The submartingale problem for a class of degenerate elliptic operators

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November 8, 2006

Abstract

We consider the degenerate elliptic operator acting on C_b^2 functions on $[0,\infty)^d$:

$$\mathcal{L}f(x) = \sum_{i=1}^d a_i(x) x_i^{\alpha_i} \frac{\partial^2 f}{\partial x_i^2}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x),$$

where the a_i are continuous functions that are bounded above and below by positive constants, the b_i are bounded and measurable, and the $\alpha_i \in (0,1)$. We impose Neumann boundary conditions on the boundary of $[0,\infty)^d$. There will not be uniqueness for the submartingale problem corresponding to $\mathcal{L}$. If we consider, however, only those solutions to the submartingale problem for which the process spends 0 time on the boundary, then existence and uniqueness for the submartingale problem for $\mathcal{L}$ holds within this class. Our result is equivalent to establishing weak uniqueness for the system of stochastic differential equations

$$dX_t^i = \sqrt{2a_i(X_t)}(X_t^i)^{\alpha_i/2} dW_t^i + b_i(X_t)\,dt + dL_t^{X^i}, \qquad X_t^i \geq 0,$$

where W_t^i are independent Brownian motions and $L_t^{X_i}$ is a local time at 0 for X^i.

*Research partially supported by NSF grant DMS-0244737.

Keywords: martingale problem, stochastic differential equations, degenerate elliptic operators, speed measure, perturbation, Bessel process, Littlewood-Paley

Subject Classification: Primary 60H10; Secondary 60H30

1 Introduction

We consider the degenerate elliptic operator acting on C_b^2 functions on $[0,\infty)^d$ defined by

$$\mathcal{L}f(x) = \sum_{i=1}^d a_i(x) x_i^{\alpha_i} \frac{\partial^2 f}{\partial x_i^2}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x). \tag{1.1}$$

We assume here that the b_i are bounded and measurable, the a_i are continuous and bounded above and below by positive constants, and each $\alpha_i \in (0,1)$. We impose zero Neumann boundary conditions on $\partial(\mathbb{R}_+^d)$, where we write $\mathbb{R}_+ = [0,\infty)$. In this paper we investigate whether there is at most one process corresponding to the operator $\mathcal{L}$.

We formulate this question in terms of a submartingale problem. Let $\Omega = C([0,\infty); \mathbb{R}_+^d)$, the continuous functions from $[0,\infty)$ to $\mathbb{R}_+^d$. Define the canonical process X by $X_t(\omega) = \omega(t)$ and let $\mathcal{F}_t$ be the filtration generated by X. Let $x \in \mathbb{R}_+^d$. We say that a probability measure $\mathbb{P}$ on Ω is a solution to the submartingale problem for $\mathcal{L}$ started at x if $\mathbb{P}(X_0 = x) = 1$ and whenever $f \in C_b^2(\mathbb{R}_+^d)$ such that for each i we have in addition $\partial f/\partial x_i \geq 0$ on $\{x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d : x_i = 0\}$, then $f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$ is a submartingale with respect to $\mathbb{P}$.

If Y_t is a one dimensional process on $[0,\infty)$, by local time at 0 of Y we mean any continuous nondecreasing process L_t^Y such that L^Y increases only when Y is at 0. Closely related to the operator $\mathcal{L}$ is the system of equations

$$dX_t^i = \sqrt{2a_i(X_t)}(X_t^i)^{\alpha_i/2} dW_t^i + b_i(X_t)dt + dL_t^{X^i}, \qquad i = 1, \ldots, d, \tag{1.2}$$

where $X_0^i = x_i$, $X_t^i \geq 0$ for all t, L^{X^i} is a local time at 0 for X^i, and the W^i are independent one dimensional Brownian motions started at 0.

We say a weak solution to (1.2) exists if there is a probability $\mathbb{P}$ such that (1.2) holds and the W^i are independent Brownian motions under $\mathbb{P}$. Weak uniqueness holds if given any two solutions $(X_j, W_j, \mathbb{P}_j)$, $j = 1, 2$, the joint law of (X_1, W_1) under $\mathbb{P}_1$ is equal to the joint law of (X_2, W_2) under $\mathbb{P}_2$. For weak uniqueness to hold for (1.2) it is only necessary that the law of X_1 under $\mathbb{P}_1$ equals the law of X_2 under $\mathbb{P}_2$; see Remark 2.2.

We have assumed that each α_i is in the interval $(0, 1)$, so in fact uniqueness for the submartingale problem for $\mathcal{L}$ does **not** hold. This can be seen even in one dimension: if one looks at the one-dimensional diffusion on natural scale with speed measure m, where $m(dx) = x^{-\alpha}\, dx$ for x positive, m is 0 on $(-\infty, 0)$, and m has an atom of mass λ at 0, one can let λ be an arbitrary non-negative number and obtain different processes.

If, however, one restricts attention to those solutions to the submartingale problem $\mathcal{L}$ for which the process spends zero time at the boundary, then uniqueness of the submartingale problem does hold. Our main theorem is the following. Let $\Delta = \partial(\mathbb{R}^d_+)$.

Theorem 1.1 *Let $x \in \mathbb{R}^d_+$*

(a) There exists one and only one solution to the submartingale problem for $\mathcal{L}$ started at x that spends zero time in Δ, i.e.,

$$\int_0^\infty 1_\Delta(X_s)ds = 0, \qquad \mathbb{P} - a.s.$$

(b) A weak solution to (1.2) exists that spends zero time in Δ. Weak uniqueness holds if we restrict attention to those weak solutions that spend zero time in Δ.

Our paper continues the study of degenerate diffusions in the positive orthant begun in [1] and [5]. Those papers concerned the operator $\mathcal{L}$ where all the α_i were equal to 1. In some sense, if all the α_i are equal to 1, we are in the critical case, in that then the exact values of the drift coefficients b_i make a large difference to the behavior of the resulting process. When $\alpha_i > 1$ and the drift coefficients are zero, then either the process never attains the boundary, or if it starts on the boundary, never leaves, so the problem then becomes a lower dimensional one. This paper deals with the case $\alpha_i < 1$.

Although the values of the drift coefficients play less of a role, the results of this paper are not a subset of those in [1]. In fact, they could not be, because here we need the additional assumption that the process spends zero time on the boundary in order to have uniqueness, while no such assumption is needed in [1].

If $\alpha_i < 1/2$, one can check that a Girsanov transformation allows one to assume that the corresponding b_i can be taken to zero; see Remark 7.1. We wanted to allow the full range of α_i and drift coefficients, so we did not restrict the values of the α_i to $(0, 1/2)$. As is often the case, uniqueness for a martingale or submartingale problem is often related to the existence of a solution to a PDE problem. That is also the case here, but we do not pursue this connection.

We believe our techniques could also be applied to diffusions on $\mathbb{R}^d$ whose coefficients decay near the i^{th} axis like $|x_i|^{\alpha_i}$. Consider the diffusion on $\mathbb{R}^d$ which corresponds to the operator

$$\sum_{i=1}^d a_i(x)|x_i|^{\alpha_i}\frac{\partial^2 f}{\partial x_i^2}(x) + \sum_{i=1}^d b_i(x)\frac{\partial f}{\partial x_i}(x).$$

The drift coefficients play much less of a role than in [1]. The drift can be handled either by a Girsanov transformation when all the $\alpha_i < 1/2$ (see Remark 7.1) or by the perturbation approach of Section 5 in the more general case when all the $\alpha_i < 1$. The arguments of Section 6 should go through with little if any change.

The papers [14], [10], and [6] also consider diffusions with reflection; the latter two consider pathwise uniqueness. Although some smoothness of the domain is needed in these papers, the key difference is that degeneracies of the type given in (1.1) and (1.2) are not allowed.

Our methods differ substantially from those in [1]. That paper used L^2 estimates only, and required a use of the Krylov-Safonov Harnack inequality. The proof of Lemma 5.1 of [1] does not extend to the present situation, and we were forced to use a different approach. In this paper we proceed by proving an analogue of Krylov's inequality (see [8] and Theorem 3.2) and then using Littlewood-Paley theory to obtain L^p estimates. The use of Littlewood-Paley theory is of independent interest, and could potentially have applications to many other types of martingale problems. (The paper [5] uses C^α estimates and is quite different, both in results and in methods.)

We give a brief overview of the proof. Existence is done by taking smooth approximations to (1.1), showing the laws of the corresponding processes are tight, and looking at a convergent subsequence. Both in order to show that the weak limit of this subsequence spends zero time on the boundary of $\mathbb{R}_+^d$ and that the limit corresponds to (1.1), we need an inequality of Krylov type that gives a bound on the amount of time spent in small sets. To prove uniqueness, we suppose we have two solutions $\mathbb{P}_1$ and $\mathbb{P}_2$ to the martingale problem and define resolvent functionals S_λ^1 and S_λ^2 by (7.6). We let Θ be the norm of $S_\lambda^1 - S_\lambda^2$ as a functional on $L^{p_0}(\mu)$ for a certain positive real p_0 and a certain measure μ. The Krylov inequality that we prove guarantees that $\Theta < \infty$. Some stochastic calculus shows that

$$\Theta \leq \Theta \|\mathcal{B}R_\lambda\|_{p_0}$$

for certain operators $\mathcal{B}$ and R_λ defined by (7.3) and (6.1). If we knew that $\|\mathcal{B}R_\lambda\|_{p_0} < \frac{1}{2}$, we would conclude that $\Theta = 0$ and therefore $S_\lambda^1 = S_\lambda^2$; from here on the proof follows a familiar path. The principal work in this paper is to establish this bound on the norm of $\mathcal{B}R_\lambda$. The first order terms are handled by calculations with transition densities of the process corresponding to $\mathcal{L}_0$ defined in (7.2) together with a variant of Young's inequality (Proposition 5.2). The second order terms are where Littlewood-Paley theory comes in. Littlewood-Paley theory refers to a large body of literature where L^p estimates on operators are obtained by looking at square functions and quadratic variation. Our application is a mixture of analytic and probabilistic techniques.

After a section on preliminaries, we prove our inequality of Krylov type in Section 3. Section 4 concerns existence of solutions. Sections 5 and 6 give the required estimates for the first and second order terms, respectively. The proof of Theorem 1.1 is completed in Section 7. Some of the calculations needed in Section 5 are deferred to Section 8, which is an appendix.

2 Preliminaries

We often write f_i and f_{ii} for the first and second partial derivatives of f with respect to x_i. The Lebesgue measure of a Borel set B will be written $|B|$. We use $\mathbb{R}_+ = [0, \infty)$ and for our state space we will use $\mathbb{R}_+^d$. We use $C_b^2(\mathbb{R}_+^d)$ to refer to the collection of functions which are C^2 on $\mathbb{R}_+^d$ and such that the function and its first and second partial derivatives are bounded on $\mathbb{R}_+^d$; for points on $\partial(\mathbb{R}_+^d)$ we use instead the appropriate one-sided partial derivatives. We will later introduce a measure μ (see (3.14)) and$\|f\|_p$ will denote the L^p norm with respect to μ. When we want the L^p norm with respect to Lebesgue measure we will write $\|f\|_{L^p(dx)}$. We use the letter c with or without subscripts to denote finite positive constants whose values are unimportant and may change from place to place.

Let

$$\Delta_i = \{x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d : x_i = 0\}, \qquad \Delta = \cup_{i=1}^d \Delta_i. \tag{2.1}$$

For any process Z and any Borel set C we let

$$T_C(Z) = T_C = \inf\{t : Z_t \in C\}, \qquad \tau_C(Z) = \tau_C = \inf\{t : Z_t \notin C\}. \tag{2.2}$$

When C is a single point $\{y\}$, we write instead $T_y = T_y(Z)$ and $\tau_y = \tau_y(Z)$.

