

# Regularity results for stable-like operators

Richard F. Bass\*

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## Abstract

For  $\alpha \in [1, 2)$  we consider operators of the form

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} [f(x+h) - f(x) - 1_{(|h| \leq 1)} \nabla f(x) \cdot h] \frac{A(x, h)}{|h|^{d+\alpha}}$$

and for  $\alpha \in (0, 1)$  we consider the same operator but where the  $\nabla f$  term is omitted. We prove, under appropriate conditions on  $A(x, h)$ , that the solution  $u$  to  $\mathcal{L}u = f$  will be in  $C^{\alpha+\beta}$  if  $f \in C^\beta$ .

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## 1 Introduction

Many models in mathematical physics, financial mathematics, and mathematical economics are based on diffusions corresponding to second order elliptic differential operators. In the last decade or so, though, researchers in these areas have found that frequently real world phenomena are better fitted if one allows jumps. To give a very simple example, an outbreak of war or a new discovery may cause the price of a stock to make a sudden jump. Since the operators corresponding to jump processes are non-local, one would like to consider operators that are the sum of an elliptic operator and a non-local term.

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Such operators are not yet well understood. In order to study them and the influence of the non-local part, it is quite natural to first look at the extreme case, that is, where the operator has no differential part, and to begin by understanding the potential theory, existence and uniqueness questions, and stochastic differential equations for non-local operators and the associated pure jump processes.

The first such purely non-local operator one would want to study is the fractional Laplacian  $-(-\Delta)^{\alpha/2}$ , where  $\Delta$  is the Laplacian and  $\alpha \in (0, 2)$ . Such operators have been much studied; the stochastic processes associated to these operators are known as symmetric stable processes. See [13], [10], and [12] for a sampling of research on these processes and operators.

The next simplest class of operators  $\mathcal{L}$  is a class introduced in [7], known as stable-like operators. These are operators  $\mathcal{L}$  defined by

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d \setminus \{0\}} [f(x+h) - f(x) - 1_{(|h| \leq 1)} \nabla f(x) \cdot h] \frac{A(x, h)}{|h|^{d+\alpha}} dh \quad (1.1)$$

for  $f \in C^2(\mathbb{R}^d)$  when  $\alpha \in [1, 2)$  and

$$\int_{\mathbb{R}^d \setminus \{0\}} [f(x+h) - f(x)] \frac{A(x, h)}{|h|^{d+\alpha}} dh \quad (1.2)$$

when  $\alpha \in (0, 1)$ . We use  $x \cdot y$  for the inner product in  $\mathbb{R}^d$ . These stable-like operators bear the same relationship to the fractional Laplacian as elliptic operators in non-divergence form do to the usual Laplacian. The name stable-like (which was introduced in [2] and also used in [15]) refers to the fact that the jump intensity measure  $A(x, h)/|h|^{d+\alpha} dh$  is comparable to that of the jump intensity measure of a symmetric stable process. See [7], [9], [24], and [25] for some additional results on these operators. See [1], [4], [5], [6], [8], [15], [16], [18], [19], [22], [23], and [28] for results on operators that are very closely related to (1.1) and (1.2) and which are also sometimes known as stable-like operators.

Two of the first questions one might ask about stable-like operators given by (1.1) and (1.2) are the Hölder continuity of harmonic functions and whether a Harnack inequality holds for non-negative functions that are harmonic with respect to  $\mathcal{L}$  when the function  $A(x, h)$  only satisfies some boundedness and measurability conditions. These questions were answered in [7]; see also [22] and [25]. A natural question one might then ask is whether one

can assert additional smoothness for the solution  $u$  to the equation  $\mathcal{L}u = f$  if  $A(x, h)$  and  $f$  also satisfy some continuity conditions. The answer to this last question is the subject of this paper.

Let  $\alpha \in (0, 2)$ . We impose the following conditions on  $A(x, h)$ .

**Assumption 1.1** *Suppose*

1. *There exist positive finite constants  $c_1, c_2$  such that*

$$c_1 \leq A(x, h) \leq c_2, \quad x, h \in \mathbb{R}^d.$$

2. *There exist  $\beta \in (0, 1)$  and a positive constant  $c_3$  such that*

$$\sup_x \sup_h |A(x + k, h) - A(x, h)| \leq c_3 |k|^\beta, \quad k \in \mathbb{R}^d.$$

3. *Neither  $\beta$  nor  $\alpha + \beta$  is an integer.*

The assumption that  $A(x, h)$  is uniformly bounded above and below is the analog of strict ellipticity for an elliptic operator in non-divergence form. The uniform Hölder continuity of  $A(x, h)$  in  $x$  is the analog of the usual assumptions of Hölder continuity in the Schauder theory; see [20, Chapter 6]. Note that no continuity in  $h$  is required here. Finally, the requirement that neither  $\beta$  nor  $\alpha + \beta$  be an integer is quite reasonable; in the theory of elliptic operators, most estimates break down when the coefficients are not in a Hölder space of non-integer order.

Our main result is the following. We let  $C^\beta$  and  $C^{\alpha+\beta}$  be the usual Hölder spaces. (We recall the definition in (2.3).)

**Theorem 1.2** *Let  $\mathcal{L}$  be given by (1.1) or (1.2) and suppose Assumptions 1.1 hold. If  $u \in C^{\alpha+\beta}(\mathbb{R}^d)$  satisfies  $\mathcal{L}u = f$ , then the following a priori estimate holds: there exists  $c_1$  not depending on  $f$  such that*

$$\|u\|_{C^{\alpha+\beta}} \leq c_1 \|u\|_{L^\infty} + c_1 \|f\|_{C^\beta}. \tag{1.3}$$

This is the exact analog of the corresponding estimate for elliptic operators; see [20, Chapter 6].

Lim [24] has obtained some partial results along the lines of Theorem 1.2. Our result here extends his results by weakening the hypotheses and strengthening the conclusions. We show in Section 7 that our result is sharp in several respects.

Two additional motivations for Theorem 1.2 are the following. In [7] harmonic functions for  $\mathcal{L}$  were discussed. There a probabilistic definition of harmonic functions was given because in general a harmonic function, although Hölder continuous, will not be smooth enough to be in the domain of  $\mathcal{L}$ . This is not surprising, because for elliptic operators this is also the case. Theorem 1.2 gives a sufficient condition for the harmonic function to be in the domain of  $\mathcal{L}$ . Secondly, when one considers the process associated with  $\mathcal{L}$ , an essential tool is, as might be expected, Ito's formula. However the hypotheses of Ito's formula require the function to be  $C^2$ . Therefore it would be useful to have conditions under which a class of functions associated with the process are at least  $C^2$ .

Our proof follows roughly along the lines of the Schauder theory for elliptic equation as presented in [20, Chapter 6]. There are some major differences, however. The estimates for the case when  $A(x, h)$  is constant in  $x$  are much more difficult than the corresponding estimates for the Laplacian. In addition, because we are dealing with non-local operators, our localization procedure is necessarily quite different.

In Section 2 we define the Hölder spaces and prove a few estimates that we will need. Section 3 investigates the derivatives of the semigroup corresponding to the operator  $\mathcal{L}$  in the case when  $A(x, h)$  does not depend on  $x$ , while Section 4 is concerned with the smoothing properties of the corresponding potential operator. In Section 5 we obtain estimates on the integrands in (1.1) and (1.2), and we prove Theorem 1.2 in Section 6.

We prove a number of results related to Theorem 1.2 in Section 7. For example we examine what happens when we add to  $\mathcal{L}$  a zero order term or a first order differential term and what happens when  $A(x, h)$  has further smoothness in  $x$ . We also discuss there a number of directions for further research, including the Dirichlet problem for bounded domains, boundary estimates for bounded domains, the parabolic case, the symmetric jump process case, and the case of variable order operators.

The letter  $c$  with subscripts denotes a finite positive constant whose value may vary from place to place.

## 2 Hölder spaces

Let  $\beta \in (0, 1)$ . We define the seminorm

$$[f]_{C^\beta} = \sup_{x \in \mathbb{R}^d} \sup_{|h| > 0} \frac{|f(x+h) - f(x)|}{|h|^\beta} \quad (2.1)$$

and the norm

$$\|f\|_{C^\beta} = \|f\|_{L^\infty} + [f]_{C^\beta}, \quad (2.2)$$

and say  $f$  is Hölder continuous of order  $\beta$  if  $\|f\|_{C^\beta} < \infty$ .

We write  $D_i f$  for  $\partial f / \partial x_i$ ,  $D_{ij} f$  for  $\partial^2 f / \partial x_i \partial x_j$ , and so on. Suppose  $\beta > 1$  is not an integer and let  $m$  be the largest integer strictly less than  $\beta$ . We define

$$\|f\|_{C^\beta} = \|f\|_{L^\infty} + \sum_{j_1, \dots, j_m=1}^d [D_{j_1 \dots j_m} f]_\beta \quad (2.3)$$

and say  $f \in C^\beta$  if  $\|f\|_{C^\beta} < \infty$ . It is well known (see the proof of Proposition 2.2 below, for example) that this norm is equivalent to the norm

$$\begin{aligned} \|f\|_{L^\infty} + \sum_{j_1=1}^d \|D_{j_1} f\|_{L^\infty} + \sum_{j_1, j_2=1}^d \|D_{j_1 j_2} f\|_{L^\infty} + \cdots + \sum_{j_1, \dots, j_m=1}^d \|D_{j_1 \dots j_m} f\|_{L^\infty} \\ + \sum_{j_1, \dots, j_m=1}^d [D_{j_1 \dots j_m} f]_\beta. \end{aligned} \quad (2.4)$$

(When we say two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent, we mean that there exist constants  $c_1, c_2$  such that

$$c_1 \|f\|_1 \leq \|f\|_2 \leq c_2 \|f\|_1$$

for all  $f$ .)

We also use the fact that the  $C^\beta$  norm is equivalent to a second difference norm: by [26, Proposition 8 of Chapter V], we have

**Proposition 2.1** *For  $\beta \in (0, 1) \cup (1, 2)$ ,  $f \in C^\beta$  if and only if  $f \in L^\infty$  and there exists  $c_1$  such that*

$$|f(x+h) + f(x-h) - 2f(x)| \leq c_1 |h|^\beta, \quad h, x \in \mathbb{R}^d.$$

The norm

$$\|f\|_{L^\infty} + \sup_x \sup_{|h|>0} \frac{|f(x+h) + f(x-h) - 2f(x)|}{|h|^\beta} \quad (2.5)$$

is equivalent to the  $C^\beta$  norm.

