Electronic Journal of Probability

Electron. J. Probab. **28** (2023), article no. 20, 1–12.
ISSN: 1083-6489 https://doi.org/10.1214/23-EJP916

The rate of escape of the most visited site of Brownian motion

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Abstract

Let $\{L_t^z\}$ be the jointly continuous local times of a one-dimensional Brownian motion and let $L_t^* = \sup_{z\in\mathbb{R}} L_t^z$. Let V_t be any point z such that $L_t^z = L_t^*$, a most visited site of Brownian motion. We prove that if $\gamma > 1$, then

$$\liminf_{t\to\infty} \frac{|V_t|}{\sqrt{t}/(\log t)^\gamma} = \infty, \qquad \text{a.s.,}$$

with an analogous result for simple random walk. This proves a conjecture of Lifshits and Shi.

Keywords: Most visited site; favorite point; Brownian motion; rate of escape.
MSC2020 subject classifications: 60J55.
Submitted to EJP on November 19, 2021, final version accepted on February 3, 2023.
Supersedes arXiv:1303.2040v5.

1 Introduction

Let S_n be a simple random walk, let $N_n^k = \sum_{j=0}^n 1_{(S_j=k)}$ be the number of visits by the random walk to the point k by time n, and let $N_n^* = \sup_{k\in\mathbb{Z}} N_n^k$. Let $\mathcal{U}_n = \{k \in \mathbb{Z} : N_n^k = N_n^*\}$, the set of values k where N_n^k takes its maximum, and let U_n be any element of $\mathcal{U}_n$. We call $\mathcal{U}_n$ the set of most visited sites of the random walk at time n. This concept was introduced in [4], and was simultaneously and independently defined by [13], who called U_n a favorite point of the random walk. In [4] it was proved that U_n is transient, and in fact

$$\liminf_{n\to\infty} \frac{|U_n|}{\sqrt{n}/(\log n)^\gamma} = \infty \tag{1.1}$$

if $\gamma > 11$ and

$$\liminf_{n\to\infty} \frac{|U_n|}{\sqrt{n}/(\log n)^\gamma} = 0 \tag{1.2}$$

if $\gamma < 1$. It has been of considerable interest since that time to prove that there exists γ_0 such that (1.1) holds if $\gamma > \gamma_0$ and (1.2) holds if $\gamma < \gamma_0$ and to find the value of γ_0.

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One can state the analogous problem for Brownian motion, and [4] used Brownian motion techniques and an invariance principle for local times to derive the results for random walk from those of Brownian motion. Let $\{L_t^z\}$ be the jointly continuous local times of a Brownian motion and let $\mathcal{V}_t(\omega)$ be the set of values of z where the function $z \to L_t^z(\omega)$ takes its maximum. We call $\mathcal{V}_t$ the set of most visited points or the set of favorite points of Brownian motion at time t. In [4] it was proved that if V_t is any element of $\mathcal{V}_t$, then

$$\liminf_{t\to\infty} \frac{|V_t|}{\sqrt{t}/(\log t)^\gamma} = \infty \tag{1.3}$$

if $\gamma > 11$ and

$$\liminf_{t\to\infty} \frac{|V_t|}{\sqrt{t}/(\log t)^\gamma} = 0 \tag{1.4}$$

if $\gamma < 1$.

The bounds in (1.2) and (1.4) have been improved somewhat. Lifshits and Shi [20] proved that the lim inf is 0 when $\gamma = 1$ as well as when $\gamma < 1$.

In [3] the most visited sites of symmetric stable processes of order α for $\alpha > 1$ were studied. As a by-product of the results there, the value of γ in (1.3) was improved from 11 to 9.

In Lifshits and Shi [20] it was asserted that the value of γ in (1.1) and (1.3) could be any value larger than 1, or equivalently, that γ_0 exists and is equal to 1. However, as Prof. Shi kindly informed us, there is a subtle but serious error in the proof; see Remark 2.5 for details.

Marcus and Rosen [22] subsequently showed that γ in (1.3) could be any value larger than 3.