The connection between Theorem 1.1(a) and (b) is the following.

Proposition 2.1 *There exists a solution to the submartingale problem for $\mathcal{L}$ started at x if and only if there exists a weak solution to (1.2). The solution to the submartingale problem will be unique if and only if there is weak uniqueness to (1.2). These assertions continue to hold if we restrict attention to probability measures $\mathbb{P}$ such that X_t spends zero time in Δ, $\mathbb{P}$-a.s.*

Proof. The proof of this proposition is very similar to the nondegenerate diffusion case (see [3], Theorem V.1.1) and we give only a brief sketch. If $\mathbb{P}$ is a weak solution to (1.2), then an application of Ito's formula shows that $\mathbb{P}$ will be a solution to the submartingale problem. If $\mathbb{P}$ is a solution to the submartingale problem, $M > 0$, and we take $f(x)$ to be a C_b^2 function which equals x_i on $[0, M]^d$, then by the definition of submartingale problem,

$$f(X_t) - \int_0^t \mathcal{L}f(X_s)ds = X_t^i - \int_0^t b_i(X_s)ds$$

is a submartingale, and so by the Doob-Meyer decomposition can be written as a martingale M_t^i plus an increasing process L_t^i. Similarly to the nondegenerate diffusion case, one can show that $X_t^i - \int_0^t b_i(X_s)\,ds$ is a martingale away from the boundary with quadratic variation $\int_0^t a_i(X_s)(X_s^i)^{\alpha_i} ds$ up until the time of first exiting $[0, M]^d$. We then let $M \to \infty$. This implies that L_t^i increases only when X_t^i is at 0, and hence is a local time at 0 for X^i. We then proceed as in the proof of the nondegenerate case. $\square$

Remark 2.2 To show that the laws of (X, W, L) under $\mathbb{P}_1$ and $\mathbb{P}_2$ are the same, it is enough to show equality of the laws of X under $\mathbb{P}_1$ and $\mathbb{P}_2$. To see this, observe that we can recover L and W from X by the following formulas:

$$\begin{aligned} L_t^{X_i} &= \int_0^t 1_{\Delta_i}(X_s)\, dX_s^i, \\ W_t^i &= \int_0^t (2a_i(X_s))^{-1/2}(X_s^i)^{-\alpha_i/2}\,(dX_s^i - b_i(X_s)\, ds - dL_s^{X^i}). \end{aligned}$$

Hence the joint law of (X, W, L) is determined by the law of X. In this argument, we use the fact that X spends zero time on the boundary to verify that W is a Brownian motion.

Any weak solution to (1.2) satisfies a uniform tightness estimate. By this we mean the following.

Proposition 2.3 *If $M > 0, \delta > 0, \eta > 0$, and $t_0 > 0$, there exists $\varepsilon > 0$ such that if $(X, W, \mathbb{P})$ is any weak solution to (1.2) and $S_M = \inf\{t : X_t \notin [0, M]^d\}$, then*

$$\mathbb{P}(\sup_{s,t \le S_M \wedge t_0, |t-s| < \varepsilon} |X_t - X_s| > \delta) < \eta.$$

Proof. It suffices to consider each component of X separately. Fix i. Let

$$A_\varepsilon = \{\sup_{s \le t \le s+\varepsilon \le S_M \wedge (t_0+\varepsilon)} |(X_t^i - L_t^{X^i}) - (X_s^i - L_s^{X^i})| > \delta/4\}. \tag{2.3}$$

By standard estimates for stochastic integals (see, e.g., [3], Proposition I.8.1) we can find ε such that if X is any solution to (1.2), then $\mathbb{P}(A_\varepsilon) < \eta$.

We claim that if $\omega \in A_\varepsilon^c$, then $\sup_{s \le t \le s+\varepsilon \le t_0+\varepsilon} |X_t - X_s| \le \delta$. Proving this claim will complete the proof. Suppose $\omega \in A_\varepsilon^c$ and suppose there exist $s \le t \le t_0 + \varepsilon$ with $t - s < \varepsilon$ such that $|X_t^i - X_s^i| > \delta$. Then $L_t^{X^i} - L_s^{X^i} \ge 3\delta/4$.

Since L^{X^i} increases only when X^i is at 0, there exist $s' < t'$ such that $s \le s' < t' \le t$, $X^i_{s'} = X^i_{t'} = 0$, and $L^{X^i}_{t'} - L^{X^i}_{s'} > \delta/2$. But then

$$\begin{aligned} 0 &= X^i_{t'} - X^i_{s'} = (X^i_{t'} - L^{X^i}_{t'}) - (X^i_{s'} - L^{X^i}_{s'}) + (L^{X^i}_{t'} - L^{X^i}_{s'}) \\ &\ge \tfrac{\delta}{2} - \tfrac{\delta}{4}, \end{aligned}$$

a contradiction; the claim is proved. □

When it comes to proving uniqueness of the submartingale problem, it suffices to consider only solutions defined on the canonical probability space $C([0,\infty);\mathbb{R}^d_+)$. If S is a stopping time, we let $\mathbb{Q}_S$ be a regular conditional probability $\mathbb{P}(\,\cdot \circ \theta_S \mid \mathcal{F}_S)$, where θ_S is the usual shift operator of Markov process theory. We denote the corresponding expectation by $\mathbb{E}_{\mathbb{Q}_S}$. Just as in the nondegenerate case, it is easy to see that with probability one $\mathbb{Q}_S$ will be a solution to the submartingale problem started at X_S; cf. [3], Proposition VI.2.1.

3 Occupation time estimates

We start with an estimate on how long the solution to a one dimensional SDE can spend near 0. We proceed by proving that the process X is always at least as large as U_t defined by (3.5). Since U_t is a one-dimensional strong Markov process, we can calculate occupation times and number of crossings, and we then use the comparison with X to derive the corresponding facts about X.

Theorem 3.1 *Suppose $x_0 \in [0,\infty)$, W_t is a Brownian motion, a_t and b_t are adapted processes, $c_1^{-1} \le a_t \le c_1$ a.s. for each t, and $|b_t| \le c_1$ a.s. for each t. Suppose either (a) X solves*

$$dX_t = \sqrt{a_t}\, X_t^{\alpha/2} dW_t + b_t\, dt + dL^X_t, \qquad X_t \ge 0, \quad X_0 = x_0, \tag{3.1}$$

and X spends zero time at 0

or (b) for some $\varepsilon > 0$

$$dX_t = \sqrt{a_t}\, (X_t + \varepsilon)^{\alpha/2} dW_t + b_t\, dt + dL^X_t, \qquad X_t \ge 0, \quad X_0 = x_0, \tag{3.2}$$

where L_t^X is a continuous nondecreasing process that increases only when X_t is at 0.

Let $K > 0$. There exists c_2, c_3, c_4 depending only on K and c_1 such that for each $\gamma \le K$ the probability of more than m upcrossings of $[\gamma/2, \gamma]$ by X before time $T_K(X)$ is bounded by

$$c_2(1 - c_3\gamma)^m \tag{3.3}$$

and if $\eta \le K$, then

$$\mathbb{E} \int_0^{T_K(X)} 1_{[0,\eta]}(X_s)\, ds \le c_4 \eta^{1-\alpha}. \tag{3.4}$$

Both (3.3) and (3.4) are used in the proof of Theorem 3.2, and (3.4) is also used in Proposition 4.1.

Proof. By first performing a nondegenerate time change, we may without loss of generality suppose that $a_t \equiv 1$. In case (b), Girsanov's theorem for general continuous semimartingales ([2], Theorem I.6.4) and the fact that the diffusion coefficient is bounded below away from 0 tell us that the solution to (3.2) will spend zero time at 0. So we can consider both cases at once if we let $\varepsilon \ge 0$ and specify that X_t spends 0 time at 0.

We next define U_t. Let U_t be the solution to

$$dU_t = (U_t + \varepsilon)^{\alpha/2}\, dW_t - c_1\, dt + dL_t^U, \tag{3.5}$$

where $U_t \ge 0$ for all t and L_t^U is a local time at 0 for U. If $a_n(x) = (x+\varepsilon)^{\alpha/2} \wedge n$, the strong existence and pathwise uniqueness of the solution to

$$dU_t^n = a_n(U_t^n)\, dW_t - c_1\, dt + dL_t^{U^n},$$

where $U_t^n \ge 0$ for all t and L^{U^n} is a local time at 0 for U^n is guaranteed for all times by Theorem 1.3 of [4]. The strong existence and pathwise uniqueness of the solution to (3.5) up to time $T_K(U)$ follows easily. Let $\mathbb{P}^x$ be the law of the process U started at the point x; then the family $\{\mathbb{P}^x\}$ forms a continuous strong Markov process corresponding to the operator $\frac{1}{2}(x+\varepsilon)^\alpha f''(x) - c_1 f'(x)$ with Neumann boundary conditions at 0 (i.e., reflecting at 0).

We want to show that if $U_0 \le X_0$, then $U_t \le X_t$ for all $t \le T_0(X)$. We show this by applying a stochastic comparison theorem. If $U_0 \le X_0$, by

Theorem VI.1.1 of [7] we see that $U_t \le X_t$ for all $t \le T_0(X) \wedge T_0(U)$. Let $V_1 = \inf\{t : U_t \ge \frac{1}{2}X_t\}$ and for $i \ge 1$, let $\widetilde{V}_i = \inf\{t > V_i : U_t = 0\}$ and $V_{i+1} = \inf\{t > \widetilde{V}_i : U_t \ge \frac{1}{2}X_t\}$. Recall that we use $\mathbb{Q}_{V_i}$ for the regular conditional probability for $\mathbb{P}(\cdot \circ \theta_{V_i} \mid \mathcal{F}_{V_i})$. Under $\mathbb{Q}_{V_i}$ the process X satisfies the hypotheses of Theorem 3.1 and $U_0 \circ \theta_{V_i} = \frac{1}{2}X_0 \circ \theta_{V_i} \le X_0 \circ \theta_{V_i}$. Applying [7], Theorem VI.1.1, to $X_t \circ \theta_{V_i}$ and $U_t \circ \theta_{V_i}$ under the probability measure $\mathbb{Q}_{V_i}$, we conclude that $U_t \le X_t$ for $t \le (T_0(X) \circ \theta_{V_i}) \wedge (T_0(U) \circ \theta_{V_i})$. Therefore

$$\mathbb{P}(U_t \le X_t \text{ for } t \in [V_i, \widetilde{V}_i] \mid \mathcal{F}_{V_i}) = 1, \qquad \mathbb{P} - \text{a.s.}$$

So if $t < T_0(X)$ and $t \in [V_i, \widetilde{V}_i]$, then $U_t \le X_t$, $\mathbb{P}$-almost surely. By the definition of V_i and $\widetilde{V}_i$, if $t < T_0(X)$ and $t \in [\widetilde{V}_i, V_{i+1}]$, then again $U_t \le X_t$. If $V_i \to \infty$, we conclude $U_t \le X_t$ for all $t \le T_0(X)$. Suppose, on the other hand, that $V_i \uparrow V_\infty < \infty$. Since $U_{\widetilde{V}_i} = 0$, by the continuity of the paths of U we see that $U_{V_\infty} = 0$. By the continuity of the paths of X,

$$X_{V_\infty} = \lim_{i\to\infty} X_{V_i} \le 2 \lim_{i\to\infty} U_{V_i} = 2U_{V_\infty} = 0,$$

so $T_0(X) \le V_\infty$. Therefore in this case too, we again have $U_t \le X_t$ for all $t \le T_0(X)$.

