Z. Wahrscheinlichkeitstheorie verw. Gebiete
70, 591–608 (1985)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete
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Law of the Iterated Logarithm for Set-indexed Partial Sum Processes with Finite Variance

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1. Introduction

Let $X_{\mathbf{j}}$, $\mathbf{j} \in \{1, 2, \ldots\}^d$ be independent, identically distributed random variables that are mean 0 with variance 1. If $\mathscr{A}$ is a collection of Borel subsets of $[0, 1]^d$, define the partial sum process $\{S_n(B); B \in \mathscr{A}\}$ by

$$S_n(B) = \sum_{\mathbf{j} \in nB} X_{\mathbf{j}}.$$

The purpose of this paper is to prove a functional law of the iterated logarithm (LIL) for a suitably smoothed and normalized version of S_n.

The first functional LIL is due to Strassen [8]. If B_t is one-dimensional Brownian motion, he showed that the set of functions $\beta_t(\cdot)$ defined by

$$\beta_t(s) = B_{st}/(2t \log\log t)^{\frac{1}{2}}, \qquad 0 \leqq s \leqq 1,$$

is relatively compact in the uniform topology and the set of limit points is the set of functions which are absolutely continuous and whose derivatives have L_2 norm less than or equal to 1. Using Skorokhod embedding, Strassen showed that random walks also obeyed a functional LIL and that the usual LIL, as well as many other interesting results, could be obtained as a consequence.

The extension of Strassen's results to Brownian motion indexed by points in $\mathbb{R}^d_+$ and to partial sum processes indexed by points in $\mathbb{R}^d_+$ was done by Pyke [7] and Wichura [9], respectively. For a Brownian motion indexed by a large collection $\mathscr{A}$ of sets, the functional law was proved by Bass and Pyke [2], and then a type of Skorokhod embedding was used to extend this for partial sum processes. However, if $H(\delta)$ is the log-entropy of $\mathscr{A}$ (defined in Sect. 2), it was necessary that $H(\delta) \leqq \delta^{-r}$ for some $r < 1$ and that the $X_{\mathbf{j}}$'s have at least $(2+\varepsilon)/(1-r)$ moments for some $\varepsilon > 0$. Moreover, the sets in $\mathscr{A}$ had to have uniformly smooth boundaries. Morrow and Philipp [6] were able to wea-

* Research partially supported by NSF Grant MCS 83-00581

ken the smooth boundary condition somewhat, but required the $X_{\jmath}$'s to have at least $(2+\varepsilon)(1+r)/(1-r)$ moments.

In this paper we prove a functional LIL that requires only finite variance for the $X_{\jmath}$'s. Our restriction on $\mathscr{A}$ is that its log-entropy with inclusion, H, satisfy

$$\int_0^1 (H(x)/x)^{\frac{1}{2}}\, dx < \infty. \tag{1.1}$$

That is the weakest one could expect, since (1.1) (using entropy without inclusion) is the condition usually used to show that the Brownian motion indexed by $\mathscr{A}$ has sample continuous paths [5]. Furthermore, no smoothness whatsoever is required on the boundaries of the sets in $\mathscr{A}$. When we take $d=1$, $\mathscr{A}=\{[0,t]\colon 0\leqq t\leqq 1\}$, we get Strassen's LIL, while $d\geqq 2$, $\mathscr{A}=\{[0,\ell]\colon \ell\in[0,1]^d\}$ yields Wichura's results for partial sums of iid random variables.

The key to our method is a modification of the metric entropy technique. Instead of truncating the $X_{\jmath}$'s once at the beginning and then doing the chaining argument, we perform a truncation at each stage of the chaining. This gives us an exponential estimate, and then more or less standard techniques are used to get our functional LIL. Some simplifications due to de Acosta [4] are used in this latter stage of the argument.

Our methods can also be used to prove the central limit theorem for set-indexed partial sum processes with finite variance. This gives a new proof of the recent result of Alexander and Pyke [1], which was done by completely different methods.

In Sect. 2 we introduce notation and state our assumptions and results. Section 3 contains the entropy argument. Section 4 gives the proof of relative compactness, while Sect. 5 identifies the limit points. A sketch of the proof of the central limit theorem is given in Sect. 6.

2. Statement of Results

Let $I^d=[0,1]^d$. If A is any Borel set, let $|A|$ denote the Lebesgue measure of A, and let $rA=\{r\ell\colon \ell\in A\}$. For $\jmath=(j_1,\dots,j_d)$, let

$$R_{\jmath}=(j_1-1,j_1]\times\dots\times(j_d-1,j_d],$$

the cube with upper corner at $\jmath$. Suppose $X_{\jmath}$, $\jmath\in\{1,2,\dots\}^d$ are iid random variables with mean 0 and variance 1. Our smoothed and normalized version of S_n will be Z_n, given by

$$Z_n(B)=n^{-d/2}\sum_{\jmath}|nB\cap R_{\jmath}|\,X_{\jmath}. \tag{2.1}$$

In the case where $d=1$ and we identify the point t with the set $[0,t]$, $n^{d/2}Z_n(t)$ is just the usual process obtained by linearly interpolating between the points $\{(i,S_i)\colon i\leqq n\}$. As a function of B, Z_n is well defined for all Borel sets $B\subseteq I^d$ and for each n is uniformly continuous with respect to the metric $d(A,B)=|A\,\triangle B|$.

Let

$$\mathrm{LL}\, n = \begin{cases} \log\log n & \text{if } n \geqq 16 \\ 1 & \text{if } n \leqq 15. \end{cases}$$

Let $\varphi_n = (2\,\mathrm{LL}\, n)^{1/2}$.

We will suppose that $\mathscr{A}$ is a collection of Borel subsets of I^d that is totally bounded with inclusion:

(2.2) for each δ, there exists a finite subcollection $\mathscr{A}(\delta)$ of $\mathscr{A}$ with the following property:

if $A \in \mathscr{A}$, there exists B, B^+ (depending on A) in $\mathscr{A}(\delta)$ such that $B \subseteq A \subseteq B^+$ and $|B^+ - B| \leqq \delta$.

For each δ, let $H(\delta)$, the log-entropy, be the log of the cardinality of the smallest such subcollection $\mathscr{A}(\delta)$.

We suppose $\mathscr{A}$ is entropy-integrable:

$$\int_0^1 (H(x)/x)^{\frac{1}{2}}\, dx < \infty. \tag{2.3}$$

For any $F\colon \mathscr{A} \to \mathbb{R}$, let

$$\|F\|_{\mathscr{A}} = \sup_{B \in \mathscr{A}} |F(B)|.$$

Let $\mathscr{K}$ be the subset of the functions from $\mathscr{A}$ to $\mathbb{R}$ given by

(2.4) $\mathscr{K} = \{F\colon$ for some $f\colon I^d \to \mathbb{R}$ with $\int_{I^d} f^2(t)\, dt \leqq 1$, $F(B) = \int_B f(t)\, dt$ for all $B \in \mathscr{A}\}$.

Given $F \in \mathscr{K}$, we extend the domain of F to all Borel subsets B of I^d by defining

$$F(B) = \int_B f(t)\, dt.$$

We can now state our theorem.

Theorem 2.1. *$\{Z_n/\varphi_n\}$ is relatively compact with respect to the metric $\|\ \|_{\mathscr{A}}$, and the set of limit points is exactly $\mathscr{K}$,* a.s.

Before proceeding to the proof of Theorem 2.1, we introduce a bit more notation and make some observations that will be useful later.