In this paper we prove that the assertion of Lifshits and Shi is correct, that (1.1) and (1.3) hold whenever $\gamma > 1$. See Theorems 2.1 and 2.2. Our method relies mainly on the Ray-Knight theorems and a moving boundary estimate due to Novikov [23].

A few words about when $\mathcal{U}_n$ and $\mathcal{V}_t$ consist of more than one point are in order. Eisenbaum [10] and Leuridan [18] have shown that at any time t there are at most two values where L_t^z takes its maximum. Toth [27] has shown that for n sufficiently large, depending on ω, there are at most 3 values of k which are most visited sites for S_n, and more recently Ding and Shen [9] have shown that almost surely $\mathcal{U}_n$ consists of 3 distinct points infinitely often. It turns out that the values of the lim inf in (1.1)-(1.4) do not depend on which value of the most visited site is chosen.

There are many results on the most visited sites of Brownian motion and of various other processes. See [5], [8], [11], [12], [14], [16], [19], [21], [24], and [26] for some of these.

In Section 2 we state our main theorems precisely and give some preliminaries. Section 3 contains some estimates on local times and squared Bessel processes of dimension 0. These are used in Section 4 to establish a lower bound on the supremum of local time at certain random times, and in Section 5 we move from random times to fixed times to obtain our result for Brownian motion. Finally in Section 6 we prove the result for random walks.

2 Preliminaries

Let W_t be a one-dimensional Brownian motion and let $\{L_t^z\}$ be a jointly continuous version of its local times. Let

$$L_t^* = \sup_{z\in\mathbb{R}} L_t^z.$$

We define the collection of most visited sites of W by

$$\mathcal{V}_t = \{x \in \mathbb{R} : L_t^x = L_t^*\}.$$

Let $V_t^s = \inf\{|x| : x \in \mathcal{V}_t\}$ and $V_t^\ell = \sup\{|x| : x \in \mathcal{V}_t\}$.

Our main theorem can be stated as follows.

Theorem 2.1. *(1) If $\gamma > 1$, then*

$$\liminf_{t\to\infty} \frac{V_t^s}{\sqrt{t}/(\log t)^\gamma} = \infty, \qquad \text{a.s.}$$

(2) If $\gamma \le 1$,

$$\liminf_{t\to\infty} \frac{V_t^\ell}{\sqrt{t}/(\log t)^\gamma} = 0, \qquad \text{a.s.}$$

We have the corresponding theorem for a simple random walk S_n. Let

$$N_n^k = \sum_{j=0}^n 1_{(S_j = k)},$$

the number of times S_j is equal to k up to time n. Let $N_n^* = \max_{k\in\mathbb{Z}} N_n^k$ and let

$$\mathcal{U}_t = \{k \in \mathbb{Z} : N_n^k = N_n^*\}.$$

Let $U_t^s = \inf\{|x| : x \in \mathcal{N}_t\}$ and $U_t^\ell = \sup\{|x| : x \in \mathcal{N}_t\}$.

Our second theorem is the following.

Theorem 2.2. *(1) If $\gamma > 1$, then*

$$\liminf_{n\to\infty} \frac{U_n^s}{\sqrt{n}/(\log n)^\gamma} = \infty, \qquad \text{a.s.}$$

(2) If $\gamma \le 1$,

$$\liminf_{n\to\infty} \frac{U_n^\ell}{\sqrt{n}/(\log n)^\gamma} = 0, \qquad \text{a.s.}$$

A process X_t is called the square of a Bessel process of dimension 0 started at $x \ge 0$, denoted $BES(0)^2$, if it is the unique solution to the stochastic differential equation

$$X_t = x + 2\sqrt{X_t}\, dW_t,$$

where $X_t \ge 0$ a.s. for each t and W is a one-dimensional Brownian motion with filtration $\{\mathcal{F}_t\}$. When X_t hits 0, which it does almost surely, it then stays there forever. X has a scaling property: for $r > 0$ and X is started at x, the process $\frac{1}{r}X_t$ has the same law as the process $X_{t/r}$ started at x/r. If Y_t is the nonnegative square root of X_t and $x > 0$, then Y is the unique solution to the stochastic differential equation

$$Y_t = \sqrt{x} + W_t - \frac{1}{2Y_t}\, dt.$$

See [25] for details.