Recall that the scale function for U on $(0,\infty)$ is any function $s(x)$ satisfying for $x > 0$ the differential equation

$$\tfrac{1}{2}(x+\varepsilon)^\alpha s''(x) - c_1 s'(x) = 0.$$

We compute the scale function $s(x)$ for the process U_t and find that it is determined by

$$\log s'(x) = \int_0^x \frac{2c_1}{(y+\varepsilon)^\alpha} dy + c_5.$$

We can take $c_5 = 0$ and $s(x) = \int_0^x e^{2c_1(y+\varepsilon)^{1-\alpha}/(1-\alpha)}\, dy$. Note that $s(x)/x \to e^{2c_1\varepsilon^{1-\alpha}/(1-\alpha)}$ since $\alpha < 1$. It follows that $s(U_t)$ corresponds to the operator

$$A_\varepsilon(x)(x+\varepsilon)^\alpha f''(x)$$

where $A_\varepsilon(x)$ is a function of x satisfying

$$c_6 \le A_\varepsilon(x) \le c_7, \qquad |x| \le K,$$

and $0 < c_6 < c_7 < \infty$ do not depend on ε; furthermore the speed measure for the process has no atom at 0. Moreover, we see that $s(\eta)/\eta$ is bounded by c_8 for η small.

We now prove (3.3). For any process R on $\mathbb{R}$, let $\widetilde{S}_0^R = 0$, $S_i^R = \inf\{t > \widetilde{S}_{i-1}^R : R_t = \gamma/2\}$, $\widetilde{S}_i^R = \{t > S_i^R : R_t = \gamma\}$, $i \geq 1$. Since $s(U)$ is on natural scale,

$$\begin{aligned}\mathbb{P}^{\gamma}(S_1^U < T_K(U)) = \mathbb{P}^{s(\gamma)}(S_1^{s(U)} < T_{s(K)}(s(U))) = 1 - \frac{s(\gamma)}{s(K)} \\ \leq 1 - c_9\gamma. \end{aligned} \tag{3.6}$$

If $X_0 = \gamma$, then

$$\mathbb{P}(S_1^X < T_K(X)) \leq \mathbb{P}^{\gamma}(S_1^U < T_K(U)) \leq 1 - c_9\gamma.$$

Under $\mathbb{Q}_{\widetilde{S}_i^X}$ the process X_t satisfies the hypotheses of Theorem 3.1, and so by what has just been proved

$$\begin{aligned}\mathbb{P}(\widetilde{S}_{i+1}^X \leq T_K(X)) &\leq \mathbb{E}[\mathbb{Q}_{\widetilde{S}_i^X}(S_1^X < T_K(X)); \widetilde{S}_i^X \leq T_K(X)] \\ &\leq (1 - c_9\gamma)\mathbb{P}(\widetilde{S}_i^X \leq T_K(X)).\end{aligned}$$

By induction,

$$\mathbb{P}(\widetilde{S}_i^X \leq T_K(X)) \leq c_{10}(1 - c_9\gamma)^i,$$

which implies (3.3).

We turn to the proof of (3.4). First of all, very similarly to the proof of (3.3), the probability of more than m upcrossings of $[\eta/4, \eta/2]$ before time $T_{6\eta}(X)$ is bounded by $c_{11}c_{12}^m$ for some $c_{12} < 1$ independent of m and η. The main difference here is that $s(\eta/2)/s(6\eta)$ is bounded below independently of η provided η is small; cf. (3.6).

Next we obtain a bound on the amount of time X spends in $[\eta/2, \eta]$ before hitting 6η. Let $\widetilde{B}_0 = 0$, $B_i = \inf\{t > \widetilde{B}_{i-1} : X_t = \eta/2\}$, and $\widetilde{B}_i = \inf\{t > B_i : X_t = \eta/4\}$. Observe

$$\mathbb{E}\int_0^{T_{6\eta}(X)} 1_{[\eta/2,\eta]}(X_r)\, dr = \mathbb{E}\sum_{i=0}^{\infty} \int_{B_i \wedge T_{6\eta}(X)}^{\widetilde{B}_i \wedge T_{6\eta}(X)} 1_{[\eta/2,\eta]}(X_r)\, dr. \tag{3.7}$$

Suppose $X_0 = \eta/2$ and U is the solution to (3.5) with $U_0 = \eta/2$. If $r \leq \widetilde{B}_1 \wedge T_{6\eta}(X)$, then $X_r \geq U_r$, and so $T_{6\eta}(X) \leq T_{6\eta}(U)$. Therefore

$$\mathbb{E} \int_0^{\widetilde{B}_1 \wedge T_{6\eta}(X)} 1_{[\eta/2,\eta]}(X_r)\, dr \leq \mathbb{E}[T_{6\eta}(U)]. \tag{3.8}$$

The expected amount of time until U hits 6η is the same as the expected amount of time until $s(U)$ hits $s(6\eta)$, and by symmetry this is the same as the expected amount of time until $\overline{U}$ exits $[-s(6\eta), s(6\eta)]$, where $\overline{U}$ is the process on $\mathbb{R}$ on natural scale corresponding to the operator $A_\varepsilon(|x|)(|x| + \varepsilon)^\alpha f''(x)$, with speed measure not having an atom at 0. Using [3, Corollary 2.4, equation (2.1) and Theorem 3.2], we have that

$$\mathbb{E} \int_0^{\widetilde{B}_1 \wedge T_{6\eta}(X)} 1_{[\eta/2,\eta]}(X_r)\, dr \leq c_{13}\eta^{2-\alpha}. \tag{3.9}$$

Under $\mathbb{Q}_{B_i}$, $X_t \circ \theta_{B_i}$ is a solution to (1.2) and therefore we conclude

$$\begin{aligned}\mathbb{E} \int_{B_i \wedge T_{6\eta}(X)}^{\widetilde{B}_i \wedge T_{6\eta}(X)} 1_{[\eta/2,\eta]}(X_r)\, dr &= \mathbb{E}\Big[\mathbb{E}_{\mathbb{Q}_{B_i}} \int_0^{\widetilde{B}_1 \wedge T_{6\eta}(X)} 1_{[\eta/2,\eta]}(X_r)\, dr; B_i \leq T_{6\eta}(X)\Big] \\ &\leq c_{13}\mathbb{P}(B_i \leq T_{6\eta}(X)). \end{aligned} \tag{3.10}$$

For B_i to be less than or equal to $T_{6\eta}(X)$, the process must have at least $i-1$ upcrossings of $[\eta/4, \eta/2]$ by time $T_{6\eta}(X)$, and the probability of this is bounded by $c_{11}c_{12}^{i-1}$. Combining this, (3.7), (3.9), and (3.10), we have

$$\mathbb{E} \int_0^{T_{6\eta}(X)} 1_{[\eta/2,\eta]}(X_r)\, dr \leq c_{14}\eta^{2-\alpha}. \tag{3.11}$$

Another upcrossing argument gives us a bound on the amount of time X spends in $[\eta/2, \eta]$ before time $T_K(X)$. Let $\widetilde{C}_0 = 0$, $C_i = \inf\{t > \widetilde{C}_{i-1} : X_t = 3\eta\}$, and $\widetilde{C}_i = \inf\{t > C_i : X_t = 6\eta\}$. Similarly to the above and using (3.11), we have

$$\begin{aligned}\mathbb{E} \int_0^{T_K(X)} 1_{[\eta/2,\eta]}(X_r)\, dr &\leq \sum_{i=0}^\infty \mathbb{E} \int_{C_i \wedge T_K(X)}^{\widetilde{C}_i \wedge T_K(X)} 1_{[\eta/2,\eta]}(X_r)\, dr \\ &\leq \sum_{i=0}^\infty c_{14}\eta^{2-\alpha}\mathbb{P}(C_i \leq T_K(X)). \end{aligned} \tag{3.12}$$

In order for C_i to be less than or equal to $T_K(X)$, we must have at least $i-1$ upcrossings of $[3\eta, 6\eta]$ by time $T_K(X)$, and by (3.3) the probability of this is bounded by $c_{15}(1-c_{16}\eta)^{i-1}$. Substituting in (3.12), we conclude

$$\mathbb{E}\int_0^{T_K(X)} 1_{[\eta/2,\eta]}(X_r)\,dr \le c_{14}\eta^{2-\alpha}\sum_{i=0}^{\infty} c_{15}(1-c_{16}\eta)^{i-1} \le c_{17}\eta^{1-\alpha}. \tag{3.13}$$

To conclude the proof of (3.4), we apply (3.13) with η replaced by $\eta 2^{-k}$, $k = 0, 1, \ldots$, and sum over k. Since X spends zero time at 0, we obtain (3.4). □

Define μ on $[0,\infty)^d$ by

$$\mu(dx) = \prod_{i=1}^{d} x_i^{-\alpha_i}\,dx. \tag{3.14}$$

This measure will be used later on; it is characterized by the fact that the operator $\sum_{i=1}^d x_i^{\alpha_i} f_{ii}$ is self-adjoint with respect to μ. As mentioned in Section 2, $\|f\|_p$ denotes the L^p norm of f with respect to μ while $\|f\|_{L^p(dx)}$ is the L^p norm with respect to Lebesgue measure.

Now we turn to our inequality of Krylov type. Theorem 3.1 shows that the process does not spend too much time near the boundary of $\mathbb{R}^d_+$. So we can restrict our attention to functions whose support are a positive distance from the boundary. We do a transformation of the state space so that we have a uniformly elliptic operator with a large drift, and use Krylov's inequality ([8]).

Theorem 3.2 *Let $M > 1$ be fixed and $\lambda > 0$. There exist p_0 and c_1 depending on $\alpha_1, \ldots, \alpha_d, M, \lambda$, and d such that if f is supported in $[0, M]^d$, then for any solution X to (1.2)*

$$\left|\mathbb{E}\int_0^\infty e^{-\lambda s} f(X_s)\,ds\right| \le c_1\|f\|_{p_0} \tag{3.15}$$

and

$$\left|\mathbb{E}\int_0^\infty e^{-\lambda s} f(X_s)\,ds\right| \le c_1\|f\|_{L^{p_0}(dx)}. \tag{3.16}$$

Proof. Let us set $D = [0, 2M]^d$ and $E = [0, M]^d$. Let $A \subset E$ and let $\varepsilon = |A|$. Let $\delta > 0$ be a small positive real to be chosen later and let $F = [\varepsilon^\delta, 2M]^d$. Let $A' = A \cap F$.