Let $\mathscr{K}^\varepsilon$ be the subset of the functions from $\mathscr{A}$ to $\mathbb{R}$ given by

$$\mathscr{K}^\varepsilon = \{F\colon \text{for some } F_0 \in \mathscr{K},\ \|F - F_0\|_{\mathscr{A}} \leqq \varepsilon\}. \tag{2.5}$$

By Cauchy-Schwartz, if $F \in \mathscr{K}$, B Borel,

$$|F(B)| = \left|\int_B f(t)\, dt\right| \leqq \left(\int_{I^d} 1_B(t)\, dt\right)^{\frac{1}{2}} \left(\int_{I^d} f^2(t)\, dt\right)^{\frac{1}{2}} \leqq |B|^{\frac{1}{2}}, \tag{2.6}$$

and so if $F \in \mathscr{K}^\varepsilon$, $B \in \mathscr{A}$,

$$|F(B)| \leqq \varepsilon + |B|^{\frac{1}{2}}. \tag{2.7}$$

Throughout, c, with or without subscripts, will denote a constant whose value may change from place to place.

For the rest of the paper, it will be technically convenient to assume $\mathscr{A}$ is contraction closed:

$$\text{if } B\in\mathscr{A},\ 0\leqq r\leqq 1, \text{ then } rB\in\mathscr{A}. \tag{2.8}$$

There is no loss of generality in making this assumption, because if $\mathscr{A}^*=\{rB\colon B\in\mathscr{A},\ 0\leqq r<1\}$ has log-entropy $H^*(\delta)$, then replacing $\mathscr{A}$ in Theorem 2.1 by the larger $\mathscr{A}^*$ only creates a better theorem, while

Lemma 2.2. $\int_0^1 (H^*(x)/x)^{\frac{1}{2}}\,dx<\infty$.

Proof. Let $\mathscr{A}^*(2\delta)=\{sB: s/\delta$ is an integer, $0\leqq s\leqq 1$, $B\in\mathscr{A}(\delta)\}$. It is easy to check that if $rA\in\mathscr{A}^*$, there exist $B, B^+\in\mathscr{A}^*(2\delta)$ with $B\subseteq rA\subseteq B^+$, $|B^+-B|\leqq 2\delta$ and that $H^*(2\delta)\leqq 2\log\delta^{-1}+H(\delta)$. The result follows. □

The following lemma will make working with (2.3) easier.

Lemma 2.3. *Suppose $G(x)$ is a strictly decreasing continuous function with $G(x)\geqq x^{-\frac{1}{2}}$ and $\int_0^1 (G(x)/x)^{\frac{1}{2}}\,dx<\infty$. Suppose $x_0<\frac{1}{2}$, and define x_i to satisfy*

$$G(x_i)/x_i=(G(x_0)/x_0)\,2^i,\qquad i=1,2,\ldots.$$

Then

(a) $x_i/x_{i+1}\leqq 2$;

(b) $G(x_{i+1})/G(x_i)\leqq 2$;

(c) $\sum_{i=0}^{\infty}(x_i\,G(x_i))^{\frac{1}{2}}\leqq 5\int_0^1 (G(x)/x)^{\frac{1}{2}}\,dx$; *and*

(d) *if* $\zeta\leqq\frac{1}{2}$, $\sum_{i=0}^{\infty}\zeta^{+x_i^{-\frac{1}{2}}}\leqq 15\zeta\int_0^1 (G(x)/x)^{\frac{1}{2}}\,dx$.

Proof. Since $G(x)/x$ is strictly decreasing and $G(x)\geqq x^{-\frac{1}{2}}$, x_i decreases to 0. Then

$$x_i/x_{i+1}=\frac{x_i}{G(x_i)}\,\frac{G(x_{i+1})}{x_{i+1}}\,\frac{G(x_i)}{G(x_{i+1})}\leqq 2,$$

and

$$G(x_{i+1})/G(x_i)=\frac{G(x_{i+1})}{x_{i+1}}\,\frac{x_i}{G(x_i)}\,\frac{x_{i+1}}{x_i}\leqq 2.$$

Observe that

$$(x_n\,G(x_n))^{\frac{1}{2}}=x_n(G(x_n)/x_n)^{\frac{1}{2}}\leqq\int_0^{x_n}(G(x)/x)^{\frac{1}{2}}\,dx\to 0\qquad\text{as } n\to\infty, \tag{2.9}$$

and

$$\begin{aligned}(G(x_i)/x_i)^{\frac{1}{2}}-(G(x_{i-1})/x_{i-1})^{\frac{1}{2}}&=(G(x_0)/x_0)^{\frac{1}{2}}\,(2^{i/2}-2^{(i-1)/2})\\&\geqq(G(x_i)/x_i)^{\frac{1}{2}}/5.\end{aligned} \tag{2.10}$$

Using (2.10) and summing by parts, we have

$$\int_0^1 (G(x)/x)^{\frac{1}{2}} \geqq \sum_{i=0}^{\infty} \int_{x_{i+1}}^{x_i} (G(x)/x)^{\frac{1}{2}}\, dx \geqq \sum_{i=0}^{n} (x_i - x_{i+1})(G(x_i)/x_i)^{\frac{1}{2}}$$

$$= x_0 (G(x_0)/x_0)^{\frac{1}{2}} + \sum_{i=1}^{n} x_i [(G(x_i)/x_i)^{\frac{1}{2}} - (G(x_{i-1})/x_{i-1})^{\frac{1}{2}}] - x_{n+1}(G(x_n)/x_n)^{\frac{1}{2}}$$

$$\geqq \sum_{i=0}^{n} (x_i\, G(x_i))^{\frac{1}{2}}/5 - (x_n\, G(x_n))^{\frac{1}{2}}.$$

This with (2.9) gives (c).

Since $(x^{-\frac{1}{2}}-1)|\log \zeta| \geqq x^{-\frac{1}{2}}/10 \geqq |\log x|/4 - \log 3$ for $\zeta,\ x \leqq \frac{1}{2}$,

$$\sum_{i=0}^{\infty} \zeta^{x_i^{-\frac{1}{2}}} \leqq 3\zeta \sum_{i=0}^{\infty} x_i^{\frac{1}{4}} \leqq 3\zeta \sum_{i=0}^{\infty} (x_i\, G(x_i))^{\frac{1}{4}},$$

and (d) follows. □

A comment about measurability: since by (2.2) $\mathscr{A}$ is totally bounded, then $\mathscr{A}$ is separable. Since for each n, Z_n is uniformly continuous with respect to the metric $d(A,B) = |A \triangle B|$, it follows that $Z_n^{-1}(\mathscr{B}) \subseteq \sigma(X_{\jmath} : \jmath \in \{1, 2, \ldots\}^d)$, where $\mathscr{B}$ is the Borel σ-field on $\mathscr{C}(\mathscr{A})$, the set of uniformly continuous functions on $\mathscr{A}$ with norm $\|\cdot\|_{\mathscr{A}}$. A similar comment applies to each of the other processes we will define.

3. Entropy

This section gives an exponential bound for partial sums of truncated random variables. For $0 \leqq a \leqq b \leqq \infty$, define

$$X_{\jmath}(n,a,b) = X_{\jmath}\, 1_{(a n^{d/2}/\varphi_n \leqq |X_{\jmath}| < b n^{d/2}/\varphi_n)}, \tag{3.1}$$

$$Z_n(A,a,b) = n^{-d/2} \sum_{\jmath \in \{1,2,\ldots\}^d} |nA \cap R_{\jmath}|\, (X_{\jmath}(n,a,b) - EX_{\jmath}(n,a,b)), \tag{3.2}$$

and

$$U_n(A,a,b) = n^{-d/2} \sum_{\jmath \in \{1,2,\ldots\}^d} |nA \cap R_{\jmath}|\, |X_{\jmath}(n,a,b)|. \tag{3.3}$$

Note $Z_n(A,a,b)$ is mean 0, and if A and B are any two sets,

$$\begin{aligned} \operatorname{Var}(Z_n(A,a,b) - Z_n(B,a,b)) &\leqq n^{-d} \sum_{\jmath} (|nA \cap R_{\jmath}| - |nB \cap R_{\jmath}|)^2 \operatorname{Var} X_{\jmath} \\ &\leqq n^{-d} \sum_{\jmath} |nA \triangle B) \cap R_{\jmath}| = |A \triangle B|, \end{aligned} \tag{3.4}$$

since $||nA \cap R_{\jmath}| - |nB \cap R_{\jmath}|| \leqq |n(A \triangle B) \cap R_{\jmath}| \leqq 1$. Note also that $Z_n(A,a,b) - Z_n(B,a,b)$ is the sum of independent random variables, each of which is bounded by $2b/\varphi_n$.