We will sometimes use the notation

$$\|Df\|_{L^\infty} = \sum_{i=1}^d \|D_i f\|_{L^\infty}, \quad \|D^2 f\|_{L^\infty} = \sum_{i,j=1}^d \|D_{ij} f\|_{L^\infty}.$$

In order to be able to include the case of integer  $\beta$  in the next two results, we introduce the following notation. If  $a$  is not an integer, set  $N(f, a) = \|f\|_{C^a}$ ; if  $a = 1$ , set  $N(f, a) = \|f\|_{L^\infty} + \|Df\|_{L^\infty}$ ; and if  $a = 2$ , set  $N(f, a) = \|f\|_{L^\infty} + \|Df\|_{L^\infty} + \|D^2 f\|_{L^\infty}$ . The following proposition is similar to known results.

**Proposition 2.2** *If  $0 < a < b < 3$  and  $\varepsilon > 0$ , there exists  $c_1$  depending only on  $a, b$ , and  $\varepsilon$  such that*

$$N(f, a) \leq c_1 \|f\|_{L^\infty} + \varepsilon N(f, b). \quad (2.6)$$

**Proof.** We first do the case when  $0 < a < b \leq 1$ . Let  $h_0 = \varepsilon^{1/(b-a)}$ . If  $|h| < h_0$ , then

$$|f(x+h) - f(x)| \leq N(f, b) |h|^b < N(f, b) |h|^a \varepsilon.$$

If  $|h| \geq h_0$ , then

$$|f(x+h) - f(x)| \leq \frac{2}{h_0^a} \|f\|_{L^\infty} |h|^a.$$

Combining, we have

$$\sup_{|h|>0} \frac{|f(x+h) - f(x)|}{|h|^a} \leq \varepsilon N(f, b) + c_2 \|f\|_{L^\infty}.$$

Taking the supremum over  $x$ , (2.6) follows immediately.

Second, we do the case  $a = 1$  and  $b \in (1, 2]$ . Fix  $1 \leq i \leq d$  and let  $x_0$  be a point in  $\mathbb{R}^d$ . The case when  $N(f, b) = 0$  is trivial, so we suppose not. Let  $R = (\|f\|_{L^\infty}/N(f, b))^{1/b}$ . By the mean value theorem, there exists  $x'$  on the line segment between  $x_0$  and  $x_0 + Re_i$  such that

$$|D_i f(x')| = \frac{|f(x_0 + Re_i) - f(x_0)|}{R} \leq 2 \frac{\|f\|_{L^\infty}}{R}.$$

Then

$$|D_i f(x_0)| \leq |D_i f(x')| + |D_i f(x') - D_i f(x_0)| \leq \frac{2\|f\|_{L^\infty}}{R} + N(f, b)R^{b-1}.$$

With our choice of  $R$ ,

$$|D_i f(x_0)| \leq c_3 \|f\|_{L^\infty}^{1-1/b} N(f, b)^{1/b}. \quad (2.7)$$

Taking the supremum over  $x_0 \in \mathbb{R}^d$  and then applying the inequality

$$x^\theta y^{1-\theta} \leq x + y, \quad x, y > 0, \quad \theta \in (0, 1), \quad (2.8)$$

we obtain

$$\|D_i f\|_{L^\infty} \leq \frac{c_4}{\varepsilon} \|f\|_{L^\infty} + \varepsilon N(f, b).$$

Third, suppose  $a = 2$  and  $b \in (2, 3)$ . Applying (2.7) with  $f$  replaced by  $D_j f$  and  $b$  replaced by  $b - 1$  and setting  $\gamma = 1/(b - 1)$ , we have

$$\|D_{ij} f\|_{L^\infty} \leq c_3 \|D_j f\|_{L^\infty}^{1-\gamma} \|D_j f\|_{C^{b-1}}^\gamma.$$

Using the well known inequality  $\|g'\|_{L^\infty} \leq c_5 \|g\|_{L^\infty}^{1/2} \|g''\|_{L^\infty}^{1/2}$  (this is a special case of (2.7)) and summing over  $i$  and  $j$ , we have

$$\|D^2 f\|_{L^\infty} \leq c_6 \|f\|_{L^\infty}^{(1-\gamma)/2} \|D^2 f\|_{L^\infty}^{(1-\gamma)/2} \|f\|_{C^b}^\gamma,$$

and therefore

$$\|D^2 f\|_{L^\infty} \leq c_7 \|f\|_{L^\infty}^{(1-\gamma)/(1+\gamma)} \|f\|_{C^b}^{2\gamma/(1+\gamma)}.$$

Applying (2.8) with  $\theta = (1 - \gamma)/(1 + \gamma)$ , we obtain (2.6).

For the case  $a \in (0, 1]$  and  $b \in (1, 2]$ , using the first and second cases above we have

$$N(f, a) \leq c_8 \|f\|_{L^\infty} + c_8 \|Df\|_{L^\infty} \leq c_8 \|f\|_{L^\infty} + c_9 \|f\|_{L^\infty} + \varepsilon N(f, b),$$

and the remaining cases are treated similarly.  $\square$

**Lemma 2.3** *If  $a \in (0, 3)$ , there exists  $c_1$  such that*

$$N(fg, a) \leq c_1 N(f, a) N(g, a).$$

**Proof.** Clearly  $\|fg\|_{L^\infty} \leq \|f\|_{L^\infty} \|g\|_{L^\infty}$ . If  $a \in (0, 1)$ , we write

$$f(x+h)g(x+h) - f(x)g(x) = f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)],$$

and it follows that

$$[fg]_{C^a} \leq \|f\|_{L^\infty} \|g\|_{C^a} + \|g\|_{L^\infty} \|f\|_{C^a}.$$

If  $a \in (1, 2)$ , we use  $D_i(fg) = f(D_i g) + (D_i f)g$ . As in the above paragraph, we bound

$$[(D_i f)g]_{C^{a-1}} \leq \|D_i f\|_{L^\infty} \|g\|_{C^{a-1}} + \|g\|_{L^\infty} \|D_i f\|_{C^{a-1}} \leq c_2 \|f\|_{C^a} \|g\|_{C^a},$$

and we bound  $[f(D_i g)]_{C^a}$  similarly. Doing this for each  $i$  takes care of the case  $a \in (1, 2)$ .

Similarly, if  $a \in (2, 3)$ , we use

$$D_{ij}(fg) = f(D_{ij}g) + g(D_{ij}f) + (D_i f)(D_j g) + (D_j f)(D_i g) \quad (2.9)$$

As in the first paragraph,

$$[(D_i f)(D_j g)]_{C^{a-2}} \leq c_3 \|D_i f\|_{C^{a-2}} \|D_j g\|_{C^{a-2}} \leq c_4 \|f\|_{C^a} \|g\|_{C^a},$$

The other terms in (2.9) are similar.

The remaining cases, when  $a = 1$  and  $a = 2$ , are easy and are left to the reader.  $\square$

We will need the following lemma.

**Lemma 2.4** *Let  $\beta \in (0, 1)$ . Let  $\varphi$  be a nonnegative  $C^\infty$  symmetric function with compact support such that  $\int \varphi(x) dx = 1$ , and let  $\varphi_\varepsilon(x) = \varepsilon^{-d} \varphi(x/\varepsilon)$ . Define  $f_\varepsilon = f * \varphi_\varepsilon$ . Then there exists  $c_1$  such that for each  $i$  and  $j$*

$$\|f - f_\varepsilon\|_{L^\infty} \leq c_1 \|f\|_{C^\beta} \varepsilon^\beta, \quad (2.10)$$

$$\|D_i f_\varepsilon\|_{L^\infty} \leq c_1 \|f\|_{C^\beta} \varepsilon^{\beta-1}, \quad \text{and} \quad (2.11)$$

$$\|D_{ij} f_\varepsilon\|_{L^\infty} \leq c_1 \|f\|_{C^\beta} \varepsilon^{\beta-2}. \quad (2.12)$$



**Proof.** The first inequality follows from

$$\begin{aligned} |f(x) - f_\varepsilon(x)| &= \left| \int [f(x) - f(x-y)]\varphi_\varepsilon(y) dy \right| \\ &\leq \|f\|_{C^\beta} \int |y|^\beta \varphi_\varepsilon(y) dy \\ &= c_2 \|f\|_{C^\beta} \varepsilon^\beta. \end{aligned}$$

Since  $\int D_i \varphi_\varepsilon(y) dy = 0$ ,

$$\begin{aligned} |D_i f_\varepsilon(x)| &= \left| \int [f(x-y) - f(x)] D_i \varphi_\varepsilon(y) dy \right| \\ &\leq \|f\|_{C^\beta} \int |y|^\beta |D_i \varphi_\varepsilon(y)| dy \\ &= c_3 \|f\|_{C^\beta} \varepsilon^{\beta-1}. \end{aligned}$$

Similarly, since  $\int D_{ij} \varphi_\varepsilon(y) dy = 0$ , then

$$\begin{aligned} |D_{ij} f_\varepsilon(x)| &= \left| \int [f(x-y) - f(x)] D_{ij} \varphi_\varepsilon(y) dy \right| \\ &\leq \|f\|_{C^\beta} \int |y|^\beta |D_{ij} \varphi_\varepsilon(y)| dy \\ &= c_4 \|f\|_{C^\beta} \varepsilon^{\beta-2}. \end{aligned}$$

□

### 3 Derivatives of semigroups

Let  $Q_t$  be the semigroup of a symmetric stable process of order  $\alpha$  and let  $q(t, x)$  be the density, that is, the function such that  $Q_t f(x) = \int f(y) q(t, x-y) dy$ . It is well known that  $q$  can be taken to be  $C^\infty$  in  $x$ .

**Proposition 3.1** *For each  $k > 0$  and each  $j_1, \dots, j_k = 1, \dots, d$ , we have*

$$\int |D_{j_1 \dots j_k} q(1, x)| dx < \infty.$$

This can be proved by generalizing the ideas of [23, Proposition 2.6], which considers the case of first derivatives. See also [28]. It can also be proved using Fourier transforms and complex analytic techniques; see [27], for example. We give a simple proof based on subordination.

**Proof.** Let  $W_t$  be a  $d$ -dimensional Brownian motion and let  $T_t$  be a one-dimensional one-sided stable process of index  $\alpha/2$  independent of  $W$ . Then it is well known, by the principle of subordination [17, Section X.7], that  $X_t = W_{T_t}$  is a symmetric stable process of index  $\alpha$ . Hence

$$q(1, x) = \int_0^\infty r(t, x) \mathbb{P}(T_1 \in dt), \quad (3.1)$$

where  $r(t, x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$  is the density of  $W_t$ .