Our first goal is to show that there exists K_1 and γ depending only on $d, \alpha_1, \ldots, \alpha_d, M$, and λ, but not ε, such that

$$\mathbb{E} \int_0^{\tau_F} 1_{A'}(X_s)\, ds \le c_2 \varepsilon^{-\delta K_1 + \gamma}. \tag{3.17}$$

Define the map $\Gamma : \mathbb{R}_+^d \to \mathbb{R}_+^d$ by

$$\Gamma(x_1, \ldots, x_d) = ([1 - \tfrac{\alpha_1}{2}]^{-1}(x_1)^{1-(\alpha_1/2)}, \ldots, [1 - \tfrac{\alpha_d}{2}]^{-1}(x_d)^{1-(\alpha_d/2)}).$$

Let $Y_t = \Gamma(X_t)$. We use Ito's formula to see that for $t \le \tau_F(X)$

$$dY_t^i = \sqrt{2} a_i(X_t)\, dW_t^i + \Big[((1 - \tfrac{\alpha_i}{2})(Y_t^i))^{-\alpha_i/(2-\alpha_i)} b_i(X_t) - (1 - \tfrac{\alpha_i}{2})^{-1} \frac{\alpha_i}{4Y_t^i}\Big]\, dt.$$

Since X is in the set F, there is no issue of the degeneracy of X_t^i at 0 causing problems. Notice that for $X \in F$ the drift coefficient of Y^i is bounded by $c_3 \varepsilon^{-K_2 \delta}$. To obtain (3.17) it suffices to bound

$$\mathbb{E} \int_0^{\tau_{\Gamma(F)}} 1_{\Gamma(A')}(Y_s)\, ds. \tag{3.18}$$

Note

$$|\Gamma(A')| \le c_4 \prod_{i=1}^d (\varepsilon^\delta)^{-\alpha_i/2} |A'| \le c_5 \varepsilon^{-\delta K_3 + 1}$$

for some $K_3 > 0$. Let $\overline{Y}$ be the process whose coefficients agree with those of Y up until the time that the process Y leaves $\Gamma(F)$ and is a d-dimensional Brownian motion thereafter. Let G be a ball of radius $c_6 M$ that contains F.

We need a bound on the expected time to leave G. By (3.4), $\mathbb{E}[\tau_{\Gamma(F)}(Y)] = \mathbb{E}[\tau_F(X)]$ is bounded by a constant not depending on ε or δ. Since $\overline{Y}$ is a Brownian motion for $t > \tau_{\Gamma(F)}$, it is then clear that $\mathbb{E}[\tau_G(\overline{Y})]$ is also bounded by a constant independent of ε and δ.

We use Krylov [8] to obtain the bound

$$\mathbb{E} \int_0^{\tau_G(\overline{Y})} 1_{\Gamma(A')}(\overline{Y}_s)\, ds \le c_7 (1 + c_8 \varepsilon^{-K_2 \delta} \mathbb{E}[\tau_G(\overline{Y})]) |\Gamma(A')|^{1/d}.$$

This inequality follows from a passage to the limit (as $t \to \infty$) in equation (4) of [8]. We have

$$\mathbb{E}\int_0^{\tau_F(X)} 1_{A'}(X_s)\,ds \le \mathbb{E}\int_0^{\tau_{\Gamma(F)}(Y)} 1_{\Gamma(A')}(Y_s)\,ds \qquad (3.19)$$
$$\le \mathbb{E}\int_0^{\tau_{G(\overline{Y})}} 1_{\Gamma(A')}(\overline{Y}_s)\,ds.$$

We deduce

$$\mathbb{E}\int_0^{\tau_F(X)} 1_{A'}(X_s)\,ds \le c_9(1+\varepsilon^{-K_2\delta})\varepsilon^{(-\delta K_3+1)/d}.$$

This proves (3.17) with $K_1 = K_2 + K_3/d$ and $\gamma = 1/d$.

Next we will show that if $A \subset E$, then there exist K_4 and K_5 depending only on $d, \alpha_i, \ldots, \alpha_d, \lambda$, and M, such that

$$\mathbb{E}\int_0^{\tau_D} 1_A(X_s)\,ds \le c_{10}(\varepsilon^{-K_4\delta+\gamma} + \varepsilon^{\delta K_5}). \qquad (3.20)$$

Write $A = A_1 \cup A_2$, where $A_1 = A \cap (\cup_{i=1}^d \{0 \le x_i \le \varepsilon^\delta\})$ and $A_2 = A \setminus A_1$. Note $A_2 \subset F$. By Theorem 3.1, we know

$$\mathbb{E}\int_0^{\tau_D} 1_{A_1}(X_s)\,ds \le \sum_{i=1}^d \mathbb{E}\int_0^{\tau_D} 1_{[0,\varepsilon^\delta]}(X_s^i)\,ds \le c_{11}\varepsilon^{\delta K_5}. \qquad (3.21)$$

So we need to bound

$$\mathbb{E}\int_0^{\tau_D} 1_{A_2}(X_s)\,ds.$$

We need to insure that X does not visit F too many times before exiting D. Let $T_0 = 0$, $S_i = \inf\{t > T_{i-1} : X_t \in F\}$, and $T_i = \inf\{t > S_i : X_t \notin [\varepsilon^\delta/2, 2M]^d\}$, $i \ge 1$. Recall that $\mathbb{Q}_{S_i}$ is used for a regular conditional probability. Then using (3.17)

$$\mathbb{E}\int_0^{\tau_D} 1_{A_2}(X_s)\,ds = \sum_{i=1}^\infty \mathbb{E}\Big[\int_{S_i}^{T_i} 1_{A_2}(X_s)\,ds; S_i < \tau_D\Big] \qquad (3.22)$$
$$= \sum_{i=1}^\infty \mathbb{E}\Big[\mathbb{E}_{\mathbb{Q}_{S_i}}\int_0^{T_1} 1_{A_2}(X_s)\,ds; S_i < \tau_D\Big]$$
$$\le c_{12}\varepsilon^{-K_1\delta+\gamma}\sum_{i=1}^\infty \mathbb{P}(S_i < \tau_D).$$

Now in order for S_i to be less than τ_D, we must have for some $j \le d$ at least $(i-1)/d$ upcrossings of X^j from $\varepsilon^\delta/2$ to ε^δ before hitting the level $2M$. We know by Theorem 3.1 that the probability of this happening is less than $c_{13}(1-c_{14}\varepsilon^\delta)^{(i-1)/d}$. Therefore

$$\sum_{i=1}^\infty \mathbb{P}(S_i < \tau_D) \le c_{15}\varepsilon^{-K_6\delta}.$$

Combining with (3.21), we have

$$\mathbb{E}\int_0^{\tau_D} 1_A(X_s)\,ds \le c_{16}(\varepsilon^{\delta K_5} + \varepsilon^{-K_1\delta+\gamma-K_6\delta}), \tag{3.23}$$

which yields (3.20) with $K_4 = K_1 + K_6$.

The next step is to show that there exists K_7 such that if $A \subset E$, then

$$\mathbb{E}\int_0^\infty e^{-\lambda s} 1_A(X_s)\,ds \le c_{17}(\varepsilon^{-K_7\delta+\gamma} + \varepsilon^{\delta K_5}). \tag{3.24}$$

Our approach is to show that if X is not in D, then X will not reach E too quickly. Let $V_0 = 0$, $U_i = \inf\{t > V_{i-1} : X_t \in E\}$, $V_i = \inf\{t > U_i : X_t \notin D\}$, $i \ge 1$. Suppose X starts at $x \notin [0,2M)^d$. Then there exist c_{18} and c_{19} not depending on x such that $\mathbb{P}(U_1 > c_{18}) > c_{19}$. This is because if $x \notin [0,2M)^d$, then at least one coordinate, say, $X_0^{i_0}$, of X_0 is greater than or equal to $2M$, and in the range $(M,3M)$, X^{i_0} is a diffusion whose diffusion coefficients are bounded above and below and whose drift coefficient is bounded above; therefore there exist c_{18} and c_{19} such that

$$\mathbb{P}(\sup_{s\le c_{18}} |X_s^{i_0} - X_0^{i_0}| > M/2) \le 1 - c_{19}.$$

We conclude

$$\begin{aligned}\mathbb{E}e^{-\lambda U_1} &\le \mathbb{P}(U_1 \le c_{18}) + e^{-\lambda c_{18}}\mathbb{P}(U_1 > c_{18})\\ &= (1-e^{-\lambda c_{18}})\mathbb{P}(U_1 \le c_{18}) + e^{-\lambda c_{18}}\\ &\le (1-e^{-\lambda c_{18}})(1-c_{19}) + e^{-\lambda c_{18}} := \rho.\end{aligned}$$

Note that $\rho < 1$. We then have

$$\begin{aligned}\mathbb{E}e^{-\lambda U_i} &= \mathbb{E}\Big[e^{-\lambda V_{i-1}}\mathbb{E}\Big[e^{-\lambda(U_i-V_{i-1})} \mid \mathcal{F}_{V_{i-1}}\Big]\Big]\\ &\le \mathbb{E}\Big[e^{-\lambda U_{i-1}}\mathbb{E}_{\mathbb{Q}_{V_{i-1}}} e^{-\lambda U_1}\Big] \le \rho\mathbb{E}e^{-\lambda U_{i-1}}.\end{aligned}$$

So by induction

$$\mathbb{E}e^{-\lambda U_i} \le \rho^{i-1}.$$

Then using (3.20)

$$\begin{aligned}
\mathbb{E}\int_0^\infty e^{-\lambda s} 1_A(X_s)\,ds &= \sum_{i=1}^\infty \mathbb{E}\int_{U_i}^{V_i} e^{-\lambda s} 1_A(X_s)\,ds \\
&= \sum_{i=1}^\infty \mathbb{E}\Big[e^{-\lambda U_i}\mathbb{E}_{\mathbb{Q}_{U_i}}\int_0^{V_1} e^{-\lambda s} 1_A(X_s)\,ds\Big] \\
&\le c_{20}(\varepsilon^{-K_4\delta+\gamma} + \varepsilon^{K_5\delta}) \sum_{i=1}^\infty \mathbb{E}e^{-\lambda U_i} \\
&\le c_{21}(\varepsilon^{-K_4\delta+\gamma} + \varepsilon^{K_5\delta}),
\end{aligned}$$

which proves (3.24).

We now choose $\delta = \gamma/(2K_7)$. Substituting this value of δ in (3.24), we obtain

$$\mathbb{E}\int_0^\infty e^{-\lambda s} 1_A(X_s)\,ds \le c_{22}\varepsilon^{\gamma'}, \tag{3.25}$$

where $\gamma' = (\gamma/2)(1 + K_5/K_7)$.

Some real analysis finishes the proof. Let $p_0 = 2/\gamma'$. Now suppose $f \in L^{p_0}(\mathbb{R}_+^d)$ with support in $[0, M]^d$. By multiplying by a constant, it suffices to consider the case where $\|f\|_{L^{p_0}(dx)} = 1$. Without loss of generality we may also suppose $f \ge 0$. Let $A_n = \{x : f(x) \ge 2^n\}$. Then

$$|A_n| \le \frac{\|f\|^{p_0}_{L^{p_0}(dx)}}{(2^n)^{p_0}} = 2^{-np_0}.$$

We then have

$$\mathbb{E}\int_0^\infty e^{-\lambda s} 1_{A_n}(X_s)\,ds \le c_{22}2^{-np_0\gamma'}.$$

Thus

$$\begin{aligned}
\mathbb{E}\int_0^\infty e^{-\lambda s} f(X_s)\,ds &\le \frac{1}{\lambda} + \sum_{n=0}^\infty 2^{n+1}\mathbb{E}\int_0^\infty e^{-\lambda s} 1_{A_n}(X_s)\,ds \\
&\le \frac{1}{\lambda} + \sum_{n=0}^\infty 2^{n+1} c_{22} 2^{-np_0\gamma'} \le c_{23} < \infty.
\end{aligned}$$

Since $\|f\|_{L^{p_0}(dx)} = 1$, the proof of (3.15) is complete.

If $A \subset [0, M]^d$, then

$$|A| = \int_{[0,M]^d} \prod_{i=1}^{d} x_i^{\alpha_i} 1_A(x)\, \mu(dx) \le M^d \mu(dx).$$

From this and (3.25) we see

$$\mathbb{E} \int_0^\infty e^{-\lambda s} 1_A(X_s)\, ds \le c_{22} |A|^{\gamma'} \le c_{24} \mu(A)^{\gamma'}.$$

Proceeding exactly as in the previous paragraph, we obtain (3.16). □

Remark 3.3 Once uniqueness is established, this theorem shows that the resolvent of X maps L^{p_0} functions with support in $[0, M]^d$ into L^∞.