A similar calculation gives

$$\operatorname{Var} U_n(A,a,b) \leqq |A|, \tag{3.5}$$

while

$$E|X_{\mathscr{J}}(n,a,b)|\leqq EX_{\mathscr{J}}^2/(a\,n^{d/2}/\varphi_n),$$

and hence

$$EU_n(A,a,b)\leqq n^{-d/2}\sum_{\mathscr{J}}|nA\cap R_{\mathscr{J}}|\,\varphi_n/a\,n^{d/2}=\varphi_n|A|/a. \tag{3.6}$$

Our estimate is the following.

Theorem 3.1. *Given λ, δ, and $\varepsilon>0$, there exists constants c, α, and n_0 (depending on λ, δ, and ε) such that if $a\leqq\alpha$ and $n\geqq n_0$,*

$$P\big(\sup_{B,C\in\mathscr{A},\,|B\triangle C|\leqq\delta}|Z_n(B,0,a)-Z_n(C,0,a)|>\lambda\varphi_n\big)\leqq c\exp\left(\frac{-\lambda^2\varphi_n^2}{(2+\varepsilon)\delta}\right).$$

Proof. We will select numbers $\bar\lambda$, λ_i, δ_i, a_i, α later so that δ_i, a_i, $\lambda_i\downarrow 0$, $\alpha<a_0$ and

$$\lambda\geqq\bar\lambda+4\sum_{i=0}^{\infty}\lambda_i. \tag{3.7}$$

If B and C are any sets in $\mathscr{A}$, there exists sets B_i, B_i^+, C_i, C_i^+ in $\mathscr{A}(\delta_i)$ such that $B_i\subseteq B\subseteq B_i^+$ and $|B_i^+-B_i|\leqq\delta_i$ and similarly for C, C_i, C_i^+, for each i. Since for each $a\leqq\alpha$ and each n, $Z_n(A,0,a)$ is a continuous function of A with respect to $d(A,B)=|A\triangle B|$, we can write

$$\begin{aligned}Z_n(B,0,a)=Z_n(B_0,0,a)&+\sum_{i=0}^{\infty}[Z_n(B_{i+1},0,a_i\wedge a)-Z_n(B_i,0,a_i\wedge a)]\\&+\sum_{i=1}^{\infty}[Z_n(B,a_i\wedge a,a_{i-1}\wedge a)-Z_n(B_i,a_i\wedge a,a_{i-1}\wedge a],\end{aligned}$$

and similarly for $Z_n(C,0,a)$, provided $\alpha<a_0$. (It will be apparent from the remainder of the proof that both series are absolutely convergent, a.s.) Then the only way $Z_n(C,0,a)-Z_n(B,0,a)$ can be larger than $\lambda\varphi_n$ for some B and C in $\mathscr{A}$ with $|C\triangle B|\leqq\delta$ is if at least one of the following holds:

(i) for some B_0, $C_0\in\mathscr{A}(\delta_0)$ with $B_0\subseteq B$, $C_0\subseteq C$, $|B-B_0|\leqq\delta_0$, $|C-C_0|\leqq\delta_0$,

$$|Z_n(C_0,0,a)-Z_n(B_0,0,a)|\geqq\bar\lambda\varphi_n;$$

(ii) for some i, for some $B_i\in\mathscr{A}(\delta_i)$, $B_{i+1}\in\mathscr{A}(\delta_{i+1})$, $|B_{i+1}\triangle B_i|\leqq 2\delta_i$,

$$|Z_n(B_{i+1},0,a_i\wedge a)-Z_n(B_i,0,a_i\wedge a)|\geqq\lambda_i\varphi_n;$$

(iii) same as (ii) with B's replaced by C's;

(iv) for some i, for some B_i, $B_i^+\in\mathscr{A}(\delta_i)$, $B\in\mathscr{A}$, $B_i\subseteq B\subseteq B_i^+$, $|B_i^+-B_i|\leqq\delta_i$,

$$|Z_n(B,a_i\wedge a,a_{i-1}\wedge a)-Z_n(B_i,a_i\wedge a,a_{i-1}\wedge a)|\geqq\lambda_i\varphi_n;$$

or (v) same as (iv) with B's replaced by C's.

The number of pairs B_i, B_i^+ in $\mathscr{A}(\delta_i)$ is $\leqq\exp(2H(\delta_i))$, while the number of pairs $B_i\in\mathscr{A}(\delta_i)$, $B_{i+1}\in\mathscr{A}(\delta_{i+1})$ is $\leqq\exp(2H(\delta_{i+1}))$, since $H(x)$ increases as x decreases.

We then have

(3.8) $$P\Big(\sup_{B,C\in\mathscr{A},\,|B\triangle C|\leqq\delta}|Z_n(C,0,a)-Z_n(B,0,a)|>\lambda\varphi_n\Big)\leqq p_0+\sum_{i=0}^{\infty}r_i+\sum_{i=1}^{\infty}s_i,$$

where

$$p_0=\exp(2H(\delta_0))\max_{B_0,C_0\in\mathscr{A}(\delta_0),\,|C_0\triangle B_0|\leqq\delta+2\delta_0}P(|Z_n(C_0,0,a)-Z_n(B_0,0,a)|\geqq\bar{\lambda}\varphi_n),$$

$$r_i=2\exp(2H(\delta_{i+1}))\max_{\substack{B_i\in\mathscr{A}(\delta_i),\,B_{i+1}\in\mathscr{A}(\delta_{i+1})\\|B_i\triangle B_{i+1}|\leqq 2\delta_i}}P(|Z_n(B_{i+1},0,a_i\wedge a)-Z_n(B_i,0,a_i\wedge a)|\geqq\lambda_i\varphi_n),$$

and

$$s_i\leqq 2\exp(2H(\delta_i))\max_{B_i,B_i^+\in\mathscr{A}(\delta_i),\,|B_i^+-B_i|\leqq\delta_i}P\Big(\sup_{B_i\subseteq B\subseteq B_i^+}|Z_n(B,a_i\wedge a,a_{i-1}\wedge a)-Z_n(B_i,a_i\wedge a,a_{i-1}\wedge a)|\geqq\lambda_i\varphi_n\Big).$$

Here p_0 bounds the terms in (i), r_i the terms in (ii) and (iii), and s_i the terms in (iv) and (v).

