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The Most Visited Site of Brownian Motion and Simple Random Walk

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Summary. Let $L(t,x)$ be the local time at x for Brownian motion and for each t, let $\bar{V}(t)=\inf\{x\geqq 0\colon L(t,x)\vee L(t,-x)=\sup_y L(t,y)\}$, the absolute value of the most visited site for Brownian motion up to time t. In this paper we prove that $\bar{V}(t)$ is transient and obtain upper and lower bounds for the rate of growth of $\bar{V}(t)$. The main tools used are the Ray-Knight theorems and William's path decomposition of a diffusion. An invariance principle is used to get analogous results for simple random walks. We also obtain a law of the iterated logarithm for $\bar{V}(t)$.

1. Introduction

Let S_n be a simple random walk in $\mathbb{R}^1$ started at 0. The basic question asked in this paper is as follows: is 0 the most visited site infinitely often? To formulate this precisely define

$$N(n,k)=\sum_{j=0}^{n} 1_{\{k\}}(S_j)$$
$$\bar{N}(n)=\sup_k N(n,k)$$
$$\bar{U}(n)=\inf\{k\geqq 0\colon N(n,k)\vee N(n,-k)=\bar{N}(n)\}.$$

The question then becomes, does $P(\bar{U}(n)=0 \text{ i.o.})=1$? Rather surprisingly the answer is no as we show in Sect. 5.

Our approach to this problem is to consider the analogous question for a one dimensional Brownian motion $W(t)$ and then use an invariance principle. Thus let $L(t,x)$ be the local time of W at x up to time t (for a precise definition

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see (2.3)) and define

$$L^*(t)=\sup_{x\geqq 0} L(t,x)$$
$$V(t)=\inf\{x\geqq 0: L(t,x)=L^*(t)\}.$$

(Observe that $V(t)$ is the "most visited site" on the positive x-axis, the results for the entire x-axis will follow easily from our analysis of this case.) We will prove that $V(t)$ is a transient process and furthermore obtain estimates on how quickly it escapes to infinity (see Theorems 3.6 and 3.7). Set

$$T_r=\inf\{t: L(t,0)>r\}.$$

Our main tool is a theorem of Knight [12] which describes $L(T_r,z)$ as a Markov process in the space variable, (for closely related results see Ray [14]). This enables us to obtain information about $V(t)$ at the random times T_r, from which the results follow fairly easily. To give some indication of how the proof goes, define for any $K>0$

$$I(t,K)=\sup_{0\leqq x\leqq K} L(t,x). \tag{1.1}$$

Obviously, for any t, $L(t,0)\leqq I(t,K)\leqq L^*(t)$. It is not too difficult to show that for any $K>0$

$$\liminf_{r\to\infty}\frac{L^*(T_r)}{r}=1, \qquad \limsup_{r\to\infty}\frac{I(T_r,K)}{r}=1$$

and so to distinguish between them, we investigate the second order terms. We prove that for any $\varepsilon>0$ and $K>0$, if r is sufficiently large

$$L^*(T_r)-r\geqq r^{1-\varepsilon}, \qquad I(T_r,K)-r\leqq r^{(1/2)+\varepsilon}$$

(see Propositions 3.3 and 3.5 for more precise results). From this it is immediate that $V(T_r)$ is transient, and a little further analysis shows that $V(t)$ is also. The main difficulty in this approach is to obtain a good upper bound for $P(L^*(T_r)-r\leqq\lambda$ for some $r\in[1, 2])$, which is done in Lemma 3.2.

In addition to investigating the small values of $V(t)$ we also investigate its large values. By the Law of the Iterated Logarithm (L.I.L.) for $W(t)$, it is immediate that

$$\limsup_{t\to\infty}\frac{V(t)}{(2t\log\log t)^{1/2}}\leqq 1 \quad \text{a.s.} \tag{1.2}$$

We prove in Sect. 4 that there is in fact equality in (1.2). To convince oneself that this should indeed be the case recall Strassen's L.I.L. [16]. Given $\varepsilon\in(0,1)$ let

$$x(s)=\begin{cases} s & \text{if } 0\leqq s\leqq 1-\varepsilon \\ 1-\varepsilon & \text{if } 1-\varepsilon\leqq s\leqq 1\end{cases}.$$

Then $x(s)$ is a limit point of $W(st)/(2t\log\log t)^{1/2}$ with probability one. Now at those times t for which $W(st)/(2t\log\log t)^{1/2}\sim x(s)$, it is intuitively clear that $V(t)$ should be approximately $(1-\varepsilon)(2t\log\log t)^{1/2}$. The proof given in Sect. 4 makes this idea precise.

In concluding the introduction we should point out that other properties of $L^*(t)$ have been studied by several authors, for example see [1, 4, 7, 10].

2. Preliminaries

Let $Y(t)$ be a Bessel process of index 0, abbreviated BES(0), [11]. That is, $Y(t)$ is a Markov process on $[0,\infty)$ with 0 as a trap and with generator $\frac{1}{2}y'' - \frac{1}{2x}y'$.

Set $X(t) = Y^2(t)$, so that $X(t)$ is a $(\mathrm{BES}(0))^2$. One readily checks that $X(t)$ is a diffusion on natural scale with 0 as a trap and with generator $2xy''$. Further its transition densities are given by ([11] page 100)

$$p_t(x,y) = (x/y)^{1/2}(2t)^{-1}\exp(-(x+y)/2t)\, I_1((xy)^{1/2}/t) \qquad x,y,t>0 \tag{2.1}$$

where I_1 is the modified Bessel function of the first kind and

$$I_1(x) \sim e^x(2\pi x)^{-1/2} \quad \text{as} \quad x\to\infty. \tag{2.2}$$

Let $W(t)$ be a one-dimensional Brownian motion. The local time of $W(t)$ defined by

$$L(t,x) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t 1_{(x-\varepsilon,x+\varepsilon)}(W(s))\,ds \tag{2.3}$$

exists and is jointly continuous in t and x except on a set of probability zero [17]. Define, for $r \geqq 0$,

$$T_r = \inf\{t: L(t,0) > r\}.$$

It is well-known that $\{T_r: r\geqq 0\}$ is a stable subordinator of index 1/2.

Theorem 2.1 (*Knight*). *For any $r\geqq 0$, the process $\{L(T_r,z): z\geqq 0\}$ is a* $(\mathrm{BES}(0))^2$ *started at $L(T_r,0)=r$.*

We now state three results which give most of the properties of $L(T_r,z)$ that are needed in the remainder of the paper.

Lemma 2.2. (a) *For each $z\geqq 0$, $\{L(T_r,z): r\geqq 0\}$ is a stationary independent increment process.*

(b) *For each $r\geqq 0$, $z\geqq 0$, $u<(2z)^{-1}$,*

$$E\exp(uL(T_r,z)) = \exp(ur(1-2uz)^{-1}).$$

(c) *For each $r\geqq 0$, $\{L(T_r,z): z\geqq 0\}$ is a martingale.*

(d) *For each $z\geqq 0$, $\{L(T_r,z)-r: r\geqq 0\}$ is a martingale.*

Proof. (a) This follows immediately from the strong Markov and stationary independent increment properties of Brownian motion.

(b) Fix $z\geqq 0$. We begin by computing the Lévy measure $n(dy)$ of the Lévy process $L(T_r,z)$. Let X_i be a sequence of i.i.d. random variables with distribution $P(X_i>y) = \exp(-y/2z)$ and $N(r)$ an independent Poisson process of intensity $(2z)^{-1}$. Then by comparing with (2.1) a direct calculation shows that $X_1 + \ldots + X_{N(r)}$ has the same distribution as $L(T_r,z)$, and since they both have stationary independent increments, they are identical as processes. Now for $\gamma \geqq 0$

$$h\, n((\gamma, \infty)) = E \sum_{r \leqq h} 1_{(\gamma, \infty)}(L(T_r, z) - L(T_{r-}, z))$$
$$= P(N(h)=1)\, P(X_1 > \gamma) + O(h^2) \quad \text{as} \quad h \to 0,$$

thus

$$n((\gamma, \infty)) = (2z)^{-1} \exp(-\gamma/2z).$$

Hence

$$E \exp(u\, L(T_r, z)) = \exp\left(r \int_0^\infty (e^{u\gamma} - 1)\, n(d\gamma)\right)$$
$$= \exp(u r (1 - 2uz)^{-1})$$

provided $u < (2z)^{-1}$.

