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Probability Theory and Related Fields

Local Times for Brownian Motion on the Sierpinski Carpet

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Summary. Jointly continuous local times are constructed for Brownian motion on the Sierpinski carpet. A consequence is that the Brownian motion hits points. The method used is to analyze a sequence of eigenvalue problems.

1. Introduction

The Sierpinski carpet is the fractal formed by dividing the unit square into 9 equal squares, removing the central one, dividing each of the 8 smaller squares into 9 equal pieces, and continuing the process. In [1] Brownian motions on the Sierpinski carpet were constructed. These are strong Markov processes with continuous paths whose state space is the Sierpinski carpet. The justification for the name "Brownian motion" is that the processes are continuous and are invariant under the appropriate class of translations, rotations, and reflections. We refer to Brownian motion *s* because there is at present no uniqueness result available, and it is conceivable that there might be more than one such process.

In this paper we study some of the properties of these processes. We prove that the Brownian motions have jointly continuous local times. A consequence is that these processes hit points. We concentrate exclusively on the Sierpinski carpet in this paper, but our methods and results apply equally well to the other fractals considered in [1].

In [3] local times were constructed for Brownian motion on the Sierpinski gasket. However the construction in [3] relied heavily on the fact that the Sierpinski gasket is a finitely ramified fractal, i.e., it can be disconnected by removing finitely many points. The Sierpinski carpet, however, is infinitely ramified, and quite different techniques are necessary.

For other work on diffusions on fractals, see [11, 13, 14], and see [12] for a survey of the vast physics literature on random walks on fractals. We found [8] useful as a guide to our intuition.

The idea of our method is this: let F_n be the nth stage of the construction of the Sierpinski carpet, and let W_t^n be Brownian motion in F_n with normal reflection at

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the boundaries. In Sect. 2 we tie together the lifetime of W_t^n with the first eigenvalue λ_n of the Laplacian on F_n with appropriate boundary conditions. By an analysis of the eigenvalue problems, we show that λ_n is nearly decreasing in n. In Sect. 3 we use a scaling argument to show that each of the Brownian motions constructed in [1] has a bounded Hölder continuous Green function. Then in Sect. 4 we construct our local times and obtain the point recurrence by use of a combination of potential theory and stochastic calculus.

The letter c will denote constants whose value is unimportant and may change from line to line. Let $B_\varepsilon(x)$ be the ball of radius ε about x. Other notation will be introduced as needed.

2. Eigenvalues

Let $F_0 = [0,1]^2$, $F_1 = [0,1]^2 - (1/3, 2/3)^2$, and let F_n be the nth stage of the construction of the Sierpinski carpet. Let μ_n be Lebesgue measure on F_n normalized so that $\mu_n(F_n) = 1$. Let $\partial_a F_n = \{(x_1, x_2) \in [0,1]^2 : x_1 = 1 \text{ or } x_2 = 1\}$ (a stands for absorbing); let $\partial_r F_n = \partial F_n - \partial_a F_n$ (r stands for reflecting).

Define W_t^n to be Brownian motion on F_n with absorption on $\partial_a F_n$ and normal reflection on $\partial_r F_n$. (There are a number of equivalent ways to define W_t^n starting at a corner of one of the removed squares; we choose to do it by conformal mapping: see the proof of Lemma 2.1.) Let

$$\tau_n = \inf\{t : W_t^n \in \partial_a F_n\} . \tag{2.1}$$

Let

$$\alpha_n = \sup_{x \in F_n} E^x \tau_n , \quad \beta_n = \inf_{x \in F_n \cap [0, 1/2]^2} E^x \tau_n . \tag{2.2}$$

Define

$$\lambda_n = \inf\left\{ \int_{F_n} |\nabla u|^2(x)\mu_n(dx) \Big/ \int u^2(x)\mu_n(dx) : u = 0 \text{ on } \partial_a F_n \right\} . \tag{2.3}$$

Then, as is well-known [6, Chap. 6], the inf is attained for some function v_n that is nonnegative, bounded, and continuous on F_n, solves the equation $\Delta v_n = -\lambda_n v_n$ in the interior of F_n, and has zero normal derivative a.e. on $\partial_r F_n$ with respect to surface measure on $\partial_r F_n$. If $U_n f(x)$ is defined to be $E^x \int_0^{\tau_n} f(W_t^n)dt$ for f bounded, then

$$U_n v_n = 2\lambda_n^{-1} v_n . \tag{2.4}$$

One advantage of expressing the smallest eigenvalue λ_n in terms of a variational problem such as (2.3) is that v_n automatically has Neumann boundary conditions on $\partial_r F_n$.

We want to show $\lambda_n^{-1} \approx \alpha_n$. First, define

$$h_n(x) = E^x \tau_n .$$

Lemma 2.1. $\int |\nabla h_n|^2(x)\mu_n(dx) = 2\int h_n(x)\mu_n(dx)$.

Proof. Since $h_n = U_n 1$, then $\frac{1}{2}\Delta h_n = -1$. Trivially $h_n(x) \leqq \alpha_n$.
Suppose we had proved:

(i) $\nabla h_n \in L^2(F_n)$ (with respect to μ_n); and

(ii) $\nabla h_n \in L^1(\partial_r F_n)$, $\partial h/\partial v = 0$, a.e. (with respect to surface measure on $\partial_r F_n$), where $\partial h/\partial v$ denotes normal derivative. (2.5)

Applying Green's first identity to a C^∞ function g on F_n, we get

$$\int_{F_n} |\nabla g|^2 = -\int_{F_n} g\Delta g + \int_{\partial F_n} \frac{\partial g}{\partial v} g\,. \tag{2.6}$$

Taking a sequence g_m of such functions such that $g_m \to h_n$ uniformly on F_n, $\Delta g_m \to -2$ boundedly pointwise, $\nabla g_m \to \nabla h_n$ in $L^2(F_n)$ and in $L^1(\partial F_n)$, and $\partial g_m/\partial v \to 0$ in $L^1(\partial F_n)$, we get our result. So it suffices to prove (2.5).

