity of X_t, we conclude

$$\mathbb{E}\,_1^x g(X_t) = \mathbb{E}\,_2^x g(X_t)$$

for all x and all t whenever g is continuous and bounded. By a limit argument this equality holds for all bounded g. Finally, by using the Markov property, the finite dimensional distributions under $\mathbb{P}_1^x$ and $\mathbb{P}_2^x$ are the same for each x.

The last step is to remove the use of Assumption 4.3. This is a standard localization argument. Because of Assumption 1.1, there exists $\widetilde{A}(x,h)$ such that $\widetilde{A}$ agrees with A in a neighborhood of x_0 and such that Assumption 1.1 holds for $\widetilde{A}$. If $\widetilde{\mathcal{L}}$ is the operator defined in terms of $\widetilde{A}$ in the same way as $\mathcal{L}$ is defined in terms of A, the above shows we have uniqueness for the martingale problem for $\widetilde{\mathcal{L}}$ started at x_0. From this point on, we proceed exactly as in the diffusion case – see [5, Chapter VI]; see also [4, Section 6] for a similar argument. This completes the proof of Theorem 1.2. □

Remark 4.7 It is clear that ψ_η defined by (1.3) can be replaced by any decreasing function ψ such that

$$\int_{|h|\le 1} \frac{1}{\psi(|h|)|h|^d}\,dh < \infty,$$

or equivalently,

$$\int_0^1 \frac{1}{\psi(r)r} dr < \infty.$$

Remark 4.8 Just as in the case of diffusions, we do not really need continuity of $A(x, h)$ in x, just that each point x_0 has a neighborhood in which $\overline{A}(x, \cdot)$ is sufficiently close to $\overline{A}(x_0, \cdot)$.

Remark 4.9 In [12] Komatsu considers uniqueness for operators of the form $\mathcal{L}_1 + \mathcal{L}_2$, where $\mathcal{L}_1$ is a stable process of index α (not necessarily symmetric, but he requires that the jump kernel for $\mathcal{L}_1$ be d times continuously differentiable in h away from the origin) and

$$\mathcal{L}_2 f(x) = \int [f(x + h) - f(x)] n(x, dh)$$

(with the appropriate modification when $\alpha \geq 1$) where there exists a measure n^* such that $|n(x, dh)| \leq n^*(dh)$ and $\int (1 \wedge |h|^\alpha)\, n^*(dh) < \infty$. If we write the kernel for $\mathcal{L}_1$ as $A_0(h)/|h|^{d+\alpha}$ and if in addition we assume n^* has a density with respect to Lebesgue measure, we can fit his framework into ours by setting

$$A(x, h) = A_0(h) + \frac{n(x, dh)}{dh} |h|^{d+\alpha}.$$

Remark 4.10 We have not tried to find the weakest possible conditions possible, particularly with regard to "large jumps." One will still have uniqueness with minimal assumptions on the intensity of the jumps above some size δ. This is apparent from the stochastic differential equations representation of X: there are only finitely many jumps of size larger than δ in any finite time interval, and so one can consider them sequentially. Our results will imply uniqueness up to the time of the first jump of size larger than δ, the law of that jump is uniquely determined by the location the process jumps from, and one then has uniqueness up to the time of the second large jump, and so on.

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