The number of jumps of  $T_t$  of size larger than  $\lambda$  is a Poisson process with parameter  $c_1 \lambda^{-\alpha/2}$ . So the probability that  $T_t$  has no jumps of size  $\lambda$  or larger by time 1 is bounded by  $\exp(-c_1 \lambda^{-\alpha/2})$ . Because  $T_t$  is non-decreasing, this implies

$$\mathbb{P}(T_1 \leq \lambda) \leq \exp(-c_1 \lambda^{-\alpha/2}).$$

Hence for any  $N > 0$ ,

$$\begin{aligned} \int_0^\infty (1 + t^{-N}) \mathbb{P}(T_1 \in dt) &\leq 2 + \int_0^1 t^{-N} \mathbb{P}(T_1 \in dt) \\ &\leq 2 + \sum_{n=0}^\infty 2^{N(n+1)} \mathbb{P}(T_1 \in [2^{-n-1}, 2^{-n}]) \\ &\leq 2 + \sum_{n=0}^\infty 2^{N(n+1)} \mathbb{P}(T_1 \leq 2^{-n}) \\ &\leq 2 + \sum_{n=0}^\infty 2^{N(n+1)} e^{-c_1 (2^{-n})^{\alpha/2}} < \infty. \end{aligned} \quad (3.2)$$

It is easy to see that for each  $a > 0$  there exist  $b$  and  $c_2$  depending on  $a$  such that

$$\sup_x (1 + |x|^a) r(t, x) \leq c_2 (1 + t^{-b}), \quad t > 0.$$

This and (3.2) allow us to use dominated convergence to differentiate under the integral sign in (3.1), and we obtain

$$D_{j_1 \dots j_k} q(1, x) = \int_0^\infty D_{j_1 \dots j_k} r(t, x) \mathbb{P}(T_1 \in dt).$$

Then, using (3.2) again and Fubini,

$$\begin{aligned} \int |D_{j_1 \dots j_k} q(1, x)| dx &\leq \int_0^\infty \int |D_{j_1 \dots j_k} r(t, x)| dx \mathbb{P}(T_1 \in dt) \\ &\leq c_3 \int_0^\infty t^{-k/2} \mathbb{P}(T_1 \in dt) < \infty. \end{aligned}$$

□

If  $f \in L^\infty$ , it follows easily that  $Q_1 f$  is  $C^\infty$  for  $t > 0$  and for each  $j_1, \dots, j_k$

$$|D_{j_1 \dots j_k} Q_1 f(x)| \leq c_1 \|f\|_{L^\infty}.$$

By scaling we have

$$|D_{j_1 \dots j_k} Q_t f(x)| \leq ct^{-k/\alpha} \|f\|_{L^\infty}. \quad (3.3)$$

Now we consider Lévy processes whose Lévy measure is comparable to that of a symmetric stable process of index  $\alpha$ . Suppose  $A_0 : \mathbb{R}^d \setminus \{0\} \rightarrow [\kappa_1, \kappa_2]$ , where  $\kappa_1, \kappa_2$  are finite positive constants. Define

$$\mathcal{L}_0 f(x) = \int_{\mathbb{R}^d \setminus \{0\}} [f(x+h) - f(x) - 1_{(|h| \leq 1)} \nabla f(x) \cdot h] \frac{A_0(h)}{|h|^{d+\alpha}} dh \quad (3.4)$$

for  $C^2$  functions  $f$  when  $\alpha \geq 1$ , and without the  $\nabla f(x)$  term when  $\alpha < 1$ . Let  $P_t$  be the semigroup corresponding to the generator  $\mathcal{L}_0$ .

**Theorem 3.2** *If  $f \in L^\infty$ , then  $P_t f$  is  $C^\infty$  for  $t > 0$  and for each  $j_1, \dots, j_k = 1, \dots, d$ , there exists  $c_1$  (depending on  $k$ ) such that*

$$|D_{j_1 \dots j_k} P_t f(x)| \leq c_1 t^{-k/\alpha} \|f\|_{L^\infty}.$$

**Proof.** Let  $\mathcal{L}_1$  be defined by (3.4) but with  $A_0(h)$  replaced by  $\kappa_1$  and let  $\mathcal{L}_2 = \mathcal{L}_0 - \mathcal{L}_1$ . Let  $Q_t^1$  and  $Q_t^2$  be the semigroups for the Lévy processes with generators  $\mathcal{L}_1, \mathcal{L}_2$ , resp., and let  $X^1, X^2$  be the corresponding Lévy processes. If we take  $X^1$  independent of  $X^2$ , then  $X^1 + X^2$  has the law of the Lévy process corresponding to the generator  $\mathcal{L}$ . Therefore  $P_t = Q_t^2 Q_t^1$ . We know that  $Q_t^1 f$  satisfies the desired estimate by (3.3) and the fact that the process associated with  $\mathcal{L}_1$  is a deterministic time change of the process

considered in Proposition 3.1. By translation invariance,  $Q_t^2$  commutes with differentiation. Therefore  $P_t f = Q_t^2 Q_t^1 f$  also satisfies the desired estimate, since

$$\|D_{j_1 \dots j_k} P_t f\|_{L^\infty} = \|Q_t^2 D_{j_1 \dots j_k} Q_t^1 f\|_{L^\infty} \leq \|D_{j_1 \dots j_k} Q_t^1 f\|_{L^\infty} \leq c_1 t^{-k/\alpha} \|f\|_{L^\infty}.$$

□

## 4 Potentials and Hölder continuity

Let  $P_t$  continue to be the semigroup corresponding to the Lévy process in  $\mathbb{R}^d$  with infinitesimal generator  $\mathcal{L}_0$  given by (3.4) and define the potential

$$Rf(x) = \int_0^\infty P_t f(x) dt$$

when the function  $t \rightarrow P_t f(x)$  is integrable. We want to prove that  $R$  takes functions in  $C^\beta$  into functions in  $C^{\alpha+\beta}$ , provided neither  $\beta$  nor  $\alpha + \beta$  is an integer and that  $Rf$  is bounded.

**Proposition 4.1** *Suppose  $\beta \in (0, 1)$ ,  $f \in C^\beta$ ,  $Rf \in L^\infty$ , and  $\alpha + \beta < 1$ . Then  $Rf \in C^{\alpha+\beta}$  and there exists  $c_1$  not depending on  $f$  such that  $\|Rf\|_{C^{\alpha+\beta}} \leq c_1 \|f\|_{C^\beta} + c_1 \|Rf\|_{L^\infty}$ .*

**Proof.** We first prove that

$$|P_s f(x) - P_s f(y)| \leq \frac{c_2}{s^{(1-\beta)/\alpha}} |y - x| \|f\|_{C^\beta}. \quad (4.1)$$

Define  $f_\varepsilon$  as in Lemma 2.4.

We have, using Theorem 3.2 and (2.10),

$$\begin{aligned} |P_s(f - f_\varepsilon)(y) - P_s(f - f_\varepsilon)(x)| &\leq \|\nabla P_s(f - f_\varepsilon)\|_{L^\infty} |y - x| \\ &\leq \frac{c_3}{s^{1/\alpha}} \|f - f_\varepsilon\|_{L^\infty} |y - x| \\ &\leq \frac{c_3}{s^{1/\alpha}} \varepsilon^\beta \|f\|_{C^\beta} |y - x|. \end{aligned} \quad (4.2)$$

Also, using (2.11),

$$\begin{aligned}
|P_s f_\varepsilon(y) - P_s f_\varepsilon(x)| &\leq c_4 \|\nabla P_s f_\varepsilon\|_{L^\infty} |y - x| \\
&\leq c_4 \|\nabla f_\varepsilon\|_{L^\infty} |y - x| \\
&\leq c_5 \varepsilon^{\beta-1} \|f\|_{C^\beta} |y - x|.
\end{aligned} \tag{4.3}$$

Setting  $\varepsilon = s^{1/\alpha}$  and combining (4.2) and (4.3) yields (4.1).

If  $x, y \in \mathbb{R}^d$  and we define  $g(z) = f(y - x + z)$ , then by the translation invariance of  $P_s$  (that is,  $P_s$  commutes with translation),  $P_s g(x) = P_s f(y)$ , and then

$$|P_s f(y) - P_s f(x)| = |P_s(g - f)(x)| \leq \|g - f\|_{L^\infty} \leq \|f\|_{C^\beta} |y - x|^\beta.$$

So putting  $t_0 = |y - x|^\alpha$ , we have

$$\int_0^{t_0} |P_s f(y) - P_s f(x)| ds \leq t_0 \|f\|_{C^\beta} |y - x|^\beta = \|f\|_{C^\beta} |y - x|^{\alpha+\beta}. \tag{4.4}$$

Using (4.1) and noting  $(1 - \beta)/\alpha > 1$ ,

$$\begin{aligned}
\int_{t_0}^\infty |P_s f(y) - P_s f(x)| ds &\leq \int_{t_0}^\infty \frac{c_6}{s^{(1-\beta)/\alpha}} \|f\|_{C^\beta} |y - x| ds \\
&= c_7 t_0^{1-(1-\beta)/\alpha} \|f\|_{C^\beta} |y - x| \\
&= c_7 \|f\|_{C^\beta} |y - x|^{\alpha+\beta}.
\end{aligned}$$

Combining this with (4.4) and the fact that

$$|Rf(y) - Rf(x)| \leq \int_0^{t_0} |P_s f(y) - P_s f(x)| ds + \int_{t_0}^\infty |P_s f(y) - P_s f(x)| ds, \tag{4.5}$$

our result follows.  $\square$

Next we consider the case when  $0 < \beta < 1$  and  $1 < \alpha + \beta < 2$ .

**Proposition 4.2** *Suppose  $\beta \in (0, 1)$ ,  $f \in C^\beta$ ,  $\|Rf\|_{L^\infty} < \infty$ , and  $\alpha + \beta \in (1, 2)$ . Then  $Rf \in C^{\alpha+\beta}$  and there exists  $c_1$  not depending on  $f$  such that  $\|Rf\|_{C^{\alpha+\beta}} \leq c_1 \|f\|_{C^\beta} + c_1 \|Rf\|_{L^\infty}$ .*

**Proof.** Define

$$V_{hs}(f)(x) = P_s f(x+h) + P_s f(x-h) - 2P_s f(x).$$

First we show

$$|V_{hs}(f)(x)| \leq c_2 |h|^2 \|f\|_{C^\beta} s^{-(2-\beta)/\alpha}. \quad (4.6)$$

By Theorem 3.2, (2.10), and Taylor's theorem,

$$\begin{aligned} |V_{hs}(f - f_\varepsilon)(x)| &\leq c_3 |h|^2 \|D^2 P_s(f - f_\varepsilon)\|_{L^\infty} \\ &\leq \frac{c_4}{s^{2/\alpha}} |h|^2 \|f - f_\varepsilon\|_{L^\infty} \\ &\leq \frac{c_5}{s^{2/\alpha}} |h|^2 \varepsilon^\beta \|f\|_{C^\beta}. \end{aligned} \quad (4.7)$$

If we set  $g_{1\varepsilon}(z) = f_\varepsilon(z+h)$  and  $g_{2\varepsilon}(z) = f_\varepsilon(z-h)$ , by the translation invariance of  $P_s$ ,  $V_{hs}(f_\varepsilon)(x) = P_s g_{1\varepsilon}(x) + P_s g_{2\varepsilon}(x) - 2P_s f_\varepsilon(x)$ , and therefore by (2.12)

$$\begin{aligned} |V_{hs}(f_\varepsilon)(x)| &= |P_s(g_{1\varepsilon} + g_{2\varepsilon} - 2f_\varepsilon)(x)| \leq \|g_{1\varepsilon} + g_{2\varepsilon} - 2f_\varepsilon\|_{L^\infty} \\ &\leq c_6 |h|^2 \|D^2 f_\varepsilon\|_{L^\infty} \leq c_7 |h|^2 \varepsilon^{\beta-2} \|f\|_{C^\beta}. \end{aligned} \quad (4.8)$$

Letting  $\varepsilon = s^{1/\alpha}$  and combining with (4.7), we obtain (4.6).