(c) By (b), $EL(T_r, z) < \infty$ and by Theorem 2.1, $L(T_r, z)$ is a diffusion on natural scale, hence it is a martingale.

(d) By (c) $EL(T_r, z) = EL(T_r, 0) = r$. Then for each z, $L(T_r, z) - r$ is a zero mean, stationary independent increment process and hence a martingale. □

Remark. The explanation for why $L(T_r, z)$ can be represented as the sum of a Poisson number of independent exponentials is part of the "folklore" of excursion theory. We very briefly sketch the argument. By Itô's theory of excursions [9], the number $N(r)$, of excursions from 0 that reach the level z by time T_r is Poisson. By the strong Markov property, one can show that the ith excursion that reaches z will increase local time at z by X_i, where X_i is exponential and independent of $X_1, \ldots, X_{i-1}$ (cf. [13]).

To find the parameters of X_i and $N(r)$, use Tanaka's formula. If $\tau_z = \inf\{t: W(t) = z\}$,

$$E(W(\tau_z \wedge t))^+ = \tfrac{1}{2} E L(\tau_z \wedge t, 0)$$

for each t. Letting $t \to \infty$, using bounded convergence on the left, and monotone convergence on the right, we get

$$EL(\tau_z, 0) = 2E(W(\tau_z))^+ = 2z.$$

By the translation invariance and symmetry of Brownian motion,

$$E(L(\tau_0, z) \mid W(0) = z) = EL(\tau_z, 0) = 2z,$$

from which one can determine the parameters of the X_i and of $N(r)$.

A slight modification of the above argument enables one to compute the distribution of $L(T_r, z)$ given $L(T_r, y)$ for $0 \leqq y \leqq z$. This, together with the first eight pages of [18] give, in fact, an excursion theory proof of Knight's theorem.

The following result, which we will later apply to the process of Theorem 2.1, gives a path decomposition of a $(\mathrm{BES}(0))^2$ at its maximum and is a corollary of Theorem 2.4 of Williams [19].

Theorem 2.3. *Consider a probability space $(\Omega, \mathscr{F}, Q)$ on which the following independent random elements are defined,*

(a) *$Y(t)$ a diffusion on $[0,\infty)$ with generator $2xy''+4y'$ started at $x=1$,*

(b) *a random variable M with distribution satisfying $P(M>m)=m^{-1}$ for $m\geqq 1$,*

(c) *for each $m\geqq 1$ a diffusion $Z^m(t)$ on $[0,m]$ with generator $2xy''-4x(m-x)^{-1}y'$ started at the entrance boundary point m and absorbed at 0.*

Define $\rho=\inf\{t\colon Y(t)=M\}$ and

$$X(t)=\begin{cases} Y(t) & t\leqq\rho \\ Z^M(t-\rho) & t\geqq\rho. \end{cases}$$

Then $X(t)$ is a $(\mathrm{BES}(0))^2$ started at $x=1$.

Remark. We will only use the premaximum representation of a $(\mathrm{BES}(0))^2$. It is easy to check that in fact $Y(t)$ is a $(\mathrm{BES}(4))^2$.

Lemma 2.4. *For all $a>0$*

(a) $\{L(t,x)\colon t\geqq 0,\ x\in\mathbb{R}\}\overset{d}{=}\{a^{-1/2}L(at,a^{1/2}x)\colon t\geqq 0, x\in\mathbb{R}\}$,

(b) $\{L(T_r,z)\colon r\geqq 0, z\in\mathbb{R}\}\overset{d}{=}\{a^{-1}L(T_{ar},az)\colon r\geqq 0, z\in\mathbb{R}\}$.

Proof. For any $a>0$, $X(t)=a^{-1/2}W(at)$ is also a Brownian motion. Let $L^X(t,x)$ and $L^W(t,x)$ denote the local times of $X(t)$ and $W(t)$, respectively. Then a simple change of variable in (2.3) gives

$$L^X(t,x)=a^{-1/2}L^W(at,a^{1/2}x) \quad \text{a.s.} \tag{2.4}$$

which proves (a).

Let $T_r^X=\inf\{t\colon L^X(t,0)>r\}$ and $T_r^W=\inf\{t\colon L^W(t,0)>r\}$. Then by (2.4)

$$T_r^X=a^{-1}T^W_{a^{1/2}r} \quad \text{a.s.} \tag{2.5}$$

Thus upon replacing a by a^2 in (2.4) and (2.5) we obtain

$$L^X(T_r,z)=a^{-1}L^W(T_{ar},az) \quad \text{a.s.}$$

which proves (b). □

If we now define

$$L^*(t)=\sup_{x\geqq 0}L(t,x)$$

we have as an immediate consequence of the above Lemma that

$$L^*(t)\overset{d}{=}t^{1/2}L^*(1) \tag{2.6}$$

and

$$L^*(T_r)\overset{d}{=}rL^*(T_1). \tag{2.7}$$

Remark. Note that we are using L^* for the maximum over the positive x-axis rather than the entire real line.

In concluding this section we would like to point out that in the remainder of this paper the symbols c, c_1, c_2 etc. will denote generic constants which may change from line to line. In addition we will freely interchange between the commonly used notations $P(\cdot\,|\,X(0)=x)$ and $P^x(\cdot)$.

3. Transience and Lower Growth Rates

In this section we will prove that $V(t)$ is a transient process and obtain estimates on how quickly it escapes to infinity. Recalling the path decomposition of a $(\mathrm{BES}(0))^2$ in Theorem 2.3, we begin by proving,

Lemma 3.1. *Let $Y(t)$ be a diffusion with generator $2xy''+4y'$ and $Y(0)=1$. Then for all $m>1$, $u>0$*

$$P(\sup_{0\leqq t\leqq u} Y(t)<m)\leqq(2\pi u)^{-1/2}(m-1).$$

Proof. If $X(t)$ is a diffusion with generator $\frac{1}{2}y''+\frac{3}{2x}y'$ and $X(0)=1$ (so X is a BES(4)), then (cf. [8]) the distribution of $X(t)^2$ is the same as that of $Y(t)$. Moreover, there exists a Brownian motion $W(t)$ started at $W(0)=0$ such that

$$X(t)=1+W(t)+\tfrac{3}{2}\int_0^t X(s)^{-1}\,ds.$$

Since $X(t)>0$ for all t, we see that $X(t)\geqq 1+W(t)$ and so

$$\begin{aligned}P(\sup_{0\leqq t\leqq u} Y(t)<m)&\leqq P(\sup_{0\leqq t\leqq u} W(t)<m^{1/2}-1)\\&\leqq P(\sup_{0\leqq t\leqq u} W(t)<m-1).\end{aligned}$$

By the reflection principle and scaling property

$$\begin{aligned}P(\sup_{0\leqq t\leqq u} W(t)<m-1)&=P(|W(u)|<m-1)\\&=P(|W(1)|<u^{-1/2}(m-1))\\&\leqq(2\pi u)^{-1/2}(m-1).\quad\square\end{aligned}$$

Next we obtain the main estimate in the proof of transience.