By the usual properties of Brownian motion in a domain, h_n is C^∞ in the interior of F_n and on $\partial_a F_n - \{(1, 1)\}$. Suppose $x_0 \in \partial_r F_n$ but x_0 is not the corner of a removed square nor the origin. Take ε sufficiently small so that $B_\varepsilon(x_0) \cap F_n$ is a semicircle, and let $S = \inf\{t : |W_t^n - x_0| \geqq \varepsilon\}$. Let $h_n'(x) = E^x S$, $h_n''(x) = E^x h_n(W_S^n)$. If $x \in B_n(x_0) \cap F_n$, the strong Markov property gives

$$h_n(x) = E^x S + E^x E^{W_S^n}\tau_n = h_n'(x) + h_n''(x)\,. \tag{2.7}$$

Now h_n' is the time for reflecting Brownian motion to leave a semicircle, or equivalently, for Brownian motion to leave a circle. As is well-known [9, Sect. 1.11], this is C^∞ in the interior of the circle, and by symmetry, $\partial h_n'(x_0)/\partial v = 0$.

Also, $h_n''(x)$ is harmonic for reflecting Brownian motion in a semicircle. By the Schwartz reflection principle, h_n'' can be extended to a harmonic function in $B_\varepsilon(x_0)$. Hence h_n'' is C^∞ in the interior, and by symmetry, $\partial h_n''(x_0)/\partial v = 0$.

Next suppose $x_0 \in \partial_r F_n$ is the corner of one of the removed squares, and choose ε small enough so that $B_\varepsilon(x_0) \cap F_n$ is a 3/4-circle. Define S as before, and again write $h_n = h_n' + h_n''$. Let us introduce complex coordinates so that $x_0 = 0$ and $B_\varepsilon(x_0) \cap F_n = \{z = (r, \theta) : 0 \leqq r \leqq \varepsilon, 0 \leqq \theta \leqq 3\pi/2\}$. Using the conformal mapping $z \to z^{2/3}$, the domain $B_\varepsilon(x_0) \cap F_n$ gets mapped into $A = \{(r, \theta) : 0 \leqq r \leqq \varepsilon^{2/3}, 0 \leqq \theta \leqq \pi\}$. By Lévy's theorem [9, Sect. 5.1] $(W_t^n)^{2/3}$ is the time change of Brownian motion in the interior of A. Since the mapping is conformal, angles are preserved, so normal reflection is preserved, hence $(W_t^n)^{2/3}$ is the time change of reflecting Brownian motion in A. Sorting out the time change, we get

$$h_n'(z) = E^{z^{2/3}} \int_0^T |W_t|^{-2/3}\,dt = \int_{B_{\varepsilon^{2/3}}(0)} g(z^{2/3}, x)|x|^{-2/3}\,dx\,,$$

where W_t is standard Brownian motion, $T = \inf\{t : |W_t| \geqq \varepsilon^{2/3}\}$, and g is the Green function for the ball of radius $\varepsilon^{2/3}$. The function $|x|^{-2/3}$ is in $L^1(B_{\varepsilon^{2/3}}(0))$, and known estimates [10, Sect. 4.2] imply that $\int_{B_{\varepsilon^{2/3}}(0)} g(z, x)|x|^{-2/3}\,dx$ is C^1 in the interior of $B_{\varepsilon^{2/3}}(0)$. It follows that $\nabla h'(z) \in L^2(B_\varepsilon(x_0) \cap F_n) \cap L^1(B_\varepsilon(x_0) \cap \partial_r F_n)$. A similar argument holds for h_n''.

The origin and (1, 1) are treated in the same manner. Thus (2.5) (i, ii) holds, and the lemma is proved. □

Remark. Note that the only way the geometry of F_n enters the proof of Lemma 2.1 is through the fact that ∂F_n is piecewise linear.

Proposition 2.2. *There exists c_1 independent of n such that*

$$c_1 \alpha_n \leqq \lambda_n^{-1} \leqq \alpha_n/2 .$$

Proof. Let us normalize v_n so that $\sup_x v_n(x) = 1$. Pick x_0 so that $v_n(x_0) = 1$. Then

$$1 = v_n(x_0) = \tfrac{1}{2}\lambda_n U_n v_n(x_0) \leqq \tfrac{1}{2}\lambda_n U_n 1(x_0) \leqq \tfrac{1}{2}\lambda_n \sup_x h_n(x) = \tfrac{1}{2}\lambda_n \alpha_n .$$

To get the other inequality, using Lemma 2.1 observe that

$$\lambda_n \leqq \int_{F_n} |\nabla h_n|^2 \Big/ \int_{F_n} h_n^2 = 2 \int_{F_n} h_n \Big/ \int_{F_n} h_n^2 .$$

But $h_n(x) \geqq \beta_n$ on $F_n \cap [0, 1/2]^2$, hence $\int_{F_n} h_n^2 \geqq \frac{1}{4}\beta_n^2$. So

$$\lambda_n \leqq 8\alpha_n/\beta_n^2 .$$

And by [1, Prop. 4.2], there exists c independent of n such that

$$c\alpha_n \leqq \beta_n \leqq \alpha_n . \quad \square \tag{2.8}$$

We now replace h_n by a function with more manageable boundary values. Let

$$g_n(x_1, x_2) = \beta_n^{-1} h_n(1 - x_1, x_2) \wedge 1 \quad \text{for} \quad x_2 \leqq 1/2 , \tag{2.9}$$

and define $g_n(x_1, x_2) = g_n(x_1, 1 - x_2)$ for $x_2 \geqq 1/2$. Since $h_n(0, x_2) \geqq \beta_n$ if $x_2 \leqq 1/2$ and $h_n(1, x_2) = 0$, we get

$$g_n(0, x_2) \equiv 0 , \quad g_n(1, x_2) \equiv 1 . \tag{2.10}$$

We next consider the variational problem

$$\rho_n = \inf\left\{ \int_{F_n} |\nabla u|^2(x)\mu_n(dx) : u(0, x_2) \equiv 0, u(1, x_2) \equiv 1 \right\} . \tag{2.11}$$

