THE SUPREMUM OF BROWNIAN LOCAL TIMES ON HÖLDER CURVES

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LE SUPREMUM DU TEMPS LOCAUX D'UN MOUVEMENT BROWNIEN SUR LES COURBES HOLDERIENNES

Short title: **Brownian local times**

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Abstract. For $f:[0,1]\to\mathbb{R}$, we consider L^f_t, the local time of space-time Brownian motion on the curve f. Let $\mathcal{S}_\alpha$ be the class of all functions whose Hölder norm of order α is less than or equal to 1. We show that the supremum of L^f_1 over f in $\mathcal{S}_\alpha$ is finite if $\alpha>\frac{5}{6}$ and infinite if $\alpha<\frac{1}{2}$.

Abstrait: Soit W_t un mouvement brownien et soit L^f_t le temps local du processus (t,W_t) pour le courbe $f:[0,1]\to\mathbb{R}$, c'est à dire, $L^f_t = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{]f(s)-\varepsilon,f(s)+\varepsilon[}(W_s)ds$. Soit $\mathcal{S}_\alpha$ la classe de toutes fonctions telle que la norme holderienne du ordre α est moins de 1. Nous démontrons que $\sup_{f\in\mathcal{S}_\alpha} L^f_1<\infty$ p.s. si $\alpha>\frac{5}{6}$ et ce supremum est infini p.s. si $\alpha<\frac{1}{2}$.

AMS subject classification: 60J65, 60J55

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0. Preface to corrected version.

The original version of this paper (*Ann. Inst. Henri Poincaré* **37** (2001) 627–642) contained an error. Lemma 3.1 of that paper was incorrectly applied in Proposition 3.2. We would like to thank Alice Vatamanelu for pointing out the mistake. The differences between this version of the paper and the published one are in Theorem 1.2, Lemma 3.1, Propositions 3.2 and 3.3, and Theorem 3.6. The main difference is that the finiteness of the supremum in Theorem 1.2 requires the assumption $\alpha > 5/6$. This is a stronger assumption than $\alpha > 1/2$ which appeared in the original paper. Heuristic arguments suggest that the original version of Theorem 1.2 is true, but at this time we do not have a rigorous argument to back up this claim.

1. Introduction.

Let W_t be one-dimensional Brownian motion and let $f : [0,1] \to \mathbb{R}$ be a Hölder continuous function. There are a number of equivalent ways to define the local time of W_t along the curve f. We will show the equivalence below, but for now define L_t^f as the limit in probability of

$$\frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(f(s)-\varepsilon, f(s)+\varepsilon)}(W_s)\, ds$$

as $\varepsilon \to 0$. Let

$$\mathcal{S}_\alpha = \{f : \sup_{0 \le t \le 1} |f(t)| \le 1, |f(s) - f(t)| \le |s-t|^\alpha \text{ if } s,t \le 1\}.$$

We were led to the results in this paper by the following question.

Question 1.1. *Is* $\sup_{f \in \mathcal{S}_1} L_1^f$ *finite or infinite?*

Our interest in this problem arose when we were working on Bass and Burdzy (1999). A positive answer to Question 1.1 at that time would have provided a proof of uniqueness for a certain stochastic differential equation; we ended up using different methods.

However, probably the greatest interest in Question 1.1 has to do with questions about metric entropy. The metric entropy of $\mathcal{S}_1$ is known to be of order $1/\varepsilon$; see, e.g., Clements (1963). That is, if one takes the cardinality of the smallest ε-net for $\mathcal{S}_1$ (with respect to the supremum norm) and takes the logarithm, the resulting number will be bounded above and below by positive constants times $1/\varepsilon$. It is known (see Ledoux and Talagrand (1991)) that this is too large for standard chaining arguments to be used to prove finiteness of $\sup_{f \in \mathcal{S}_1} L_1^f$. Nevertheless, the supremum in Question 1.1 is finite.

It is a not uncommon belief among the probability community that metric entropy estimates are almost always sharp: the supremum of a process is finite if the metric entropy is small enough, and infinite otherwise. That is not the case here. Informally, our main result is

Theorem 1.2. *The supremum of $f \to L_1^f$ over $\mathcal{S}_\alpha$ is finite if $\alpha > \frac{5}{6}$ and infinite if $\alpha < \frac{1}{2}$.*

See Theorems 3.6 and 3.8 for formal statements.

The metric entropy of $\mathcal{S}_\alpha$ when $\alpha \in (\frac{5}{6}, 1]$ is far beyond what chaining methods can handle. Sometimes the method of majorizing measures provides a better result than that of metric entropy. We do not know if this is the case here.

For previous work on local times for space-time curves, see Burdzy and San Martín (1995) and Davis (1998). For some results on local times on Lipschitz curves for two-dimensional Brownian motion, see Bass and Khoshnevisan (1992) and Marcus and Rosen (1996).

In Section 2 we prove the equivalence of various definitions of L_t^f as well as some lemmas of independent interest. In Section 3 we prove finiteness of the supremum over $\mathcal{S}_\alpha$ when $\alpha > \frac{5}{6}$ and that this fails when $\alpha < \frac{1}{2}$. We also show that $(f, t) \to L_t^f$ is jointly continuous on $\mathcal{S}_\alpha \times [0, 1]$ when $\alpha > \frac{5}{6}$.

The letter c with subscripts will denote finite positive constants whose exact values are unimportant. We renumber them in each proof.

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2. Preliminaries.

We discuss three possible definitions of L_t^f.

(i) $L_t^f = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(f(s)-\varepsilon, f(s)+\varepsilon)}(W_s) ds$;

(ii) L_t^f is the continuous additive functional of space-time Brownian motion associated to the potential $U^f(x,t) = \int_0^{1-t} p(s, x, f(t+s)) ds$, where p is the transition density for one-dimensional Brownian motion;

(iii) (for $f \in \mathcal{S}_1$ only) L_t^f is the local time in the semimartingale sense at 0 of the process $W_t - f(t)$.

One of the goals of this section is to show the equivalence of these definitions. We begin with the following lemma which will be used repeatedly throughout the paper.