For any process ξ_t let

$$\tau_a = \tau_a^\xi = \inf\{t > 0 : \xi_t = a\}, \tag{2.1}$$

the hitting time of a by the process ξ_t.

Let

$$T_r = T(r) = \inf\{t > 0 : L_t^0 \ge r\}, \tag{2.2}$$

the inverse local time at 0.

The main preliminary result we need is the following version of a special case of the Ray-Knight theorems. See [17], [22], and [25].

Theorem 2.3. *Suppose $r > 0$. The processes $\{L^z_{T_r}, z \ge 0\}$ and $\{L^{-z}_{T_r}, z \ge 0\}$ are each $BES(0)^2$ processes with time parameter z started at r and are independent of each other.*

We also need

Proposition 2.4. *Let $0 < r < s$. The processes $\{L^z_{T_s} - L^z_{T_r}, z \ge 0\}$ and $\{L^{-z}_{T_s} - L^{-z}_{T_r}, z \ge 0\}$ are each $BES(0)^2$ processes started at $s - r$, are independent of each other, and are independent of the processes $\{L^z_{T_r}, z \ge 0\}$ and $\{L^{-z}_{T_r}, z \ge 0\}$.*

Proof. Since the local time at 0 of a Brownian motion increases only when the Brownian motion is at 0, then $W_{T_r} = 0$ for all $r > 0$. Proposition 2.4 follows easily from this, the strong Markov property applied at time T_r, and Theorem 2.3. □

We use the letter c with or without subscripts to denote finite positive constants whose exact value is unimportant and whose value may change from line to line.

Remark 2.5. The error in [20] is that inequality (2.12) of that paper need not hold. Let $a > 0$. Note that $\sup_{y > a\sqrt{t}} L^y_t$ can be decreasing in t at some times because the supremum is over decreasing sets. This can happen even when $W_t > a\sqrt{t}$. Similarly, $\sup_{x < a\sqrt{t}} L^x_t$ can be increasing in t at some times even when $W_t > a\sqrt{t}$ because the supremum is over increasing sets.

3 Some estimates

Define

$$I^+(t, h) = \sup_{0 \le z \le h} L^z_t.$$

Proposition 3.1. *Let $\theta > 0$. There exists a positive real number M depending on θ such that*

$$\limsup_{t \to \infty} \frac{\sup_{s \le t}[I^+(s, \sqrt{t}/(\log t)^\theta) - L^0_s]}{\sqrt{t} \log \log t / (\log t)^{\theta/2}} \le M, \qquad \text{a.s.}$$

Proof. Let A_n be the event

$$A_n = \Big\{ \sup_{s \le 2^{n+1}} [I^+(s, 2^{(n+1)/2}/(\log 2^n)^\theta) - L^0_s] \ge M \frac{2^{n/2} \log \log 2^n}{(\log 2^{n+1})^{\theta/2}} \Big\},$$

where M is a positive real to be chosen in a moment. By scaling, the probability of A_n is the same as the probability of

$$B_n = \Big\{ \sup_{s \le 1} [I^+(s, 1/(\log 2^n)^\theta) - L^0_s] \ge M \frac{2^{-1/2} \log \log 2^n}{(\log 2^{n+1})^{\theta/2}} \Big\}.$$

Lemma 5.2 of [4] says that if $\delta \le 1$ and $t \ge 1$, then

$$\mathbb{P}(\sup_{s \le t} \sup_{0 \le x, y \le 1, |x-y| \le \delta} |L^y_s - L^x_s| \ge \lambda) \le \frac{c_1}{\delta} e^{-\lambda / c_2 \delta^{1/2} t^{1/4}}.$$