Let f_n be the function at which the minimum is attained. We do not need this fact, but one can show using calculus of variations that

$$f_n(x) = P^x(Z_t^n \text{ hits} \{1\} \times [0, 1] \text{ before hitting } \{0\} \times [0, 1]) ,$$

and ρ_n is the energy of the harmonic function f_n, where Z_t^n is Brownian motion on F_n with normal reflection on ∂F_n.

Proposition 2.3. *There exist constants c_2 and c_3 independent of n such that $c_2 \lambda_n \leqq \rho_n \leqq c_3 \lambda_n$.*

Proof. Using Lemma 2.1, (2.10), and the fact that g_n satisfies the constraints of the variational problem (2.11),

$$\rho_n \leqq \int_{F_n} |\nabla g_n|^2 = 2 \int_{F_n \cap [0,1]\times[0,1/2]} |\nabla(1 \wedge \beta_n^{-1} h_n)|^2$$

$$\leqq 2\beta_n^{-2} \int_{F_n} |\nabla h_n|^2 = 4\beta_n^{-2} \int_{F_n} h_n \leqq 4\alpha_n/\beta_n^2 .$$

Applying (2.8) gives the second inequality.

To get the first, define ψ_n on $F_n \cap \{(x_1, x_2): x_2 \leqq x_1\}$ by

$$\psi_n(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \leqq 1/3 \\ f_{n-1}(2 - 3x_1, 3x_2) & \text{if } 1/3 \leqq x_1 \leqq 2/3 \\ 0 & \text{if } 2/3 \leqq x_1 , \end{cases}$$

and define ψ_n on $F_n \cap \{(x_1, x_2): x_2 > x_1\}$ by symmetry about the line $x_1 = x_2$. Since $\psi_n = 1$ on $[0, 1/3]^2$, then $\int_{F_n} \psi_n^2 \geqq 1/8$. Using scaling,

$$\int_{F_n} |\nabla\psi_n|^2 \leqq 2 \cdot \frac{9}{8} \int_{F_{n-1}} |\nabla f_{n-1}|^2 = \frac{9}{4}\rho_{n-1} .$$

Hence

$$\lambda_n \leqq 18\rho_{n-1} . \tag{2.12}$$

By [1, Prop. 4.2] there exists constants c_4 and c_5 such that

$$c_4\alpha_n \leqq \alpha_{n-1} \leqq c_5\alpha_n . \tag{2.13}$$

Then by Prop. 2.2,

$$\lambda_{n-1} \leqq c\alpha_{n-1}^{-1} \leqq c'\alpha_n^{-1} \leqq c''\lambda_n . \tag{2.14}$$

Combining (2.12) and (2.14) gives the first inequality. □

The main result of this section is

Theorem 2.4. $\rho_n \leqq \frac{27}{28}\rho_{n-1}$.

Proof. Let $a = 2/7$, and let

$$e_n(x_1, x_2) = \begin{cases} af_{n-1}(3x_1, 3x_2 - [3x_2]) & \text{if } 0 \leqq x_2 \leqq 1/3 \\ a + (1-2a)f_{n-1}(3x_1 - 1, 3x_2 - [3x_2]) & \text{if } 1/3 \leqq x_1 \leqq 2/3 \\ 1 - a + af_{n-1}(3x_1 - 2, 3x_2 - [3x_2]) & \text{if } 2/3 \leqq x_1 \leqq 1 . \end{cases}$$

Then

$$\rho_n \leqq \int_{F_n} |\nabla e_n|^2 = 3a^2 \cdot \frac{9}{8}\rho_{n-1} + 2(1-2a)^2 \cdot \frac{9}{8}\rho_{n-1} + 3a^2\frac{9}{8}\rho_{n-1}$$

$$= \frac{27}{28}\rho_{n-1} .$$

(Of course, a was chosen to minimize the right hand side.) □

Corollary 2.5. *There exists c_6 independent of n and r such that*

$$a_{n-r}/\alpha_n \leqq c_6(27/28)^r \leqq c_6 .$$

We can also give a lower bound for ρ_n/ρ_{n-1}.

Theorem 2.6 $\rho_n \geqq \frac{3}{4}\rho_{n-1}$.

Proof. (cf. [4]) Let $A = F_n \cap ([0,1] \times [0,1/3])$, $B = F_n \cap ([0,1] \times [2/3,1])$, $C = F_n \cap [0,1/3]^2$, and $\Lambda_i = \{i/3\} \times [0,1/3]$ for $i = 0,1,2,3$. Let Z_t^n be Brownian motion in A with normal reflection on the boundary of A.