Lemma 2.1. *Suppose A_t^1 and A_t^2 are two nondecreasing continuous processes with $A_0^1 = A_0^2 = 0$. Let $B_t = A_t^1 - A_t^2$. Suppose that for all $s \le t$, and some right-continuous filtration $\{\mathcal{F}_t\}$,*

$$\mathbb{E}[A_t^i - A_s^i \mid \mathcal{F}_s] \le M, \qquad \text{a.s.} \qquad i = 1, 2,$$

and for all $s \le t$

$$\big|\,\mathbb{E}[B_t - B_s \mid \mathcal{F}_s]\,\big| \le \gamma, \qquad \text{a.s.}$$

There exist c_1, c_2 such that for all $\lambda > 0$,

$$\mathbb{P}(\sup_{s\le t} |B_s| > \lambda\sqrt{\gamma M}) \le c_1 e^{-c_2\lambda}.$$

Proof. We have

$$(B_t - B_s)^2 = 2\int_s^t (B_t - B_r) dB_r.$$

Using a Riemann sum approximation (cf. Bass (1995), Exercise I.8.28) we obtain

$$\begin{aligned}\mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s] &= 2\mathbb{E}\Big[\int_s^t (B_t - B_r)\, dB_r \mid \mathcal{F}_s\Big] \\ &= 2\mathbb{E}\Big[\int_s^t \mathbb{E}[B_t - B_r \mid \mathcal{F}_r]\, dB_r \mid \mathcal{F}_s\Big] \\ &\le 2\mathbb{E}\Big[\int_s^t \gamma(dA_r^1 + dA_r^2) \mid \mathcal{F}_s\Big] \le 4\gamma M.\end{aligned}$$

This inequality holds a.s. for each s. The left hand side is equal to

$$\mathbb{E}[B_t^2 \mid \mathcal{F}_s] - 2B_s\mathbb{E}[B_t \mid \mathcal{F}_s] + B_s^2$$

and hence is right continuous. Therefore there is a null set outside of which

$$\mathbb{E}[(B_t - B_s)^2 \mid F_s] \le 4\gamma M$$

for all s. In particular, if T is a stopping time, by Jensen's inequality we obtain

$$\mathbb{E}[|B_t - B_T| \mid \mathcal{F}_T] \le (\mathbb{E}[(B_t - B_T)^2 \mid \mathcal{F}_T])^{1/2} \le (4\gamma M)^{1/2}.$$

Our result now follows by Bass (1995, Theorem I.6.11), and Chebyshev's inequality. □

Let W_t be one-dimensional Brownian motion. Define

$$p(t,x,y) = (2\pi t)^{-1/2}\exp(-|x-y|^2/2t), \tag{2.1}$$

the transition density of one dimensional Brownian motion. In the rest of the paper, $\mathcal{F}_t$ will denote the (right-continuous) filtration generated by W_t.

For a measurable function $f : [0,1] \to \mathbb{R}$ set $\|f\| = \sup_{t\le 1}|f(t)|$. Let

$$D_t^f(\varepsilon) = \frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{(f(s)-\varepsilon, f(s)+\varepsilon)}(W_s)\, ds.$$

Proposition 2.2. *For f measurable on $[0,1]$, there exists a nondecreasing continuous process L_t^f such that $\mathbb{E}\|D^f(\varepsilon) - L^f\|^2 \to 0$ as $\varepsilon \to 0$.*

Proof. Let $\mathbb{E}^{(x,t)}$ denote the expectation corresponding to the distribution of Brownian motion starting from x at time t, i.e., satisfying $W_t = x$. For any x and any $t \le 1$,

$$\begin{aligned}\mathbb{E}^{(x,t)} \frac{1}{2\varepsilon} \int_0^{1-t} \mathbf{1}_{(f(t+s)-\varepsilon, f(t+s)+\varepsilon)}(W_{t+s})\, ds &= \frac{1}{2\varepsilon} \int_0^{1-t} \int_{f(t+s)-\varepsilon}^{f(t+s)+\varepsilon} p(s,x,y)\, dy\, ds \\ &\le c_1 \int_0^{1-t} \frac{1}{\sqrt{s}}\, ds \le c_2 \sqrt{1-t} \le c_2. \end{aligned} \tag{2.2}$$

This implies that,

$$\mathbb{E}[D_1^f(\varepsilon) - D_t^f(\varepsilon) \mid \mathcal{F}_t] = \mathbb{E}^{(W_t,t)} \frac{1}{2\varepsilon} \int_0^{1-t} \mathbf{1}_{(f(t+s)-\varepsilon, f(t+s)+\varepsilon)}(W_{t+s})\, ds \le c_2. \tag{2.3}$$

The supremum of

$$\frac{1}{2\varepsilon} \int_{f(t+s)-\varepsilon}^{f(t+s)+\varepsilon} p(s,x,y)\, dy$$

over $\varepsilon > 0$, $t \le 1$ and $s \le 1-t$ is bounded. By the continuity of $p(s,x,y)$ in y and the bounded convergence theorem, as $\varepsilon \to 0$,

$$\frac{1}{2\varepsilon} \int_0^{1-t} \int_{f(t+s)-\varepsilon}^{f(t+s)+\varepsilon} p(s,x,y)\, dy\, ds \to \int_0^{1-t} p(s,x,f(t+s))\, ds$$

uniformly over x and t. Calculations similar to those in (2.2) and (2.3) yield the following estimate: for any $\eta > 0$,

$$\left| \mathbb{E}[(D_1^f(\varepsilon_1) - D_1^f(\varepsilon_2)) - (D_t^f(\varepsilon_1) - D_t^f(\varepsilon_2)) \mid \mathcal{F}_t] \right| \le \eta, \qquad \text{a.s.}, \tag{2.4}$$

for all $t \le 1$ provided ε_1 and ε_2 are small enough.

Because of (2.3) and (2.4), we can apply Lemma 2.1 with $A_t^1 = D_t^f(\varepsilon_1)$ and $A_t^2 = D_t^f(\varepsilon_2)$. The estimate in that lemma shows that, in a sense, the supremum of the difference between $D_t^f(\varepsilon_1)$ and $D_t^f(\varepsilon_2)$ is of order $\sqrt{\eta}$. We see that $\mathbb{E}(\|D^f(\varepsilon_1) - D^f(\varepsilon_2)\|^2) \to 0$ as $\varepsilon_1, \varepsilon_2 \to 0$. This implies that $\{D^f(\varepsilon_n)\}$ is a Cauchy sequence, and therefore $D^f(\varepsilon_n)$ converges as $n \to \infty$, for any sequence $\{\varepsilon_n\}$ converging to 0. Denote the limit by L_t^f; it is routine to check that the limit does not depend on the sequence $\{\varepsilon_n\}$. Since the convergence is uniform over t and $t \to D_t^f(\varepsilon)$ is continuous for every ε, then L_t^f is continuous in t. For a similar reason, $t \to L_t^f$ is nondecreasing. □

