

# The supremum of Brownian local times on Hölder curves revisited

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**Abstract.** For  $f : [0, 1] \rightarrow \mathbb{R}$ , we consider  $L_t^f$ , the local time of space-time Brownian motion on the curve  $f$ . Let  $\mathcal{S}_\alpha$  be the class of all functions whose Hölder norm of order  $\alpha$  is less than or equal to 1. We show that the supremum of  $L_1^f$  over  $f$  in  $\mathcal{S}_\alpha$  is finite if  $\alpha > \frac{1}{2}$ .

**Résumé.** Soit  $W_t$  un mouvement brownien et soit  $L_t^f$  le temps local du processus  $(t, W_t)$  pour le courbe  $f : [0, 1] \rightarrow \mathbb{R}$ . Soit  $\mathcal{S}_\alpha$  la classe de toutes fonctions telle que la norme holderienne du ordre  $\alpha$  est moins de 1. Nous démontrons que  $\sup_{f \in \mathcal{S}_\alpha} L_1^f < \infty$  p.s. si  $\alpha > \frac{1}{2}$ .

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## 1. Introduction

The main claim of [1] was that the supremum of Brownian local times over all  $\alpha$ -Hölder curves is finite if  $\alpha > 1/2$  (see Theorem 1.1 below for the precise statement). An error in the proof was pointed out to us by A. Vatamanelu; however we were able to establish the claim for  $\alpha > 5/6$  in [2]. The purpose of this note is to prove the original claim from [1], that finiteness of the supremum indeed holds for all  $\alpha \in (1/2, 1]$ . We also showed in [1] that  $\alpha = 1/2$  is the critical value; see Theorem 3.8 of that paper for the precise statement.

Let  $W_t$  be one-dimensional Brownian motion and let  $f : [0, 1] \rightarrow \mathbb{R}$  be a Hölder continuous function. There are a number of equivalent ways to define  $L_t^f$ , the local time of  $W_t$  along the curve  $f$ , one of which is as the limit in probability of

$$\frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(f(s)-\varepsilon, f(s)+\varepsilon)}(W_s) ds$$

as  $\varepsilon \rightarrow 0$ . See [1, Sect. 2] for a discussion of other ways of defining  $L_t^f$ . Let

$$\mathcal{S}_\alpha = \{f : \sup_{0 \leq t \leq 1} |f(t)| \leq 1, |f(s) - f(t)| \leq |s - t|^\alpha \text{ if } s, t \leq 1\}.$$

Our main result in this paper is the following.

**Theorem 1.1.** *For any  $\alpha \in (1/2, 1]$ , there exists  $\tilde{L}_t^f$  such that*

(i) *for each  $f \in \mathcal{S}_\alpha$ , we have  $\tilde{L}_t^f = L_t^f$  for all  $t$ , a.s.,*

(ii) *with probability one,  $f \rightarrow \tilde{L}_1^f$  is a continuous map on  $\mathcal{S}_\alpha$  with respect to the supremum norm, and*

(iii) *with probability one,  $\sup_{f \in \mathcal{S}_\alpha} \tilde{L}_1^f < \infty$ .*

The interest in Theorem 1.1 has several sources. One is that the metric entropy of  $\mathcal{S}_\alpha$  is far too large for chaining arguments to work; nevertheless the supremum is finite a.s. Another is the work of Holden and Sheffield [4] on scaling limits of the Schelling model, where they used some of the techniques in [1] to analyze local times of random fields over Lipschitz surfaces. A recent interesting paper [3] is devoted to similar questions involving local times.

In the interests of space we present only the changes needed to [1] to prove our result and refer to the original paper for the unchanged part of the proof.

## 2. The finiteness of the supremum

Let  $W_t$  be a Brownian motion. A key ingredient in our proof is Lemma 3.1 of [1]. The proof there is correct; the error in [1] was in how this lemma was applied further on.

We replace Propositions 3.2 and 3.3 in [1] by the following.