Lemma 3.2. *For any $\varepsilon>0$, there exists a constant c such that for all λ sufficiently small*

$$P(\inf_{1\leqq r\leqq 2}(L^*(T_r)-r)<\lambda)\leqq c\,\lambda^{\frac{1}{3}-\varepsilon}$$

Proof. By Lemma 2.4, for any n

$$\begin{aligned}P(\inf_{1\leqq r\leqq 2}(L^*(T_r)-r)<\lambda)&\leqq\sum_{k=0}^{n-1}P(\inf_{1+\frac{k}{n}\leqq r\leqq 1+\frac{k+1}{n}}(L^*(T_r)-r)<\lambda)\\&=\sum_{k=0}^{n-1}P\left(\inf_{1\leqq r\leqq 1+\frac{1}{n+k}}(L^*(T_r)-r)<\frac{\lambda n}{(n+k)}\right)\\&\leqq n\,P(\inf_{1\leqq r\leqq 1+\frac{1}{n}}(L^*(T_r)-r)<\lambda).\end{aligned}$$

Now for any $x>0$

$$
\begin{aligned}
&P\Big(\inf_{1\leqq r\leqq 1+\frac{1}{n}} (L^*(T_r)-r)<\lambda\Big)\\
&\leqq P\Big(\inf_{1\leqq r\leqq 1+\frac{1}{n}} (L^*(T_r)-r)<\lambda,\ L^*(T_1)>1+2\lambda,\ V(T_1)<x\Big)\\
&\quad +P\Big(\inf_{1\leqq r\leqq 1+\frac{1}{n}} (L^*(T_r)-r)<\lambda,\ V(T_1)>x\Big)\\
&\quad +P(L^*(T_1)<1+2\lambda)\\
&=I+II+III.
\end{aligned}
$$

Observe that

$$
I=\int_{z=0}^{x} P\Big(\inf_{1\leqq r\leqq 1+\frac{1}{n}} (L^*(T_r)-r)<\lambda,\ L^*(T_1)>1+2\lambda \mid V(T_1)=z\Big)\,P(V(T_1)\in dz)
$$

and that for $0\leqq z\leqq x$

$$
\begin{aligned}
&P\Big(\inf_{1\leqq r\leqq 1+\frac{1}{n}} (L^*(T_r)-r)<\lambda,\ L^*(T_1)>1+2\lambda \mid V(T_1)=z\Big)\\
&\leqq P\Big(\inf_{1\leqq r\leqq 1+\frac{1}{n}} (L(T_r,z)-r)<\lambda,\ L(T_1,z)>1+2\lambda \mid V(T_1)=z\Big)\\
&\leqq P\Big(\inf_{0\leqq r\leqq \frac{1}{n}} (L(T_r,z)-r)<-\lambda\Big)
\end{aligned}
$$

since $L(T_r,z)-r$ as a process in r has stationary independent increments. Furthermore since $L(T_r,z)-r$ is a martingale, we have by Doob's inequality and Lemma 2.2(b)

$$
\begin{aligned}
P\Big(\inf_{0\leqq r\leqq \frac{1}{n}} (L(T_r,z)-r)<-\lambda\Big)&=P\Big(\sup_{0\leqq r\leqq \frac{1}{n}} (r-L(T_r,z))>\lambda\Big)\\
&\leqq e^{-u\lambda}\,E\exp\left[u\left(\frac{1}{n}-L(T_{\frac{1}{n}},z)\right)\right]\\
&=\exp\left[-u\lambda+\frac{u}{n}-\frac{u}{n(1+2uz)}\right]\\
&\leqq \exp(-n\lambda^2/8z)
\end{aligned}
$$

if $u=n\lambda/4z$. Thus

$$
I\leqq \int_0^x \exp(-n\lambda^2/8z)\,P(V(T_1)\in dz)\leqq \exp(-n\lambda^2/8x).
$$

Next, since $L^*(T_r)$ is increasing, if we assume that $\lambda < n^{-1}$, then

$$II \leqq P(V(T_1) > x, L^*(T_1) < 1 + 2/n).$$

From this, Theorem 2.1, Theorem 2.3 and Lemma 3.1 we see that (using the notation of Theorem 2.3)

$$\begin{aligned} II &\leqq Q(\rho > x, M < 1 + 2/n) \\ &= \int_1^{1+(2/n)} Q(\rho > x \mid M = m)\, Q(M \in dm) \\ &= \int_1^{1+(2/n)} Q(\sup_{0 \leqq z \leqq x} Y(z) < m \mid M = m)\, Q(M \in dm) \\ &= \int_1^{1+(2/n)} Q(\sup_{0 \leqq z \leqq x} Y(z) < m)\, m^{-2}\, dm \\ &\leqq c\, x^{-1/2} \int_1^{1+(2/n)} (m-1)\, m^{-2}\, dm \\ &\leqq c\, x^{-1/2}\, n^{-2}. \end{aligned}$$

Finally

$$III = 2\lambda(1+2\lambda)^{-1} \leqq 2\lambda$$

since $L(T_1, z)$ is a diffusion on natural scale. Thus for any n and any $x > 0$

$$P(\inf_{1 \leqq r \leqq 2} (L^*(T_r) - r) < \lambda) = n \exp(-n\lambda^2/8x) + c\, x^{-1/2} n^{-1} + 2n\lambda.$$

Now given any $\varepsilon > 0$ if we set $x = \lambda^{\frac{6}{5} + 4\varepsilon}$ and $n = \lambda^{-(\frac{4}{5} + \varepsilon)}$ the result follows. □

Proposition 3.3. *For all* $\beta > 5$,

$$\liminf_{r \to \infty} \frac{L^*(T_r) - r}{r(\log r)^{-\beta}} = \infty \quad \text{a.s.}$$

Proof. Fix $\beta > 5$ and let $\lambda(r) = r(\log r)^{-\beta}$. Then by Lemmas 2.4 and 3.2 for any $\varepsilon > 0$, if n is sufficiently large,

$$\begin{aligned} &P(L^*(T_r) - r < \lambda(r) \text{ for some } r \in [2^n, 2^{n+1})) \\ &\quad \leqq P(\inf_{1 \leqq r \leqq 2} (L^*(T_r) - r) < \lambda(2^{n+1})\, 2^{-n}) \\ &\quad \leqq c\, n^{-\beta(\frac{1}{5} - \varepsilon)}. \end{aligned}$$

The result now follows from the Borel-Cantelli Lemma. □

We will now obtain estimates on the growth of $I(T_r, h(r))$ where $I(t, K)$ is defined by (1.1).

Lemma 3.4. *There exists a constant* c *such that for all* h *and* λ *sufficiently small,*

$$P(\sup_{1 \leqq r \leqq 2} \sup_{0 \leqq z \leqq h} (L(T_r, z) - r) > \lambda) \leqq c\, \lambda\, h^{-1} \exp(-\lambda^2/32h).$$

Proof. Set $Y(r) = \sup_{0 \leqq z \leqq h} (L(T_r, z) - r)$. By Doob's inequality and Lemma 2.2(b), if $u < (2h)^{-1}$

$$P(Y(r)>x) \leqq e^{-ux} E \exp(u(L(T_r, h)-r))$$
$$= \exp(-ux - ur + ur(1-2uh)^{-1}).$$

Setting $u = x(4rh + 2xh)^{-1} < (2h)^{-1}$ gives

$$P(Y(r)>x) \leqq \exp(-x^2(8rh+4xh)^{-1}). \tag{3.1}$$

From this it follows immediately that $E|Y(r)|<\infty$, and consequently $Y(r)$ is a submartingale, being the supremum of a collection of martingales. Thus by Doob's inequality and (3.1), if $u<1/h$

$$P(\sup_{1 \leqq r \leqq 2} Y(r) > \lambda) \leqq e^{-u\lambda} E \exp(u Y(2))$$
$$\leqq e^{-u\lambda} \int_0^\infty u \exp(ux - x^2(16h+4xh)^{-1})\, dx.$$

By splitting up the range of integration at $x=4$ one can easily show that the integral is bounded above by

$$u(4\exp(8u^2 h) + 8h(1-8uh)^{-1} \exp(-4(1-8uh)(8h)^{-1}))$$

provided $u<(8h)^{-1}$. Thus if h is sufficiently small, then

$$P(\sup_{1 \leqq r \leqq 2} Y(r) > \lambda) \leqq c\,u \exp(-u\lambda + 8u^2 h).$$

The result now follows by setting $u = \lambda/16h$. □

Proposition 3.5. *Assume that $\lambda(r)$ and $h(r)$ are increasing functions which satisfy*

i(i) $\lambda(r)/r \to 0$, $h(r)/r \to 0$ *as* $r \to \infty$,

(ii) *for some* γ, $\lambda(r)/h(2r)(\log r)^\gamma$ *and* $r h(2r) \log\log r/\lambda^2(r)$ *remain bounded as* $r \to \infty$.