Using (4.6) and noting  $(2-\beta)/\alpha > 1$ ,

$$\int_{|h|^\alpha}^\infty |V_{hs}(f)(x)| \leq c_8 |h|^2 \|f\|_{C^\beta} \int_{|h|^\alpha}^\infty s^{-(2-\beta)/\alpha} ds = c_9 \|f\|_{C^\beta} |h|^{\alpha+\beta}. \quad (4.9)$$

Let  $g_{10}(x) = f(x+h)$ ,  $g_{20}(x) = f(x-h)$ . By translation invariance and the Hölder continuity of  $f$ ,

$$\begin{aligned} |V_{hs}(f)(x)| &= |P_s(g_{10} + g_{20} - 2f)(x)| \leq \|g_{10} + g_{20} - 2f\|_{L^\infty} \\ &\leq 2\|f\|_{C^\beta} |h|^\beta, \end{aligned}$$

and thus

$$\int_0^{|h|^\alpha} |V_{hs}(f)(x)| ds \leq 2\|f\|_{C^\beta} |h|^{\alpha+\beta}. \quad (4.10)$$

Adding (4.9) and (4.10) we conclude

$$|Rf(x+h) + Rf(x-h) - 2Rf(x)| \leq c\|f\|_{C^\beta} |h|^{\alpha+\beta}.$$

This with Proposition 2.1 completes the proof.  $\square$

Finally we consider the case when  $\alpha + \beta \in (2, 3)$ .

**Proposition 4.3** *Suppose  $\beta \in (0, 1)$ ,  $f \in C^\beta$ ,  $\|Rf\|_{L^\infty} < \infty$ , and  $\alpha + \beta \in (2, 3)$ . Then  $Rf \in C^{\alpha+\beta}$  and there exists  $c_1$  not depending on  $f$  such that  $\|Rf\|_{C^{\alpha+\beta}} \leq c_1\|f\|_{C^\beta} + c_1\|Rf\|_{L^\infty}$ .*

**Proof.** Necessarily  $\alpha > 1$ . In view of Proposition 2.2 it suffices to show

$$\|D_i Rf\|_{C^{\alpha+\beta-1}} \leq c_2\|f\|_{C^\beta}, \quad i = 1, \dots, d. \quad (4.11)$$

Fix  $i$  and let  $Q_t = D_i P_t$ . From Theorem 3.2 we have

$$\|D_{j_1 j_2} Q_t f\|_{L^\infty} \leq c_3 t^{-3/\alpha} \|f\|_{L^\infty}, \quad j_1, j_2 = 1, \dots, d.$$

Define  $W_{hs}(f)(x) = Q_s f(x+h) + Q_s f(x-h) - 2Q_s f(x)$ . Note that  $Q_s$  is translation invariant. Analogously to (4.7) and (4.8),

$$\begin{aligned} |W_{hs}(f - f_\varepsilon)(x)| &\leq c_4 |h|^2 \|D^2 Q_s (f - f_\varepsilon)\|_{L^\infty} \\ &\leq \frac{c_5 |h|^2}{s^{3/\alpha}} \|f - f_\varepsilon\|_{L^\infty} \\ &\leq \frac{c_6 |h|^2}{s^{3/\alpha}} \varepsilon^\beta \|f\|_{C^\beta} \end{aligned}$$

and

$$\begin{aligned} |W_{hs}(f_\varepsilon)(x)| &\leq c_7 |h|^2 \|D^2 Q_s f_\varepsilon\|_{L^\infty} = c_7 |h|^2 \|Q_s D^2 f_\varepsilon\|_{L^\infty} \\ &\leq \frac{c_8 |h|^2}{s^{1/\alpha}} \|D^2 f_\varepsilon\|_{L^\infty} \leq \frac{c_9}{s^{1/\alpha}} |h|^2 \varepsilon^{\beta-2} \|f\|_{C^\beta}. \end{aligned}$$

Taking  $\varepsilon = s^{1/\alpha}$  we obtain

$$|W_{hs}(f)(x)| \leq c_{10} |h|^2 s^{(\beta-3)/\alpha} \|f\|_{C^\beta}.$$

Integrating this bound over  $[|h|^\alpha, \infty)$  yields  $c_{11} |h|^{\alpha+\beta-1} \|f\|_{C^\beta}$ .

On the other hand, if  $g_{10}$  and  $g_{20}$  are defined as in the proof of Proposition 4.2,

$$\begin{aligned} |W_{hs}(f)(x)| &\leq \|Q_s(g_{10} + g_{20} - 2f)\|_{L^\infty} \leq c_{12} s^{-1/\alpha} \|g_{10} + g_{20} - 2f\|_{L^\infty} \\ &\leq c_{13} s^{-1/\alpha} |h|^\beta \|f\|_{C^\beta}, \end{aligned}$$

and integrating this bound over  $s$  from 0 to  $|h|^\alpha$  yields  $c_{14}|h|^{\alpha+\beta-1}\|f\|_{C^\beta}$ ; we use the fact that  $1/\alpha < 1$  here. Therefore

$$|W_{hs}(D_i Rf)(x)| \leq c_{14}|h|^{\alpha+\beta-1}\|f\|_{C^\beta},$$

which with Proposition 2.1 yields (4.11).  $\square$

We reformulate and summarize the preceding propositions in the following theorem. Let  $\mathcal{L}_0$  be defined as in (3.4).

**Theorem 4.4** *Suppose  $\beta \in (0, 1)$  and  $\alpha + \beta \in (0, 1) \cup (1, 2) \cup (2, 3)$ . There exists  $c_1$  such that if  $u$  is in the domain of  $\mathcal{L}_0$  and  $\mathcal{L}_0 u = f$  with  $\|u\|_{L^\infty} < \infty$ , then*

$$\|u\|_{C^{\alpha+\beta}} \leq c_1\|f\|_{C^\alpha} + c_1\|u\|_{L^\infty}. \quad (4.12)$$

**Proof.** If  $\mathcal{L}_0 u = f$  and  $\|u\|_{L^\infty} < \infty$ , then we have  $u = -Rf$ , and so the result follows by Propositions 4.1, 4.2, and 4.3.  $\square$

## 5 First and second differences

For  $f$  bounded define

$$E_h f(x) = f(x+h) - f(x). \quad (5.1)$$

For  $f \in C^1$  define

$$F_h f(x) = f(x+h) - f(x) - \nabla f(x) \cdot h. \quad (5.2)$$

Observe that if  $g : \mathbb{R} \rightarrow \mathbb{R}$  is in  $C^\gamma$  with  $\gamma \in (1, 2)$ , then

$$\begin{aligned} |g(t) - g(0) - g'(0)t| &= \left| \int_0^t [g'(s) - g'(0)] ds \right| \\ &\leq \|g\|_{C^\gamma} \int_0^t s^{\gamma-1} ds \leq c_1 \|g\|_{C^\gamma} t^\gamma, \end{aligned} \quad (5.3)$$



while if  $\gamma \in (2, 3)$ , then

$$\begin{aligned}
|g(t) - g(0) - g'(0)t - \frac{1}{2}g''(0)t^2| &= \left| \int_0^t [g'(s) - g'(0)] ds - \frac{1}{2}g''(0)t^2 \right| \quad (5.4) \\
&= \left| \int_0^t \int_0^s [g''(r) - g''(0)] dr ds \right| \\
&\leq \|g\|_{C^\gamma} \int_0^t \int_0^s r^{\gamma-2} dr ds = c_2 \|g\|_{C^\gamma} t^\gamma.
\end{aligned}$$

Let  $Hf$  be the Hessian of  $f$ , so that

$$h \cdot Hf(x)k = \sum_{i,j=1}^d h_i D_{ij}f(x)k_j$$

if  $h = (h_1, \dots, h_d)$  and  $k = (k_1, \dots, k_d)$ .

**Theorem 5.1** *Suppose  $f \in C^\gamma$  for  $\gamma \in (0, 1) \cup (1, 2) \cup (2, 3)$ . There exists  $c_1$  not depending on  $f$  such that the following estimates hold.*

(a) For all  $\gamma$ ,

$$|E_h f(x)| \leq c_1 (|h|^{\gamma \wedge 1} \wedge 1) \|f\|_{C^\gamma} \quad (5.5)$$

and if  $\gamma > 1$ ,

$$|F_h f(x)| \leq c_1 (|h|^{\gamma \wedge 2} \wedge 1) \|f\|_{C^\gamma}. \quad (5.6)$$

(b) For all  $\gamma$ ,

$$|E_h f(x+k) - E_h f(x)| \leq c_1 (|h|^{\gamma \wedge 1} \wedge |k|^{\gamma \wedge 1}) \|f\|_{C^\gamma}. \quad (5.7)$$

(c) If  $\gamma \in (1, 2)$ , then

$$|E_h f(x+k) - E_h f(x)| \leq c_1 ((|h|^{\gamma-1}|k|) \wedge (|h||k|^{\gamma-1})) \|f\|_{C^\gamma}. \quad (5.8)$$

(d) If  $\gamma \in (1, 2)$ , then

$$|F_h f(x+k) - F_h f(x)| \leq c_1 ((|h|^\gamma) \wedge (|h||k|^{\gamma-1})) \|f\|_{C^\gamma}. \quad (5.9)$$

(e) If  $\gamma \in (2, 3)$ , then

$$|F_h f(x+k) - F_h f(x)| \leq c_1 ((|k|^{\gamma-2}|h|^2) \wedge (|h|^{\gamma-1}|k|)) \|f\|_{C^\gamma}. \quad (5.10)$$

**Proof.** (a) The estimate for  $E_h f$  follows by the definition of  $C^\gamma$ . The one for  $F_h f$  follows from (5.3) or (5.4) applied to  $g(s) = f(x + sh/|h|)$  with  $t = |h|$ .

(b) Write

$$E_h f(x+k) - E_h f(x) = [f(x+h+k) - f(x+k)] - [f(x+h) - f(x)], \quad (5.11)$$

and note that because  $f \in C^\gamma$ , this is bounded by  $2|h|^{\gamma \wedge 1} \|f\|_{C^\gamma}$ . We can also write  $E_h f(x+k) - E_h f(x)$  as

$$[f(x+h+k) - f(x+h)] - [f(x+k) - f(x)], \quad (5.12)$$

so we also get the bound  $2|k|^{\gamma \wedge 1} \|f\|_{C^\gamma}$ .