Consider an integer  $N > 0$ . For  $0 \leq \ell \leq N$ ,  $-N^\alpha - 1 \leq m \leq N^\alpha$ , let  $R_{\ell m}$  be the rectangle defined by

$$R_{\ell m} = [\ell/N, (\ell + 1)/N] \times [m/N^\alpha, (m + 1)/N^\alpha].$$

**Proposition 2.1.** *Let  $\alpha \in (1/2, 1]$  and  $\varepsilon \in (0, 1/16)$ . There exist  $c_1, c_2$ , and  $c_3$  such that:*

- (i) *there exists a set  $D_N$  with  $\mathbb{P}(D_N) \leq c_1 \exp(-c_2 N^{\varepsilon/2})$ ;*
- (ii) *if  $\omega \notin D_N$  and  $f \in \mathcal{S}_\alpha$ , then there are at most  $c_3 N^{(1/2)+2\varepsilon}$  rectangles  $R_{\ell m}$  in  $[0, 1] \times [-1, 1]$  which contain both a point of the graph of  $f$  and a point of the graph of  $W_t(\omega)$ .*

**Proof.** Let  $M = \lfloor N^\varepsilon \rfloor$  and set

$$Q_{ik} = [i/M, (i + 1)/M] \times [k/M^{1/2}, (k + 1)/M^{1/2}],$$

for  $0 \leq i \leq M$  and  $-M^{1/2} - 1 \leq k \leq M^{1/2}$ . Let  $J = \lceil N/M \rceil$ .

Let

$$I_{ikj} = \{\exists t \in [i/M + (j - 1)/N, i/M + j/N] : \\ k/M^{1/2} \leq W_t \leq (k + 1)/M^{1/2}\},$$

$$A_{ik} = \sum_{j=1}^J \mathbf{1}_{I_{ikj}},$$

and

$$C_{ik} = \{A_{ik} \geq J^{(1/2)+\varepsilon}\}.$$

By Lemma 3.1 of [1] with  $\lambda = J^\varepsilon$  and the Markov property applied at  $i/M$  we have  $\mathbb{P}(C_{ik}) \leq c_4 \exp(-c_5 J^\varepsilon)$ .

There are at most  $c_6 M^{3/2}$  rectangles  $Q_{ik}$ , so if  $D_N = \cup_{i,k} C_{ik}$ , where  $0 \leq i \leq M$  and  $-M^{1/2} - 1 \leq k \leq M^{1/2}$ , then

$$\mathbb{P}(D_N) \leq c_7 M^{3/2} \exp(-c_5 J^\varepsilon) \leq c_7 \exp(-c_8 N^{\varepsilon/2}).$$

Let  $f$  be any function in  $\mathcal{S}_\alpha$ . If  $f$  intersects  $Q_{ik}$  for some  $i$  and  $k$ , then  $f$  might intersect  $Q_{i,k-1}$  and  $Q_{i,k+1}$ . But because  $f \in \mathcal{S}_\alpha$  and  $\alpha > 1/2$ , it cannot intersect  $Q_{ir}$  for any  $r$  such that  $|r - k| > 1$ . Therefore  $f$  can intersect at most  $3(M + 1)$  of the  $Q_{ik}$ .

Now suppose  $\omega \notin D_N$ . Look at any one of the  $Q_{ik}$  that  $f$  intersects. Since  $\omega \notin D_N$ , then there are at most  $J^{(1/2)+\varepsilon}$  integers  $j$  that are less than  $J$  and for which the path of  $W_t(\omega)$  intersects  $([i/M + (j - 1)/N, i/M + j/N] \times [-1, 1]) \cap Q_{ik}$ . If  $f$  intersects a rectangle  $R_{\ell m}$ , then it can intersect a rectangle  $R_{\ell r}$  only if  $|r - m| \leq 1$ , since  $f \in \mathcal{S}_\alpha$ . Therefore there are at most  $3J^{(1/2)+\varepsilon}$  rectangles  $R_{\ell m}$  contained in  $Q_{ik}$  which contain both a point of the graph of  $f$  and a point of the graph of  $W_t(\omega)$ .

Since there are at most  $3(M + 1)$  rectangles  $Q_{ik}$  which contain a point of the graph of  $f$ , there are therefore at most

$$3(M + 1)3J^{(1/2)+\varepsilon} \leq c_9 N^{(1/2)+2\varepsilon}$$

rectangles  $R_{\ell m}$  that contain both a point of the graph of  $f$  and a point of the graph of  $W_t(\omega)$ . □

Our present Proposition 2.1 is almost identical to Proposition 3.3 in [1], so the latter can be omitted. With this change, the remainder of [1], beyond Proposition 3.3, can proceed as in the original.

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## References

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