Then

$$\limsup_{r\to\infty} \frac{I(T_r, h(r)) - r}{\lambda(r)} < \infty \quad \text{a.s.}$$

Proof. By Lemmas 2.2 and 3.4, for any $a>0$

$$P(I(T_r, h(r)) - r > a\lambda(r) \text{ for some } r \in [2^n, 2^{n+1}))$$
$$\leqq P(I(T_r, h(2^{n+1})2^{-n}) - r > a\lambda(2^n)2^{-n} \text{ for some } r \in [1,2))$$
$$\leqq c \frac{\lambda(2^n)}{h(2^{n+1})} \exp\left(-\frac{a^2 \lambda^2(2^n)}{32 h(2^{n+1}) 2^n}\right).$$

Since this gives rise to a convergent series if a is chosen large enough, the result follows from the Borel-Cantelli Lemma. □

We now put together Proposition 3.3 and 3.5 to prove the transience of $V(t)$ and also to obtain estimates on its rate of escape.

Theorem 3.6. (a) *For all* $\gamma > 10$

$$\liminf_{r\to\infty} \frac{V(T_r)}{r(\log r)^{-\gamma}} > 0 \quad \text{a.s.}$$

(b) *For all* $\gamma > 11$

$$\liminf_{t\to\infty} \frac{V(t)}{t^{1/2}(\log t)^{-\gamma}} > 0 \quad \text{a.s.}$$

Proof. Fix $\alpha>5$, $\beta\in(5,\alpha)$ and set $h(r)=r(\log r)^{-2\alpha}$. By Proposition 3.5 with $\lambda(r)=r(\log\log r)^{1/2}(\log r)^{-\alpha}$, for r sufficiently large (depending on ω)

$$I(T_r, h(r))\leqq r+c\,r(\log\log r)^{1/2}(\log r)^{-\alpha}$$

while by Proposition 3.3 for r sufficiently large

$$L^*(T_r)\geqq r+r(\log r)^{-\beta}.$$

Thus for all $\alpha>5$, $\{L(T_r, z): z\geqq 0\}$ takes its maximum on $[h(r),\infty)$ if r is sufficiently large and so

$$\liminf_{r\to\infty}\frac{V(T_r)}{r(\log r)^{-2\alpha}}>0.$$

Next, since $L^*(t)$ is continuous and $T_s\uparrow T_{r-}$ as $s\uparrow r$, we see that for r sufficiently large

$$L^*(T_{r-})\geqq r+r(\log r)^{-\beta}.$$

Now fix t; then for some r, $T_{r-}\leqq t\leqq T_r$. Thus if t is sufficiently large

$$L^*(t)\geqq L^*(T_{r-})\geqq r+r(\log r)^{-\beta} \tag{3.2}$$

while for any increasing function $k(t)$,

$$I(t, k(t))\leqq I(T_r, k(T_r))\leqq r+r(\log\log r)^{1/2}(\log r)^{-\alpha} \tag{3.3}$$

provided $k(T_r)\leqq h(r)$. However by [6], for any $\varepsilon>0$, $T_r/r^2(\log r)^{2+\varepsilon}\to 0$ as $r\to\infty$. Thus if we choose $k(t)=t^{1/2}(\log t)^{-(2\alpha+1+\varepsilon)}$, we have that for all t sufficiently large

$$L^*(t)>I(t, k(t)),$$

and so $V(t)\geqq k(t)$ by the argument given above. Since $\varepsilon>0$ was arbitrary, this completes the proof of (b). □

Remark. As pointed out earlier, it follows easily from this result that

$$\liminf_{t\to\infty}\frac{\overline{V}(t)}{t^{1/2}(\log t)^{-\gamma}}>0$$

for all $\gamma>11$ where

$$\overline{V}(t)=\inf\{x\geqq 0: L(t,x)\vee L(t,-x)=\sup_{x\in\mathbb{R}^1} L(t,x)\}, \tag{3.4}$$

i.e., $\overline{V}(t)$ is the absolute value of the most visited site on the entire x-axis.

In view of part (b) of the preceding theorem, it is natural to ask whether $t^{1/2}(\log t)^{-\gamma}$ is a good approximation to the lower growth rate of $V(t)$. The following theorem shows that, up to a power of $\log t$, it is. Since it will be needed in the proof, we point out here that if $Y(t)$ is a $(\mathrm{BES}(0))^2$ then from (2.1) and (2.2)

$$\liminf_{t\to 0} P^1(Y(t)<1-g(t))>0 \tag{3.5}$$

whenever $g(t)=o(t^{1/2})$.

Theorem 3.7. *For all* $\gamma<2$

$$\liminf_{t\to\infty}\frac{V(t)}{t^{1/2}(\log t)^{-\gamma}}<\infty \quad \text{a.s.}$$

Proof. Let $p>1$ and set $r_n=\exp(n^p)$. Given $\gamma<2$, choosen $\beta\in(\gamma,2)$ and let $h(r)=r(\log r)^{-\beta}$. Let $h_n=h(r_n)$ and define

$$\hat{I}_n=\sup_{0\leqq z\leqq h_{n+1}}(L(T_{r_{n+1}},z)-L(T_{r_n},z))$$

and

$$\hat{J}_n=\sup_{z>h_{n+1}}(L(T_{r_{n+1}},z)-L(T_{r_n},z)).$$

Recall that if $Y(t)$ is a $(\mathrm{BES}(0))^2$, then $Y(t)$ is on natural scale, thus if $\delta<1$ and $x\leqq(1-3n^{-\delta})$

$$P^x(\sup_{t\geqq 0}Y(t)<(1-2n^{-\delta}))=1-x(1-2n^{-\delta})^{-1}\geqq n^{-\delta}.$$

Hence by the Markov property, Lemma 2.4 and (3.5)

$$\begin{aligned} P(\hat{J}_n<(1-2n^{-\delta})\hat{I}_n)&\geqq P(\hat{J}_n<(1-2n^{-\delta})(r_{n+1}-r_n))\\ &=P^{r_{n+1}-r_n}(\sup_{t\geqq h_{n+1}}Y(t)<(1-2n^{-\delta})(r_{n+1}-r_n))\\ &=P^1(\sup_{t\geqq h_{n+1}(r_{n+1}-r_n)^{-1}}Y(t)<(1-2n^{-\delta}))\\ &\geqq\int_{x=0}^{1-3n^{-\delta}}P^x(\sup_{t\geqq 0}Y(t)<(1-2n^{-\delta}))\\ &\quad P^1(Y(h_{n+1}(r_{n+1}-r_n)^{-1})\in dx)\\ &\geqq n^{-\delta}P^1(Y(h_{n+1}(r_{n+1}-r_n)^{-1})\leqq 1-3n^{-\delta})\\ &\geqq cn^{-\delta} \end{aligned} \tag{3.6}$$

for n large provided δ and p are chosen so that $p\beta<2\delta$. Since the events $(\hat{J}_n<(1-2n^{-\delta})\hat{I}_n)$, $n=1,2,\ldots$ are independent, the Borel-Cantelli Lemma gives

$$P(\hat{J}_n<(1-2n^{-\delta})\hat{I}_n \text{ i.o.})=1. \tag{3.7}$$

Next

$$\begin{aligned} P(L^*(T_{r_n})>n^{-\delta}\hat{I}_n)&\leqq P(L^*(T_{r_n})>n^{-\delta}(r_{n+1}-r_n))\\ &=\frac{r_n}{n^{-\delta}(r_{n+1}-r_n)}. \end{aligned}$$