(c) Using (5.3)

$$f(x+h+k) - f(x+k) = \nabla f(x+k) \cdot h + R_1$$

and

$$f(x+h) - f(x) = \nabla f(x) \cdot h + R_2,$$

where  $R_1$  and  $R_2$  are both bounded by  $c_2 \|f\|_{C^\gamma} |h|^\gamma$ . By (5.11)

$$E_h f(x+k) - E_h f(x) = [\nabla f(x+k) - \nabla f(x)] \cdot h + R_1 - R_2,$$

and the right hand side is bounded by

$$c_3 \|f\|_{C^\gamma} (|k|^{\gamma-1} |h| + |h|^\gamma). \quad (5.13)$$

Starting with (5.12) instead of (5.11) we also get the bound

$$c_3 \|f\|_{C^\gamma} (|h|^{\gamma-1} |k| + |k|^\gamma). \quad (5.14)$$

Using (5.13) when  $|h| \leq |k|$  and (5.14) when  $|h| > |k|$  proves (5.8).

(d) By (5.4)

$$|F_h f(x)| \leq c_3 \|f\|_{C^\gamma} |h|^\gamma,$$

and the same bound holds for  $F_h f(x+k)$ , so

$$|F_h f(x+k) - F_h f(x)| \leq c_3 \|f\|_{C^\gamma} |h|^\gamma. \quad (5.15)$$

On the other hand

$$f(x+k+h) - f(x+h) = \nabla f(x+h) \cdot k + R_3$$

and

$$f(x+k) - f(x) = \nabla f(x) \cdot k + R_4,$$

where  $R_3$  and  $R_4$  are both bounded by  $c_4 \|f\|_{C^\gamma} |k|^\gamma$ . Also

$$|\nabla f(x+k) \cdot h - \nabla f(x) \cdot h| \leq c_5 \|f\|_{C^\gamma} |h| |k|^{\gamma-1}$$

and

$$|\nabla f(x+h) \cdot k - \nabla f(x) \cdot k| \leq c_5 \|f\|_{C^\gamma} |k| |h|^{\gamma-1}.$$

Combining and using the fact that  $\gamma < 2$ ,

$$|F_h f(x+k) - F_h f(x)| \leq c_6 \|f\|_{C^\gamma} (|k|^\gamma + |h| |k|^{\gamma-1} + |k| |h|^{\gamma-1}),$$

which together with (5.15) proves (5.9).

(e) Applying (5.4)

$$|F_h f(x) - \frac{1}{2} h \cdot Hf(x)h| \leq c_7 \|f\|_{C^\gamma} |h|^\gamma \quad (5.16)$$

and we obtain the same bound for  $|F_h f(x+k) - \frac{1}{2} h \cdot Hf(x+k)h|$ . Since

$$|h \cdot (Hf(x+k) - Hf(x))h| \leq c_8 \|f\|_{C^\gamma} |h|^2 |k|^{\gamma-2},$$

then

$$|F_h f(x+k) - F_h f(x)| \leq c_9 \|f\|_{C^\gamma} (|h|^2 |k|^{\gamma-2} + |h|^\gamma). \quad (5.17)$$

On the other hand, using (5.3) and (5.4),

$$f(x+k+h) - f(x+k) = \nabla f(x+h) \cdot k + \frac{1}{2} k \cdot Hf(x+h)k + R_5,$$

$$f(x+k) - f(x) = \nabla f(x) \cdot k + \frac{1}{2} k \cdot Hf(x)k + R_6,$$

$$\nabla f(x+k) \cdot h - \nabla f(x) \cdot h = k \cdot Hf(x)h + R_7,$$

and

$$\nabla f(x+h) \cdot k - \nabla f(x) \cdot k = h \cdot Hf(x)k + R_8,$$

where  $R_5$  and  $R_6$  are both bounded by  $c_{10} \|f\|_{C^\gamma} |k|^\gamma$ ,  $R_7$  is bounded by  $c_{10} \|f\|_{C^\gamma} |k|^{\gamma-1} |h|$ , and  $R_8$  is bounded  $c_{10} \|f\|_{C^\gamma} |h|^{\gamma-1} |k|$ . Therefore

$$|F_h f(x+k) - F_h f(x) - \frac{1}{2} k \cdot (Hf(x+h) - Hf(x))k| \leq |R_5| + |R_6| + |R_7| + |R_8|,$$

which implies

$$|F_h f(x+k) - F_h f(x)| \leq c_{11} \|f\|_{C^\gamma} (|k|^\gamma + |k|^{\gamma-1}|h| + |h|^{\gamma-1}|k| + |k|^2|h|^{\gamma-2}). \quad (5.18)$$

Using (5.17) if  $|h| \leq |k|$  and (5.18) if  $|h| > |k|$  proves (5.10).  $\square$

We have the following corollary.

**Corollary 5.2** *Suppose  $f \in C^{\alpha+\beta}$  for some  $\beta \in (0, 1)$  and  $\alpha + \beta \in (0, 1) \cup (1, 2) \cup (2, 3)$ . There exists  $c_1$  not depending on  $f$  such that*

(a) *If  $\alpha < 1$ , then*

$$\int |E_h f(x+k) - E_h f(x)| \frac{dh}{|h|^{d+\alpha}} \leq c_1 |k|^\beta \|f\|_{C^{\alpha+\beta}}. \quad (5.19)$$

(b) *If  $\alpha \in [1, 2)$ , then*

$$\begin{aligned} \int_{|h| \leq 1} |F_h f(x+k) - F_h f(x)| \frac{dh}{|h|^{d+\alpha}} + \int_{|h| > 1} |E_h f(x+k) - E_h f(x)| \frac{dh}{|h|^{d+\alpha}} \\ \leq c_1 |k|^\beta \|f\|_{C^{\alpha+\beta}}. \end{aligned} \quad (5.20)$$

**Proof.** If  $|k| > 1$ , the left hand side of (5.19) is less than or equal to

$$\int |E_h f(x+k)| \frac{dh}{|h|^{d+\alpha}} + \int |E_h f(x)| \frac{dh}{|h|^{d+\alpha}},$$

which is bounded using Theorem 5.1(a). We treat (5.20) similarly.

If  $|k| \leq 1$ , we use the bounds in Theorem 5.1(b)–(e), breaking the integrals into three: where  $|h| < |k|$ , where  $|k| \leq |h| \leq 1$ , and where  $|h| > 1$ . The rest is elementary calculus.  $\square$

**Remark 5.3** By Theorem 5.1(a), the integrals defining  $\mathcal{L}u$  are thus absolutely convergent if  $u \in C^{\alpha+\beta}$  for some  $\beta > 0$ . In particular, the domain of  $\mathcal{L}$  contains  $C^{\alpha+\beta}$  for each  $\beta > 0$ .

The following is immediate from Corollary 5.2.

**Corollary 5.4** *Suppose  $u \in C^{\alpha+\beta}$  for some  $\beta \in (0, 1)$  and  $\alpha + \beta \in (0, 1) \cup (1, 2) \cup (2, 3)$ . Let  $\mathcal{L}_0$  be defined by (3.4). Then  $\mathcal{L}_0 u \in C^\beta$  and there exists  $c_1$  such that*

$$\|\mathcal{L}_0 u\|_{C^\beta} \leq c_1 \|u\|_{C^{\alpha+\beta}}.$$

## 6 Proof of Theorem 1.2

Let  $B(x, r)$  denote the ball of radius  $r$  centered at  $x$ . Let  $\bar{\varphi}$  be a cut-off function that is 1 on  $B(0, 1)$ , 0 on  $B(0, 2)^c$ , takes values in  $[0, 1]$ , and is  $C^\infty$ . Let  $\varphi_{r, x_0}(x) = r^{-d} \bar{\varphi}((x - x_0)/r)$ . When  $r$  and  $x_0$  are clear, we will write just  $\varphi$  for  $\varphi_{r, x_0}$ .

**Proposition 6.1** *Suppose  $\|u\|_{C^{\alpha+\beta}} < \infty$ . Suppose for each  $\delta > 0$  there exists  $r$  and  $c_1$  (depending on  $\delta$ ) such that*

$$\|u \varphi_{r, x_0}\|_{C^{\alpha+\beta}} \leq c_1 \|f\|_{C^\beta} + c_1 \|u\|_{L^\infty} + \delta \|u\|_{C^{\alpha+\beta}} \quad (6.1)$$

*Then there exists  $c_2$  depending on  $\delta$  such that*

$$\|u\|_{C^{\alpha+\beta}} \leq c_2 \|f\|_{C^\beta} + c_2 \|u\|_{L^\infty}. \quad (6.2)$$

**Proof.** First we do the case where  $\alpha + \beta \in (0, 1) \cup (1, 2)$ . Recall from Proposition 2.2 that there exist  $c_3$  and  $c_4$  such that

$$\begin{aligned} c_3 \|g\|_{C^{\alpha+\beta}} &\leq \|g\|_{L^\infty} + \sup_x \sup_{|h|>0} \frac{g(x+h) + g(x-h) - 2g(x)}{|h|^{\alpha+\beta}} \\ &\leq c_4 \|g\|_{C^{\alpha+\beta}} \end{aligned} \quad (6.3)$$

for all  $g \in C^{\alpha+\beta}$ . Choose  $\delta = c_3/2c_4$  and then choose  $r$  and  $c_1$  using (6.1). If  $x_0 \in \mathbb{R}^d$ , let  $v = u \varphi_{r, x_0}$ , and note that  $u = v$  in the ball  $B(x_0, r)$ . If  $|h| < r$ ,

$$\begin{aligned} |u(x_0 + h) + u(x_0 - h) - 2u(x_0)| &= |v(x_0 + h) + v(x_0 - h) - 2v(x_0)| \\ &\leq c_4 \|v\|_{C^{\alpha+\beta}} |h|^{\alpha+\beta}. \end{aligned} \quad (6.4)$$

On the other hand, if  $|h| \geq r$ ,

$$|u(x_0 + h) + u(x_0 - h) - 2u(x_0)| \leq \frac{4}{r^{\alpha+\beta}} \|u\|_{L^\infty} |h|^{\alpha+\beta} = c_5 \|u\|_{L^\infty} |h|^{\alpha+\beta}. \quad (6.5)$$

Combining (6.4) and (6.5) and using (6.1),

$$\begin{aligned} |u(x_0 + h) + u(x_0 - h) - 2u(x_0)| &\leq (c_4 \|v\|_{C^{\alpha+\beta}} + c_5 \|u\|_{L^\infty}) |h|^{\alpha+\beta} \\ &\leq (c_1 c_4 \|f\|_{C^\beta} + (c_1 c_4 + c_5) \|u\|_{L^\infty} + c_4 \delta \|u\|_{C^{\alpha+\beta}}) |h|^{\alpha+\beta}. \end{aligned}$$

This and (6.3) yield

$$\|u\|_{C^{\alpha+\beta}} \leq c_6 \|f\|_{C^\beta} + c_6 \|u\|_{L^\infty} + \frac{1}{2} \|u\|_{C^{\alpha+\beta}}.$$

Subtracting  $\frac{1}{2} \|u\|_{C^{\alpha+\beta}}$  from both sides and multiplying by 2 gives (6.1).