If $Z_0^n \in \Lambda_1$, then by symmetry Z_t^n is equally likely to hit Λ_0 and Λ_2 first, with a similar statement if $Z_0^n \in \Lambda_2$. Let $S_0 = \inf\left\{t: Z_t^n \in \bigcup_{k=0}^{3} \Lambda_k\right\}$, let V_0 be the value of k such that $Z_{S_0}^n \in \Lambda_k$, let $S_{i+1} = \inf\left\{t > S_i: Z_t^n \in \bigcup_{k \neq V_i} \Lambda_k\right\}$, and let V_{i+1} be the value of k such that $Z_{S_{i+1}}^n \in \Lambda_k$. So V_i is the sequence of Λ_k's that Z_t^n visits. Then V_i is a simple random walk on $\{0,1,2,3\}$ with reflection at 0 and 3. Therefore,

$$P^x(Z_t^n \text{ hits } \Lambda_3 \text{ before hitting } \Lambda_0) = k/3 \quad \text{if} \quad x \in \Lambda_k . \tag{2.15}$$

Now consider the variational problem

$$\omega_n = \inf\left\{\int_A |\nabla u|^2(x)\mu_n(dx): u = 0 \quad \text{on} \quad \Lambda_0, u = 1 \quad \text{on} \quad \Lambda_3\right\}, \tag{2.16}$$

and let g_n be the minimizing function. By standard calculus of variations techniques, we see that g_n is the harmonic function having boundary values 0 on Λ_0 and 1 on Λ_3 and having zero normal derivative on the remainder of the boundary of A. Hence

$$g_n(x) = P^x(Z_t^n \text{ hits } \Lambda_3 \text{ before hitting } \Lambda_0) .$$

Let

$$\pi_n = \inf\left\{\int_C |\nabla u|^2(x)\mu_n(dx): u = 0 \quad \text{on} \quad \Lambda_0, u = 1/3 \quad \text{on} \quad \Lambda_1\right\}. \tag{2.17}$$

Using (2.15) and the fact that A is the union of C and two of its translates, we see

$$\omega_n = 3\pi_n .$$

By scaling, the function that minimizes the variational problem (2.17) is $u(x) = \frac{1}{3}f_{n-1}(3x)$, and so

$$\pi_n = \rho_{n-1}/8 .$$

Finally, using symmetry,

$$\rho_n = \int_{F_n} |\nabla f_n|^2 \geqq 2\int_A |\nabla f_n|^2 \geqq 2\omega_n = 6\pi_n = \frac{3}{4}\rho_{n-1} . \quad \square$$

Corollary 2.7 *There exists c_7 independent of n and r such that*

$$\alpha_{n-r}/\alpha_n \geqq c_7(3/4)^r .$$

Remark. A paper in preparation [2] will show that Corollaries 2.5 and 2.7 lead to upper and lower bounds for the spectral dimension of the Sierpinski carpet. Specifically, Corollary 2.5 leads to the upper bound $2\log 8/\log(28/3) \approx 1.862$, while Corollary 2.7 leads to the lower bound $2\log 8/\log 12 \approx 1.674$. Refinements to Theorems 2.4 and 2.6 lead to slight improvement of both the upper and lower bounds. Numerical calculations suggest that the actual value of the spectral dimension is approximately 1.81.

3. Green Functions

We need to introduce some more notation. Let $F = \bigcap_{n=0}^{\infty} F_n$, $\partial_a F = \partial_a F_1$. Let $d_f = \log 8/\log 3$, the Hausdorff dimension of F. Let μ be the weak limit of the μ_n; μ is a multiple of the Hausdorff x^{d_f}-measure on F.

For $x = (x_1, x_2)$, define $D_r(x) = [(j-1)/3^r, \ (j+1)/3^r) \times [(k-1)/3^r, (k+1)/3^r)$ if $(j-\frac{1}{2})3^{-r} \leqq x_1 < (j+\frac{1}{2})3^{-r}$, $(k-\frac{1}{2})3^{-r} \leqq x_2 < (k+\frac{1}{2})3^{-r}$, j, k integers. Note that

$$\inf_{r,x} 3^r \operatorname{dist}(\partial D_r(x), \partial D_{r-1}(x)) > 0 . \tag{3.1}$$

Let Ω be the collection of continuous paths in $[0, 1]^2$ and let X_t be the canonical coordinate process. Let P_n^x be the law of $W_{\alpha_n t}^n$ starting at x. Let $\tau = \inf\{t: X_t \in \partial_a F\}$ and let

$$\sigma_r(x) = \inf\{t: X_t \notin D_r(x)\} \wedge \tau .$$

Let n_j be any sequence tending to infinity such that for each $x \in F$, $P_{n_j}^x$ converges weakly, say to P^x, and (P^x, X_t) forms a strong Markov process on F with continuous paths. The existence of such sequences n_j is one of the main results of [1].

Finally, let U_n, $u_n(x, y)$ and U, $u(x, y)$ be the Green potential and function for (P_n^x, X_t) and (P^x, X_t), respectively. So, for instance,

$$E_n^x \int_0^\tau 1_A(X_s)ds = U_n 1_A(x) = \int_A u_n(x, y)\mu_n(dy) .$$

By [1, Sect. 7], given ε, there exists $c_8(\varepsilon)$, $c_9(\varepsilon)$ and α (independent of n) such that

$$u_n(x, y), u(x, y) \leqq c_8 \quad \text{whenever} \quad x, y \in F, |x - y| > \varepsilon \tag{3.2}$$

and

$$|u_n(x, y) - u_n(x, z)| \leqq c_9 |y - z|^\alpha , \quad |u(x, y) - u(x, z)| \leqq c_9 |y - z|^\alpha$$
$$\text{whenever } x, y, z \in F, |x - y|, |x - z| > \varepsilon . \tag{3.3}$$

Remark. Note that if $x \neq y$, $E^y \int_0^\tau 1_{\{x\}}(X_s)ds = \int_{\{x\}} u(y, z)\mu(dz) = 0$. And by [1, Sect. 6], X_t started at x leaves $\{x\}$ immediately. So by the strong Markov property,

$$E^x \int_0^\tau 1_{\{x\}}(X_s)ds = 0 . \tag{3.4}$$

A consequence of (3.2) is that if $\mu(A) = 0$, $\varepsilon > 0$, then

$$E^x \int_0^\tau 1_{A \cap B_\varepsilon^c(x)}(X_s)\,ds = \int_{A \cap B_\varepsilon^c(x)} u(x, y)\,\mu(dy) \leqq c_8\,\mu(A \cap B_\varepsilon^c(x)) = 0\,.$$

Letting $\varepsilon \to 0$ and using (3.4) then tells us that if $\mu(A) = 0$, $U1_A \equiv 0$.