4 Existence

In this section we prove existence of a solution to the submartingale problem. for the operator $\mathcal{L}$ defined in (1.1) with Neumann boundary conditions on Δ.

There are two complications that are not present in the usual case: we need to show that our solution spends zero time on the set Δ defined in (2.1); and unless the α_i are small, we cannot use the Girsanov transformation to reduce to the case of zero drift.

For $\varepsilon > 0$ let $\mathcal{L}^\varepsilon$ be the operator defined by

$$\mathcal{L}^\varepsilon f(x) = \sum_{i=1}^{d} a_i(x)(x_i + \varepsilon)^{\alpha_i} f_{ii}(x) + \sum_{i=1}^{d} b_i(x) f_i(x),$$

again with Neumann boundary conditions on $\Delta = \partial(\mathbb{R}_+^d)$. The diffusion coefficients are uniformly positive definite and continuous and are of at most linear growth, so there exists a unique solution to the submartingale problem for $\mathcal{L}^\varepsilon$ started from x_0 for every x_0; let us denote it $\mathbb{P}_\varepsilon$. (We reflect the coefficients over the coordinate axes, construct the solution to the corresponding martingale problem on $\mathbb{R}^d$, and then look at the law of $(|X^1|, \ldots, |X^d|)$.)

We remark that the statement and proof of Proposition 2.3 apply equally well to solutions to (3.2). It is now standard (cf. Section VI.1 of [3]) that the $\mathbb{P}_\varepsilon$ are a tight sequence of probability measures on $C([0,\infty);\mathbb{R}^d_+)$ and there must exist a subsequence ε_j such that $\mathbb{P}_{\varepsilon_j}$ converges weakly. Denote the limit measure by $\mathbb{P}$ and the corresponding expectation by $\mathbb{E}$. It is obvious that $\mathbb{P}(X_0 = x_0) = 1$.

Proposition 4.1 *Under $\mathbb{P}$ the process spends zero time in Δ, i.e.,*

$$\int_0^\infty 1_\Delta(X_s)ds = 0, \qquad \mathbb{P} - a.s.$$

Proof. Under $\mathbb{P}_\varepsilon$, the ith component of X_t will satisfy an SDE of the form

$$dX_t^i = \sqrt{2a_i(X_t)}(X_t^i + \varepsilon)^{\alpha_i/2} dW_t^i + b_i(X_t)\,dt + dL_t^{X^i},$$

where L^{X^i} is a local time at 0. Applying Theorem 3.1 the amount of time X_t^i spends in $[0,\eta]$ before exceeding K under $\mathbb{P}_\varepsilon$ is bounded by $c_1\eta^{1-\alpha_i}$, where c_1 may depend on K but not ε. Taking a limit as $\varepsilon \to 0$

$$\mathbb{E}_\mathbb{P} \int_0^{\tau_K(X^i)} 1_{[0,\eta]}(X_s^i)ds \le c_1\eta^{1-\alpha_i}.$$

Letting $\eta \to 0$, we have

$$\mathbb{E}_\mathbb{P} \int_0^{\tau_K(X^i)} 1_{\Delta_i}(X_s)\,ds = 0.$$

Since K is arbitrary, and using this argument for each $i = 1, \ldots, d$, we obtain

$$\mathbb{E}_\mathbb{P} \int_0^\infty 1_\Delta(X_s)ds = 0, \tag{4.1}$$

and hence the amount of time spent in Δ is 0 almost surely. □

Proposition 4.2 *$\mathbb{P}$ is a solution to the submartingale problem for $\mathcal{L}$ started at x_0.*

Proof. The proof of this proposition is very similar to the proof of Theorem VI.1.3 in [3] and we give only a brief sketch, leaving the details to the reader. Note that Theorem 3.2 is needed because the drift coefficients are not assumed to be continuous.

If $f \in C_b^2$ is such that $f_i \geq 0$ on Δ_i, then

$$M_t^\varepsilon = f(X_t) - f(X_0) - \int_0^t \mathcal{L}^\varepsilon f(X_r)\, dr \tag{4.2}$$

is a submartingale. The goal is to show that one can take a limit as $\varepsilon \to 0$. It suffices to show that if $Y = \prod_{i=1}^n g_i(X_{r_i})$, where the g_i are bounded continuous functions with compact support and $r_1 \leq r_2 \leq \cdots \leq r_n \leq s \leq t$, then

$$\mathbb{E}[M_t Y] \geq \mathbb{E}[M_s Y],$$

where $M_t = f(X_t) - f(X_0) - \int_0^t \mathcal{L} f(X_r)\, dr$. We know $\mathbb{E}_\varepsilon[M_t^\varepsilon Y] \geq \mathbb{E}_\varepsilon[M_s^\varepsilon Y]$, where $\mathbb{E}_\varepsilon$ is the law of the process corresponding to $\mathcal{L}^\varepsilon$. The key step is to show

$$\mathbb{E}_\varepsilon\Big[Y \int_0^t \mathcal{L} f(X_r)\, dr\Big] \to \mathbb{E}\Big[Y \int_0^t \mathcal{L} f(X_r)\, dr\Big]; \tag{4.3}$$

because the b_i are not continuous, then $\mathcal{L} f$ is not continuous. We approximate $\mathcal{L} f$ by a bounded continuous function G, that is, so that $\|\mathcal{L}F - G\|_{L^{p_0}(dx)}$ is small. Then by the definition of weak convergence,

$$\mathbb{E}_\varepsilon\Big[Y \int_0^t G(X_r)\, dr\Big] \to \mathbb{E}\Big[Y \int_0^t G(X_r)\, dr\Big], \tag{4.4}$$

provided we take the limit along the appropriate subsequence of ε's. Theorem 3.2 shows that the left and right hand sides of (4.4) are close to the left and right hand sides of (4.3), resp. □

5 First order estimates

We first consider the continuous strong Markov process Z_t on $[0, \infty)$ associated with the operator

$$\mathcal{A}_Z f(x) = x^\alpha f''(x).$$

Here $\alpha \in (0,1)$ and we impose Neumann (i.e., reflecting) boundary conditions at 0. More precisely, we have a process on natural scale whose speed measure has no atom at 0 and does not charge $(-\infty, 0]$.

Let
$$b = 1 - \frac{\alpha}{2},$$
and note that $b \in (\frac{1}{2}, 1)$. If we set
$$Y_t = \frac{1}{b\sqrt{2}} Z_t^b,$$
a straightforward calculation shows that Y is a continuous strong Markov process on $[0,\infty)$ associated to the operator
$$\mathcal{A}_Y f(x) = \tfrac{1}{2} f''(x) + \frac{b-1}{2bx} f'(x)$$
with reflection at 0, i.e., a Bessel process of order $\delta = \frac{b-1}{b} + 1$ with reflection at 0. By [13] the transition densities of Y (with respect to Lebesgue measure) are given by
$$p_Y(t,x,y) = \Big(\frac{y}{x}\Big)^{\nu} \frac{y}{t} e^{-(x^2+y^2)/2t} I_\nu(xy/t),$$
where $\nu = \frac{\delta}{2} - 1 = -\frac{1}{2b}$ and I_ν is the standard modified Bessel function.

A change of variables then gives
$$p_Z(t,x,y) = \frac{c_1}{t} y^{2b-\frac{3}{2}} e^{-c_2 y^{2b}/2t} x^{\frac{1}{2}} e^{-c_2 x^{2b}/2t} I_\nu(c_2 x^b y^b/t) \tag{5.1}$$
and we have the scaling relationship
$$p_Z(t,x,y) = t^{-1/2b} p_Z(1, xt^{-1/2b}, yt^{-1/2b}). \tag{5.2}$$

We will need the following lemma, the proof of which is given in the appendix. The proof consists of lengthy calculation.

Lemma 5.1 *There exists a constant c_1 such that*
$$\sup_x \int_0^\infty \Big| \frac{\partial}{\partial x} p_Z(t,x,y) \Big| dy \le c_1 t^{-\frac{1}{2b}}; \tag{5.3}$$
$$\sup_y y^\alpha \int_0^\infty \Big| \frac{\partial}{\partial x} p_Z(t,x,y) \Big| x^{-\alpha} dx \le c_1 t^{-\frac{1}{2b}}. \tag{5.4}$$

Let P_t be the semigroup for Z_t, i.e., $P_t f(x) = \mathbb{E}^x f(Z_t)$. Let $\mu(dx) = x^{-\alpha} dx$ and we consider the space $L^2(\mathbb{R}_+, \mu)$. We use the above estimates to prove the following proposition, which is a variant of the classical Young inequality (cf. [2], Theorem IV.5.1).

Proposition 5.2 *Suppose $p \in (1, \infty)$. There exists a constant c_1 depending only on p such that*

$$\|(P_t f)'\|_p \le c_1 t^{-1/2b} \|f\|_p, \qquad f \in L^2(\mathbb{R}_+, \mu).$$

Proof. Fix $t > 0$ and write $K(x, y)$ for $|\frac{\partial}{\partial x} p(t, x, y)|$. Let q be the conjugate exponent to p. Then by Lemma 5.1 we have

$$\begin{aligned}
\|(P_t f)'\|_p^p &\le \int_{\mathbb{R}_+} \Big[\int_{\mathbb{R}_+} K(x,y) |f(y)| \, dy \Big]^p x^{-\alpha} dx \\
&= \int_{\mathbb{R}_+} \Big[\int_{\mathbb{R}_+} K(x,y)^{1/q} K(x,y)^{1/p} |f(y)| \, dy \Big]^p x^{-\alpha} dx \\
&\le \int_{\mathbb{R}_+} \Big[\int_{\mathbb{R}_+} K(x,y) \, dy \Big]^{p/q} \Big[\int_{\mathbb{R}_+} K(x,y) |f(y)|^p \, dy \Big] x^{-\alpha} dx \\
&\le c_2 t^{-(1/2b)(p/q)} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K(x,y) x^{-\alpha} \, dx \, |f(y)|^p \, dy \\
&\le c_3 t^{-(1/2b)((p/q)+1)} \int_{\mathbb{R}_+} |f(y)|^p y^{-\alpha} \, dy.
\end{aligned}$$

Now take p^{th} roots of both sides. □

Now let us turn to the d-dimensional case. The analogue of Proposition 5.2 follows readily from that proposition. We suppose Z_t^i is the process on $\mathbb{R}$ corresponding to the operator

$$\mathcal{A}_i f(x) = x^{\alpha_i} f''(x), \tag{5.5}$$

with the speed measure having no atom at 0 and not charging $(-\infty, 0)$ and $\alpha_i \in (0, 1)$. Then $Z_t^i \ge 0$ for all t and is reflecting at 0. Set $b_i = 1 - \frac{\alpha_i}{2}$. We let $p_i(t, x, y)$ denote the transition densities of Z_t^i, P_t^i the corresponding semigroups, let $Z_t = (Z_t^1, \ldots, Z_t^d)$, and let $p(t, x, y) = \prod_{i=1}^d p_i(t, x_i, y_i)$ be the transition densities for Z when $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$. Let P_t now denote the semigroup for Z. Let μ_i be the measure on $\mathbb{R}_+$ whose Radon-Nikodym derivative with respect to d-dimensional Lebesgue measure is $x_i^{-\alpha_i}$ and let μ be the measure on $\mathbb{R}_+^d$ given by

$$\mu(dx) = \prod_{i=1}^{d} \mu_i(dx_i). \tag{5.6}$$

We have the analogue of Proposition 5.2.