Now we consider the case when  $\alpha + \beta \in (2, 3)$ . Since  $u \in C^{\alpha+\beta}$  if  $u \in L^\infty$  and each  $D_i u \in C^{\alpha+\beta-1}$ , by (2.3), (2.4), and Propositions 2.1 and 2.2 there exists  $c_7$  such that

$$\|u\|_{C^{\alpha+\beta}} \leq c_7 \left( \|u\|_{L^\infty} + \sup_i \sup_x \sup_{|h|>0} \frac{|D_i u(x+h) + D_i u(x-h) - 2D_i u(x)|}{|h|^{\alpha+\beta-1}} \right).$$

Let  $\delta = 1/2c_7(1 + c_4)$ , choose  $r$  using (6.1), and let  $v = u\varphi_{r,x_0}$ . If  $|h| < r$ , then for any  $i$ ,

$$\begin{aligned} &|D_i u(x_0 + h) + D_i u(x_0 - h) - 2D_i u(x_0)| \\ &= |D_i v(x_0 + h) + D_i v(x_0 - h) - 2D_i v(x_0)| \\ &\leq c_4 \|v\|_{C^{\alpha+\beta}} |h|^{\alpha+\beta-1} \\ &\leq (c_1 c_4 \|f\|_{C^\beta} + c_1 c_4 \|u\|_{L^\infty} + \delta c_4 \|u\|_{C^{\alpha+\beta}}) |h|^{\alpha+\beta-1}. \end{aligned}$$

On the other hand, if  $|h| \geq r$ , then

$$|D_i u(x_0 + h) + D_i u(x_0 - h) - 2D_i u(x_0)| \leq \frac{4}{r^{\alpha+\beta-1}} \|D_i u\|_{L^\infty} |h|^{\alpha+\beta-1}. \quad (6.6)$$

Choose  $\varepsilon = r^{\alpha+\beta-1} \delta / 4$  and then use Proposition 2.2 to see there exists  $c_8$  such that

$$\|D_i u\|_{L^\infty} \leq c_8 \|u\|_{L^\infty} + \varepsilon \|u\|_{C^{\alpha+\beta}}.$$

Substituting this in (6.6),

$$|D_i u(x_0 + h) + D_i u(x_0 - h) - 2D_i u(x_0)| \leq (c_9 \|u\|_{L^\infty} + \delta \|u\|_{C^{\alpha+\beta}}) |h|^{\alpha+\beta-1}.$$

Therefore

$$\begin{aligned} & |D_i u(x_0 + h) + D_i u(x_0 - h) - 2D_i u(x_0)| \\ & \leq (c_{10} \|f\|_{C^\beta} + c_{10} \|u\|_{L^\infty} + (1 + c_4) \delta \|u\|_{C^{\alpha+\beta}}) |h|^{\alpha+\beta-1}, \end{aligned}$$

and hence

$$\|u\|_{C^{\alpha+\beta}} \leq c_{11} \|f\|_{C^\beta} + c_{11} \|u\|_{L^\infty} + \frac{1}{2} \|u\|_{C^{\alpha+\beta}}.$$

Subtracting  $\frac{1}{2} \|u\|_{C^{\alpha+\beta}}$  from both sides, and multiplying by 2 yields our result.  $\square$

**Proof of Theorem 1.2.** *Step 1.* In this step we define a certain function  $F$ . Let us suppose for now that  $\alpha < 1$ , leaving the case  $\alpha \geq 1$  until later. Fix  $\delta > 0$  and let  $\varepsilon > 0$  be chosen later. Let  $x_0 \in \mathbb{R}^d$  be fixed and choose  $r$  such that

$$\sup_{|h|>0} |A(x, h) - A(x_0, h)| < \varepsilon$$

if  $|x - x_0| \leq 4r$ . Let  $b(x, h) = A(x, h) - A(x_0, h)$ ,

$$\mathcal{L}_0 u(x) = \int [u(x+h) - u(x)] \frac{A(x_0, h)}{|h|^{d+\alpha}} dh,$$

and  $\mathcal{B} = \mathcal{L} - \mathcal{L}_0$ . Let  $\varphi = \varphi_{r, x_0}$  be as in the paragraph preceding Proposition 6.1 and let  $v = u\varphi$ .

We have

$$\begin{aligned} v(x+h) - v(x) &= u(x)[\varphi(x+h) - \varphi(x)] + \varphi(x)[u(x+h) - u(x)] \\ &\quad + [u(x+h) - u(x)][\varphi(x+h) - \varphi(x)], \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{L}v(x) &= u(x)\mathcal{L}\varphi(x) + \varphi(x)\mathcal{L}u(x) + H(x) \\ &= u(x)\mathcal{L}\varphi(x) + \varphi(x)f(x) + H(x), \end{aligned}$$

where

$$H(x) = \int [u(x+h) - u(x)][\varphi(x+h) - \varphi(x)] \frac{A(x, h)}{|h|^{d+\alpha}} dh.$$

On the other hand,

$$\mathcal{L}v(x) = \mathcal{L}_0v(x) + \mathcal{B}v(x),$$

and so we have

$$\begin{aligned} \mathcal{L}_0v(x) &= u(x)\mathcal{L}\varphi(x) + \varphi(x)f(x) + H(x) - \mathcal{B}v(x) \\ &= J_1(x) + J_2(x) + J_3(x) + J_4(x). \end{aligned} \quad (6.7)$$

Set

$$F(x) = \sum_{i=1}^4 J_i(x). \quad (6.8)$$

By Theorem 4.4 we have

$$\|v\|_{C^{\alpha+\beta}} \leq c_1(\|F\|_{C^\beta} + \|v\|_{L^\infty}) \leq c_1(\|F\|_{C^\beta} + \|u\|_{L^\infty}).$$

So if, given  $\varepsilon$ , we can show

$$\|F\|_{C^\beta} \leq c_2(\|f\|_{C^\beta} + \|u\|_{L^\infty} + \varepsilon\|u\|_{C^{\alpha+\beta}}), \quad (6.9)$$

we take  $\varepsilon = \delta/c_2$ , we then have (6.1), we apply Proposition 6.1, and we are done.

*Step 2.* We first look at the  $L^\infty$  norm of  $F$ . Since

$$\int [\varphi(x+h) - \varphi(x)] \frac{1}{|h|^{d+\alpha}} dh \leq \int |E_h\varphi(x)| \frac{1}{|h|^{d+\alpha}} dh \leq c_3 < \infty,$$

where  $E_h$  is defined in (5.1), then

$$|u(x)\mathcal{L}\varphi(x)| \leq c_3\|u\|_{L^\infty}.$$

Similarly

$$\begin{aligned} |H(x)| &= \left| \int [u(x+h) - u(x)] [\varphi(x+h) - \varphi(x)] \frac{A(x_0, r)}{|h|^{d+\alpha}} dh \right| \\ &\leq c_4\|u\|_{L^\infty} \int |E_h\varphi(x)| \frac{1}{|h|^{d+\alpha}} dh \\ &\leq c_5\|u\|_{L^\infty}. \end{aligned}$$

We also have

$$|\varphi(x)f(x)| \leq \|f\|_{L^\infty} \leq \|f\|_{C^\beta}.$$



It remains to bound  $\mathcal{B}v(x)$ . If  $x \notin B(x_0, 3r)$ , then since  $v(x) = 0$  and  $v(x+h) = 0$  unless  $|h| > r$ , we see

$$|\mathcal{B}v(x)| = \left| \int_{|h|>r} v(x+h) \frac{b(x,h)}{|h|^{d+\alpha}} dh \right| \leq c_6 \|u\|_{L^\infty} \int_{|h|>r} |h|^{-d-\alpha} dh = c_7 \|u\|_{L^\infty}.$$

We have

$$\|v\|_{C^{\alpha+\beta}} \leq c_8 \|\varphi\|_{C^{\alpha+\beta}} \|u\|_{C^{\alpha+\beta}} \leq c_9 \|u\|_{C^{\alpha+\beta}},$$

since  $\varphi$  is smooth. By Theorem 5.1(a),

$$|E_h v(x)| \leq c_{10} (|h|^{(\alpha+\beta) \wedge 1} \wedge 1) \|v\|_{C^{\alpha+\beta}},$$

and so

$$\begin{aligned} |\mathcal{B}v(x)| &= \left| \int E_h v(x) \frac{b(x,h)}{|h|^{d+\alpha}} dh \right| \\ &\leq c_{10} \varepsilon \int (|h|^{(\alpha+\beta) \wedge 1} \wedge 1) \frac{1}{|h|^{d+\alpha}} dh \|v\|_{C^{\alpha+\beta}} \\ &\leq c_{11} \varepsilon \|v\|_{C^{\alpha+\beta}} \leq c_{12} \varepsilon \|u\|_{C^{\alpha+\beta}}. \end{aligned}$$

We used the fact that we chose  $r$  small so that  $|b(x,h)| \leq \varepsilon$ . To summarize, in this step we have shown

$$\|F\|_{L^\infty} \leq c_{13} (\|f\|_{C^\beta} + \|u\|_{L^\infty} + \varepsilon \|u\|_{C^{\alpha+\beta}}). \quad (6.10)$$

*Step 3.* We next estimate  $[F]_{C^\beta}$ . Since we have

$$|F(x+k) - F(x)| \leq 2\|F\|_{L^\infty} \leq (2^\beta/r^\beta) \|F\|_{L^\infty} |k|^\beta$$

when  $|k| \geq r/2$  and we have an upper bound of the correct form for  $\|F\|_{L^\infty}$  in (6.10), to bound  $[F]_{C^\beta}$  it suffices to look at  $F(x+k) - F(x)$  when  $|k| \leq r/2$ . We look at the differences for  $J_i$  for  $i = 1, \dots, 4$ .