We now proceed to estimate p_0, r_i, and s_i. Recalling (3.4) and the fact that $Z_n(A,0,a)-Z_n(B,0,a)$ is the sum of terms bounded by $2a/\varphi_n$, we have by Bernstein's inequality (see [3]),

(3.9) $$\begin{aligned}p_0&\leqq 2\exp(2H(\delta_0))\exp\left(\frac{-\bar{\lambda}^2\varphi_n^2}{2(\delta+2\delta_0)+2(2a/\varphi_n)(\bar{\lambda}\varphi_n)/3}\right)\\&=2\exp(2H(\delta_0))\exp(-\bar{\lambda}^2\varphi_n^2/(2\delta+4\delta_0+4a\bar{\lambda}/3)).\end{aligned}$$

Similarly,

(3.10) $$r_i\leqq 4\exp(2H(\delta_{i+1}))\exp(-\lambda_i^2\varphi_n^2/(4\delta_i+4a_i\lambda_i/3)).$$

To estimate s_i, note that

$$\begin{aligned}&\sup_{B_i\subseteq B\subseteq B_i^+}|Z_n(B,a_i\wedge a,a_{i-1}\wedge a)-Z_n(B_i,a_i\wedge a,a_{i-1}\wedge a)|\\&\qquad\leqq U_n(B_i^+-B_i,a_i,a_{i-1})+EU_n(B_i^+-B_i,a_i,a_{i-1}).\end{aligned}$$

By the definition of U_n, $U_n(B_i^+-B_i,a_i,a_{i-1})$ is the sum of independent random variables each of which is bounded by a_{i-1}/φ_n, and by (3.6), if $|B_i^+-B_i|\leqq\delta_i$,

$$EU_n(B_i^+-B_i,a_i,a_{i-1})\leqq\varphi_n\delta_i/a_i,$$

Then, provided

(3.11) $$\lambda_i\varphi_n\geqq 3\varphi_n\delta_i/a_i,$$

by Bernstein's inequality,

(3.12) $$\begin{aligned}s_i&\leqq 2\exp(2H(\delta_i))\max_{B_i,B_i^+\in\mathscr{A}(\delta_i),\,|B_i^+-B_i|\leqq\delta_i}P(U_n(B_i^+-B_i,a_i,a_{i-1})\\&\qquad-EU_n(B_i^+-B_i,a_i,a_{i-1})\geqq\lambda_i\varphi_n/3)\\&\leqq 2\exp(2H(\delta_i))\exp\left(\frac{-\lambda_i^2\varphi_n^2/9}{2\delta_i+2(a_{i-1}/\varphi_n)(\lambda_i\varphi_n/3)/3}\right)\\&=2\exp(2H(\delta_i))\exp(-\lambda_i^2\varphi_n^2/(18\delta_i+2a_{i-1}\lambda_i)).\end{aligned}$$

We now choose δ_i, λ_i, a_i, $\bar{\lambda}$, and α. Let $G(x)$ be a continuous, strictly decreasing function so that $G(x) \geqq x^{-\frac{3}{4}}$, $G(x) \geqq H(x)$, but

$$\gamma = \int_0^1 (G(x)/x)^{\frac{1}{2}}\, dx < \infty.$$

For $x \leqq 1$, note that $G(x) \geqq x^{-\frac{1}{2}}$. Take $\bar{\lambda}$ less than but sufficiently close to λ so that $\bar{\lambda}^2 \geqq \lambda^2/(1+\varepsilon/4)$. Let $\beta = (\lambda - \bar{\lambda})/20\gamma$. Choose $\delta_0 < 1$ small enough so that $\delta_0 < \varepsilon\,\delta/8$ and $\delta_0^{-\frac{1}{4}} \beta^2/60 \geqq \lambda^2/2\delta$. Let δ_i be defined by $G(\delta_i)/\delta_i = (G(\delta_0)/\delta_0)\, 2^i$, and let

$$\lambda_i = (\delta_i\, G(\delta_i))^{\frac{1}{2}} \beta, \qquad i = 0, 1, \ldots.$$

Let $a_i = 3\delta_i/\lambda_i$, $i = 0, 1, \ldots$. Take $\alpha = \min(a_0/2, \varepsilon\,\delta/4(1+\lambda))$.

By Lemma 2.3(c), (3.7) is satisfied. By the definition of a_i, (3.11) is satisfied. From our choices of $\bar{\lambda}$, α, and δ_0 and (3.9) we get

$$p_0 \leqq c \exp(-\lambda^2 \varphi_n^2/(2+\varepsilon)\,\delta) \tag{3.13}$$

provided $a \leqq \alpha$. By Lemma 2.3(b), $H(\delta_{i+1}) \leqq G(\delta_{i+1}) \leqq 2G(\delta_i)$. Since $\lambda_i^2 \varphi_n^2/\delta_i = G(\delta_i)\,\beta^2 \varphi_n^2 \geqq 64 G(\delta_i)$ for n sufficiently large, we have

$$\begin{aligned} r_i &\leqq 4 \exp(-\lambda_i^2 \varphi_n^2/8\delta_i + 4G(\delta_i)) \\ &\leqq 4 \exp(-G(\delta_i)\,\beta^2 \varphi_n^2/16) \\ &\leqq 4 \exp(-\delta_i^{-\frac{1}{4}} \lambda^2 \varphi_n^2/2\delta) \end{aligned}$$

for n sufficiently large, uniformly in i. By Lemma 2.3(d) with $\zeta = \exp(-\lambda^2 \varphi_n^2/2\delta)$, we have

$$\sum_{i=0}^{\infty} r_i \leqq c \exp(-\lambda^2 \varphi_n^2/2\delta). \tag{3.14}$$

By our choice of δ_i and λ_i, $a_{i-1}/a_i = 2^{\frac{1}{2}}$, so

$$\begin{aligned} s_i &\leqq 2 \exp(-\lambda_i^2 \varphi_n^2/30\delta_i + 2G(\delta_i)) \\ &\leqq 2 \exp(-G(\delta_i)\,\beta^2 \varphi_n^2/60) \\ &\leqq 2 \exp(-\delta_i^{-\frac{1}{4}} \lambda^2 \varphi_n^2/2\delta) \end{aligned}$$

for n sufficiently large, uniformly in i. By Lemma 2.3(d) again,

$$\sum_{i=0}^{\infty} s_i \leqq c \exp(-\lambda^2 \varphi_n^2/2\delta). \tag{3.15}$$

Substituting (3.13), (3.14), and (3.15) in (3.8) proves the theorem. □

4. Relative Compactness

We now proceed to prove part of Theorem 2.1, that Z_n/φ_n has limit points, all contained in $\mathscr{K}$. We separate out some preliminary propositions for convenience. First we have

Proposition 4.1. *For each* a,

$$\|Z_n(\cdot, a, \infty)/\varphi_n\|_{\mathscr{A}} \to 0 \text{ a.s.} \quad \textit{as } n\to\infty.$$

Proof. Since for each A,

$$|Z_n(A, a, \infty)| \leqq U_n(I^d, a, \infty) + EU_n(I^d, a, \infty),$$

it suffices to show

$$U_n(I^d, a, \infty)/\varphi_n \to 0, \text{ a.s.} \tag{4.1}$$

and

$$EU_n(I^d, a, \infty)/\varphi_n \to 0. \tag{4.2}$$

If $\jmath = (j_1, \ldots, j_d)$, let $\mu(\jmath) = \max(j_1, \ldots, j_d)$. Given i and a, let

$$\psi(i, a) = \sup\{k \text{ integer: } a\,k^{d/2}/\varphi_k < i+1\}.$$

Since the number of $\jmath$'s in $\{1, 2, \ldots\}^d$ with $\mu(\jmath) = k$ is less than or equal to $c\,k^{d-1}$ for a constant c, we have

$$\begin{aligned}\text{(4.3)}\quad \sum_{\jmath} 1_{(i+1 > a\mu(\jmath)^{d/2}/\varphi_{\mu(\jmath)})}(\mu(\jmath)^{d/2}\varphi_{\mu(\jmath)})^{-1} &= \sum_k \sum_{\{j: \mu(\jmath)=k\}} 1_{(i+1 > ak^{d/2}/\varphi_k)}(k^{d/2}\varphi_k)^{-1} \\ &\leqq c \sum_k 1_{(i+1 > ak^{d/2}/\varphi_k)}(k^{d/2-1}/\varphi_k) \\ &\leqq c\,\psi(i, a)^{d/2}/\varphi_{\psi(i,a)} \\ &\leqq c(i+1)/a.\end{aligned}$$