Proposition 5.3 *There exists a constant c_1 such that for each i*

$$\Big\| \frac{\partial(P_t f)}{\partial x_i} \Big\|_p \le c_1 t^{-1/2b_i} \|f\|_p, \qquad f \in L^p(\mathbb{R}_+^d, \mu).$$

Proof. We will prove this in the case $i = 1$, the case for other i's being exactly similar. Let $\overline{x} = (x_2, \dots, x_d)$. Let $f \in L^2(\mathbb{R}_+^d, \mu)$ and set

$$F(x_1; \overline{x}) = \int \cdots \int \prod_{j=2}^{d} p_j(t, x_j, y_j) f(x_1, y_2, \dots y_d)\, dy_2 \, \cdots \, dy_d.$$

Then

$$\Big| \frac{\partial}{\partial x_1} P_t f(x) \Big| = \Big| \frac{\partial}{\partial x_1} P_t^1 F(x_1; \overline{x}) \Big|,$$

and so by Proposition 5.2 we have

$$\int \Big| \frac{\partial P_t f(x)}{\partial x_1} \Big|^p \mu_1(dx_1) \le c_2 t^{-p/2b_1} \int |F(x_1; \overline{x})|^p \mu_1(dx_1).$$

If we integrate both sides with respect to $\overline{\mu}(dx_2 \cdots dx_d) = \prod_{j=2}^{d} \mu_j(dx_j)$, we will have our result provided we show

$$\int |F(x_1; \overline{x})|^p \mu(dx) \le \int |f|^p \mu(dx). \tag{5.7}$$

To prove (5.7) let $\overline{P}_t$ be the semigroup corresponding to $(Z_t^2, \dots, Z_t^d)$. It is easy to check that $\sum_{j=2}^{d} \mathcal{A}_j$ is self-adjoint with respect to the measure $\overline{\mu}$. Therefore, using Jensen's inequality,

$$\|\overline{P}_t g\|_{L^p(\overline{\mu})} \le \|g\|_{L^p(\overline{\mu})}, \qquad g : \mathbb{R}_+^{d-1} \to \mathbb{R},$$

or

$$\int |\overline{P}_t g(x)|^p \overline{\mu}(dx) \leq \int |g(x)|^p \overline{\mu}(dx). \tag{5.8}$$

We hold x_1 fixed and apply this to $g(\overline{x}; x_1) = f(x_1, \ldots, x_d)$. Note $\overline{P}_t g(\cdot; x_1) = F(x_1; \overline{x})$. So applying (5.8) to this g, we have

$$\int |F(x_1; \overline{x})|^p \overline{\mu}(dx) \leq \int |f(x_1, \ldots, x_d)|^p \overline{\mu}(dx).$$

(5.7) follows by integrating both sides of this equation with respect to $\mu_1(dx_1)$.
□

Our main result of this section is the following. Let

$$R_\lambda f = \int_0^\infty e^{-\lambda t} P_t f \, dt.$$

Theorem 5.4 *There exists c_1 such that for each i*

$$\|\partial(R_\lambda f)/\partial x_i\|_p \leq c_1 \lambda^{\frac{1}{2b_i}-1} \|f\|_p.$$

Proof. Since $-1/(2b_i) > -1$, the result follows from Proposition 5.3, dominated convergence, and Minkowski's inequality for integrals:

$$\begin{aligned} \Big\| \frac{\partial}{\partial x_i} \int_0^\infty e^{-\lambda t} P_t f \, dt \Big\|_p &= \Big\| \int_0^\infty e^{-\lambda t} \frac{\partial P_t f}{\partial x_i} dt \Big\|_p \\ &\leq \int_0^\infty e^{-\lambda t} c_2 t^{-1/(2b_i)} dt \|f\|_p. \end{aligned}$$

□

Remark 5.5 Only very minor changes are needed to get the same conclusion as in Theorem 5.4 if we instead set R_λ to be the resolvent for the operator $\sum_{i=1}^d a_i \mathcal{A}_i$, where the a_i are strictly positive finite constants.

We need one more estimate.

Lemma 5.6 *If g is in C^2 with compact support contained in $(0,\infty)^d$, then for each $t>0$ and $\lambda>0$ we have that $P_t g$ and $P_t R_\lambda g$ are C^2 on $\mathbb{R}^d_+$ and for each i we have $(P_t R_\lambda g)_i = 0$ on Δ_i.*

Proof. We have a formula for the derivative of the transition density in the one-dimensional case in (8.6) below in the appendix. If we differentiate once more and use the fact that the transition densities for the process factor as a product of transition densities of one-dimensional processes, then tedious calculations show that $P_t g$ is C^2 with normal derivative 0 on the boundary. (This is somewhat easier than in the proof of Lemma 5.1 since we can use the fact that g has compact support.) Moreover one can show that the second derivatives of $P_t g$ grow with t at most polynomially. Since $P_t R_\lambda g = \int_0^\infty e^{-\lambda s} P_{s+t} g\, ds$ by the semigroup property, the lemma follows. □

6 Second order estimates

Let $\mathcal{A}_i$ be defined by (5.5), and let P^i_t be the semigroup corresponding to the process Z^i_t associated with $\mathcal{A}_i$ that spends zero time at 0. We let P_t be the semigroup corresponding to the process $Z_t = (Z^1_t, \dots, Z^d_t)$, where the Z^i_t are independent. The independence implies that if $f(z) = \prod_{i=1}^d f^{(i)}(z_i)$ and $z = (z_1, \dots, z_d)$, then $P_t f(z) = \prod_{i=1}^d P^i_t f^{(i)}(z_i)$. Let

$$R_\lambda f(z) = \int_0^\infty e^{-\lambda t} P_t f(z)\, dt, \qquad \lambda \geq 0, \tag{6.1}$$

be the resolvent for Z.

We let U_t be the Poisson semigroup defined in terms of P_t:

$$U_t = \int_0^\infty \Big(\frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2} \Big) P_s\, ds; \tag{6.2}$$

see [11], p. 127; U_t is also known as the Cauchy semigroup.

The semigroup P^i_t is self-adjoint on $(\mathbb{R}_+, \mu_i)$, where $\mu_i(dz) = |z_i|^{-\alpha_i}\, dz_i$. We let $\mu(dz) = \prod_{i=1}^d \mu_i(dz_i)$ be the product measure on $\mathbb{R}^d$.

We use spectral theory to prove the following.

Lemma 6.1 *Let f, h be C^2 on $\mathbb{R}_+^d$ with compact support in $(0,\infty)^d$ we have the identity*

$$\int (\mathcal{A}_i R_0 f(z)) h(z)\,\mu(dz) = \int \int_0^\infty t(\mathcal{A}_i U_{t/2} f(z))(U_{t/2} h(z))\,dt\,\mu(dz). \quad (6.3)$$

Proof. Using the spectral theorem, there exists (see [12], Theorem 13.30) a spectral representation

$$P_t^i = \int_0^\infty e^{-\lambda_i t} dE_{\lambda_i}^i, \qquad i = 1, \dots, d.$$

Write $s(\lambda) = \sum_{i=1}^d \lambda_i$ if $\lambda = (\lambda_1, \dots, \lambda_d)$. If $f(z) = \prod_{i=1}^d f^{(i)}(z_i)$, then

$$P_t f = \int_0^\infty \cdots \int_0^\infty e^{-ts(\lambda)} dE_{\lambda_1}^1(f^{(1)}) \cdots dE_{\lambda_d}^d(f^{(d)}),$$

and since $\int_0^\infty e^{-ts(\lambda)}\,dt = 1/s(\lambda)$, then

$$R_0 f = \int_0^\infty \cdots \int_0^\infty \frac{1}{s(\lambda)} dE_{\lambda_1}^1(f^{(1)}) \cdots dE_{\lambda_d}^d(f^{(d)}).$$

We have by (6.2) and ([11], p. 127, Equation (5)) that

$$U_t f = \int_0^\infty \cdots \int_0^\infty e^{-t\sqrt{s(\lambda)}} dE_{\lambda_1}^1(f^{(1)}) \cdots dE_{\lambda_d}^d(f^{(d)}).$$

Note also

$$\mathcal{A}_i = \int_0^\infty \lambda_i dE_{\lambda_i}^i.$$

Therefore, if $h(z) = \prod_{i=1}^d h^{(i)}(z_i)$, the left hand side of (6.3) is

$$\int_0^\infty \cdots \int_0^\infty \frac{\lambda_i}{s(\lambda)} d(E_{\lambda_1}^1(f^{(1)}), E_{\lambda_1}^1(h^{(1)})) \cdots d(E_{\lambda_d}^d(f^{(d)}), E_{\lambda_d}^d(h^{(d)})). \quad (6.4)$$

We use here $(\,\cdot, \cdot\,)$ for the inner product in $L^2(\mu)$.

Similarly, the right hand side of (6.3) is

$$\begin{aligned}
&\int_0^\infty \Big[\int_0^\infty \cdots \int_0^\infty t\,\lambda_i e^{-(t/2)\sqrt{s(\lambda)}} e^{-(t/2)\sqrt{s(\lambda)}} d(E_{\lambda_1}^1(f^{(1)}), E_{\lambda_1}^1(h^{(1)})) \\
&\qquad \cdots d(E_{\lambda_d}^d(f^{(d)}), E_{\lambda_d}^d(h^{(d)})) \Big]\, dt \\
&= \int_0^\infty \Big[\int_0^\infty \cdots \int_0^\infty t\,\lambda_i e^{-t\sqrt{s(\lambda)}} d(E_{\lambda_1}^1(f^{(1)}), E_{\lambda_1}^1(h^{(1)})) \\
&\qquad \cdots d(E_{\lambda_d}^d(f^{(d)}), E_{\lambda_d}^d(h^{(d)})) \Big]\, dt.
\end{aligned}$$

Since
$$\int_0^\infty te^{-Kt}\,dt = \frac{1}{K^2},$$
this is equal to (6.4). Linear combinations of functions of the form $\prod_{i=1}^d f^{(i)}(z_i)$ are dense in $L^2(\mu)$, and an approximation argument completes the proof. □

We use the notation $u_f(z,t) = U_t f(z)$ and similarly with f replaced by h. For $t > 0$ and $f \in C^2(\mathbb{R}_+^d)$ with support disjoint from the boundary, we have that $\partial u_f/\partial z_i$ exists by Lemma 5.6.

The main theorem of this section is the following.

Theorem 6.2 *Let $1 < p < \infty$. There exists a constant c_1 depending only on p such that*
$$\|\mathcal{A}_i R_0 f\|_p \le c_1 \|f\|_p.$$

Proof. Let f and h be C^2 with compact support contained in $(0,\infty)^d$. Using Lemma 6.1, integration by parts, and a change of variables, we have
$$\begin{aligned}
\Big|\int (\mathcal{A}_i R_0 f(z)) h(z)\,\mu(dz)\Big| \qquad\qquad (6.5)\\
&= \Big|\int\int_0^\infty t(\mathcal{A}_i U_{t/2} f(z))(U_{t/2}h(z))\,dt\,\mu(dz)\Big|\\
&= \Big|\int\int_0^\infty t|z_i|^{\alpha_i}\frac{\partial U_{t/2}f}{\partial z_i}(z)\frac{\partial U_{t/2}h}{\partial z_i}(z)\,dt\,\mu(dz)\Big|\\
&= 4\Big|\int\int_0^\infty t|z_i|^{\alpha_i}\frac{\partial U_t f}{\partial z_i}(z)\frac{\partial U_t h}{\partial z_i}(z)\,dt\,\mu(dz)\Big|\\
&\le 4\int\int_0^\infty t|z_i|^{\alpha_i}\Big|\frac{\partial U_t f}{\partial z_i}(z)\Big|\,\Big|\frac{\partial U_t h}{\partial z_i}(z)\Big|\,dt\,\mu(dz).
\end{aligned}$$
Let W_t be a Brownian motion independent of Z and define the measure $\xi_s(dz\,dt)$ on $\mathbb{R}_+^d \times [0,\infty)$ by $\xi_s(dz\,dt) = \mu(dz)\delta_s(dt)$, where δ_s is point mass at s. Let $\tau = \inf\{t : W_t = 0\}$.