Remark 2.3. A very similar proof shows that L_t^f is the limit in L^2 of

$$\frac{1}{\varepsilon} \int_0^t \mathbf{1}_{[f(s), f(s)+\varepsilon)}(W_s) ds.$$

Remark 2.4. Let

$$U^f(x,t) = \int_0^{1-t} p(s,x,f(t+s))\,ds.$$

A straightforward limit argument shows that

$$\mathbb{E}[L_1^f - L_t^f \mid \mathcal{F}_t] = \int_0^{1-t} p(s,W_t,f(t+s))\,ds. \tag{2.5}$$

It follows that $U^f(W_t,t)$ is a potential for the space-time Brownian motion $t \to (W_t,t)$. Hence the function $U^f(x,t)$ is excessive with respect to space-time Brownian motion, and therefore L_t^f can also be viewed as the continuous additive functional for the space-time Brownian motion (W_t,t) whose potential is U^f.

Corollary 2.5. *Suppose $f_n \to f$ uniformly. Then $\|L^{f_n} - L^f\|$ converges to 0 in L^2.*

Proof. From (2.5),

$$\mathbb{E}[L_1^f - L_u^f \mid \mathcal{F}_u] \le c_1 \int_0^{1-u} \frac{1}{\sqrt{s}}\,ds \le c_2\sqrt{1-u} \le c_2$$

and

$$\begin{aligned}\big|\,\mathbb{E}[L_1^{f_n} - L_u^{f_n} \mid \mathcal{F}_u] - \mathbb{E}[L_1^f - L_u^f \mid \mathcal{F}_u]\,\big| \\ &= \Big|\int_0^{1-u} [p(s,W_u,f_n(u+s)) - p(s,W_u,f(u+s))]\,ds\Big| \\ &\le \int_0^{1-u} |p(s,W_u,f_n(u+s)) - p(s,W_u,f(u+s))|\,ds.\end{aligned}$$

The right hand side tends to 0 by the assumption that $f_n \to f$ uniformly, and the result now follows by Lemma 2.1, using the same argument as at the end of the proof of Proposition 2.2. □

If f is a Lipschitz function, then $W_t - f(t)$ is a semimartingale. We can therefore define a local time for W_t along the curve f by setting K_t^f to be the local time (in the semimartingale sense) at 0 of $Y_t = W_t - f(t)$. That is,

$$K_t^f = |Y_t| - |Y_0| - \int_0^t \operatorname{sgn}(Y_s)\,dY_s.$$

Proposition 2.6. *With probability one, $K_t^f = L_t^f$ for all t.*

Proof. By Revuz and Yor (1994) Corollary VI.1.9,

$$K_t^f = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^t \mathbf{1}_{[0,\varepsilon)}(Y_s)d\langle Y\rangle_s. \tag{2.6}$$

Since $Y_t = W_t - f(t)$, then $\langle Y \rangle_t = \langle W \rangle_t = t$, and so by Remark 2.3, $K_t^f = L_t^f$ a.s. Since both K_t^f and L_t^f are continuous in t, the result follows. □

3. The supremum of local times.

We will assume that $\alpha \in [1/2, 1]$ in the first part of this section. Our first goal is to obtain an estimate on the number of rectangles of size $(1/N) \times (2/N^{\alpha/2})$ that are hit by a Brownian path. Fix any $a \in \mathbb{R}$ and $b \in (a, a + 2/N^{\alpha/2}]$. Let

$$I_j = \{\exists t \in [(j-1)/N, j/N] : a \le W_t \le b)\},$$

and

$$A_k = \sum_{j=1}^{k} \mathbf{1}_{I_j}.$$

Lemma 3.1. *There exist c_1 and c_2 such that for all $\lambda > 0$,*

$$\mathbb{P}(A_k \ge \lambda \sqrt{k} N^{(1-\alpha)/2}) \le c_1 e^{-c_2 \lambda}.$$

Proof. There is probability $c_3 > 0$ independent of x such that

$$\mathbb{P}^x(\sup_{s \le 1/N} |W_s - W_0| < 1/\sqrt{N}) > c_3.$$

So by the strong Markov property applied at the first $t \in [(j-1)/N, j/N]$ such that $a \le W_t \le b$,

$$c_3 \mathbb{P}^x(I_j) \le \mathbb{P}^x(W_{j/N} \in [a - (1/\sqrt{N}), a + (3/\sqrt{N})]).$$

This and the standard bound

$$\mathbb{P}^x(W_t \in [c, d]) = \int_c^d \frac{1}{\sqrt{2\pi t}} e^{-|y-x|^2/2t} dy \le \frac{1}{\sqrt{2\pi t}} |d - c|,$$

imply that

$$\mathbb{P}^x(I_j) \le c_4 \frac{1}{N^{\alpha/2}} \frac{1}{\sqrt{j/N}} = \frac{c_4}{\sqrt{j}} N^{(1-\alpha)/2}.$$

Therefore

$$\mathbb{E}^x A_k = \sum_{j=1}^{k} \mathbb{P}^x(I_j) \le c_5 \sqrt{k} N^{(1-\alpha)/2}. \tag{3.1}$$

By the Markov property,

$$\mathbb{E}[A_k - A_i \mid \mathcal{F}_{i/n}] \le 1 + \mathbb{E}^{W(i/n)} A_k \le c_6 \sqrt{k} N^{(1-\alpha)/2}. \tag{3.2}$$

Corollary I.6.12 of Bass (1995) can be applied to the sequence $A_k/(c_7\sqrt{k}N^{(1-\alpha)/2})$, in view of (3.1) and (3.2). That result says that $\mathbb{E}\exp(c_8 \sup_k A_k/(c_7\sqrt{k}N^{(1-\alpha)/2})) \le 2$ for some $c_8 > 0$. This easily implies our lemma. $\square$

Fix an integer $N > 0$. Let $R_{\ell m} = R_{\ell m}(N)$ be the rectangle defined by

$$R_{\ell m} = [\ell/N, (\ell+1)/N] \times [m/N^\alpha, (m+1)/N^\alpha], \qquad 0 \le \ell \le N, \quad -N^\alpha - 1 \le m \le N^\alpha.$$

Let K be such that N/K is an integer and $\sqrt{N} < N/K \le \sqrt{N} + 1$. Set

$$Q_{ik} = Q_{ik}(N) = [iK/N, (i+1)K/N] \times [k(K/N)^\alpha, (k+1)(K/N)^\alpha],$$

for $0 \le i \le K$ and $-(N/K)^\alpha - 1 \le k \le (N/K)^\alpha$. Note that $Q_{ik}(N) = R_{ik}(N/K)$ but it will be convenient to use both notations.