We see from this that

$$\begin{aligned}\text{(4.4)}\quad & \sum_{\jmath} E|X_{\jmath}(\mu(\jmath), a, \infty)|/\mu(\jmath)^{d/2}\varphi_{\mu(\jmath)} \\ &\leqq \sum_{\jmath} \sum_{i+1 > a\mu(\jmath)^{d/2}/\varphi_{\mu(\jmath)}} (i+1)\,P(i \leqq |X_{\jmath}| \leqq i+1)\,(\mu(\jmath)^{d/2}\varphi_{\mu(\jmath)})^{-1} \\ &\leqq \sum_{i=0}^{\infty} \Big[\sum_{\jmath} 1_{(i+1 > a\mu(\jmath)^{d/2}/\varphi_{\mu(\jmath)})}(\mu(\jmath)^{d/2}\varphi_{\mu(\jmath)})^{-1}\Big](i+1)\,P(i \leqq |X_1| \leqq i+1) \\ &\leqq (c/a)\sum_{i=0}^{\infty}(i+1)^2 P(i \leqq |X_1| \leqq i+1) \leqq (c/a)\,E(|X_1|+1)^2 < \infty,\end{aligned}$$

where we use the fact that the $X_{\jmath}$'s are iid, $X_{\jmath}(\mu(\jmath), a, \infty)$ is defined by (3.1), and $1 = (1, 1, \ldots, 1)$.

An immediate consequence of (4.4) is that

$$\sum_{\jmath} |X_{\jmath}(\mu(\jmath), a, \infty)|\mu(\jmath)^{d/2}\varphi_{\mu(\jmath)} < \infty, \text{ a.s.} \tag{4.5}$$

Given ε, pick $n_1(\omega)$ such that

$$\sum_{\mu(\jmath) > n_1} |X_{\jmath}(\mu(\jmath), a, \infty)|/\mu(\jmath)^{d/2}\varphi_{\mu(\jmath)} < \varepsilon.$$

Then

$$U_n(I^d, a, \infty)/\varphi_n \leqq \sum_{\mu(j)\leqq n_1} |X_j(n, a, \infty)|/n^{d/2}\,\varphi_n + \sum_{n\geqq \mu(j)>n_1} |X_j(\mu(j), a, \infty)|/\mu(j)^{d/2}\,\varphi_{\mu(j)},$$

or $\limsup_{n\to\infty} U_n(I^d, a, \infty)/\varphi_n \leqq 2\varepsilon$. Since ε is arbitrary, this proves (4.1).

Just as (4.1) follows from (4.5), we see that (4.2) follows from (4.4), and the proof is complete. □

For notational convenience, given m and $k \in \{1, \ldots, m\}^d$, let

$$Q_{km} = m^{-1} R_k. \tag{4.6}$$

If n is a multiple of m, let

$$Y_{nm}(A, a, b) = \sum_{k\in\{1,\ldots,m\}^d} \frac{|A \cap Q_{km}|}{|Q_{km}|} Z_n(Q_{km}, a, b). \tag{4.7}$$

Proposition 4.2. *Given $1/8 > \varepsilon > 0$, m fixed, and n a multiple of m, there exists α such that if $a < \alpha$ and n is sufficiently large,*

$$P(Y_{nm}(\cdot, 0, a)/\varphi_n \notin \mathcal{K}^{4\varepsilon}) \leqq c \exp(-(1+\varepsilon)\varphi_n^2/2).$$

Proof. If we let

$$y_{nm}(t) = \sum_{k\in\{1,\ldots,m\}^d} |Q_{km}|^{-1} Z_n(Q_{km}, 0, a)\, 1_{Q_{km}}(t),$$

we see that for all A

$$Y_{nm}(A, 0, a) = \int_A y_{nm}(t)\, dt.$$

So to show $Y_{nm}(\cdot, 0, a)/\varphi_n \in \mathcal{K}^{4\varepsilon}$, we need to consider

$$\varphi_n^{-2} \int_{I^d} (y_{nm}(t))^2\, dt = \varphi_n^{-2} \sum_{k\in\{1,\ldots,m\}^d} |Z_n(Q_{km}, 0, a)|^2/|Q_{km}|. \tag{4.8}$$

Let

$$T_{nk} = |Z_n(Q_{km}, 0, a)|^2/|Q_{km}|,$$

and let

$$\bar{T}_{nk} = \begin{cases} T_{nk} & \text{if } T_{nk} \leqq (1+2\varepsilon)\varphi_n^2 \\ 0 & \text{otherwise.} \end{cases}$$

By Bernstein's inequality,

$$\begin{aligned} P(\bar{T}_{nk} \neq T_{nk} \text{ for some } k) &\leqq m^d P(T_{n1} \geqq (1+2\varepsilon)\varphi_n^2) \\ &= m^d P(|Z_n(Q_{1m}, 0, a)| \geqq (1+2\varepsilon)^{\frac12} \varphi_n |Q_{1m}|^{\frac12}) \\ &\leqq 2m^d \exp\left(-\frac{(1+2\varepsilon)\varphi_n^2 |Q_{1m}|}{2|Q_{1m}| + 2(2a/\varphi_n)((1+2\varepsilon)^{\frac12}\varphi_n |Q_{1m}|^{\frac12})/3}\right) \\ &\leqq 2m^d \exp(-(1+\varepsilon)\varphi_n^2/2) \end{aligned} \tag{4.9}$$

if a is sufficiently small, $1 = (1, \ldots, 1)$.

By the same argument, if $x \leqq 2\varphi_n^2$ and a is sufficiently small,

$$P(\bar{T}_{n\ell} > x) \leqq P(T_{n\ell} > x) \leqq 2\exp\left(\frac{-x|Q_{1m}|}{2|Q_{1m}| + 2(2a/\varphi_n)x^{\frac{1}{2}}|Q_{1m}|^{\frac{1}{2}}/3}\right)$$
$$\leqq 2\exp(-x(1-\varepsilon)/2),$$

and so

$$E\,e^{u\bar{T}_{n\ell}} = u \int_0^{(1+2\varepsilon)\varphi_n^2} e^{ux} P(\bar{T}_{n\ell} > x)\,dx \tag{4.10}$$
$$\leqq 2u \int_0^{(1+2\varepsilon)\varphi_n^2} e^{ux} e^{-x(1-\varepsilon)/2}\,dx$$
$$\leqq 2/\varepsilon < \infty$$

if $u = (1-2\varepsilon)/2$.

Since the $\bar{T}_{n\ell}$ are independent (recall n is a multiple of m), from (4.10) we get

$$P\Big(\sum_{\ell\in\{1,\ldots,m\}^d} \bar{T}_{n\ell} \geqq (1+4\varepsilon)\varphi_n^2\Big) \leqq \exp(-u(1+4\varepsilon)\varphi_n^2)\,E\exp\Big(u\sum_{\ell}\bar{T}_{n\ell}\Big) \tag{4.11}$$
$$= \exp(-u(1+4\varepsilon)\varphi_n^2)\prod_{\ell} E\exp(u\,\bar{T}_{n\ell})$$
$$\leqq c\exp(-(1+\varepsilon)\varphi_n^2/2),$$

and hence

$$P\Big(\sum_{\ell\in\{1,\ldots,m\}^d} T_{n\ell} \geqq (1+4\varepsilon)\varphi_n^2\Big) \leqq P(T_{n\ell} \neq \bar{T}_{n\ell} \text{ for some } \ell) \tag{4.12}$$
$$+ P\Big(\sum_{\ell}\bar{T}_{n\ell} \geqq (1+4\varepsilon)\varphi_n^2\Big)$$
$$\leqq c\exp(-(1+\varepsilon)\varphi_n^2/2).$$

By (2.6), $rF\in\mathscr{K}$ implies $\|F - rF\|_{\mathscr{A}} = (1-r)\|rF\|_{\mathscr{A}}/r \leqq (1-r)/r$, which implies $F\in\mathscr{K}^{(1-r)/r}$. By (4.8), if $\sum_{\ell} T_{n\ell}(\omega) \leqq (1+4\varepsilon)\varphi_n^2$, then $(1+4\varepsilon)^{-\frac{1}{2}} Y_{nm}(\cdot, 0, a)(\omega)/\varphi_n \in \mathscr{K}^{4\varepsilon}$, and so, letting $r = (1+4\varepsilon)^{-\frac{1}{2}}$, $Y_{nm}(\cdot, 0, a)(\omega)/\varphi_n \in \mathscr{K}^{4\varepsilon}$. This with (4.12) proves the proposition. □

We need one more preliminary proposition.