Another consequence of (3.4) is that we may define $u(x, x)$, i.e., u on the diagonal, arbitrarily, without violating (3.2) or (3.3). We choose to define

$$u(x, x) = \limsup_{y \to x} u(x, y)\,. \tag{3.5}$$

We now prove that the restriction $|x - y| > \varepsilon$ in (3.2) can be dispensed with for $u(x, y)$.

Theorem 3.1. *There exists c_{10} such that*

$$u(x, y) \leqq c_{10} \quad \text{whenever } x, y \in F \tag{3.6}$$

Proof. Fix $x, y \in F$, $x \neq y$. Let $\varepsilon > 0$ and set $A = A(\varepsilon) = F_0 \cap B_\varepsilon(x)$. Note that for $n \geqq 1$

$$\mu_n(A) \leqq 12\mu(A)\,.$$

The main step of the proof is to get an upper bound for

$$q_{nr}(y, A) = E_n^y \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s)\,ds \tag{3.7}$$

for $n > r \geqq 0$. Using the strong Markov property we have

$$q_{nr}(y, A) \leqq \sup_{z \in \partial D_{r+1}(x)} E_n^z \int_0^{\sigma_r(x)} 1_A(X_s)\,ds\,. \tag{3.8}$$

Also, $q_{nr}(y, A) = q_{nr}(y, A \cap D_r(x))$, and $q_{nr}(y, A) = 0$ for $y \notin D_r(x)$.

First consider the case $r \leqq 4$. Then by (3.8)

$$q_{nr}(y, A) \leqq \sup_{z \in \partial D_{r+1}(x)} E_n^z \int_0^\tau 1_A(X_s)\,ds\,.$$

If $A \subseteq D_{r+1}(x)$ then by (3.1) there exists a $\delta_1 > 0$ (independent of r and x) such that $\operatorname{dist}(\partial D_{r+1}(x), D_{r+2}(x)) > \delta_1$. So, by (3.2),

$$\sup_{z \in \partial D_{r+1}(x)} E_n^z \int_0^\tau 1_A(X_s)\,ds \leqq \mu_n(A) \sup_{\substack{z \in \partial D_{r+1}(x) \\ w \in A}} u_n(z, w) \leqq 12\mu(A)\,c_8(\delta_1)\,.$$

On the other hand, if $A \nsubseteq D_{r+1}(x)$ then $\mu(A) > \delta_2 > 0$ for some constant δ_2 independent of x and r. So

$$\begin{aligned} E_n^z \int_0^\tau 1_A(X_s)\,ds &\leqq E_n^z \tau \\ &\leqq 1 \leqq \delta_2^{-1}\mu(A)\,. \end{aligned}$$

Combining the last two inequalities we have that, for some constant $c_{10} < \infty$,

$$q_{nr}(y, A) \leqq c_{10}\,\mu(A)\,, \quad 0 \leqq r \leqq 4\,, \quad n \geqq r\,. \tag{3.9}$$

We now use scaling to generalize (3.9). Let $r > 4$, and $p = r - 3$. Suppose for the moment that $D_{r+1}(x) \subseteq [0, 3^{-r+1})^2$. The law of $W^n(t)$ started at x is the same as the law of $3^{-p} W^{n-p}(9^p t)$ starting at $3^{-p}x$. So X_t under P_n^x has the same law as $3^{-p} X(t 9^p \alpha_n / \alpha_{n-p})$ under $P_{n-p}^{3^p x}$. Hence, writing $\theta_{np} = 9^p \alpha_n / \alpha_{n-p}$,

$$\begin{aligned} q_{nr}(y, A) &= E_{n-p}^{3^p y} \int_{\theta_{np}\sigma_4(3^p x)}^{\theta_{np}\sigma_3(3^p x)} 1_A(3^{-p} X(t\theta_{np}))\,dt \\ &= \theta_{np}^{-1} E_{n-p}^{3^p y} \int_{\sigma_4(3^p x)}^{\sigma_3(3^p x)} 1_A(3^{-p} X_s)\,ds \\ &= \theta_{np}^{-1} q_{n-p,3}(3^p y, 3^p A)\,. \end{aligned}$$

Thus by (3.9)

$$q_{nr}(y, A) \leqq \theta_{np}^{-1} c_{10}\, \mu(3^p A)\,.$$

Now $\mu(3^p A) \leqq 3^{pd_f} \mu(A)$, and so, using Corollary 2.5,

$$\begin{aligned} q_{nr}(y, A) &\leqq c_{10} 9^{-p} \alpha_{n-p} \alpha_n^{-1} 3^{pd_f} \mu(A) \\ &\leqq c_6 c_{10} 9^{-p} (27/28)^p 3^{pd_f} \mu(A) \\ &= c_6 c_{10} (6/7)^p \mu(A) = c(6/7)^r \mu(A)\,. \end{aligned} \tag{3.10}$$

Let $n \to \infty$ along the sequence n_j. Then since $U 1_{\partial A} \equiv 0$ by the remark following (3.3),

$$E^y \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s)\,ds \leqq c(6/7)^r \mu(A)\,. \tag{3.11}$$

We now remove the restriction $D_r(x) \subseteq [0, 3^{-r+1})^2$. If $D_r(x) \nsubseteq [0, 3^{-r+1})^2$, we can perform suitable translations, rotations, and reflections to find $\hat{x}$, $\hat{y}$, and $\hat{A}$ so that

$$q_{nr}(y, A) \leqq q_{nr}(\hat{y}, \hat{A}) = E_n^{\hat{y}} \int_{\sigma_{r+1}(\hat{x})}^{\sigma_r(\hat{x})} 1_{\hat{A}}(X_s)\,ds$$

with $D_r(\hat{x}) \subset [0, 3^{-r+1})^2$ and $\mu_n(A) \geqq \mu_n(\hat{A})$. Applying the above argument to $q_{nr}(\hat{y}, \hat{A})$, we get the bound (3.11) as before.