We look at  $J_4$  first, since this is the most difficult one. First suppose  $x \notin B(x_0, 3r)$ . Then  $v(x+h+k)$ ,  $v(x+h)$ ,  $v(x+k)$ , and  $v(x)$  are all zero if

$|h| \leq r/2$ . So

$$\begin{aligned}
& |\mathcal{B}v(x+k) - \mathcal{B}v(x)| \\
&= \left| \int_{|h|>r/2} [v(x+h+k)b(x+k,h) - v(x+h)b(x,h)] \frac{dh}{|h|^{d+\alpha}} \right| \\
&\leq \int_{|h|>r/2} |v(x+h+k) - v(x+h)| |b(x+k,h)| \frac{dh}{|h|^{d+\alpha}} \\
&\quad + \int_{|h|>r/2} |v(x+h)| |b(x+k,h) - b(x,h)| \frac{dh}{|h|^{d+\alpha}} \\
&\leq c_{14} \|v\|_{C^\beta} |k|^\beta \int_{|h|>r/2} \frac{dh}{|h|^{d+\alpha}} + c_{11} \|v\|_{L^\infty} |k|^\beta \int_{|h|>r/2} \frac{dh}{|h|^{d+\alpha}}.
\end{aligned}$$

Since  $\|v\|_{L^\infty} \leq \|u\|_{L^\infty}$  and

$$\begin{aligned}
\|v\|_{C^\beta} &\leq c_{15} \|u\|_{C^\beta} \|\varphi\|_{C^\beta} \leq c_{16} \|u\|_{C^\beta} \\
&\leq c_{17} \|u\|_{L^\infty} + \varepsilon \|u\|_{C^{\alpha+\beta}}
\end{aligned}$$

by Proposition 2.2, we have our required estimate when  $x \notin B(x_0, 3r)$ .

Now suppose  $x \in B(x_0, 3r)$ . Since  $|k| \leq r/2$ , then  $x+k \in B(x_0, 4r)$ , and so  $|b(x,h)| \leq \varepsilon$  and  $|b(x+k,h)| \leq \varepsilon$  for all  $h$ . We write

$$\begin{aligned}
& |\mathcal{B}v(x+k) - \mathcal{B}v(x)| \\
&\leq \int |E_h v(x+k) - E_h v(x)| \frac{|b(x+k,h)|}{|h|^{d+\alpha}} dh \\
&\quad + \int_{|h| \leq \zeta} |E_h v(x)| \frac{|b(x+k,h) - b(x,h)|}{|h|^{d+\alpha}} dh \\
&\quad + \int_{|h| > \zeta} |E_h v(x)| \frac{|b(x+k,h) - b(x,h)|}{|h|^{d+\alpha}} dh \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where  $\zeta$  will be chosen in a moment. By Theorem 5.1,

$$I_1 \leq \varepsilon \int (|h|^{(\alpha+\beta)\wedge 1} \wedge |k|^{(\alpha+\beta)\wedge 1}) \frac{dh}{|h|^{d+\alpha}} \leq c_{18} \varepsilon \|v\|_{C^{\alpha+\beta}} |k|^\beta.$$

Suppose for the moment that  $\alpha + \beta < 1$ . For  $I_2$  we have

$$I_2 \leq c_{19} \|v\|_{C^{\alpha+\beta}} \int_{|h| \leq \zeta} (|h|^{\alpha+\beta} \wedge 1) \frac{|k|^\beta}{|h|^{d+\alpha}} dh \leq \varepsilon \|v\|_{C^{\alpha+\beta}},$$

provided we take  $\zeta$  small; note that the choice of  $\zeta$  can be made to depend only on  $d, \alpha, \beta$ , and  $\varepsilon$ . For  $I_3$  we have

$$I_3 \leq c_{20} \|v\|_{C^\beta} \int_{|h|>\zeta} (|h|^\beta \wedge 1) \frac{|k|^\beta}{|h|^{d+\alpha}} dh \leq c_{21} \|v\|_{C^\beta} |k|^\beta.$$

We now use

$$\|v\|_{C^{\alpha+\beta}} \leq c_{22} \|u\|_{C^{\alpha+\beta}} \|\varphi\|_{C^{\alpha+\beta}}$$

and

$$\begin{aligned} \|v\|_{C^\beta} &\leq c_{23} \|u\|_{C^\beta} \|\varphi\|_{C^\beta} \\ &\leq \varepsilon \|u\|_{C^{\alpha+\beta}} + c_{24} \|u\|_{L^\infty}. \end{aligned}$$

Summing the estimates for  $I_1, I_2$ , and  $I_3$ , we have the desired bound for  $J_4$  when  $\alpha + \beta < 1$ . The case  $\alpha + \beta \in (1, 2)$  is very similar; the details are left to the reader.

Next we look at  $J_1$ . Similarly to the estimates for  $J_4$ , we see that  $\|\mathcal{L}\varphi\|_{C^\beta} \leq c_{25}$ . We then have

$$\|J_2\|_{C^\beta} \leq c_{26} \|u\|_{C^\beta} \|\mathcal{L}\varphi\|_{C^\beta},$$

and then Proposition 2.2 gives our estimate.

The estimate for  $J_2$  is quite easy. By Lemma 2.3

$$\|\varphi f\|_{C^\beta} \leq c_{26} \|\varphi\|_{C^\beta} \|f\|_{C^\beta} \leq c_{27} \|f\|_{C^\beta}.$$

It remains to handle  $J_3$ . We have

$$\begin{aligned} H(x+k) - H(x) &= \int [E_h u(x+k) - E_h u(x)] E_h \varphi(x+k) \frac{A(x+k, h)}{|h|^{d+\alpha}} dh \\ &\quad + \int E_h u(x) [E_h \varphi(x+k) - E_h \varphi(x)] \frac{A(x+k, h)}{|h|^{d+\alpha}} dh \\ &\quad + \int E_h u(x) E_h \varphi(x) \frac{A(x+k, h) - A(x, h)}{|h|^{d+\alpha}} dh \\ &= I_4 + I_5 + I_6. \end{aligned}$$

By Theorem 5.1

$$\begin{aligned} |I_4| &\leq c_{28} |k|^\beta \|u\|_{C^\beta} \int (|h|^\beta \wedge 1) \frac{dh}{|h|^{d+\alpha}} \\ &\leq c_{29} |k|^\beta \|u\|_{C^\beta}. \end{aligned}$$

Also by Theorem 5.1

$$\begin{aligned} |I_5| &\leq c_{30} \|u\|_{L^\infty} \int (|h|^\beta \wedge |k|^\beta \wedge 1) \frac{dh}{|h|^{d+\alpha}} \\ &\leq c_{31} \|u\|_{L^\infty} |k|^\beta; \end{aligned}$$

to get the second inequality we split the integral into  $|h| \leq |k|$ ,  $|k| < |h| \leq 1$ , and  $|h| > 1$ . Using Theorem 5.1 a third time

$$|I_6| \leq c_{32} \|u\|_{L^\infty} \int (|h|^\beta \wedge 1) \frac{|k|^\beta}{|h|^{d+\alpha}} dh \leq c_{33} \|u\|_{L^\infty} |k|^\beta.$$

Combining yields

$$[H]_{C^\beta} \leq c_{34} \|u\|_{C^\beta},$$

and we now apply Proposition 2.2.

*Step 4.* Finally we consider the case  $\alpha \geq 1$ . This is very similar to the  $\alpha < 1$  case, but where we replace the use of  $E_h f$  by  $F_h f$ . We leave the details to the reader.  $\square$

## 7 Further results and remarks

### 7.1 An extension

We remark that the proof of Theorem 1.2 really only required that there exist  $c_1$  and  $h_0$  such that

$$\sup_x \sup_{|h| \leq h_0} |A(x+k, h) - A(x, h)| \leq c_1 |k|^\beta.$$

The observation needed is that one can bound

$$\left\| \int_{|h| > h_0} [u(x+h) - u(x)] \frac{A(x, h)}{|h|^{d+\alpha}} dh \right\|_{C^\beta} \leq c_2 \|u\|_{C^\beta} \leq c_3 \|u\|_{L^\infty} + \varepsilon \|u\|_{C^{\alpha+\beta}}.$$

## 7.2 Zero order terms

We can add a zero order term to  $\mathcal{L}$  and have the result remain valid.

**Theorem 7.1** *Let  $P$  be a function such that  $\|P\|_{C^\beta} < \infty$ . Let*

$$\mathcal{L}'u(x) = \mathcal{L}u(x) + P(x)u(x),$$

where  $\mathcal{L}$  is defined by (1.1) or (1.2) and satisfies the assumptions of Theorem 1.2. Then there exists  $c_1$  (which depends on  $\|P\|_{C^\beta}$ ) such that if  $\mathcal{L}'u(x) = f(x)$  and  $\|u\|_{C^{\alpha+\beta}} < \infty$ , then

$$\|u\|_{C^{\alpha+\beta}} \leq c_1(\|u\|_{L^\infty} + \|f\|_{C^\beta}).$$

**Proof.** We proceed as in the proof of Theorem 1.2, but now in (6.8) we write  $F(x) = J_1(x) + \cdots + J_5(x)$ , where

$$J_5(x) = P(x)v(x).$$

We have, using Proposition 2.2 and Lemma 2.3,

$$\begin{aligned} \|J_5\|_{C^\beta} &\leq c_2\|P\|_{C^\beta}\|\varphi\|_{C^\beta}\|u\|_{C^\beta} \\ &\leq c_3(\|u\|_{L^\infty} + \varepsilon\|u\|_{C^{\alpha+\beta}}). \end{aligned}$$

Other than this additional term, the rest of the proof goes through as before.

□

## 7.3 First order terms

If  $\alpha > 1$ , we can add a first order term to  $\mathcal{L}$ . (We can also keep the zero order term as in Theorem 7.1, but we omit this in the following discussion for simplicity.)

**Theorem 7.2** *Suppose  $\alpha > 1$ . For  $i = 1, \dots, d$ , let  $Q_i$  be functions such that  $\|Q_i\|_{C^\beta} < \infty$ . Let*

$$\mathcal{L}''u(x) = \mathcal{L}u(x) + \sum_{i=1}^d Q_i(x)D_iu(x),$$

where  $\mathcal{L}$  is defined by (1.1) or (1.2) and satisfies the assumptions of Theorem 1.2. Then there exists  $c_1$  (which depends on  $\sum_{i=1}^d \|Q_i\|_{C^\beta}$ ) such that if  $\mathcal{L}''u(x) = f(x)$  and  $\|u\|_{C^{\alpha+\beta}} < \infty$ , then

$$\|u\|_{C^{\alpha+\beta}} \leq c_1(\|u\|_{L^\infty} + \|f\|_{C^\beta}).$$

**Proof.** As in the proof of Theorem 7.1 we have an additional term in the definition of  $F$ , but this time the term is

$$J'_5(x) = \sum_{i=1}^d Q_i(x) D_i v(x).$$

We have

$$\begin{aligned} \|Q_i D_i v\|_{C^\beta} &\leq c_2 \|Q_i\|_{C^\beta} (\|\varphi D_i u\|_{C^\beta} + \|u D_i \varphi\|_{C^\beta}) \\ &\leq c_3 (\|\varphi\|_{C^\beta} \|D_i u\|_{C^\beta} + \|u\|_{C^\beta} \|D_i \varphi\|_{C^\beta}) \\ &\leq c_4 \|u\|_{L^\infty} + \varepsilon \|u\|_{C^{\alpha+\beta}}, \end{aligned}$$

using Lemma 2.3 and Proposition 2.2. With  $J'_5$  handled in this fashion, we proceed as before.  $\square$

## 7.4 Higher order smoothness

One would expect that if  $f$  and  $A(\cdot, h)$  have additional smoothness, then the solution  $u$  to  $\mathcal{L}u = f$  should have additional smoothness. This is indeed the case. One way to show this is to extend the estimates previously proved to  $C^\beta$  and  $C^{\alpha+\beta}$  when  $\beta > 1$ . Here is an alternate way. We do the case  $\beta \in (1, 2)$  for concreteness, but the case when  $\beta \in (m, m+1)$  for some  $m$  is similar. When we write  $D_i A(x, h)$ , we mean the  $i^{\text{th}}$  partial derivative in the variable  $x$ .