Proposition 4.3. *Suppose A is fixed. Given $\eta > 0$, there exists m_0 such that if $m \geqq m_0$, n is a multiple of m, $a > 0$, then*

$$\operatorname{Var}(Y_{nm}(A,0,a) - Z_n(A,0,a)) \leqq \eta.$$

Moreover, $Y_{nm}(A,0,a) - Z_n(A,0,a)$ is the sum of independent mean 0 random variables, each bounded by $2a/\varphi_n$.

Proof. By the definition of Y_{nm} (see (4.7)),

(4.13)
$$Y_{nm}(A,0,a)-Z_n(A,0,a)=\sum_{k\in\{1,\ldots,m\}^d}\left\{\frac{|A\cap Q_{km}|}{|Q_{km}|}Z_n(Q_{km},0,a)-Z_n(A\cap Q_{km},0,a)\right\}$$
$$=n^{-d/2}\sum_{k}\sum_{j\in nQ_{km}}\left(\frac{|A\cap Q_{km}|}{|Q_{km}|}-|n(A\cap Q_{km})\cap R_j|\right)$$
$$\cdot(X_j(n,0,a)-EX_j(n,0,a)).$$

Since $||A\cap Q_{km}|/|Q_{km}|-|n(A\cap Q_{km})\cap R_j||\leqq 1$, the second part of the proposition is clear. From (4.13),

(4.14)
$$\operatorname{Var}(Y_{nm}(A,0,a)-Z_n(A,0,a))$$
$$\leqq n^{-d}\sum_{k\in\{1,\ldots,m\}^d}\sum_{j\in nQ_{km}}(|A\cap Q_{km}|/|Q_{km}|-|n(A\cap Q_{km})\cap R_j|)^2$$
$$=n^{-d}\sum_{k}\sum_{j\in nQ_{km}}(|n(A\cap Q_{km})\cap R_j|^2-(|A\cap Q_{km}|/|Q_{km}|)^2)$$
$$\leqq\sum_{k}(|A\cap Q_{km}|-|A\cap Q_{km}|^2/|Q_{km}|)$$
$$=\sum_{k}|A\cap Q_{km}|\,|A^c\cap Q_{km}|/|Q_{km}|,$$

since $\sum_{j\in nQ_{km}}|n(A\cap Q_{km})\cap R_j|=n^d|A\cap Q_{km}|$.

If we let

$$h_m(B,t)=\sum_{k\in\{1,\ldots,m\}^d}\frac{|B\cap Q_{km}|}{|Q_{km}|}1_{Q_{km}}(t),$$

by the Lebesgue density theorem, $h_m(B,t)\to 1_B(t)$ for almost all t as $m\to\infty$. So by dominated convergence and (4.14),

$$\operatorname{Var}(Y_{nm}(A,0,a)-Z_n(A,0,a))\leqq\int_{I^d}h_m(A,t)\,h_m(A^c,t)\,dt$$
$$\to\int_{I^d}1_A(t)\,1_{A^c}(t)\,dt=0.$$

Hence if m is sufficiently large, $\operatorname{Var}(Y_{nm}(A,0,a)-Z_n(A,0,a))\leqq\eta$. □

We now prove

Theorem 4.4. *For each* ε,

$$P(Z_n(\cdot)/\varphi_n\notin\mathcal{K}^{22\varepsilon}\text{ i.o.})=0.$$

Proof. Let $1/16>\varepsilon>0$. Pick $\delta<\varepsilon^2/2$. Pick m large enough so that if n is a multiple of m,

$$\operatorname{Var}(Y_{nm}(B,0,a)-Z_n(B,0,a))\leqq\varepsilon^2/4$$

for each $B\in\mathcal{A}(\delta)$; this is possible by Proposition 4.3. Then if n is a multiple of m,

$$(4.15)\quad \begin{aligned} P(Z_n(\cdot,0,a)/\varphi_n\notin\mathcal{K}^{15\varepsilon}) &\leqq P(\sup_{B\in\mathcal{A}(\delta)}|Z_n(B,0,a)-Y_{nm}(B,0,a)|>\varepsilon\varphi_n) \\ &+P(Y_{nm}(\cdot,0,a)/\varphi_n\notin\mathcal{K}^{4\varepsilon}) \\ &+P(\sup_{A\in\mathcal{A},\,B\in\mathcal{A}(\delta),\,|A\triangle B|<\delta}|Y_{nm}(A,0,a) \\ &-Y_{nm}(B,0,a)|>9\varepsilon\varphi_n,\ Y_{nm}(\cdot,0,a)/\varphi_n\in\mathcal{K}^{4\varepsilon}) \\ &+P(\sup_{\substack{A\in\mathcal{A},\,B\in\mathcal{A}(\delta)\\ |A\triangle B|\leqq\delta}}|Z_n(A,0,a)-Z_n(B,0,a)|>\varepsilon\varphi_n) \\ &=I_1+I_2+I_3+I_4. \end{aligned}$$

By Proposition 4.3 and Bernstein's inequality,

$$\begin{aligned} I_1 &\leqq \exp(H(\delta))\sup_{B\in\mathcal{A}(\delta)}P(|Z_n(B,0,a)-Y_{nm}(B,0,a)|>\varepsilon\varphi_n) \\ &\leqq c\exp(-\varepsilon^2\varphi_n^2/(2(\varepsilon^2/4)+2(2a/\varphi_n)(\varepsilon\varphi_n)/3)) \\ &\leqq c(\log n)^{-\rho} \end{aligned}$$

for some $\rho>1$, provided a is chosen small enough.

By Proposition 4.2,

$$I_2\leqq c\exp(-(1+\varepsilon)\varphi_n^2/2)\leqq c(\log n)^{-\rho}$$

for some $\rho>1$, provided a is chosen small enough.

By (2.7), if $Y_{nm}(\cdot,0,a)/\varphi_n\in\mathcal{K}^{4\varepsilon}$ and $|A\triangle B|<\delta$,

$$|Y_{nm}(A,0,a)-Y_{nm}(B,0,a)|/\varphi_n\leqq 8\varepsilon+\delta^{\frac{1}{2}}<9\varepsilon.$$

So $I_3=0$.

Finally, provided a is sufficiently small,

$$I_4\leqq c\exp(-\varepsilon^2\varphi_n^2/(2+\varepsilon)\,\delta)\leqq c(\log n)^{-\rho}$$

for some $\rho>1$ by Theorem 3.1.