For use in Theorem 3.2, note that the only reason we may have to have $q_{nr}(y, A) < q_{nr}(\hat{y}, \hat{A})$ is because x and y may be too close to $\partial_a F$.

Finally, summing over r gives

$$E^y \int_0^\tau 1_A(X_s)\,ds \leqq c \sum_{r=0}^{\infty} (6/7)^r \mu(A) \leqq c\mu(A) \tag{3.12}$$

for all ε. By the continuity of $u(x, y)$ off the diagonal (see (3.3)), letting $\varepsilon \to 0$ gives (3.6) if $x \neq y$. Using (3.5) completes the proof. □

We now obtain the joint continuity of $u(x, y)$.

Theorem 3.2. *There exists c_{11} and $\alpha > 0$ such that*

$$|u(x, y) - u(x, z)| \leqq c_{11} |y - z|^\alpha \quad \textit{whenever } x, y, z \in F\,. \tag{3.13}$$

Proof. We prove the estimate (3.13) for the case $x \neq y$, $x \neq z$. Once we have this, we get (3.13) for all x, y, z by (3.5).

Suppose for now that $y \in F \cap [0, 8/9)^2$. In view of (3.3), we may suppose $|y - z| \leqq 3^{-4}$. By the strong Markov property, the continuity of u off the diagonal, and the fact that $u(\cdot, \cdot)$ is harmonic in each variable off the diagonal,

$$u(x, y) - u(x, z) = E^x[u(X_S, y) - u(X_S, z)] ,$$

where $S = \inf\{t: |X_t - y| < \frac{3}{2}|y - z|\}$. Hence it suffices to obtain the bound (3.13) for $|x - y|, |x - z| \leqq \frac{5}{2}|y - z|$.

Let $\varepsilon < |y - z|/5$, let $A = A(\varepsilon) = B_\varepsilon(x) \cap F_0$, and define $q_{nr}(z, A), q_{nr}(y, A)$ by (3.7).

First, consider the case: $|y - z| \leqq 3^{-r-4}$ and $r \geqq 4$. As in the proof of Theorem 3.1, we may suppose $\{x, y, z\} \cup A \subset [0, 3^{-r+1})^2$. Let $p = r - 3$. Using scaling as in (3.10),

$$q_{nr}(y, A) = E^{3^p y}_{n-p} H(X_{\sigma_4(3^p x)}) , \tag{3.14}$$

where

$$\begin{aligned} H(w) &= E^w_{n-p} \int_0^{\sigma_3(3^p x)} 1_{(3^p A)}(X(9^p t\alpha_n/\alpha_{n-p}))\, dt \\ &\leqq c 3^{r(d_f - 2)} \mu(A) , \end{aligned} \tag{3.15}$$

and similarly for $q_{nr}(z, A)$.

By [1, Sect. 3], since $3^p y, 3^p z \in F \cap [0, 8/9)^2$, there exists $\xi < (2 - d_f)/2$ so that

$$\begin{aligned} |q_{nr}(y, A) - q_{nr}(z, A)| &\leqq c \|H\| \, |3^p y - 3^p z|^\xi \\ &\leqq c 3^{r(d_f - 2 + \xi)} |y - z|^\xi \mu(A) . \end{aligned} \tag{3.16}$$

The second case: $|y - z| \leqq 3^{-r-4}$ and $r \leqq 4$, is similar, but no scaling is necessary (cf. the proof of Theor. 3.1).

In both the first and second cases, letting $n \to \infty$ along the subsequence n_j and using the fact that $U 1_{\partial A} \equiv 0$,

$$\left| E^y \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s)\, ds - E^z \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s)\, ds \right| \leqq c 3^{r(d_f - 2 + \xi)} |y - z|^\xi \mu(A) . \tag{3.17}$$

For the third case: $|y - z| > 3^{-r-4}$,

$$|q_{nr}(y, A) - q_{nr}(z, A)| \leqq |q_{nr}(y, A)| + |q_{nr}(z, A)| \leqq c 3^{r(d_f - 2)} \mu(A)$$

by (3.10). Again, let $n \to \infty$ along the subsequence n_j.

Then, summing over r, for all ε,

$$\begin{aligned} \left| E^y \int_0^\tau 1_A(X_s)\, ds - E^z \int_0^\tau 1_A(X_s)\, ds \right| &\leqq \sum_{\{r: |y - z| \leqq 3^{-r-4}\}} c 3^{r(d_f - 2 + \xi)} |y - z|^\xi \mu(A) \\ &\quad + \sum_{\{r: |y - z| > 3^{-r-4}\}} c 3^{r(d_f - 2)} \mu(A) \\ &\leqq c \mu(A) [|y - z|^\xi + |y - z|^{-(d_f - 2)}] . \end{aligned}$$

Dividing both sidea by $\mu(A)$, letting $\varepsilon \to 0$, and recalling $u(x, y)$ is continuous off the diagonal gives (3.13) if $y \in F \cap [0, 8/9)^2$.