Thus by Borel-Cantelli, with probability one

$$L^*(T_{r_n})\leqq n^{-\delta}\hat{I}_n \tag{3.8}$$

for n sufficiently large. Now set $J(t,z)=\sup_{x>z}L(t,x)$. Then by (3.7) and (3.8), infinitely often with probability one

$$\begin{aligned} J(T_{r_{n+1}},h_{n+1})&\leqq\hat{J}_n+L^*(T_{r_n})\\ &\leqq(1-n^{-\delta})\hat{I}_n\\ &\leqq(1-n^{-\delta})I(T_{r_{n+1}},h_{n+1}). \end{aligned} \tag{3.9}$$

Thus by (3.9) and Theorem 1 of [10], infinitely often with probability one

$$\begin{aligned} V(T_{r_{n+1}}) &\leqq h(r_{n+1}) \\ &= h(L(T_{r_{n+1}}, 0)) \\ &\leqq h((3\,T_{r_{n+1}} \log\log T_{r_{n+1}})^{1/2}) \\ &\leqq T_{r_{n+1}}^{1/2} (\log T_{r_{n+1}})^{-\gamma} \end{aligned} \tag{3.10}$$

which completes the proof. □

Remark. If

$$\hat{J}_n^- = \sup_{z < -h_{n+1}} (L(T_{r_{n+1}}, z) - L(T_{r_{n+1}}, z))$$

then following the proof of (3.6) we obtain

$$P(\hat{J}_n^- < (1-2n^{-\delta})\,\hat{I}_n) \geqq P(\hat{J}_n^- < (1-2n^{-\delta})(r_{n+1} - r_n)) \geqq c\,n^{-\delta}.$$

Now $\hat{J}_n$ depends only on the positive excursions and $\hat{J}_{\bar{n}}$ only on the negative excursions of a Brownian motion, so they are independent. Thus if $\beta < 1$ we can choose $\delta < \frac{1}{2}$ and $p > 1$ to satisfy $\beta p < 2\delta$, and then

$$\begin{aligned} &\sum P(\hat{J}_n \vee \hat{J}_n^- < (1-2n^{-\delta})\,\hat{I}_n) \\ &\geqq \sum P(\hat{J}_n \vee \hat{J}_n^- < (1-2n^{-\delta})(r_{n+1} - r_n)) \\ &\geqq \sum c\,n^{-2\delta}. \end{aligned}$$

Imitating the proof above we obtain

$$\liminf_{t\to\infty} \frac{\bar{V}(t)}{t^{1/2} (\log t)^{-\gamma}} < \infty \quad \text{a.s.}$$

for all $\gamma < 1$, where $\bar{V}(t)$ is defined by (3.4).

4. Law of the Iterated Logarithm

We will now investigate the upper growth rate of $V(t)$. In this case it is easier to deal directly with $\bar{V}(t)$ as defined by (3.4). Thus correspondingly we introduce $\bar{L}$, the maximum value of $L(t,x)$ on the entire axis, defined by

$$\bar{L}(t) = \sup_{x \in \mathbb{R}^1} L(t,x).$$

Observe that by Lemma 2.4, the analogue of (2.6) also holds for $\bar{L}$, i.e.

$$\bar{L}(t) \stackrel{d}{=} t^{1/2}\, \bar{L}(1). \tag{4.1}$$

The following notation will also prove useful: for the Brownian motion $W(t)$ and any times $0 \leqq u \leqq s$, set

$$(W(s) - W(u))^* = \sup\{|W(t) - W(u)| : u \leqq t \leqq s\}$$
$$W^*(s) = \sup\{|W(t)| : 0 \leqq t \leqq s\}.$$

The proof of the L.I.L. for $\bar{V}(t)$ is based on the intuitive idea given in the introduction and is similar to the proof of Theorem 1 in [7]. We will need the following generalization of the Borel-Cantelli Lemma. It has been noted previously by Kesten [10] and follows from the proof of Corollary 1 on page 323 in Doob [5].

Lemma 4.1. *Let* $\mathscr{F}(s)=\sigma\{W(t): 0\leqq t\leqq s\}$ *and for* $k=0,1,\ldots$ *assume that* $C_k\in\sigma\{W(t): s_k\leqq t\leqq s_{k+1}\}$ *where* $0\leqq s_0<s_1<\ldots$ *and* $s_k\uparrow\infty$. *Then*

$$\sum_k P(C_k|\mathscr{F}(s_k))=\infty \quad \text{a.s. } \textit{implies } P(C_k \text{ i.o.})=1.$$

Theorem 4.2.

$$\limsup_{t\to\infty}\frac{\bar{V}(t)}{(2t\log\log t)^{1/2}}=1 \quad \text{a.s.}$$

Proof. The upper bound is an immediate consequence of the Law of the Iterated Logarithm for Brownian motion.

To prove the lower bound assume that $\alpha<1$. Now choose $\delta>0$ sufficiently small and η, K sufficiently large so that

$$P(\eta\leqq W(1)\leqq K\eta)\geqq\exp(-\eta^2/2\alpha)+2\delta. \tag{4.2}$$

This is possible since $P(W(1)>\eta)\sim(2\pi\eta^2)^{-(1/2)}\exp(-\eta^2/2)$. Next choose λ and ρ such that

$$P(\bar{L}(1)\leqq\lambda)\geqq 1-\delta, \tag{4.3}$$

$$P(W^*(1)\leqq\rho)\geqq 1-\delta. \tag{4.4}$$

Now let $\zeta=4(2\rho\eta^{-1}+K)\lambda$ and set $b=P(\bar{L}(1)\geqq\zeta)$. Finally choose $\bar{\rho}$ such that

$$P(W^*(1)\leqq\bar{\rho})\geqq 1-(b/2). \tag{4.5}$$

Fix $t>0$ and let $t_k=kat/\log\log t$ where $a=\eta^2/2\alpha^2$. Define, for $k=0,1,\ldots,[(\log\log t)/a]$

$$E_k^1=(\sup_{x\in\mathbb{R}^1}(L(t_{k+1},x)-L(t_k,x))\leqq\lambda\gamma(t))$$

$$E_k^2=((W(t_{k+1})-W(t_k))^*\leqq\rho\gamma(t))$$

$$E_k^3=((k+1)\eta\gamma(t)\leqq W(t_{k+1})\leqq(K+k)\eta\gamma(t))$$

where $\gamma(t)=(at/\log\log t)^{1/2}$. Now set

$$E_k=E_k^1\cap E_k^2\cap E_k^3 \qquad \text{for } k=0,\ldots,[(\log\log t)/a]-2,$$

$$E_k=((W(t_{k+1})-W(t_k))^*\leqq\bar{\rho}\gamma(t);$$
$$\sup_{x\in\mathbb{R}^1}(L(t_{k+1},x)-L(t_k,x))\geqq\zeta\gamma(t)) \qquad \text{for } k=[(\log\log t)/a]-1,$$

$$E_k=E_k^2 \qquad \text{for } k=[(\log\log t)/a].$$

Observe that for any x, by (4.1) and (4.3)

$$\begin{aligned}P(E_k^1|W(t_k)=x)&=P(\bar{L}(t_1)\leqq\lambda\gamma(t))\\&=P(\bar{L}(1)\leqq\lambda)\\&\geqq 1-\delta\end{aligned}$$

while by the scaling property of Brownian motion and (4.4)

$$\begin{aligned} P(E_k^2 \mid W(t_k)=x) &= P(W^*(t_1) \leqq \rho\, \gamma(t)) \\ &= P(W^*(1) \leqq \rho) \\ &\geqq 1-\delta. \end{aligned}$$