Choose q sufficiently small so that $1\leqq q^d\leqq 1+\varepsilon$. If we take a sufficiently small and $n_k=m[q^k]$, where $[q^k]$ is the integer part of q^k, then substituting our bounds for I_1, I_2, I_3, and I_4 in (4.15) gives

$$\sum_k P(Z_{n_k}(\cdot,0,a)/\varphi_{n_k}\notin K^{15\varepsilon})\leqq c\sum_k k^{-\rho}<\infty.$$

By Borel-Cantelli,

$$P(Z_{n_k}(\cdot,0,a)/\varphi_{n_k}\notin\mathcal{K}^{15\varepsilon}\text{ i.o.})=0.$$

Since by Proposition 4.1, $\|Z_n(\cdot,a,\infty)/\varphi_n\|_{\mathcal{A}}\to 0$ a.s., it follows that $P(Z_{n_k}(\cdot)/\varphi_{n_k}\notin\mathcal{K}^{16\varepsilon}\text{ i.o.})=0$.

Given $F\in\mathcal{K}$ with $F(A)=\int_A f(t)\,dt$, define F_r, $0\leqq r\leqq 1$, by

$$(4.16)\qquad F_r(A)=\int_A f_r(t)\,dt,$$

where

$$(4.17)\qquad f_r(t)=r^d f(r\,t).$$

Note

$$F_r(A)=\int_A r^d f(r\,t)\,dt=\int_{rA} f(t)\,dt=F(r\,A)$$

and

$$\int_{I^d} f_r^2(t)\,dt=r^{2d}\int_{I^d} f^2(r\,t)\,dt=r^d\int_{rI^d} f^2(t)\,dt\leqq 1,$$

and so $F_r\in\mathscr{K}$.

Now fix ω, and then pick k large so that there exists $F\in\mathscr{K}$ with

$$\|Z_{n_{k+1}}(\cdot)/\varphi_{n_{k+1}}-F\|_{\mathscr{A}}\leqq 16\,\varepsilon$$

and $(n_{k+1}/n_k)^{d/2}\,\varphi_{n_{k+1}}/\varphi_{n_k}\leqq 1+3\varepsilon$. Suppose $n_k\leqq n\leqq n_{k+1}$. Let $r=n/n_{k+1}$. Then

$$n^{d/2}\,Z_n(A)=\sum_{j}|n\,A\cap R_j|\,X_j=\sum_{j}|n_{k+1}(r\,A)\cap R_j|\,X_j=n_{k+1}^{d/2}\,Z_{n_{k+1}}(r\,A).$$

Since $\mathscr{A}$ is contraction closed (see (2.8)), if $A\in\mathscr{A}$,

$$\begin{aligned}|Z_n(A)/\varphi_n-F_r(A)|&\leqq|Z_{n_{k+1}}(r\,A)/\varphi_{n_{k+1}}-F(r\,A)|\\&\quad+\left(\frac{\varphi_{n_{k+1}}}{\varphi_n}\left(\frac{n_{k+1}}{n}\right)^{d/2}-1\right)|Z_{n_{k+1}}(r\,A)/\varphi_{n_{k+1}}|\\&\leqq 16\varepsilon+3\varepsilon(1+16\varepsilon)\\&\leqq 22\varepsilon,\end{aligned}$$

using (2.7). So, for all k sufficiently large, $n_k\leqq n\leqq n_{k+1}$, $Z_n(\cdot)(\omega)/\varphi_n\in\mathscr{K}^{22\varepsilon}$. □

5. Limit Points

We now determine the limit points of Z_n/φ_n.

Theorem 5.1. *For each $F\in\mathscr{K}$ and each $1>\varepsilon>0$,*

$$P(\|Z_n(\cdot)/\varphi_n-F\|_{\mathscr{A}}<6\varepsilon\text{ i.o.})=1.$$

Proof. We first suppose that for some $\tau>0$, $F(A)=\int_A f(t)\,dt$ for all $A\in\mathscr{A}$, and

$$\int_{I^d} f^2(t)\,dt\leqq 1-\tau. \tag{5.1}$$

Since $Z_n/\varphi_n\in\mathscr{K}^{\varepsilon}$ for n large by Theorem 4.1 and $F\in\mathscr{K}$, it suffices by (2.7) to show

$$P(\max_{B\in\mathscr{A}(\delta)}|Z_n(B)/\varphi_n-F(B)|<4\varepsilon\text{ i.o.})=1 \tag{5.2}$$

for $\delta=\varepsilon^2/8$.

Given a set B, there exists a closed set C with $|B\triangle C|<\delta/2$. Since $|Q_{1m}|=m^{-d}$ and $O_m(C)\downarrow C$ as $m\to\infty$, where

$$O_m(C)=\bigcup_{\substack{Q_{\ell m}\cap C\neq\phi\\ \ell\in\{1,\dots,m\}^d}} Q_{\ell m},$$

the outer rectilinear fit of C by the cubes Q_{km} defined in (4.6), then if m is sufficiently large, there is a set $A=O_m(C)-Q_{1m}$ that is made up of the union of Q_{km}'s, $k \neq 1$, with $|A \triangle B|<\delta$.

We construct such a set A for each $B \in \mathscr{A}(\delta)$: label the sets of $\mathscr{A}(\delta)$ by $B_1, \ldots, B_q$; then we can find an integer m and finitely many sets $A_1, \ldots, A_q$ such that each A_i is made up of the union of Q_{km}'s, $k \neq 1$, and $|A_i \triangle B_i| \leqq \delta$, $i=1, \ldots, q$.

Let $n_l=m^l$. By Bernstein, for each $i=1, \ldots, q$

$$\begin{aligned}\sum_l P(|Z_{n_l}(A_i, 0, a)-Z_{n_l}(B_i, 0, a)|>\varepsilon \varphi_{n_l}) &\leqq \sum_l 2 \exp(-\varepsilon^2 \varphi_{n_l}^2/(2\delta+2(2a/\varphi_{n_l})(\varepsilon \varphi_{n_l})/3) \\ &\leqq \sum_l \exp(-2\varphi_{n_l}^2)<\infty,\end{aligned}$$

provided $a<\delta$. Using Proposition 4.1 and Borel-Cantelli,

$$P(|Z_{n_l}(A_i)-Z_{n_l}(B_i)|>2\varepsilon \varphi_{n_l} \text{ i.o.})=0,$$

and so to prove (5.2), it suffices to show

$$P(\max_{1 \leqq i \leqq q} |Z_{n_l}(A_i)/\varphi_{n_l}-F(A_i)|<\varepsilon \text{ i.o.})=1. \tag{5.3}$$

Now Z_{n_l} and F are finitely additive, and each A_i, $i=1, \ldots, q$, is the union of Q_{km}'s, $k \neq 1$. So to prove (5.3), it suffices to prove

$$P(\max_{k \in \{1, \ldots, m\}^d-\{1\}} |Z_{n_l}(Q_{km})/\varphi_{n_l}-F(Q_{km})|<\varepsilon/m^d \text{ i.o.})=1. \tag{5.4}$$

We now prove (5.4). Let $D=I^d-Q_{1m}$. Let

$$\Lambda_{kl}=(|Z_{n_l}(Q_{km})/\varphi_{n_l}-F(Q_{km})|<\varepsilon/m^d), \qquad k \in \{1, \ldots, m\}^d-\{1\},$$

and let

$$\Gamma_l=\bigcap_{k \in \{1, \ldots, m\}^d-\{1\}} \Lambda_{kl}.$$

Each $Z_{n_l}(Q_{km})$ is the sum of $s_l=n_l^d|Q_{km}|$ iid mean 0, variance 1 random variables, and so by Lemma 2.4 of [4] (his α_k is our $s_l^{\frac{1}{2}} \varphi_{s_l}$, his m_k our s_l), for each $\eta>0$,

$$P(\Lambda_{kl}) \geqq \exp\left(-\left(\frac{(F(Q_{km}))^2}{2}+\eta\right) \varphi_{n_l}^2\right)$$

for l sufficiently large. By the proof of (2.6), $|F(Q_{km})| \leqq (1-\tau)^{\frac{1}{2}} |Q_{km}|^{\frac{1}{2}}$, and so

$$\sum_{k \in \{1, \ldots, m\}^d} (F(Q_{km}))^2 \leqq 1-\tau.$$

Since the $Z_{n_l}(Q_{k m})$, $k \in \{1, \ldots, m\}^d$ are independent,

$$P(\Gamma_l) = \prod_{k \in \{1, \ldots, m\}^d - \{1\}} P(A_{kl}) \geqq \exp(-((1-\tau)/2 + \eta m^d)\varphi_{n_l}^2)$$
$$= c(\log n_l)^{-\rho} = c\, l^{-\rho}$$

for some $\rho < 1$, provided η is sufficiently small and l sufficiently large.