But the restriction $y \in F \cap [0, 8/9)^2$ may be removed by a very similar proof to that of [1, Theor. 7.2]; the details are left to the reader. □

4. Local Times

Once we have $u(x, y)$ bounded and Hölder continuous, the construction of jointly continuous local times is routine.

Proposition 4.1. *For each $y \in F$, $u(x, y)$ is excessive.*

Proof. The function $x \to \int_{B_\varepsilon(y) \cap F} u(x, z)\mu(dz) = E^x \int_0^\tau 1_{B_\varepsilon(y) \cap F}(X_s)ds$ is a potential, hence excessive. By the continuity of u, the function $x \to u(x, y)$ is the uniform limit of $\mu(B_\varepsilon(y) \cap F)^{-1} \int_{B_\varepsilon(y) \cap F} u(x, z)\mu(dz)$ as $\varepsilon \to 0$, and hence is also excessive. □

Proposition 4.2. *For each $y \in F$, $u(x, y)$ is a regular potential.*

Remark. Recall that $u(x, y)$ is a regular potential if for each z, $E^z u(X_{T_n}, y) \to E^z u(X_T, y)$ whenever T_n are stopping times increasing to T.

Proof. This is clear by the boundedness and continuity of u and the continuity of X_t. □

We now use [5, Theor. IV. 3.13] to see that for each y there exists a continuous additive functional L_t^y whose potential is $u(x, y)$. Since by the Markov property

$$E^z L_t^y = u(x, y) - E^z u(X_{t \wedge \tau}, y)$$

for each z, it follows easily that

$$M_t^y = u(X_{t \wedge \tau}, y) - u(X_0, y) + L_t^y \tag{4.1}$$

is a martingale with respect to P^x for each x. Moreover, $M_0^y \equiv 0$, P^x-a.s. for each x.

We now want to show we can choose a version of L_t^y that is jointly continuous in t and y. Since $u(X_{t \wedge \tau}, y)$ is continuous in t and y jointly, we can concentrate our attention on M_t^y.

Let $U_t^y = u(X_{t \wedge \tau}, y) - u(X_0, y)$, let $N_t(y_1, y_2) = M_t^{y_1} - M_t^{y_2}$, and let $N^*(y_1, y_2) = \sup_{s \leqq \tau} |N_s(y_1, y_2)|$

Proposition 4.3. *There exists ζ and $\theta > 0$ such that for all z*

$$P^z(N^*(y_1, y_2) > \lambda) \leqq \exp\left(\frac{-\lambda}{\theta |y_1 - y_2|^\zeta}\right).$$

Proof. Fix y_1 and y_2, let $\Lambda = \sup_{x,y} u(x, y)$, and let $\delta = \sup_x |u(x, y_1) - u(x, y_2)|$. Trivially, $\delta \leqq 2\Lambda$.

By Ito's formula, for each z and t

$$E^z(U_t^{y_1} - U_t^{y_2})^2 = 2E^z \int_0^t (U_s^{y_1} - U_s^{y_2})d(U_s^{y_1} - U_s^{y_2}) + E^z \langle N(y_1, y_2) \rangle_t ,$$

and hence

$$\begin{aligned} E^z\langle N(y_1, y_2)\rangle_t &\leqq (2\delta)^2 + 2\delta E^z \int_0^t d(U_s^{y_1} + U_s^{y_2}) \\ &\leqq 4\delta^2 + 4\delta\Lambda \\ &\leqq 12\delta\Lambda\,. \end{aligned} \tag{4.2}$$

By monotone convergence, we get the same bound for $E^z\langle N(y_1, y_2)\rangle_\tau$.

Suppose S and T are bounded stopping times with $S \leqq T$. Then by the strong Markov property and Cauchy-Schwartz,

$$\begin{aligned} E^z(|N_T(y_1, y_2) - N_S(y_1, y_2)|\,|\mathscr{F}_S) &\leqq (E^z((N_T(y_1, y_2) - N_S(y_1, y_2))^2\,|\mathscr{F}_S))^{1/2} \\ &\leqq (E^z(\langle N(y_1, y_2)\rangle_\tau - \langle N(y_1, y_2)\rangle_S|\mathscr{F}_S))^{1/2} \\ &\leqq (E^z E^{X_S}(\langle N(y_1, y_2)\rangle_\tau))^{1/2} \\ &\leqq (12\delta\Lambda)^{1/2}\,. \end{aligned} \tag{4.3}$$

We can then apply [7, p. 193], and get

$$E^z \exp(\gamma N^*(y_1, y_2)) \leqq 2$$

provided $\gamma \leqq (8(12\delta\Lambda)^{1/2})^{-1}$. The proposition now follows by Chebyshev's inequality together with the fact that $\delta \leqq K|y_1 - y_2|^\alpha$ for some K and $\alpha > 0$. □

With the estimate of Proposition 4.3, we can now appeal to [3, Sect. 6] to conclude that there is a version of L_t^y that is jointly continuous in t and y.

We have

Proposition 4.4. *Except for a set N such that $P^x(N) = 0$ for all x,*

$$\int_0^{t\wedge\tau} f(X_s)ds = \int f(y)L_t^y\mu(dy)$$

for all f bounded on F.