Proposition 3.2. *Let $\alpha \in (1/2, 1]$ and $\varepsilon \in (0, 1/16)$. There exist c_1, c_2, and c_3 such that:*

(i) there exists a set D_N with $\mathbb{P}(D_N) \le c_1 N \exp(-c_2 N^{\varepsilon/2})$;

(ii) if $\omega \notin D_N$ and $f \in \mathcal{S}_\alpha$, then there are at most $c_3 N^{(5/4)-(\alpha/2)+(\varepsilon/2)}$ rectangles $R_{\ell m}$ in $[0,1] \times [-1,1]$ which contain both a point of the graph of f and a point of the graph of $W_t(\omega)$.

Proof. Let

$$I_{ikj} = \{\exists t \in [iK/N + (j-1)/N, iK/N + j/N] : k(K/N)^\alpha \le W_t \le (k+1)(K/N)^\alpha\},$$

$$A_{ik} = \sum_{j=1}^{K} \mathbf{1}_{I_{ikj}},$$

and

$$C_{ik} = C_{ik}(N) = \{A_{ik} \ge K^{(1/2)+\varepsilon} N^{(1-\alpha)/2}\}.$$

By Lemma 3.1 with $k = [K]$ and $\lambda = K^\varepsilon$, and the Markov property applied at kK/N we have $\mathbb{P}(C_{ik}) \le c_4 \exp(-c_5 K^\varepsilon)$.

There are at most $c_6 N^{(1/2)+(\alpha/2)}$ rectangles Q_{ik}, so if $D_N = \cup_{i,k} C_{ik}$, where $0 \le i \le K$ and $-(N/K)^\alpha - 1 \le k \le (N/K)^\alpha$, then

$$\mathbb{P}(D_N) \le c_7 N^{(1+\alpha)/2} \exp(-c_5 K^\varepsilon) \le c_7 N \exp(-c_8 N^{\varepsilon/2}).$$

Now suppose $\omega \notin D_N$. Let f be any function in $\mathcal{S}_\alpha$. If f intersects Q_{ik} for some i and k, then f might intersect $Q_{i,k-1}$ and $Q_{i,k+1}$. But because $f \in \mathcal{S}_\alpha$, it cannot intersect Q_{ir} for any r such that $|r-k| > 1$. Therefore f can intersect at most $3(K+1)$ of the Q_{ik}.

Look at any one of the Q_{ik} that f intersects. Since $\omega \notin D_N$, then there are at most $K^{(1/2)+\varepsilon}N^{(1-\alpha)/2}$ integers j that are less than K and for which the path of $W_t(\omega)$ intersects $([iK/N + (j-1)/N, iK/N + j/N] \times [-1,1]) \cap Q_{ik}$. If f intersects a rectangle $R_{\ell m}$, then it can intersect a rectangle $R_{\ell r}$ only if $|r-m| \le 1$, since $f \in \mathcal{S}_\alpha$. Therefore there are at most $3K^{(1/2)+\varepsilon}N^{(1-\alpha)/2}$ rectangles $R_{\ell m}$ contained in Q_{ik} which contain both a point of the graph of f and a point of the graph of $W_t(\omega)$.

Since there are at most $3(K+1)$ rectangles Q_{ik} which contain a point of the graph of f, there are therefore at most

$$3(K+1)3K^{(1/2)+\varepsilon}N^{(1-\alpha)/2} \le c_9 N^{(5/4)-(\alpha/2)+(\varepsilon/2)}$$

rectangles $R_{\ell m}$ that contain both a point of the graph of f and a point of the graph of $W_t(\omega)$. □

We can now iterate this to obtain a better estimate.

Proposition 3.3. *Fix $\alpha \in (1/2, 1]$ and $\delta, \eta > 0$. There exist c_1 and N_0 such that if $N \ge N_0$:*

(i) there exists a set E with $\mathbb{P}(E) \le \eta$;

(ii) if $\omega \notin E$ and $f \in \mathcal{S}_\alpha$, then there are at most $c_1 N^{(3/2)-\alpha+\delta}$ rectangles $R_{\ell m}(N)$ contained in $[0,1] \times [-1,1]$ which contain both a point of the graph of f and a point of the graph of $W_t(\omega)$.

Proof. For any ε, the quantity $c_1 N \exp(-c_2 N^{\varepsilon/2})$ is summable. First choose $\varepsilon \in (0, \delta/4)$ and then choose N_1 large so that, using Proposition 3.2 and its notation,

$$\sum_{N=N_1}^{\infty} \mathbb{P}(D_N) \le \sum_{N=N_1}^{\infty} c_1 N \exp(-c_2 N^{\varepsilon/2}) < \eta.$$

Let $E = \cup_{N=N_1}^{\infty} D_N$.

Fix $\omega \notin E$. Suppose N is large enough so that $\sqrt{N} \ge 2N_1$. Recall the definition of K and note that N/K differs from $\sqrt{N}$ by at most 1. Then by Proposition 3.2 applied with N/K, there are at most $c_2(\sqrt{N})^{(5/4)-(\alpha/2)+\varepsilon}$ rectangles $R_{ik}(N/K)$ that contain both a point of the graph of f and a point of the graph of $W_t(\omega)$. Recall the definitions of the events C_{ik} and D_N from Proposition 3.2 and its proof. Since we are assuming that $\omega \notin E$, we also have $\omega \notin C_{ik}(N)$ for any i, k. This implies that inside each rectangle $R_{ik}(N/K)$, there are at most $c_3(\sqrt{N})^{(1/2)+\varepsilon}N^{(1-\alpha)/2}$ rectangles $R_{\ell m}(N)$ that contain both a point of the graph of f and a point of the graph of $W_t(\omega)$. Thus there are at most

$$c_4(\sqrt{N})^{(5/4)-(\alpha/2)+\varepsilon}(\sqrt{N})^{(1/2)+\varepsilon}N^{(1-\alpha)/2} = c_4 N^{(11/8)-(3\alpha/4)+\varepsilon}$$

rectangles $R_{\ell m}(N)$ that contain both a point of the graph of f and a point of the graph of $W_t(\omega)$.