Next for $k\eta\,\gamma(t) \leqq x \leqq (K+k-1)\,\eta\,\gamma(t)$,

$$\begin{aligned} P(E_k^3 \mid W(t_k)=x) &= P((k+1)\,\eta\,\gamma(t) \leqq W(t_1) \leqq (K+k)\,\eta\,\gamma(t) \mid W(0)=x) \\ &= P((k+1)\,\eta \leqq W(1) \leqq (K+k)\,\eta \mid W(0)=x/\gamma(t)) \\ &= P(\eta \leqq W(1) \leqq K\eta \mid W(0)=y) \end{aligned}$$

where $y = x/\gamma(t) - k\eta$. Observe that $0 \leqq y \leqq (K-1)\,\eta$ and thus by the shape of the normal density we conclude that

$$\begin{aligned} P(E_k^3 \mid W(t_k)=x) &\geqq P(\eta \leqq W(1) \leqq K\eta \mid W(0)=0) \\ &\geqq \exp(-\eta^2/2\alpha) + 2\delta. \end{aligned}$$

Thus for $k=0, \ldots, [(\log\log t)/a]-2$

$$P(E_k \mid W(t_k)=x) \geqq \exp(-\eta^2/2\alpha) \tag{4.6}$$

provided $k\eta\,\gamma(t) \leqq x \leqq (K+k-1)\,\eta\,\gamma(t)$, while for $k=[(\log\log t)/a]$

$$P(E_k \mid W(t_k)=x) \geqq 1-\delta \tag{4.7}$$

independently of x. For $k=[(\log\log t)/a]-1$, essentially the same calculations show that

$$P(E_k \mid W(t_k)=x) \geqq b/2, \tag{4.8}$$

again provided that $k\eta\,\gamma(t) \leqq x \leqq (K+k-1)\,\eta\,\gamma(t)$. Let $D_n = \bigcap_{k=0}^{n} E_k$ and $\mathscr{F}(s) = \sigma\{W(r)\colon r \leqq s\}$. Then by the Markov property, for any n

$$\begin{aligned} P(D_{n+1}) &= EE\left[\prod_{k=0}^{n+1} 1_{E_k} \,\middle|\, \mathscr{F}(t_{n+1})\right] \\ &= E\left[\prod_{k=0}^{n} 1_{E_k} P(E_{n+1} \,\middle|\, W(t_{n+1}))\right]. \end{aligned}$$

But $(n+1)\,\eta\,\gamma(t) \leqq W(t_{n+1}) \leqq (K+n)\,\eta\,\gamma(t)$ on E_n if $n \leqq [(\log\log t)/a]-3$, thus by (4.6)

$$P(D_{n+1}) \geqq \exp(-\eta^2/2\alpha)\, P(D_n) \qquad \text{if } n \leqq [(\log\log t)/a]-3$$

while by (4.7) and (4.8)

$$\begin{aligned} &P(D_{n+1}) \geqq (b/2)\, P(D_n) \qquad \text{if } n=[(\log\log t)/a]-2, \\ &P(D_{n+1}) \geqq (1-\delta)\, P(D_n) \qquad \text{if } n=[(\log\log t)/a]-1. \end{aligned}$$

Thus writing $D = D_{[(\log\log t)/a]}$ we see that for some positive constant c

$$P(D) \geqq c\exp(-\eta^2 \log\log t/2a\alpha) = c\exp(-\alpha \log\log t). \tag{4.9}$$

Next observe that on D

$$\bar{L}(t) \geqq \zeta\,\gamma(t)$$

while for any $k \leqq [(\log\log t)/a]-2$, if $s \in [kat/\log\log t, (k+1)at/\log\log t]$ then $L(s,x)$ can only increase if

$$(k\eta-\rho)\gamma(t) \leqq x \leqq (K+k-1+\rho)\eta\gamma(t). \tag{4.10}$$

Now for a given x, since there are at most $(2\rho\eta^{-1}+K)$ values of k for which $k \leqq [(\log\log t)/a]-2$ and (4.10) holds, we see that on D

$$L(t,x) \leqq (2\rho\eta^{-1}+K)\lambda\gamma(t) = (\zeta/4)\gamma(t)$$

provided $x \leqq ([(\log\log t)/a]-1)\eta\gamma(t)-(\bar{\rho}+\rho)\gamma(t)$.(Observe that this restriction on x ensures that $L(s,x)$ remains constant for

$$s \in [([\log\log t)/a]-1)\,a\,t(\log\log t)^{-1}, t]).$$

Thus for any $\varepsilon>0$, if t is sufficiently large

$$D \subseteq (\bar{L}(t) \geqq \zeta\gamma(t), \sup_{x \leqq \sqrt{2}\alpha(1-\varepsilon)\varphi(t)} L(t,x) \leqq (\zeta/4)\gamma(t)) \tag{4.11}$$

where $\varphi(t)=(t\log\log t)^{1/2}$. (Recall that $\eta\alpha^{-1/2}=\sqrt{2}\alpha$).

We will now complete the proof by using the generalization of the Borel-Cantelli Lemma given in Lemma 4.1. Let $p=\alpha^{-1}$ and set $s_k=\exp(k^p)$. Define

$$C_k = (\sup_{x\in\mathbb{R}^1}(L(s_{k+1},x)-L(s_k,x)) \geqq (\zeta/2)\gamma(s_{k+1}),$$
$$\sup_{x \leqq \sqrt{2}\alpha(1-\varepsilon)\varphi(s_{k+1})}(L(s_{k+1},x)-L(s_k,x)) \leqq (\zeta/4)\gamma(s_{k+1})).$$

Observe that if $|y| \leqq (3s_k\log\log s_k)^{1/2}$ then for sufficiently large k, by (4.9) and (4.11)

$$\begin{aligned}P(C_k \mid W(s_k)=y) &= P(\sup_{x\in\mathbb{R}^1}(L(s_{k+1},x)-L(s_k,x)) \geqq (\zeta/2)\gamma(s_{k+1}),\\ &\quad \sup_{x \leqq \sqrt{2}\alpha(1-\varepsilon)\varphi(s_{k+1})-y}(L(s_{k+1},x)-L(s_k,x)) \leqq (\zeta/4)\gamma(s_{k+1}) \mid W(s_k)=0)\\ &\geqq P(\bar{L}(s_{k+1}-s_k) \geqq \zeta\gamma(s_{k+1}-s_k), \sup_{x \leqq \sqrt{2}\alpha(1-\varepsilon/2)\varphi(s_{k+1}-s_k)} L(s_{k+1}-s_k,x)\\ &\quad \leqq (\zeta/4)\gamma(s_{k+1}-s_k) \mid W(0)=0) \quad \geqq c\,k^{-1}.\end{aligned}$$

However by the L.I.L. for Brownian motion, $|W(s_k)| \leqq (3s_k\log\log s_k)^{1/2}$ eventually, and so for some (random) k_0, if $k \geqq k_0$, then with probability one

$$P(C_k \mid \mathscr{F}(s_k)) = P(C_k \mid W(s_k)) \geqq c\,k^{-1}.$$

Thus by Lemma 4.1, $P(C_k \text{ i.o.})=1$. Next observe that

$$\bar{L}(s_{k+1}) \geqq \sup_{x\in\mathbb{R}^1}(L(s_{k+1},x)-L(s_k,x))$$

while

$$\sup_{x \leqq \sqrt{2}\alpha(1-\varepsilon)\varphi(s_{k+1})} L(s_{k+1},x) \leqq \bar{L}(s_k) + \sup_{x \leqq \sqrt{2}\alpha(1-\varepsilon)\varphi(s_{k+1})}(L(s_{k+1},x)-L(s_k,x)).$$