Proof. Suppose f is continuous on F. Multiplying M_t^y by $f(y)$ and integrating with respect to μ, we see that

$$\int f(y)U_t^y\mu(dy) + \int f(y)L_t^y\mu(dy) \text{ is a } P^x\text{-martingale for all } x\,.$$

On the other hand, under P^x

$$\begin{aligned} \int f(y)U_t^y\mu(dy) &= \int f(y)[u(X_{t\wedge\tau}, y) - u(X_0, y)]\mu(dy) \\ &= E^{X_{t\wedge\tau}}\int_0^\tau f(X_s)ds - E^x\int_0^\tau f(X_s)ds \\ &= E^x\left[\int_{t\wedge\tau}^\tau f(X_s)ds\,|\mathscr{F}_{t\wedge\tau}\right] - E^x\int_0^\tau f(X_s)ds \\ &= E^x\left[\int_0^\tau f(X_s)ds\,|\mathscr{F}_{t\wedge\tau}\right] - \int_0^{t\wedge\tau} f(X_s)ds - E^x\int_0^\tau f(X_s)ds\,. \end{aligned}$$

Hence $\int f(y) L_t^y \mu(dy) - \int_0^{t \wedge \tau} f(X_s) ds$ is a P^x-martingale that is null at 0, continuous, and of bounded variation. But the only such martingale is 0, and hence if

$$N_f = \left\{ \omega \colon \int f(y) L_t^y \mu(dy) \neq \int_0^{t \wedge \tau} f(X_s) ds \text{ for some } t \leqq \tau \right\},$$

then $P^x(N_f) = 0$.

Let f_i be a countable collection of continuous functions, the closure of whose linear span is $\mathscr{C}(F)$. Let $N = \bigcup_{i=1}^{\infty} N_{f_i}$. Then for each x, $P^x(N) = 0$; while if $\omega \notin N$, the equality asserted in the statement of Proposition 4.4 holds for all continuous f. By a monotone class argument the equality holds for all bounded f when $\omega \notin N$, proving the proposition. □

Point recurrence is an easy consequence of the existence of local times. Let

$$T_y = \inf\{t \colon X_t = y\}.$$

Theorem 4.5. *If $x, y \in F - \partial_a F$, then $P^x(T_y < \infty) > 0$.*

Proof. If $x = y$, choose r large enough so that $D_r(x) \cap \partial_a F = \emptyset$, and then

$$P^x(T_y < \infty) \geqq E^x P^{X_{\sigma_r(x)}}(T_y < \infty).$$

So it suffices to consider the case $x \neq y$.

Since $\int_F u(x, z) \mu(dz) > 0$, there exists $z \neq x$ such that $u(x, z) > 0$. Since $u(x, \cdot)$ is harmonic on $F - \{x\}$, this implies by the Harnack inequality that $u(x, y) > 0$ (see [1, Sect. 3]). But $u(x, y) = E^x L_\tau^y$, which implies $P^x(L_\tau^y > 0) > 0$. And since L_t^y increases only when X_t is at y, this proves the theorem. □

Remark. If the boundary $\partial_a F$ were changed from absorbing to reflecting, a renewal argument could be used to show $P^x(T_y < \infty) = 1$.

Closely related to the notion of point recurrence is that of points being regular for themselves.

Theorem 4.6. *If $x \in F - \partial_a F$, then $P^x(T_x = 0) = 1$.*

Proof. Fix $x \in F - \partial_a F$, let $\varepsilon > 0$, and set $A(\varepsilon) = B_\varepsilon(x) \cap F$. Let

$$S_\varepsilon = \inf\{t \colon X_t \in \partial B_\varepsilon(x)\}.$$

By the strong Markov property, if $\varepsilon < |y - x|$,

$$U 1_{A(\varepsilon)}(y) = E^y(U 1_{A(\varepsilon)}(X_{S_\varepsilon}); S_\varepsilon < \tau). \tag{4.4}$$

Multiply both sides of (4.4) by $\mu(A(\varepsilon))^{-1}$ and let $\varepsilon \to 0$. Using the continuity and boundedness of u, the continuity of X_t, and dominated convergence, we get

$$u(y, x) = u(x, x) P^y(T_x < \tau). \tag{4.5}$$

Since $u(x, \cdot)$ is harmonic off $\{x\}$ and $u(x, \cdot) = 0$ on $\partial_a F$, then $u(x, \cdot)$ takes its maximum at x. Since $u(x, y) > 0$ if $y \neq x$ (see the proof of Theor. 4.5), this implies

$u(x, x) > 0$. Letting $y \to x$ in (4.5) and using the continuity of u then gives

$$\lim_{y \to x} P^y(T_x < \tau) = 1. \tag{4.6}$$

Recall that starting at x, X leaves $\{x\}$ immediately ([1, Sect. 6]), hence $S_\varepsilon \downarrow 0$, P^x-a.s. Since

$$P^x(T_x < \tau) \geqq P^x(S_\varepsilon < T_x < \tau) = E^x P^{X_{S_\varepsilon}}(T_x < \tau) \to 1$$

as $\varepsilon \to 0$, then

$$P^x(T_x < \tau) = 1 .$$

Finally, choose y_0 such that $\operatorname{dist}(y_0, \partial_a F) = 1/9$, $|y_0 - x| \geqq 1/3$. By a combination of knight's moves and corner moves (see [1, Sect. 2]), $P^{y_0}(T_x = \infty) > 0$. By the Harnack inequality of [1, Sect. 3], if $\varepsilon \in (0, \operatorname{dist}(x, \partial_a F)/2)$, there exists $c(\varepsilon)$ such that $P^y(T_x = \infty) > c(\varepsilon)$ whenever $|y - x| = \varepsilon$. But then

$$\begin{aligned} 1 = P^x(T_x < \tau) &= P^x(T_x \leqq S_\varepsilon) + E^x(P^{X_{S_\varepsilon}}(T_x < \tau); T_x > S_\varepsilon) \\ &\leqq P^x(T_x \leqq S_\varepsilon) + (1 - c(\varepsilon))P^x(T_x > S_\varepsilon) \\ &= 1 - c(\varepsilon)P^x(T_x > S_\varepsilon) . \end{aligned}$$

Hence $P^x(T_x > S_\varepsilon) = 0$, and letting $\varepsilon \to 0$ proves $P^x(T_x > 0) = 0$. □

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