**Theorem 7.3** *Suppose  $\beta \in (1, 2)$  and there exists  $c_1$  such that for each  $i = 1, \dots, d$ ,*

$$\sup_x \sup_h |D_i A(x+k, h) - D_i A(x, h)| \leq c_1 |k|^{\beta-1}.$$

*Then there exists  $c_2$  such that if  $f \in C^\beta$  and  $u \in C^{\alpha+\beta}$  with  $\mathcal{L}u = f$ , we have*

$$\|u\|_{C^{\alpha+\beta}} \leq c_1(\|u\|_{L^\infty} + \|f\|_{C^\beta}).$$

**Proof.** We sketch the proof, and we restrict our attention to  $\alpha < 1$  for simplicity. Differentiating  $\mathcal{L}u = f$  yields

$$\mathcal{L}(D_i u)(x) + \int [u(x+h) - u(x)] \frac{D_i A(x, h)}{|h|^{d+\alpha}} dh = D_i f.$$

Writing  $G_i(x)$  for the second term on the left,

$$\mathcal{L}(D_i u) = D_i f - G_i,$$

and by Theorem 1.2,

$$\|D_i u\|_{C^{\beta-1}} \leq c_3(\|D_i u\|_{L^\infty} + \|D_i f\|_{C^{\beta-1}} + \|G_i\|_{C^{\beta-1}}).$$

Note  $\|D_i f\|_{C^{\beta-1}} \leq c_4 \|f\|_{C^\beta}$  and  $\|D_i u\|_{L^\infty} \leq c_5 \|u\|_{L^\infty} + \varepsilon \|u\|_{C^{\alpha+\beta}}$ . Also  $\|u\|_{C^\beta} \leq c_6 \sum_{i=1}^d \|D_i u\|_{C^{\beta-1}}$ . So the key step is to prove that

$$\|G_i\|_{C^{\beta-1}} \leq c_7 \|u\|_{L^\infty} + \varepsilon \|u\|_{C^{\alpha+\beta}}. \quad (7.1)$$

By arguments similar to the derivation of the estimates for  $J_4$  in the proof of Theorem 1.2 but somewhat simpler,

$$\|G_i\|_{C^{\beta-1}} \leq c_8 \|u\|_{C^{\alpha+\beta-1}}.$$

By Proposition 2.2, the right hand side is bounded by the right hand side of (7.1).  $\square$

## 7.5 Sharpness

Our results are sharp in several respects. For example, one might ask if the solution  $u$  to  $\mathcal{L}u = f$  can be taken to be in  $C^{\alpha+\beta+\delta}$  for some  $\delta > 0$  when  $f \in C^\beta$ . The answer is no in general. Let  $\mathcal{L} = \mathcal{L}_0$ , where  $\mathcal{L}_0$  is defined by (3.4). Let  $f$  be a  $C^\beta$  function that is not in  $C^{\beta+\delta}$  for any  $\delta$ . If the solution to  $\mathcal{L}u = f$  satisfied

$$\|u\|_{C^{\alpha+\beta+\delta}} \leq c_1(\|u\|_{L^\infty} + \|f\|_{C^\beta}),$$

then by Corollary 5.4,  $f = \mathcal{L}_0 u$  would be in  $C^{\beta+\delta}$ , a contradiction.

Another question is whether one can still obtain our main estimate (1.3) if  $A(x, h)$  only satisfies

$$\sup_x \sup_h |A(x+k, h) - A(x, h)| \leq c_1 |k|^{\beta-\delta}, \quad k \in \mathbb{R}^d, \quad (7.2)$$

for some  $\delta > 0$ . Again the answer is no in general. Let  $f$  be a function that is in  $C^\beta$  but not in any  $C^{\beta+\zeta}$  for  $\zeta > 0$ . Let  $w$  be a function that is in  $C^{\beta-\delta}$  for some  $\delta \in (0, \beta)$  but not in  $C^{\beta-\delta+\zeta}$  for any  $\zeta > 0$ . Suppose also that  $w$  is bounded below by a positive constant. Let  $\mathcal{L}_0$  be defined as in (3.4), and define  $A(x, h) = w(x)A_0(h)$ . Then  $\mathcal{L}u(x) = w(x)\mathcal{L}_0u(x)$ , and  $A(x, h)$  satisfies (7.2). Consider the solution to  $\mathcal{L}u(x) = f(x)$ . We have  $\mathcal{L}_0u(x) = f(x)/w(x)$ . If  $u$  were in  $C^{\alpha+\beta}$ , then  $f(x)/w(x) = \mathcal{L}_0u(x)$  would be in  $C^\beta$ , a contradiction.

## 7.6 The $\|u\|_{L^\infty}$ term

Our main estimate (1.3) has a  $\|u\|_{L^\infty}$  on the right hand side. When can one dispense with this term? First we give a condition where one can do so.

Suppose one considers  $\mathcal{L}'u(x) = f(x)$ , where  $\mathcal{L}'$  is defined in Theorem 7.1 and moreover for some  $\lambda > 0$ ,  $P(x) \leq -\lambda$  for all  $x$ . If  $X_t$  is the strong Markov process associated to  $\mathcal{L}$  (that is, the infinitesimal generator of  $X$  is  $\mathcal{L}$ , for example), the solution to  $\mathcal{L}'u(x)$  is given in probabilistic terms by

$$u(x) = -\mathbb{E}^x \int_0^\infty e^{\int_0^s P(X_r) dr} f(X_s) ds.$$

Under the condition that  $P(x) \leq -\lambda$ , then

$$|u(x)| \leq \mathbb{E}^x \int_0^\infty e^{-\lambda s} |f(X_s)| ds \leq \frac{1}{\lambda} \|f\|_{L^\infty}.$$

In this case, we have the bound

$$\|u\|_{L^\infty} \leq \|f\|_{L^\infty} / \lambda \leq \|f\|_{C^\beta} / \lambda.$$

On the other hand, if there is no zero order term, there is no reason to expect that a bound of the form

$$\|u\|_{L^\infty} \leq c_1 \|f\|_{C^\beta} \quad (7.3)$$



should hold when  $\mathcal{L}u = f$ . This bound trivially fails to hold because  $u$  plus a constant is still a solution to the equation.

Even when we restrict ourselves to solutions that vanish at infinity, (7.3) cannot hold. To see this, let  $A(x, h)$  be identically 1, so that  $\mathcal{L}$  is the infinitesimal generator of a symmetric stable process, let  $\bar{\varphi}$  be defined as in the beginning of Section 6, and let  $f_r(x) = \bar{\varphi}(x/r)$ . Then  $\|f_r\|_{L^\infty} = 1$  for all  $r$ , while  $[f_r]_{C^\beta} \rightarrow 0$  as  $r \rightarrow \infty$  for each  $\beta \in (0, 1)$ . On the other hand, if  $u_r$  is the solution to  $\mathcal{L}u = f_r$ , a scaling argument shows that  $|u_r(0)| = c_1 r^\alpha \rightarrow \infty$  as  $r \rightarrow \infty$ .

## 7.7 Future research

We mention some directions for future research.

1. *Interior estimates for the Dirichlet problem.* Can one give interior estimates for the regularity of harmonic functions (the Dirichlet problem) and the regularity of potentials (the analog of Poisson's equation) in bounded domains?
2. *Boundary estimates.* To obtain a satisfactory theory, one would like estimates on harmonic functions and potentials in bounded domains that are valid up to the boundary.
3. *Symmetric processes.* Suppose instead of  $\mathcal{L}$  one works instead with the Dirichlet form

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))(g(y) - g(x)) \frac{B(x, y)}{|x - y|^{d+\alpha}}.$$

The generator associated to  $\mathcal{E}$  is the analog of an elliptic operator in divergence form. The Harnack inequality and Hölder regularity for harmonic functions are known in this setting under the assumption that  $B(x, y)$  is symmetric and bounded above and below by positive constants; see [15]. However if one adds some continuity conditions to  $B$ , one would expect the corresponding potentials and harmonic functions to have additional smoothness.

4. *The parabolic case.* One could look at the fundamental solution or heat kernel  $p(t, x, y)$ , which is equivalent to looking at the transition densities

of the associated process. One would expect that if the  $A(x, h)$  (and the  $B(x, h)$ ) have some smoothness, say, Hölder continuous of order  $\beta$ , and are bounded above and below by positive constants, then the  $p(t, x, y)$  are not only Hölder continuous in  $x$  and  $y$ , but will be  $C^{\alpha+\beta}$  in each coordinate. (In the symmetric case Hölder continuity is known, but of a smaller order.) This question could be asked about the transition densities in the whole space  $\mathbb{R}^d$  and also in bounded domains.

5. *Variable order.* Consider operators  $\mathcal{L}$  of the form

$$\mathcal{L}f(x) = \int [f(x+h) - f(x) - 1_{(|h|\leq 1)} \nabla f(x) \cdot h] n(x, h) dh, \quad (7.4)$$

where we assume

$$\frac{c_1}{|h|^{d+\alpha}} \leq n(x, h) \leq \frac{c_2}{|h|^{d+\beta}}, \quad x \in \mathbb{R}^d, 1 \geq |h| > 0,$$

$0 < \alpha < \beta < 2$ , and some appropriate condition is imposed on  $n(x, h)$  for  $|h| \geq 1$ . Such an operator is of variable order because if one writes it as a pseudo-differential operator, then the order is not fixed; see [21]. Some progress has already been made on operators of variable order; see [4] and [5] for the operators  $\mathcal{L}$  in (7.4) and see [1] and [6] for non-local Dirichlet forms of variable order. Can one give suitable assumptions on  $n(x, h)$  so that harmonic functions and potentials have additional smoothness?

6. *Diffusions with jumps.* If we consider operators that are the sum of an elliptic differential operator and a non-local operator, the same questions could be asked as for the pure jump case: higher order derivatives, regularity up to the boundary, transition density estimates. (The Harnack inequality was considered in [18] and [19].)

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Richard F. Bass  
Department of Mathematics  
University of Connecticut  
Storrs, CT 06269-3009, USA  
*bass@math.uconn.edu*