However by Theorem 1 of [10], since $\varphi(s_k)/\gamma(s_{k+1}) \to 0$

$$\limsup_{k\to\infty}\frac{\bar{L}(s_k)}{\gamma(s_{k+1})}=0.$$

Thus

$$P(\bar{L}(s_{k+1}) \geqq (\zeta/2)\,\gamma(s_{k+1}), \sup_{x \leqq \sqrt{2}\alpha(1-\varepsilon)\varphi(s_{k+1})} L(s_{k+1}, x) \leqq (\zeta/3)\,\gamma(s_{k+1}) \text{ i.o.}) = 1,$$

i.e.,

$$\limsup_{k\to\infty} \frac{\bar{V}(s_{k+1})}{(s_{k+1}\log\log s_{k+1})^{1/2}} \geqq \sqrt{2}\,\alpha(1-\varepsilon).$$

Since $\varepsilon > 0$ and $\alpha < 1$ were arbitrary, the proof is complete. □

Remarks. 1. A minor modification of the proof shows that for any $K > 0$ and any $\alpha < \sqrt{2}$

$$P(\bar{L}(t) > K \sup_{x \leqq \alpha(t\log\log t)^{1/2}} L(t,x) \text{ i.o. as } t \uparrow \infty) = 1.$$

2. It would be interesting to obtain an analogue of Kolmogorov's test for $\bar{V}(t)$. In particular, do there exist deterministic functions $\varphi(t)$ for which $P(W(t) > \varphi(t) \text{ i.o.}) = 1$ and $P(\bar{V}(t) > \varphi(t) \text{ i.o.}) = 0$? The answer to the analogous question for simple random walk is yes by Theorem 2 of Erdös and Révész [20].

3. It is an immediate consequence of the above theorem that $\limsup_{t\to\infty} V(t)/(2t\log\log t)^{1/2} = 1$ a.s.

5. Random Walks

In this section, $X_1, X_2, \ldots$, is a sequence of independent identically distributed random variables with $P(X_i = 1) = P(X_i = -1) = 1/2$ and $S_n = X_1 + \ldots + X_n$. Throughout this section k will always be restricted to the integers. Let

$$N(n,k) = \sum_{j=0}^{n} 1_{\{k\}}(S_j),$$

$$N^*(n) = \sup_{k \geqq 0} N(n,k),$$

$$U(n) = \inf\{k \geqq 0 \colon N(n,k) = N^*(n)\}.$$

To prove the analogue of Theorems 3.6 and 3.7 for random walks, the basic tool is the invariance principle of Révész [15].

Theorem 5.1. *There exists a Brownian motion $W(t)$ and a simple random walk S_n such that for all $\varepsilon > 0$*

$$\sup_k |L(n,k) - N(n,k)| = o(n^{1/4+\varepsilon}) \quad \text{a.s.}$$

We begin with some estimates on Brownian local time. Let

$$D(t,x,y) = L(t,x) - L(t,y),$$

$$M(t,x,y) = \sup_{s \leqq t} |D(s,x,y)|.$$

Lemma 5.2. *There exist constants c_1 and c_2 such that for any integer k, $\delta \leqq 1$ and $t \geqq 1$*

$$P\Big(\sup_{\substack{|x-y|\leqq\delta\\ x,y\in[k,k+1]}} M(t,x,y)>\lambda\Big)\leqq c_1\,\delta^{-1}\exp(-\lambda/c_2\,\delta^{1/2}\,t^{1/4}).$$

Proof. By Tanaka's formula, for any bounded stopping time T,

$$L(T,x)=|W(T)-x|-|x|-\int_0^T \operatorname{sgn}(W(s)-x)\,dW(s).$$

Now assume that $x\leqq y\leqq x+1$ and $R\leqq T$ are stopping times bounded by t. Then

$$\begin{aligned}
E(|D(T,x,y)-D(R,x,y)|\,|\mathscr{F}(R))&\leqq(E(|D(T,x,y)-D(R,x,y)|^2\,|\,\mathscr{F}(R)))^{1/2}\\
&\leqq 2|x-y|+2\left(E\left(\left(\int_R^T 1_{[x,y]}(W(s))\,dW(s)\right)^2\,|\mathscr{F}(R)\right)\right)^{1/2}\\
&\leqq 2|x-y|+2\left(E\left(\int_R^{R+t} 1_{[x,y]}(W(s))\,ds\,|\mathscr{F}(R)\right)\right)^{1/2}\\
&=2|x-y|+2\left(E^{W(R)}\int_0^t 1_{[x,y]}(W(s))\,ds\right)^{1/2}\\
&=2|x-y|+2\left(\int_0^t P^{W(R)}(W(s)\in[x,y])\,ds\right)^{1/2}\\
&\leqq c\,t^{1/4}|x-y|^{1/2},
\end{aligned}$$

provided $t>1$. So by [3], p. 193,

$$E\exp(M(T,x,y)/8\,c\,t^{1/4}|x-y|^{1/2})\leqq 2,$$

and consequently

$$P(M(t,x,y)>\lambda)\leqq 2\exp(-\lambda/8\,c\,t^{1/4}|x-y|^{1/2}).$$

Now fix k, λ, δ and let $\lambda_i=(1-2^{-1/4})\,2^{-i/4}\lambda$, so that $\sum_{i=0}^{\infty}\lambda_i=\lambda$. Let

$$Q(\delta)=\{x\in\mathbb{R}: x=k+j\,\delta\,2^{-i} \text{ for some } 0\leqq i<\infty,\ 0\leqq j\leqq 2^i[\delta^{-1}]+1\}.$$

Given $x\in[k,k+1]$ define $x_i=\sup\{k+j\,\delta\,2^{-i}\leqq x: 0\leqq j\leqq 2^i[\delta^{-1}]+1\}$. Now since $L(t,x)$ is jointly continuous,

$$\sup_{\substack{|x-y|\leqq\delta\\ x,y\in[k,k+1]}} M(t,x,y)>\lambda \quad \text{only if} \quad \sup_{\substack{|x-y|\leqq\delta\\ x,y\in Q(\delta)}} M(t,x,y)>\lambda,$$

while for $x,y\in Q(\delta)$ with $|x-y|\leqq\delta$

$$M(t,x,y)\leqq M(t,x_0,y_0)+\sum_{i=1}^{\infty}(M(t,x_{i-1},x_i)+M(t,y_{i-1},y_i)).$$

(Observe that eventually each term in the infinite sum will be zero.) Thus if $M(t,x,y)>\lambda$ then either $M(t,x_0,y_0)>\lambda_0$ or for some i, $M(t,x_{i-1},x_i)$

$\vee M(t, y_{i-1}, y_i) > \lambda_i/2$. Hence for λ sufficiently large

$$\begin{aligned}
&P(\sup_{\substack{|x-y|\leqq\delta \\ x,y\in[k,k+1]}} M(t,x,y)>\lambda) \\
&\leqq \sum_{i=0}^{\infty} 2(2^i[\delta^{-1}]+1) \sup_{0\leqq j\leqq 2^i[\delta^{-1}]} P(M(t,k+j\delta 2^{-i}, k+(j+1)\delta 2^{-i})>\lambda_i/2) \\
&\leqq c_1\delta^{-1}\sum_{i=0}^{\infty} 2^{i+1}\exp(-\lambda_i/16c_2 t^{1/4}(\delta 2^{-i})^{1/2}) \\
&\leqq c_1\delta^{-1}\exp(-\lambda/c_2 t^{1/4}\delta^{1/2}). \quad \square
\end{aligned}$$

Lemma 5.3. *For every* $\varepsilon>0$

$$\sup_k \sup_{\substack{s\leqq t \\ z\in[k,k+1]}} |L(s,z)-L(s,k)| = o(t^{1/4+\varepsilon}) \quad \text{a.s.}$$