Now Γ_l depends only on X_j's with $j \in m^l D$, and since D, mD, $m^2 D, \ldots$ are disjoint, the Γ_l are independent. Then by Borel-Cantelli,

$$P(\Gamma_l \text{ i.o.}) = 1,$$

which is precisely (5.4).

Finally, if F is any element of $\mathscr{K}$, $\|(1-\tau)F - F\|_{\mathscr{A}} \to 0$ as $\tau \to 0$, while

$$(1-\tau)F(A) = \int_A (1-\tau) f(t)\, dt.$$

Hence $(1-\tau)F$ is an element of $\mathscr{K}$ that satisfies (5.1). Since τ is arbitrary, the theorem follows. □

Proof of Theorem 2.1. This is immediate from Theorems 4.1 and 5.1. □

6. Central Limit Theorem

Let $\mathscr{C}(\mathscr{A})$ be the uniformly continuous functions on $\mathscr{A}$ with norm $\|\cdot\|_{\mathscr{A}}$. Let $W(\cdot)$ be Brownian motion indexed by $\mathscr{A}$: a mean 0 Gaussian process with sample paths in $\mathscr{C}(\mathscr{A})$ and $\operatorname{Cov}(W(A), W(B)) = |A \triangle B|$. In this section we show how to prove that Z_n converges weakly to W in $\mathscr{C}(\mathscr{A})$.

Since enlarging $\mathscr{A}$ just makes a stronger theorem, we may assume without loss of generality that $\mathscr{A}$ is closed with respect to $d(A,B) = |A \triangle B|$ and that $\mathscr{A}$ still satisfies (2.2), (2.3), and (2.8). One can show $\mathscr{A}$ is complete ($A \to 1_A$ maps $(\mathscr{A}, d(\cdot,\cdot))$ isometrically into a closed subset of $L_1(dt)$). With the total boundedness of $\mathscr{A}$, we conclude $\mathscr{A}$ is compact.

One can show that the finite dimensional distributions of Z_n converges to those of W; for an elegant proof of this, see [1]. Since $\mathscr{C}(\mathscr{A})$ is a complete and separable metric space, Z_n will converge weakly to W provided the Z_n's are tight, by Prohorov's theorem. Recalling that $\phi \in \mathscr{A}$ by (2.8) and that $Z_n(\phi) = 0$ for each n by (2.1), to show tightness it is easy to see that it suffices to show: given $\eta > 0$, for each k, there exists $\delta(k)$ such that

$$\sup_n P\Big(\sup_{B, C \in \mathscr{A},\, |B \triangle C| \leqq \delta(k)} |Z_n(B) - Z_n(C)| > 2^{-k}\Big) \leqq \eta\, 2^{-k}.$$

We accomplish this by proving

Theorem 6.1. *Given* $1 > \lambda$, $\varepsilon > 0$, *there exists* τ *such that*

$$\sup_n P\Big(\sup_{B, C \in \mathscr{A},\, |B \triangle C| \leqq \tau} |Z_n(B) - Z_n(C)| > 3\lambda\Big) \leqq 4\varepsilon.$$

Proof. We will use the notation of Sect. 3, except that for this proof we take $\varphi_n \equiv 1$. With this change, we follow the proof of Theorem 3.1 up through statement (3.12) word for word, but we now proceed to choose the δ_i's, λ_i's, etc. somewhat differently.

Let $G(x)$ be a continuous, strictly decreasing function such that $G(x) \geqq x^{-\frac{3}{4}}$, $H(x/2) = o(G(x))$ as $x \to 0$, and $\gamma = \int_0^1 (G(x)/x)^{\frac{1}{2}} dx < \infty$. Let $\bar{\lambda} = \lambda/2$, $\beta = \lambda/40\gamma$. Let M be an integer $\geqq 1$ chosen large enough so that $\exp(-M) \leqq \varepsilon/60(\gamma+1)$. Choose $\delta = \delta_0 < 1$ sufficiently small so that $\delta^{-\frac{1}{4}}\beta^2/60 > M$, $G(x)\beta^2 \geqq 120 H(x/2)$ for all $x \leqq \delta$, $\lambda^2/56\delta > M$, and $\delta H(\delta) \leqq \lambda^2/112$. Define δ_i by $G(\delta_i)/\delta_i = (G(\delta_0)/\delta_0)\, 2^i$, $\lambda_i = (\delta_i G(\delta_i))^{\frac{1}{2}} \beta$, $a_i = 3\delta_i/\lambda_i$, and $a = \alpha = \min(a_0/2, \delta(1+\lambda)^{-1})$.

As before, $a < a_0$ and (3.7) and (3.11) are satisfied. From (3.9),

$$(6.1) \qquad p_0 \leqq 2 \exp\left(\frac{-\lambda^2 + 56\delta H(\delta)}{28\delta}\right) \leqq 2\exp(-\lambda^2/56\delta) < \varepsilon.$$

By Lemma 2.3(a), $\delta_{i+1} \geqq \delta_i/2$, and so $H(\delta_{i+1}) \leqq H(\delta_i/2)$. From (3.10),

$$\begin{aligned} r_i &\leqq 4\exp(-\lambda_i^2/8\delta_i - 2H(\delta_i/2)) \\ &\leqq 4\exp(-G(\delta_i)\beta^2/16) \\ &\leqq 4\exp(-M\delta_i^{-\frac{1}{2}}), \end{aligned}$$

and by Lemma 2.3(d) with $\zeta = \exp(-M)$,

$$(6.2) \qquad \sum_{i=0}^{\infty} r_i \leqq \varepsilon.$$

Similarly, from (3.12),

$$s_i \leqq 2\exp(-M\delta_i^{-\frac{1}{2}}),$$

and

$$(6.3) \qquad \sum_{i=1}^{\infty} s_i \leqq \varepsilon.$$

Substituting (6.1), (6.2), and (6.3) in (3.8), for the above choices of δ and a,

$$P(\sup_{B, C \in \mathscr{A}, |B \triangle C| \leqq \delta} |Z_n(B, 0, a) - Z_n(C, 0, a)| \geqq \lambda) \leqq 3\varepsilon.$$

Taking $\varphi_n \equiv 1$, the proof of Proposition 4.1 shows that

$$P(\|Z_n(\cdot, a, \infty)\|_{\mathscr{A}} > \lambda) \to 0 \quad \text{as } n \to \infty,$$

and so for $n \geqq n_1$ for some n_1 sufficiently large,

$$P(\sup_{B, C \in \mathscr{A}, |B \triangle C| \leqq \delta} |Z_n(B) - Z_n(C)| > 3\lambda) \leqq 4\varepsilon.$$

Since Z_n is uniformly continuous for $n \leqq n_1$ and $\sup_{B, C \in \mathscr{A}, |B \triangle C| \leqq \tau} |Z_n(B) - Z_n(C)|$ decreases as τ decreases, taking τ sufficiently small and $\leqq \delta$ proves the theorem. □

Acknowledgment. I would like to thank K. Alexander and R. Pyke for helpful discussions concerning their paper [1].

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Received July 10, 1984