We continue iterating: take N large so that $N \geq (4N_1)^4$. There are $c_4(\sqrt{N})^{(11/8)-(3\alpha/4)+\varepsilon}$ rectangles $R_{\ell m}(N/K)$ that contain both a point of the graph of f and a point of the graph of $W_t(\omega)$. Each of these contains at most $c_5(\sqrt{N})^{(1/2)+\varepsilon}N^{(1-\alpha)/2}$ rectangles $R_{\ell m}(N)$ that contain both a point of the graph of f and a point of the graph of $W_t(\omega)$, for a total of

$$c_6(\sqrt{N})^{(11/8)-(3\alpha/4)+\varepsilon}(\sqrt{N})^{(1/2)+\varepsilon}N^{(1-\alpha)/2} = c_6 N^{(23/16)-(7\alpha/8)+\varepsilon}$$

rectangles $R_{\ell m}(N)$.

Continuing, if N is large enough, we can get the exponent of N as close to $(3/2) - \alpha + \varepsilon$ as we like. In particular, by a finite number of iterations, we can get the exponent less than $(3/2) - \alpha + \delta$. □

Recall the definition of $p(t, x, y)$ in (2.1).

Lemma 3.4. *If $\|f - g\| \leq \varepsilon$, then for some constant c_1 and all $\varepsilon < \frac{1}{2}$,*

$$\int_0^1 |p(t, 0, f(t)) - p(t, 0, g(t))|\, dt \leq c_1 \varepsilon \log(1/\varepsilon).$$

Proof. For $t \leq \varepsilon^2$, we use the estimate $p(t, 0, x) \leq c_2 t^{-1/2}$ and obtain

$$\int_0^{\varepsilon^2} |p(t, 0, f(t)) - p(t, 0, g(t))|\, dt \leq 2c_2 \int_0^{\varepsilon^2} \frac{1}{\sqrt{t}} dt \leq c_3 \varepsilon.$$

For $t \geq \varepsilon^2$, note that

$$\Big|\frac{\partial p(t, 0, x)}{\partial x}\Big| = c_4 t^{-1/2} \frac{|x|}{t} e^{-x^2/2t} = c_4 t^{-1} \frac{|x|}{\sqrt{t}} e^{-x^2/2t} \leq c_5 t^{-1},$$

since $|y| e^{-y^2/2}$ is bounded. We then obtain

$$\int_{\varepsilon^2}^1 |p(t, 0, f(t)) - p(t, 0, g(t))|\, dt \leq \int_{\varepsilon^2}^1 |f(t) - g(t)| c_5 t^{-1} dt \leq c_5 \varepsilon \int_{\varepsilon^2}^1 t^{-1}\, dt = c_6 \varepsilon \log(1/\varepsilon).$$

Adding the two integrals proves the lemma. □

Proposition 3.5. *Let f and g be two functions with*

$$\sup_{(j-1)/N \leq t \leq j/N} |f(t) - g(t)| \leq \delta.$$

Then, for all $\lambda > 0$,

$$\mathbb{P}\big(|(L^f_{j/N} - L^f_{(j-1)/N}) - (L^g_{j/N} - L^g_{(j-1)/N})| \geq \lambda N^{-1/4}(\delta \log(1/\delta))^{1/2}\big) \leq c_1 e^{-c_2\lambda}.$$

Proof. Write s for $(j-1)/N$ and $A^f_t = L^f_{s+t} - L^f_s$, $A^g_t = L^g_{s+t} - L^g_s$. We have for $s \leq r \leq t \leq s + (1/N)$,

$$\mathbb{E}[A^f_t - A^f_r \mid \mathcal{F}_r] = \mathbb{E}^{W_r} A^f_{t-r} \leq \sup_z \mathbb{E}^z A^f_{1/N}.$$

But for any z,

$$\mathbb{E}^z A^f_{1/N} = \int_0^{1/N} p(t, z, f(t))\, dt \leq \int_0^{1/N} \frac{1}{\sqrt{t}} dt \leq c_3 N^{-1/2}.$$

We have a similar bound for $\mathbb{E}^z A^g_{1/N}$. For the difference, we have

$$|\mathbb{E}[(A^f_t - A^g_t) - (A^f_r - A^g_r) \mid \mathcal{F}_r]| = |\mathbb{E}^{W_r}[A^f_{t-r} - A^g_{t-r}]|.$$

However, for any z,

$$\begin{aligned}|\mathbb{E}^z[A^f_{t-r} - A^g_{t-r}]| &= \Big|\int_s^{s+t-r} [p(u, z, f(u)) - p(u, z, g(u))] du\Big| \\ &\leq \int_0^1 |p(u, 0, \widetilde{f}(u)) - p(u, 0, \widetilde{g}(u))|\, du,\end{aligned}$$

where we define $\widetilde{f}(u) = f(u) - z$ for all u and we define $\widetilde{g}(u) = g(u) - z$ if $s \leq u \leq s + (t-r)$ and $\widetilde{g}(u) = \widetilde{f}(u)$ otherwise. So $\|\widetilde{f}(u) - \widetilde{g}(u)\| \leq \delta$, and by Lemma 3.4,

$$|\mathbb{E}^z[A^f_{t-r} - A^g_{t-r}]| \leq c_4 \delta \log(1/\delta).$$

Our result now follows by Lemma 2.1. □

Theorem 3.6. *For any $\alpha \in (5/6, 1]$, there exists $\widetilde{L}^f_t$ such that*

- *(i) for each $f \in \mathcal{S}_\alpha$, we have $\widetilde{L}^f_t = L^f_t$ for all t,* a.s.,
- *(ii) with probability one, $f \to \widetilde{L}^f_1$ is a continuous map on $\mathcal{S}_\alpha$ with respect to the supremum norm, and*
- *(iii) with probability one, $\sup_{f \in \mathcal{S}_\alpha} \widetilde{L}^f_1 < \infty$.*

Proof.

Step 1. In this step, we will define and analyze a countable dense family of functions in $\mathcal{S}_\alpha$.

Let $N = 2^n$ and let T_n denote the class of functions f in $\mathcal{S}_\alpha$ such that on each interval $[(j-1)/N, j/N]$ the function f is linear with slope either $N^{1-\alpha}$ or $-N^{1-\alpha}$ and $f(j/N)$ is a multiple of $1/N^\alpha$ for each j. Note that the collection of all functions which are piecewise linear with these slopes contains some functions which are not in $\mathcal{S}_\alpha$– such functions do not belong to T_n.