Proof. Let $t_n=2^n$, then since $L(t,z)=0$ for $z\geqq \sup_{s\leqq t}|W(s)|$,

$$\begin{aligned}
&P(\sup_k \sup_{\substack{s\leqq t_n \\ z\in[k,k+1]}} |L(s,z)-L(s,k)|\geqq t_n^{1/4+\varepsilon/2}) \\
&\leqq P(\sup_{s\leqq t_n}|W(s)|\geqq t_n) + 2^{n+2}\sup_{|k|\leqq t_n} P(\sup_{\substack{s\leqq t_n \\ z\in[k,k+1]}} |L(s,z)-L(s,k)|>t_n^{1/4+\varepsilon/2}) \\
&\leqq 2\exp(-t_n/2) + 2^{n+2}c_1\exp(-t_n^{1/4+\varepsilon/2}/c_2 t_n^{1/4}).
\end{aligned}$$

The result now follows from the Borel-Cantelli Lemma. $\square$

Theorem 5.4. (a) *For any* $\gamma>11$

$$\liminf_{n\to\infty}\frac{U(n)}{n^{1/2}(\log n)^{-\gamma}}=\infty \quad \text{a.s.}$$

(b) *For any* $\gamma<2$

$$\liminf_{n\to\infty}\frac{U(n)}{n^{1/2}(\log n)^{-\gamma}}=0 \quad \text{a.s.}$$

Proof. (a) From (3.2) and (3.3) we see that if r is sufficiently large, $T_{r-}\leqq t\leqq T_r$ and $\varepsilon>0$, then

$$L^*(t)\geqq I(t, t^{1/2}(\log t)^{-(2\alpha+1+\varepsilon)}) + r(\log r)^{-\beta}$$

where $5<\beta<\alpha$. Furthermore for each $\eta>0$, $T_r\leqq r^2(\log r)^{2+\eta}$ for large r, [6]; thus

$$L^*(t)\geqq I(t, t^{1/2}(\log t)^{-(2\alpha+1+\varepsilon)}) + t^{1/2-\varepsilon} \tag{5.1}$$

eventually for any $\varepsilon>0$. Now by Theorem 5.1 and Lemma 5.3

$$|L^*(n)-N^*(n)| = o(n^{1/4+\varepsilon}) \quad \text{a.s.,}$$

and

$$|I(n, n^{1/2}(\log n)^{-(2\alpha+1+\varepsilon)}) - \sup_{0\leqq k\leqq n^{1/2}(\log n)^{-(2\alpha+1+\varepsilon)}} N(n,k)| = o(n^{1/4+\varepsilon}) \quad \text{a.s.}$$

Combining this with (5.1) gives (a).

(b) Recalling the notation of Theorem 3.7, we proved in (3.9) that infinitely often with probability one,

$$J(T_{r_{n+1}}, h_{n+1}) \leqq (1-n^{-\delta}) I(T_{r_{n+1}}, h_{n+1}).$$

Thus if $j \leqq T_{r_{n+1}} \leqq j+1$,

$$\begin{aligned} \sup_{0 \leqq k \leqq h_{n+1}} N(j+1,k) &\geqq I(j+1, h_{n+1}) - o(j^{1/4+\varepsilon}) \\ &\geqq I(T_{r_{n+1}}, h_{n+1}) - o(j^{1/4+\varepsilon}) \\ &\geqq J(T_{r_{n+1}}, h_{n+1}) + n^{-\delta} I(T_{r_{n+1}}, h_{n+1}) - o(j^{1/4+\varepsilon}) \\ &\geqq \sup_{k \geqq h_{n+1}} N(j,k) + n^{-\delta} I(T_{r_{n+1}}, h_{n+1}) - o(j^{1/4+\varepsilon}). \end{aligned}$$

Now since

$$I(T_{r_{n+1}}, h_n) \geqq r_{n+1},$$

and $T_{r_{n+1}}/r_{n+1}^{2+\eta} \to 0$ for all $\eta > 0$, [6], we see that

$$j^{1/4+\varepsilon} = o(n^{-\delta} I(T_{r_{n+1}}, h_n)), \quad \text{a.s.}$$

if ε is sufficiently small. Thus for infinitely many j, with probability one

$$\sup_{0 \leqq k \leqq h_{n+1}} N(j+1,k) \geqq \sup_{k \geqq h_{n+1}} N(j,k) + \zeta(j)$$

where $\zeta(j) \to \infty$ a.s. Since for each k, $N(j+1,k)$ and $N(j,k)$ differ by at most 1, we have by the same argument as in (3.10) that infinitely often with probability one

$$\begin{aligned} U(j+1) &\leqq h_{n+1} \\ &\leqq T_{r_{n+1}}^{1/2} (\log T_{r_{n+1}})^{-\gamma} \\ &\leqq (j+1)^{1/2} (\log(j+1))^{-\gamma}. \quad \square \end{aligned}$$

Remarks. 1. There is a two sided version of Theorem 5.4 which is proved similarly (see the remarks following Theorems 3.6 and 3.7).

2. Theorem 5.4 holds if S_n is a lattice-valued random walk with lattice size 1, where X_i is mean 0 and variance 1 with finite moments of all orders. This follows by replacing Theorem 5.1 with the results of Csáki and Révész [2].

3. There is an obvious random walk analogue of Theorem 4.2. We leave the statement and proof to the reader. See also Theorem 1 of [20].

References

1. Barlow, M.T., Yor, M.: (Semi-) Martingale inequalities and local times. Z. Wahrscheinlichkeitstheor. Verw. Geb. **55**, 237–254 (1981)
2. Csáki, E., Révész, P.: Strong invariance for local times. Z. Wahrscheinlichkeitstheor. Verw. Geb. **62**, 263–278 (1983)
3. Dellacherie, C., Meyer, P.-A.: Probabilités et potentiel: théorie des martingales. Paris: Hermann 1980
4. Donsker, M.D., Varadhan, S.R.S.: On the law of the iterated logarithm for local times. Comm. Pure Appl. Math. **30**, 707–753 (1977)
5. Doob, J.L.: Stochastic Processes. New York: Wiley 1953
6. Fristedt, B.: Sample functions of stochastic processes with stationary independent increments. In: Advances in probability and related topics, vol. 3. New York: Marcel Dekker 1974

7. Griffin, P.S.: Laws of the iterated logarithm for symmetric stable processes. Z. Wahrscheinlichkeitstheor. Verw. Geb. **68**, 271–285 (1985)
8. Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. Amsterdam: North-Holland 1981
9. Itô, K.: Poisson point processes attached to Markov processes. In: Proc. Sixth Berkeley Symp., vol. 3, 225–239. Berkeley: Univ. of California Press 1972
10. Kesten, H.: An iterated logarithm law for local time. Duke Math. J. **32**, 447–456 (1965)
11. Knight, F.B.: Essentials of Brownian Motion and Diffusion. Am. Math. Soc. Mathematical Surveys **18**, 1981
12. Knight, F.B.: Random walks and a sojourn density process of Brownian motion. Trans. Am. Math. Soc. **109**, 56–86 (1963)
13. Meyer, P.-A.: Un théorème sur la répartition des temps locaux. In: Séminaire de Prob. **V**, 209–210. Berlin Heidelberg New York: Springer-Verlag 1971
14. Ray, D.B.: Sojourn times of diffusion processes. Ill. J. Math. **7**, 615–630 (1963)
15. Révész, P.: Local time and invariance. In: Analytical Methods in Probability Theory, Proceedings 1980, 128–145. Berlin Heidelberg New York: Springer-Verlag 1981
16. Strassen, V.: An invariance principle for the law of the iterated logarithm. Z. Wahrscheinlichkeitstheor. Verw. Geb. **3**, 211–226 (1964)
17. Trotter, H.F.: A property of Brownian motion paths. Ill. J. Math. **2**, 425–433 (1958)
18. Walsh, J.B.: Excursions and local time. Astérisque **52–53**, 159–192 (1978)
19. Williams, D.A.: Path. decomposition and continuity of local time for one-dimensional diffusions. Proc. Lond. Math. Soc. **28**, 738–768 (1974)
20. Erdös, P., Révész, P.: On the favourite points of a random walk. Preprint.

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