Applying this with $t = 1$, $\delta = 1/(\log 2^n)^\theta$, $x = 0$, and

$$\lambda = 2^{-1/2} M \log \log 2^n / (\log 2^{n+1})^{\theta/2},$$

and recalling $\mathbb{P}(A_n) = \mathbb{P}(B_n)$, we see that $\mathbb{P}(A_n)$ is summable provided we choose M large enough. By the Borel-Cantelli lemma, $\mathbb{P}(A_n \text{ i.o.}) = 0$. If $2^n \le t \le 2^{n+1}$ and t is large enough (depending on ω), then

$$\begin{aligned}\sup_{s\le t}[I^+(s,\sqrt{t}/(\log t)^\theta)-L^0_s] &\le \sup_{s\le 2^{n+1}}[I^+(s,2^{(n+1)/2}/(\log 2^n)^\theta)-L^0_s]\\ &\le M\frac{2^{n/2}\log\log 2^n}{(\log 2^{n+1})^{\theta/2}}\\ &\le M\sqrt{t}\log\log t/(\log t)^{\theta/2}.\end{aligned}$$

The proposition follows. □

Proposition 3.2. *Let X_t be a $BES(0)^2$ and let $\mathbb{P}^x$ denote the law of X started at x. Then*

$$\mathbb{P}^1(\tau_0<\tau_{1+a})=\frac{a}{1+a}.$$

Proof. We know $\tau_0<\infty$ a.s. Now X is a continuous martingale, hence a time change of a Brownian motion, and thus the hitting probabilities are the same as those for a Brownian motion. □

The next two propositions show that in many respects a $BES(0)^2$ is similar to a Brownian motion as long as it is not too close to 0.

Proposition 3.3. *For X a $BES(0)^2$ and $x>0$,*

$$\mathbb{P}^x(\inf_{s\le t}X_s<x-\lambda)\le c_1e^{-c_2\lambda^2/xt}.$$

Proof. Since $X\ge 0$, there is nothing to prove unless $\lambda\le x$. By a scaling argument, it suffices to suppose $x=1$.

We start by writing

$$\mathbb{P}^1(\tau^X_{1-\lambda}\le t)\le \mathbb{P}^1(\tau^X_2\le t)+\mathbb{P}^1(\tau^X_{1-\lambda}\le t,\tau^X_2>t). \tag{3.1}$$

To estimate the terms on the right hand side of (3.1) we use Doob's inequality. Recalling that $dX_t=2\sqrt{X_t}\,dW_t$, we have $d\langle X\rangle_t=4X_t\,dt$.

Suppose $a>0$. Then

$$\begin{aligned}\mathbb{P}^1(\tau^X_2\le t)=\mathbb{P}^1(\sup_{s\le t\wedge\tau^X_2}X_s\ge 2)&=\mathbb{P}^1(\sup_{s\le t\wedge\tau^X_2}a(X_s-1)\ge a)\\ &\le e^{-a}\mathbb{E}^1\exp(a(X_{t\wedge\tau^X_2}-1)).\end{aligned}$$

To bound the expectation,

$$\begin{aligned}&\mathbb{E}^1\exp(a(X_{t\wedge\tau^X_2}-1))\\ &\qquad=\mathbb{E}^1\Big[\exp(a(X_{t\wedge\tau^X_2}-1)-\tfrac12 a^2\langle X\rangle_{t\wedge\tau^X_2})\exp(\tfrac12 a^2\langle X\rangle_{t\wedge\tau^X_2})\Big]\\ &\qquad\le \mathbb{E}^1\exp(a(X_{t\wedge\tau^X_2}-1)-\tfrac12 a^2\langle X\rangle_{t\wedge\tau^X_2})e^{4a^2t}.\end{aligned}$$