Consider any element h of $\mathcal{S}_\alpha$. Let $h^{(n)}$ denote a function in T_n which approximates h in the following sense. We will define $h^{(n)}$ inductively on intervals of the form $[(j-1)/N, j/N]$. First we take the initial value $h^{(n)}(0)$ to be the closest integer multiple of $1/N^\alpha$ to $h(0)$ (we take the smaller value in case of a tie). The slope of $h^{(n)}$ is chosen to be positive on $[0, 1/N]$ if and only if $h^{(n)}(0) \le h(0)$. Once the function $h^{(n)}$ has been defined on all intervals $[(j-1)/N, j/N]$, $j = 1, 2, \dots, k$, we choose the slope of $h^{(n)}$ on $[k/N, (k+1)/N]$ to be $N^{1-\alpha}$ if and only if $h^{(n)}(k/N) \le h(k/N)$. Strictly speaking, our definition generates some functions with values in $[-1-1/N^\alpha, 1+1/N^\alpha]$ rather than in $[-1, 1]$ and so $h^{(n)}$ might not belong to $\mathcal{S}_\alpha$. We leave it to the reader to check that this does not affect our arguments.

We will argue that $|h^{(n)}(t) - h(t)| \le 2/N^\alpha$ for all t. This is true for $t = 0$ by definition. Suppose that $1/N^\alpha \le |h^{(n)}(t) - h(t)| \le 2/N^\alpha$ for some $t = j/N$. Then the fact that both functions belong to $\mathcal{S}_\alpha$ and our choice for the slope of $h^{(n)}$ easily imply that the absolute value of the difference between the two functions will not be greater at time $t = (j+1)/N$ than at time $t = j/N$. An equally elementary argument shows that in the case when $|h^{(n)}(t) - h(t)| \le 1/N^\alpha$, the distance between the two functions may sometimes increase but will never exceed $2/N^\alpha$. The induction thus proves the claim for all times t of the form $t = j/N$. An extension to all other times t is easy.

Later in the proof we will need to consider the difference between $h^{(n)}$ and $h^{(n+1)}$. First let us restrict our attention to the interval $[\ell/N, (\ell+1)/N]$. The estimates from the previous paragraph show that $|h^{(n)}(t) - h^{(n+1)}(t)| \le 4/N^\alpha$ on this interval. Let

$$F_{h,\ell} = \{|(L^{h^{(n)}}_{(\ell+1)/N} - L^{h^{(n)}}_{\ell/N}) - (L^{h^{(n+1)}}_{(\ell+1)/N} - L^{h^{(n+1)}}_{\ell/N})| \ge N^{-(1/4)-(\alpha/2)+\varepsilon}\}.$$

By Proposition 3.5 with $\lambda = N^\varepsilon$, for any $h \in \mathcal{S}_\alpha$, ℓ and n,

$$\mathbb{P}(F_{h,\ell}) \le c_1 \exp(-c_2 N^\varepsilon).$$

There are only $N+1$ integers ℓ with $0 \le \ell \le N$. For a fixed ℓ, there are no more than $3N^\alpha$ possible values of $h^{(n)}(\ell/N)$, and the same is true for $h^{(n)}((\ell+1)/N)$. The analogous upper bound for the number of possible values for each of $h^{(n+1)}(\ell/N)$, $h^{(n+1)}((\ell+1/2)/N)$ and $h^{(n+1)}((\ell+1)/N)$ is $6N^\alpha$. Hence, if we let

$$G_N = \bigcup_{h \in \mathcal{S}_\alpha} \bigcup_{0 \le \ell \le N} F_{h,\ell},$$

then

$$\mathbb{P}(G_N) \leq c_3 N^{5\alpha+1} \exp(-c_2 N^{\varepsilon}).$$

We will derive a similar estimate for $f^{(n)}$ and $h^{(n)}$, where $f, h \in \mathcal{S}_\alpha$. Let us assume that $\|f - h\| \leq 1/N^\alpha$. Then $|f^{(n)}(t) - h^{(n)}(t)| \leq 5/N^\alpha$ for all t. If we define

$$\widetilde{F}_{f,h,\ell} = \{|(L^{f^{(n)}}_{(\ell+1)/N} - L^{f^{(n)}}_{\ell/N}) - (L^{h^{(n)}}_{(\ell+1)/N} - L^{h^{(n)}}_{\ell/N})| \geq N^{-(1/4)-(\alpha/2)+\varepsilon}\}.$$

then

$$\mathbb{P}(\widetilde{F}_{f,h,\ell}) \leq c_7 \exp(-c_8 N^{\varepsilon}).$$

Next we let

$$\widetilde{G}_N = \bigcup_{f,h\in\mathcal{S}_\alpha} \bigcup_{0\leq\ell\leq N} \widetilde{F}_{f,h,\ell}.$$

Counting all possible paths $f^{(n)}$ and $h^{(n)}$ yields an estimate analogous to the one for G_N,

$$\mathbb{P}(\widetilde{G}_N) \leq c_9 N^{4\alpha+1} \exp(-c_8 N^{\varepsilon}).$$

Step 2. In this step, we will prove uniform continuity of $f \to L^f_1$ on the set $T_\infty = \bigcup_{n=1}^\infty T_n$.

Fix arbitrarily small $\eta, \beta > 0$. Choose $\varepsilon > 0$ so small that $(9/8) - (3\alpha/2) + 2\varepsilon < 0$. Recall the events D_N from Proposition 3.2. Since $\sum_N (\mathbb{P}(D_N) + \mathbb{P}(G_N) + \mathbb{P}(\widetilde{G}_N)) < \infty$, we can take N_0 sufficiently large so that $\mathbb{P}(H) \leq \eta$, where $H = \bigcup_{N=N_0}^\infty (D_N \cup G_N \cup \widetilde{G}_N)$. Without loss of generality we may take N_0 to be an integer power of 2, say $N_0 = 2^{n_0}$.