An application of Ito's formula shows that $u_f(Z_{t\wedge\tau}, W_{t\wedge\tau})$ is a martingale, and
$$\langle u_f(Z,W)\rangle_t = \int_0^{t\wedge\tau}\Big(\sum_{j=1}^d |Z_s^j|^{\alpha_j}\Big|\frac{\partial u_f}{\partial z_j}(Z_s,W_s)\Big|^2 + \Big|\frac{\partial u_f}{\partial t}(Z_s,W_s)\Big|^2\Big)\,ds. \quad (6.6)$$

We claim that if $F \geq 0$

$$\mathbb{E}^{\xi_s} \int_0^\tau F(Z_r, W_r)\, dr = \int \int_0^\infty (t \wedge s) F(z,t)\, dt\, \mu(dz). \tag{6.7}$$

To see this, it suffices to prove it for F of the form $F(z,t) = F_1(z)F_2(t)$ and then use linearity and an approximation procedure. Recall that the Green function for Brownian motion on $[0,\infty)$ killed on hitting 0 is $G(s,t) = s \wedge t$; this is easily derived from equation (2.1) of [3] by taking a limit. By the independence of Z and W, the product structure of ξ_s, and the fact that μ is an invariant measure for Z,

$$\begin{aligned}\mathbb{E}^{\xi_s} \int_0^\tau F_1(Z_r) F_2(W_r)\, dr &= \int F_1(z)\, \mu(dz)\, \mathbb{E}^s \int_0^\tau F_2(W_r)\, dr \\ &= \int F_1(z)\, \mu(dz) \int_0^\infty (t \wedge s) F_2(t)\, dt,\end{aligned}$$

and (6.7) follows. Hence

$$\begin{aligned}&4 \int \int_0^\infty (t \wedge s) |z_i|^{\alpha_i} \Big| \frac{\partial U_t f}{\partial z_i}(z) \Big|\, \Big| \frac{\partial U_t h}{\partial z_i}(z) \Big|\, dt\, \mu(dz) \\ &\quad = 4\mathbb{E}^{\xi_s} \int_0^\tau |Z_r^i|^{\alpha_i} \Big| \frac{\partial u_f}{\partial z_i}(Z_r, W_r) \Big|\, \Big| \frac{\partial u_h}{\partial z_i}(Z_r, W_r) \Big|\, dr.\end{aligned} \tag{6.8}$$

Using Cauchy-Schwarz, Hölder's inequality and (6.6), the right hand side of (6.8) is bounded by

$$\begin{aligned}&4\mathbb{E}^{\xi_s} \Big[\Big(\int_0^\tau |Z_r^i|^{\alpha_i} \Big| \frac{\partial u_f}{\partial z_i}(Z_r, W_r) \Big|^2 dr \Big)^{1/2} \Big(\int_0^\tau |Z_r^i|^{\alpha_i} \Big| \frac{\partial u_h}{\partial z_i}(Z_r, W_r) \Big|^2 dr \Big)^{1/2} \Big] \\ &\quad \leq 4\mathbb{E}^{\xi_s} [\langle u_f(Z,W) \rangle_\tau^{1/2} \langle u_h(Z,W) \rangle_\tau^{1/2}] \\ &\quad \leq 4 \Big(\mathbb{E}^{\xi_s} \langle u_f(Z,W) \rangle_\tau^{p/2} \Big)^{1/p} \Big(\mathbb{E}^{\xi_s} \langle u_h(Z,W) \rangle_\tau^{q/2} \Big)^{1/q},\end{aligned} \tag{6.9}$$

where q is the conjugate exponent to p.

Let $\nu_{n,s}$ be the restriction of ξ_s to $[0,n]^d \times [0,\infty)$, so that $\nu_{n,s}$ is a finite measure. By the Burkholder-Davis-Gundy inequalities and Doob's inequality,

$$\mathbb{E}^{\nu_{n,s}} \langle u_f(Z,W) \rangle_\tau^{p/2} \leq c_2 \mathbb{E}^{\nu_{n,s}} |u_f(Z_\tau, W_\tau)|^p + c_2 \mathbb{E}^{\nu_{n,s}} |u_f(Z_0, W_0)|^p.$$

Letting $n \to \infty$, we have

$$\mathbb{E}^{\xi_s}\langle u_f(Z,W)\rangle_\tau^{p/2} \le c_2\mathbb{E}^{\xi_s}|u_f(Z_\tau,W_\tau)|^p + c_2\mathbb{E}^{\xi_s}|u_f(Z_0,W_0)|^p.$$

Now

$$\mathbb{E}^{\xi_s}|u_f(Z_\tau,W_\tau)|^p = \int |u_f(z,0)|^p\,\mu(dz) = \int |f(z)|^p\,\mu(dz),$$

and since μ is an invariant measure for P_t, and hence for U_t, an application of Jensen's inequality yields

$$\mathbb{E}^{\xi_s}|u_f(Z_0,W_0)|^p = \int |u_f(z,s)|^p\,\mu(dz) \le \int |f(z)|^p\,\mu(dz).$$

We have a similar inequality for u_h and q. Therefore for each s

$$4\Big(\mathbb{E}^{\xi_s}\langle u_f(Z,W)\rangle_\tau^{p/2}\Big)^{1/p}\Big(\mathbb{E}^{\xi_s}\langle u_h(Z,W)\rangle_\tau^{q/2}\Big)^{1/q} \le c_3\|f\|_p\|h\|_q.$$

Let $s \to \infty$; by (6.5) and Fatou's lemma

$$\int(\mathcal{A}_i R_0 f(z))h(z)\,\mu(dz) \le c_3\|f\|_p\|h\|_q.$$

Taking the supremum over h with $\|h\|_q \le 1$, the duality of L^p and L^q implies that

$$\|\mathcal{A}_i R_0 f\|_p \le c_3\|f\|_p$$

if f is C^2 with support disjoint from the boundary. An approximation argument allows us to extend this inequality to all $f \in L^p$. □

Corollary 6.3 *Let $\lambda > 0$. Let $1 < p < \infty$. There exists a constant c_1 depending only on p such that*

$$\|\mathcal{A}_i R_\lambda f\|_p \le c_1\|f\|_p.$$

Proof. If $f \in L^p$, then $f - \lambda R_\lambda f$ is also in L^p with $\|f - \lambda R_\lambda f\|_p \le 2\|f\|_p$. Our result now follows from Theorem 6.2 because $R_\lambda f = R_0(f - \lambda R_\lambda f)$. □

Remark 6.4 As with the first order estimates, only minor changes are needed if R_λ is the resolvent for $\sum_{i=1}^d a_i\mathcal{A}_i$, where the a_i are finite positive constants.

7 Uniqueness

We now can complete the proof of Theorem 1.1.

The existence part of Theorem 1.1 was done in Section 4. It remains to prove uniqueness.

Proof of uniqueness in Theorem 1.1. Fix $x_0 \in \mathbb{R}^d_+$ and let $\varepsilon > 0$ be specified later. Let $M > 0$. As in the nondegenerate case, to prove uniqueness it suffices to localize, that is, to consider only the case where

$$\sum_{i=1}^{d} |a_i(y) - a_i(x_0)| \le \varepsilon, \qquad y \in \mathbb{R}^d_+, \tag{7.1}$$

and in addition $a_i(y) = a_i(x_0)$ and $b_i(y) = 0$ if $y \notin [0, M]^d$, $i = 1, \dots, d$; see [3], Section VI.3 and note that the continuity of the a_{ij} is used here. Let p_0 be the positive real given by Theorem 3.2. Set

$$\mathcal{L}_0 f(x) = \sum_{i=1}^{d} a_i(x_0) x_i^{\alpha_i} \frac{\partial^2 f}{\partial x_i^2}(x) \tag{7.2}$$

and let

$$\mathcal{B} = \mathcal{L} - \mathcal{L}_0. \tag{7.3}$$

Note $\mathcal{B}f(y) = 0$ if $y \notin [0, M]^d$. Let R_λ and P_t be the resolvent and semigroup, respectively, for the operator $\mathcal{L}_0$. Taking into account Remarks 5.5 and 6.4, by Theorem 5.4 and Corollary 6.3 we have

$$\|\mathcal{B} R_\lambda f\|_{p_0} \le c_1 \Big(d\varepsilon + \sum_{i=1}^{d} \lambda^{\frac{1}{2b_i} - 1} \|b_i\|_\infty \Big) \|f\|_{p_0}. \tag{7.4}$$

Let us now choose ε small enough and λ large enough so that by (7.4) we have

$$\|\mathcal{B} R_\lambda f\|_{p_0} \le \tfrac{1}{2} \|f\|_{p_0}. \tag{7.5}$$

Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be any two solutions to the submartingale problem for $\mathcal{L}$ started at x_0, where we continue to assume (7.1) holds. We also assume that under each $\mathbb{P}_i$ the process spends zero time on Δ. Define

$$S^i_\lambda h = \mathbb{E}_i \int_0^\infty e^{-\lambda t} h(X_t)\, dt, \qquad i = 1, 2. \tag{7.6}$$

Let $R_K = \inf\{t : \sum_{i=1}^d L_t^{X^i} \geq K\}$. By Ito's formula, if $f \in C_b^2$ and $f_i = 0$ on Δ_i for each i, then

$$\mathbb{E}_i f(X_{t\wedge R_K}) - f(x_0) = \mathbb{E}_i \int_0^{t\wedge R_K} \mathcal{L} f(X_s)\, ds, \qquad i = 1, 2.$$

We let $K \to \infty$, so that $R_K \to \infty$. we then multiply both sides by $\lambda e^{-\lambda t}$ and integrate over t from 0 to ∞ to obtain

$$\lambda S_\lambda^i f - f(x_0) = S_\lambda^i \mathcal{L} f = S_\lambda^i \mathcal{L}_0 f + S_\lambda^i \mathcal{B} f. \tag{7.7}$$

Now let g be C_b^2 with compact support contained in $(0, \infty)^d$ and let $f = P_t R_\lambda g$. By Lemma 5.6 we can apply (7.7) to f. Note

$$\mathcal{L}_0 f = \mathcal{L}_0 R_\lambda P_t g = \lambda R_\lambda P_t g - P_t g.$$

Therefore (7.7) becomes

$$S_\lambda^i P_t g = R_\lambda P_t g(x_0) + S_\lambda^i \mathcal{B} R_\lambda P_t g. \tag{7.8}$$

Let

$$\Theta = \sup\{|S_\lambda^1 h - S_\lambda^2 h| : h \text{ supported in } [0, M]^d, \|h\|_{p_0} \leq 1\}.$$

By Theorem 3.2 we know $\Theta < \infty$. By (7.8) and (7.5) and recalling that $\mathcal{B} R_\lambda g$ is supported in $[0, M]^d$,

$$\begin{aligned} |S_\lambda^1 P_t g - S_\lambda^2 P_t g| &\leq \Theta \|\mathcal{B} R_\lambda P_t g\|_{p_0} \\ &\leq \tfrac{1}{2}\Theta \|P_t g\|_{p_0} \\ &\leq \tfrac{1}{2}\Theta \|g\|_{p_0}. \end{aligned}$$

The last inequality follows by Jensen's inequality and the fact that P_t is self-adjoint with respect to μ. Since the support of g is disjoint from Δ, we can let $t \to 0$ and obtain

$$|S_\lambda^1 g - S_\lambda^2 g| \leq \tfrac{1}{2}\Theta \|g\|_{p_0}.$$

We now take the supremum over all such g that in addition satisfy $\|g\|_{p_0} \leq 1$. Since neither S_λ^1 nor S_λ^2 charge Δ, we then have

$$\Theta \leq \tfrac{1}{2}\Theta.$$

Since $\Theta < \infty$, we conclude $\Theta = 0$.