Fix an $\omega \notin H$. Consider any $f, h \in T_\infty$ with $\|f - h\| \leq 1/N_0^\alpha$. Note that

$$|L^h_1 - L^{h^{(n_0)}}_1| \leq \sum_{n=n_0}^\infty |L^{h^{(n+1)}}_1 - L^{h^{(n)}}_1|, \tag{3.3}$$

and

$$|L^{h^{(n+1)}}_1 - L^{h^{(n)}}_1| \leq \sum_{m=1}^{2^n} |(L^{h^{(n+1)}}_{(m+1)/2^n} - L^{h^{(n+1)}}_{m/2^n}) - (L^{h^{(n)}}_{(m+1)/2^n} - L^{h^{(n)}}_{m/2^n})|. \tag{3.4}$$

Consider $2^n = N \geq N_0$. Since $\omega \notin \bigcup_{N\geq N_0} D_N$, Proposition 3.3 implies that there are at most $c_1 N^{(3/2)-\alpha+\varepsilon}$ values of m for which there is a rectangle R_{mi} in which there is a point of the graph of $h^{(n)}$ or of $h^{(n+1)}$ and a point of the graph of $W_t(\omega)$. So there are no more than $c_1 N^{(3/2)-\alpha+\varepsilon}$ summands on the right hand side of (3.4) that are non-zero.

For a value of m for which the summand on the right hand side is nonzero, it is at most $N^{-(1/4)-(\alpha/2)+\varepsilon}$, because $\omega \notin \bigcup_{N\geq N_0} G_N$. Multiplying the number of nonzero summands by the the largest value each summand can be, we obtain

$$\begin{aligned} |L^{h^{(n+1)}}_1 - L^{h^{(n)}}_1| &\leq c_1 N^{(3/2)-\alpha+\varepsilon} N^{-(1/4)-(\alpha/2)+\varepsilon} \\ &= c_1 N^{(5/4)-(3\alpha/2)+2\varepsilon} = c_1 (2^n)^{(5/4)-(3\alpha/2)+2\varepsilon}. \end{aligned} \tag{3.5}$$

We have assumed that ε is so small that $(5/4) - (3\alpha/2) + 2\varepsilon < 0$, so the bound in (3.5) is summable in n. We increase n_0, if necessary, so that $\sum_{n \geq n_0} c_1 (2^n)^{(5/4)-(3\alpha/2)+2\varepsilon} \leq \beta/3$. Then (3.3) implies that

$$|L_1^h - L_1^{h^{(n_0)}}| \leq \beta/3.$$

Similarly,

$$|L_1^f - L_1^{f^{(n_0)}}| \leq \beta/3.$$

A similar reasoning will give us a bound for $|L_1^{f^{(n_0)}} - L_1^{h^{(n_0)}}|$. We have

$$|L_1^{f^{(n_0)}} - L_1^{h^{(n_0)}}| \leq \sum_{\ell=1}^{2^n} |(L_{(\ell+1)/N}^{f^{(n)}} - L_{\ell/N}^{f^{(n)}}) - (L_{(\ell+1)/N}^{h^{(n)}} - L_{\ell/N}^{h^{(n)}})|.$$

First, the number of non-zero summands is bounded by $c_1 N_0^{(3/2)-\alpha+\varepsilon}$, for the same reason as above. We have assumed that $\|f - h\| \leq 1/N_0^\alpha$, so, in view of the fact that $\omega \notin \bigcup_{N \geq N_0} \widetilde{G}_N$, the size of a non-zero summand is bounded by $N_0^{-(1/4)-(\alpha/2)+\varepsilon}$. Hence,

$$|L_1^{f^{(n_0)}} - L_1^{h^{(n_0)}}| \leq c_1 N_0^{(3/2)-\alpha+\varepsilon} N_0^{-(1/4)-(\alpha/2)+\varepsilon} = c_1 (2^{n_0})^{(5/4)-(3\alpha/2)+2\varepsilon} \leq \beta/3.$$

By the triangle inequality, with probability greater than $1 - \eta$,

$$|L_1^f - L_1^h| \leq \beta$$

if $f, h \in T_\infty$ and $\|f - h\| \leq 1/N_0^\alpha \stackrel{\text{df}}{=} \delta(\beta)$. We now fix an arbitrarily small $\eta_0 > 0$ and a sequence $\beta_k \to 0$, and find $\delta(\beta_k) > 0$ such that with probability greater than $1 - \eta_0/2^k$,

$$|L_1^f - L_1^h| \leq \beta_k,$$

if $f, h \in T_\infty$ and $\|f - h\| \leq \delta(\beta_k)$. This implies that, with probability greater than $1 - \eta_0$, the function $f \to L_1^f$ is uniformly continuous on T_∞. Since η_0 is arbitrarily small, the uniform continuity is in fact an almost sure property, although the modulus of continuity may depend on ω.

For an arbitrary $f \in \mathcal{S}_\alpha$, define $\widetilde{L}^f = \lim_{n\to\infty} L_1^{f^{(n)}}$. By Corollary 2.5, $L^f = \widetilde{L}^f$ a.s. Therefore $\widetilde{L}^f$ is a version of L^f.

Since the function $f \to L_1^f$ is uniformly continuous on T_∞, its extension to $\mathcal{S}_\alpha$ is uniformly continuous with the same (random) modulus of continuity. The family $\mathcal{S}_\alpha$ is equicontinuous, hence a compact set with respect to $\|\cdot\|$. Therefore the supremum of $\widetilde{L}_1^f$ over $\mathcal{S}_\alpha$ is finite, a.s. $\square$

Remark 3.7. It is rather easy to see that, with probability one, $f \to \widetilde{L}_t^f$ is actually jointly continuous on $\mathcal{S} \times [0,1]$. To see this, note that in the proof of Proposition 3.5 we used Proposition 2.1, so what we actually proved was that

$$\mathbb{P}\left(\sup_{(j-1)/n \le t \le j/n} |(L_t^f - L_{(j-1)/n}^f) - (L_t^g - L_{(j-1)/n}^g)| \ge \lambda N^{-1/4}(\delta \log(1/\delta))^{1/2}\right) \le e^{-c_1 \lambda}.$$

If we replace (3.4) by

$$\sup_t |L_t^{h^{(n+1)}} - L_t^{h^{(n)}}| \le \sum_{m=1}^{2^n} \sup_{m/2^n \le t \le (m+1)/2^n} |(L_t^{h^{(n+1)}} - L_{m/2^n}^{h^{(n+1)}}) - (L_t^{h^{(n)}} - L_{m/2^n}^{h^{(n)}})|,$$

then proceeding as in the proof of Theorem 3.6, we obtain the joint continuity.

We will show that, in a sense, $\sup_{f \in \mathcal{S}_\alpha} L_1^f = \infty$, a.s., if $\alpha < 1/2$. This statement is quite intuitive – one would like to let $f(\omega) = W_t(\omega)$ so that $L_1^f(\omega) = \infty$ – but we have not defined the local time simultaneously for all $f \in \mathcal{S}_\alpha$, and there is a difficulty with the number of null sets. Theorem 3.6 suggests that the question of joint existence is tied to the question of the finiteness of the supremum, so we have to express our result in a different way.

Theorem 3.8. *Suppose $\alpha < 1/2$. Then there exists a countable family $F \subset \mathcal{S}_\alpha$ such that $\sup_{f \in F} L_1^f = \infty$ a.s.*

Proof. Let ℓ_t^x be the ordinary local time at x for Brownian motion. It is well known that there exists a version of this process which is jointly continuous in x and t (see Karatzas and Shreve (1994) but note that their local times are half of our local times).

Suppose that a piecewise linear function f is equal to y on an interval $[s,t]$. Then Proposition 2.2 and a similar well known result for ℓ^y show that with probability one, for all $u \in [s,t]$,

$$L_u^f - L_s^f = \ell_u^y - \ell_s^y.$$

Fix $\alpha \in (0, 1/2)$. Let F be the countable family of all functions f defined on the interval $[0,1]$ such that for some integers $n = n(f)$ and $m = m(f)$, on each interval of the form $[(j-1)/n, (j-\frac{1}{2})/n]$ the function f is a constant multiple of 2^{-m}, f is linear on the intervals $[(j-\frac{1}{2})/n, j/n]$, and $f \in \mathcal{S}_\alpha$. Then, with probability one, for all j, all $f \in F$ and $n = n(f)$,

$$L_{(j-(1/2))/n}^{f((j-1)/n)} - L_{(j-1))/n}^{f((j-1)/n)} = \ell_{(j-(1/2))/n}^{f((j-1)/n)} - \ell_{(j-1))/n}^{f((j-1)/n)}. \tag{3.6}$$

In the rest of the proof we assume that this assertion and the joint continuity of ℓ_t^x hold for all ω.

Let

$$T = \inf\{t : |W_t| \ge 1 \text{ or } \exists r, s \le t \text{ such that } |W_r - W_s| \ge (\tfrac{1}{4}|r-s|)^\alpha\}. \tag{3.7}$$

By the well-known results on the modulus of continuity for Brownian motion, $T > 0$ a.s.

Let $\varepsilon > 0$. There exists δ such that $\mathbb{P}(T < \delta) < \varepsilon$. Fix n. On the interval $[(j-1)/n, (j-\frac{1}{2})/n]$, let $f_1(t) = W((j-1)/n)$. On the interval $[(j-\frac{1}{2})/n, j/n]$ let $f_1(t)$ be linear with $f_1(j/n) = W(j/n)$. Let $f_2(t) = f_1(t)$ for $t \le \delta/2$ and constant for $t \ge \delta/2$.

It is quite easy to show that $f_2 \in \mathcal{S}_\alpha$ for each ω in the set $\{T > \delta\}$ using the definition (3.6) of T. By the Markov property, the random variables

$$X_j = \ell^{f_2((j-1)/n)}_{(j-(1/2))/n} - \ell^{f_2((j-1)/n)}_{(j-1))/n}$$

form an independent sequence, and by Brownian scaling, $Y_j = \sqrt{2n} X_j$ has the same distribution as ℓ^0_1. Let $c_1 = \mathbb{E}\ell^0_1$. By Chebyshev's inequality,

$$\mathbb{P}\Big(\Big|\sum_{j=1}^{[\delta n/2]} (Y_j - c_1)\Big| \ge c_1\delta n/4\Big) \le \frac{[\delta n/2] \operatorname{Var} Y_1}{(c_1 \delta n/4)^2} \le \frac{c_2 \mathbb{E}(\ell^0_1)^2}{\delta n} = \frac{c_3}{\delta n}.$$

Take n large so that $c_3/(\delta n) < \varepsilon$. Then there exists a set A_n of probability at most 2ε such that if $\omega \notin A_n$, then $T(\omega) \ge \delta$ and

$$\sum_{j=1}^{[\delta n/2]} X_j \ge c_4 \sqrt{\delta n}.$$

We now choose m large and find $f_3 \in F$ so that on each interval $[(j-1)/n, (j-\frac{1}{2})/n]$ the function f_3 is a multiple of 2^{-m}, f_3 is linear on the intervals $[(j-\frac{1}{2})/n, j/n]$, and

$$\sum_{j=1}^{[\delta n/2]} \Big[\ell^{f_3((j-1)/n)}_{(j-(1/2))/n} - \ell^{f_3((j-1)/n)}_{(j-1))/n}\Big] \ge c_4\sqrt{\delta n}/2;$$

this is possible by the joint continuity of ℓ^x_t.

By (3.6) we can replace ℓ by L in the last formula, so

$$L^{f_3}_1 \ge \sum_{j=1}^{[\delta n/2]} \Big[L^{f_3}_{(j-(1/2))/n} - L^{f_3}_{(j-1))/n}\Big] \ge c_4 \sqrt{\delta n}/2.$$

We conclude that

$$\sup_{f \in F} L^f_1 \ge c_4 \sqrt{\delta n}/2,$$

with probability greater than or equal to $1-2\varepsilon$. Since n and ε are arbitrary, the proposition is proved. □

References

1. R.F. Bass (1995). *Probabilistic Techniques in Analysis.* Springer-Verlag, New York.
2. R.F. Bass and D. Khoshnevisan (1992). Local times on curves and uniform invariance principles. *Probab. Theory Related Fields* **92**, 465–492.
3. R.F. Bass and K. Burdzy (1999). Stochastic bifurcation models. *Ann. Probab.* **27**, 50–108.
4. K. Burdzy and J. San Martín (1995). Iterated law of iterated logarithm. *Ann. Probab.* **23**, 1627–1643.
5. G.F. Clements (1963). Entropies of several sets of real valued functions. *Pacific J. Math.* **13**, 1085–1095.
6. B. Davis, (1998). Distribution of Brownian local time on curves. *Bull. London Math. Soc.* **30**, 182–184.
7. I. Karatzas and S. Shreve (1994) *Brownian Motion and Stochastic Calculus, Second Edition.* Springer-Verlag, New York.
8. M. Ledoux and M. Talagrand (1991). *Probability in Banach spaces. Isoperimetry and Processes.* Springer-Verlag, Berlin.
9. M.B. Marcus and J. Rosen (1996). Gaussian chaos and sample path properties of additive functionals of symmetric Markov processes. *Ann. Probab.* **24**, 1130–1177.
10. D. Revuz and M. Yor (1994). *Continuous Martingales and Brownian Motion, 2nd ed.* Springer-Verlag, Berlin.

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