From this point on we follow the proof of the nondegenerate case; see [3], Theorem VI.3.2. Very briefly, the argument can be summarized as follows. Since $S^1_\lambda = S^2_\lambda$, by the uniqueness of the Laplace transform, $\mathbb{E}_1 h(X_t) = \mathbb{E}_2 h(X_t)$for almost every t. If h is continuous and bounded, the continuity implies this equality for all t. Therefore the one-dimensional distributions of X under $\mathbb{E}_1$ and $\mathbb{E}_2$ are equal. An argument using regular conditional probabilities is then used to show that the finite dimensional distributions of X under $\mathbb{E}_1$ and $\mathbb{E}_2$ are the same. Finally, one shows that the localization assumptions can be dispensed with. $\square$

Remark 7.1 We show how Girsanov's theorem allows us to dispense with the drift when all the α_i are small enough. Suppose each $\alpha_i < 1/2$ and under $\mathbb{P}$, X is a solution to

$$dX^i_t = \sqrt{2a_i(X_t)}(X^i_t)^{\alpha_i/2}\, dW^i_t + dL^{X^i}_t,$$

where L^{X^i} is a local time at 0 for X^i, the W^i are independent one-dimensional Brownian motions, and $X^i_t \geq 0$ for all t. If we define $d\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp(M_t - \tfrac{1}{2}\langle M\rangle_t),$$

where

$$M_t = -\int_0^t \sum_{i=1}^d \frac{b_i(X_t)}{\sqrt{2a_i(X_t)}(X^i_s)^{\alpha_i/2}}\, dW^i_s,$$

then

$$\begin{aligned} \mathbb{E}[\langle M\rangle_{\tau_{[0,M]^d}}] &\leq c_1 \mathbb{E}\int_0^{\tau_{[0,M]^d}(X)} \sum_{i=1}^d (X^i_s)^{-\alpha_i}\, ds \\ &\leq c_2 \sum_{i=1}^d \mathbb{E}\int_0^{\tau_{[0,M]}(X^i)} (X^i_s)^{-\alpha_i}\, ds. \end{aligned}$$

Using (3.4), this can be seen to be finite if $\alpha_i < 1/2$ for each i. It is readily checked using Girsanov's theorem for continuous semimartingales (see [2], Theorem I.6.4) that under $\mathbb{Q}$, X_t solves (1.2); cf. the proof of Theorem VI.3.1 in [3]. This simplifies the argument in the case where all the $\alpha_i < 1/2$, both for existence and for uniqueness.

8 Appendix

We prove Lemma 5.1.

We will use the well known facts (see [9], pp. 150–152):

$$I_p'(x) = I_{p+1}(x) + \frac{p}{x} I_p(x), \tag{8.1}$$

$$I_p'(x) = I_{p-1}(x) - \frac{p}{x} I_p(x), \tag{8.2}$$

$$I_p(x) \sim \frac{1}{2^p \Gamma(p+1)} x^p, \qquad x \to 0, \tag{8.3}$$

$$I_p(x) \sim \frac{1}{\sqrt{2\pi}} \frac{e^x}{\sqrt{x}}, \qquad x \to \infty. \tag{8.4}$$

In what follows we will take $p = \nu$ or $\nu + 1$ and $\nu = -1/(2b)$.

If we let $F(x) = I_{\nu+1}(x) - I_\nu(x)$, then from (8.1) and (8.2) we have

$$F'(x) = -F(x) - \frac{\nu+1}{x} I_{\nu+1}(x) - \frac{\nu}{x} I_\nu(x).$$

Using (8.4)

$$|F'(x) + F(x)| \le c_1 \frac{e^x}{x^{3/2}},$$

or

$$|(e^x F(x))'| \le c_1 \frac{e^{2x}}{x^{3/2}}.$$

Therefore

$$|e^x F(x)| \le |eF(1)| + c_1 \int_1^x \frac{e^{2y}}{y^{3/2}} dy.$$

By l'Hôpital's rule, the integral is bounded by $c_2 e^{2x}/x^{3/2}$, and so we deduce

$$|I_{\nu+1}(x) - I_\nu(x)| \le c_3 \frac{e^x}{x^{3/2}} \tag{8.5}$$

for $x \ge 1$.

Proof of Lemma 5.1. We start with (5.3). By scaling it suffices to do the case $t = 1$. Differentiating (5.1) we have

$$\frac{\partial p_X(1,x,y)}{\partial x} = c x^{b-\frac{1}{2}} y^{2b-\frac{3}{2}} e^{-K(x^{2b}+y^{2b})/2} [-x^b I_\nu(K x^b y^b) + y^b I_{\nu+1}(K x^b y^b)], \tag{8.6}$$

where K is some fixed positive constant. We write

$$\int_0^\infty \left|\frac{\partial p_X(1,x,y)}{\partial x}\right| dy = \int_0^{1/x} + \int_{1/x}^\infty := S_1 + S_2.$$

Using the bounds on I_ν,

$$S_1 \le c x^{2b-1} e^{-Kx^{2b}/2} \int_0^\infty [y^{2b-2} + y^{4b-2}] e^{-Ky^{2b}/2} dy.$$

Since $2b-2 > -1$, the integral term is finite. Since $2b-1 > 0$, the factor in front of the integral is bounded independently of x, so S_1 is bounded independently of x.

Since

$$\begin{aligned} &|-x^b I_\nu(Kx^b y^b) + y^b I_{\nu+1}(Kx^b y^b)| \\ &\qquad = |(y^b - x^b) I_\nu(Kx^b y^b) + y^b (I_{\nu+1}(Kx^b y^b) - I_\nu(Kx^b y^b))| \\ &\qquad \le c|y^b - x^b| e^{Kx^b y^b} x^{-\frac{b}{2}} y^{-\frac{b}{2}} + c y^b (e^{Kx^b y^b} x^{-b} y^{-b}) \end{aligned}$$

for $y \ge 1/x$, to bound S_2 we need to bound

$$\begin{aligned} &\int_{1/x}^\infty |y^b - x^b| x^{\frac{b-1}{2}} y^{\frac{3b-3}{2}} e^{-K(y^b - x^b)^2/2} dy \\ &\qquad + \int_{1/x}^\infty x^{-\frac{1}{2}} y^{2b-\frac{3}{2}} e^{-K(y^b - x^b)^2/2} dy \\ &= S_3 + S_4. \end{aligned}$$

For S_3 we make the substitution $z = y^b - x^b$ and then

$$S_3 = c \int_{x^{-b} - x^b}^\infty |z| x^{\frac{b-1}{2}} (x^b + z)^{\frac{b-1}{2b}} c^{-Kz^2/2} dz.$$

Since $(b-1)/(2b) < 0$ and $x^b + z \ge x^{-b}$, this is less than

$$c \int_{-\infty}^\infty |z| x^{\frac{b-1}{2}} (x^{-b})^{\frac{b-1}{2b}} e^{-Kz^2/2} dz$$

which is bounded independently of x. For S_4 we use the same substitution. Since $2b-1 > 0$, we have

$$(x^b + z)^{\frac{2b-1}{2b}} \le c(x^{\frac{2b-1}{2}} + z^{\frac{2b-1}{2b}}).$$

Hence

$$S_4 \le \int_{x^{-b}-x^b}^{\infty} x^{-\frac{1}{2}}(x^b+z)^{\frac{2b-1}{2b}} e^{-Kz^2/2} dz$$
$$\le c \int_{x^{-b}-x^b}^{\infty} (x^{b-1} + x^{-\frac{1}{2}} z^{\frac{2b-1}{2b}}) e^{-Kz^2/2} dz.$$

For each $p \ge 1$ and $q > 0$ there exists $c(p,q)$ such that

$$\int_a^{\infty} (1+z)^q e^{-Kz^2/2} dz \le c(p,q) a^{-p}, \qquad a > 1. \tag{8.7}$$

From this we see that S_4 is bounded independently of x for $x \le 1$. On the other hand, for $x \ge 1$,

$$S_4 \le c \int_{-\infty}^{\infty} (1 + |z|^{\frac{2b-1}{2b}}) e^{-Kz^2/2} dz \le c.$$

We now turn to the proof of (5.4). Again by scaling we may assume $t = 1$. Looking at $\int_0^{1/y}$ and using the bounds on I_ν,

$$y^\alpha \int_0^{1/y} \Big| \frac{\partial p_X(1,x,y)}{\partial x} \Big| x^{-\alpha} dx$$
$$\le c[(y^{\alpha+2b-2} + y^{\alpha+4b-2}) e^{-Ky^{2b}/2}] \int_0^{\infty} x^{2b-1-\alpha} e^{-Kx^{2b}/2} dx.$$

Since $\alpha + 2b - 2 = 0$, $\alpha + 4b - 2 > 0$, and $2b - 1 - \alpha = 1 - 2\alpha > -1$, the integral is finite and the expression in brackets is bounded in y.

To look at $\int_{1/y}^{\infty}$, we rewrite the integral as in S_2 and see that we have to bound

$$cy^{\frac{3}{2}b-\frac{3}{2}+\alpha} \int_{1/y}^{\infty} x^{\frac{b}{2}-\frac{1}{2}-\alpha} |y^b - x^b| e^{-K(y^b-x^b)^2/2} dx$$
$$+ cy^{2b-\frac{3}{2}+\alpha} \int_{1/y}^{\infty} x^{-\frac{1}{2}-\alpha} e^{-K(y^b-x^b)^2/2} dx$$
$$= S_5 + S_6.$$

Letting $z = x^b - y^b$ as in S_3,

$$S_5 \le cy^{\frac{3}{2}b-\frac{3}{2}+\alpha} \int_{y^{-b}-y^b}^{\infty} (y^b+z)^{-\frac{b-1+2\alpha}{2b}} z e^{-z^2/2} dz$$
$$\le cy^{b-1} \int_{y^{-b}-y^b}^{\infty} z e^{-z^2/2} dz.$$

When $y \le 1$ this is bounded using (8.7). When $y \ge 1$ this is bounded because $b - 1 < 0$.

For S_6 we have

$$S_6 \le c y^{2b - \frac{3}{2} + \alpha} \int_{y^{-b} - y^b}^{\infty} (y^b + z)^{-\frac{2b-1+2\alpha}{2b}} e^{-Kz^2/2} dz$$
$$\le c y^{b-1} \int_{y^{-b} - y^b}^{\infty} e^{-Kz^2/2} dz.$$

This is seen to be bounded in y for $y \le 1$. This is bounded in y for $y > 1$ because $b - 1$ is negative. □

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