Setting $a=1/8t$ yields

$$\mathbb{P}^1(\tau^X_2\le t)\le e^{-1/16t}.$$

The second term of (3.1) is slightly more complicated, but quite similar. Let $\widetilde{X}_t$ be X_t stopped at time τ^X_2 and use (2.1) to define $\tau^{\widetilde{X}}_{1-\lambda}$. Suppose $a>0$ and write

$$\begin{aligned}\mathbb{P}^1(\tau^X_{1-\lambda}\le t,\tau^X_2>t)&\le \mathbb{P}^1(\inf_{s\le t\wedge\tau^{\widetilde{X}}_{1-\lambda}}(\widetilde{X}_s-1)\le-\lambda)\\ &=\mathbb{P}^1(\sup_{s\le t\wedge\tau^{\widetilde{X}}_{1-\lambda}}(-a(\widetilde{X}_s-1))\ge a\lambda)\\ &\le e^{-a\lambda}\mathbb{E}^1\exp(a(-(\widetilde{X}_{t\wedge\tau^{\widetilde{X}}_{1-\lambda}}-1)))\end{aligned}$$

and the expectation on the last line is equal to

$$\mathbb{E}^1\Big[\exp(-a(\widetilde{X}_{t\wedge\tau^{\widetilde{X}}_{1-\lambda}}-1)-\tfrac{1}{2}a^2\langle\widetilde{X}\rangle_{t\wedge\tau^{\widetilde{X}}_{1-\lambda}})\exp(\tfrac{1}{2}a^2\langle\widetilde{X}\rangle_{t\wedge\tau^{\widetilde{X}}_{1-\lambda}})\Big],$$

which is bounded by e^{4a^2t}. Setting $a=\lambda/8t$ we see the second term on the right of (3.1) is bounded by $e^{-\lambda^2/16t}$.

Combining the two estimates for the terms on the right hand side of (3.1) and recalling that we are supposing $\lambda\le 1$ yields the proposition. □

Another approach to the preceding proposition is to use the results of [6].

Proposition 3.4. *Let $R>0$, let X_t be a $BES(0)^2$, and let g be a non-negative absolutely continuous function on $[0,R]$ with $g(0)>0$. Let $p>1$. Then*

$$\begin{aligned}\mathbb{P}^1(X_t\le 1+g(t),0\le t\le R)&\qquad (3.2)\\ \le c_1e^{c_2(p)R}\Big(\frac{g(0)}{\sqrt{R}}\Big)^{1/p^2}\exp\Big(\frac{1}{2(p-1)p}\int_0^R g'(s)^2\,ds\Big)+c_3e^{-c_4/R}.\end{aligned}$$

Proof. By Novikov [23], Theorem 6,

$$\begin{aligned}\mathbb{P}^0(W_t\le g(t),0\le t\le R)&\qquad (3.3)\\ \le c_1\Big(\Phi_0\Big(\frac{g(0)}{\sqrt{R}}\Big)\Big)^{1/p}\exp\Big(\frac{1}{2(p-1)}\int_0^R g'(s)^2\,ds\Big),\end{aligned}$$

where W is a Brownian motion, $\Phi_0(x)=2\Phi(x)-1$, and $\Phi(x)$ is the distribution function of a standard normal random variable. Note $\Phi_0(x)\le cx$ for $x\ge 0$.

Let Z be the unique solution to

$$dZ_t=dW_t-a(Z_t)\,dt,$$

where $a(x)=1/2x$ for $x\ge 1/2$ and $a(x)=1$ for $x<1/2$. Let $Y_t=X_t^{1/2}$.

We start by writing

$$\begin{aligned}\mathbb{P}^1(X_t\le 1+g(t),\,0\le t\le R)&\qquad (3.4)\\ \le \mathbb{P}^1(X_t\le 1+g(t),\,0\le t\le R,\tau^X_{1/4}>R)+\mathbb{P}^1(\tau^X_{1/4}\le R).\end{aligned}$$

The second term on the right is bounded by $c_1e^{-c_2/R}$ by Proposition 3.3. The first term on the right is equal to

$$\begin{aligned}\mathbb{P}^1(Y_t\le(1+g(t))^{1/2},0\le t\le R,\tau^Y_{1/2}>R)&\\ &\le \mathbb{P}^1(Y_t\le 1+\tfrac{1}{2}g(t),\,0\le t\le R,\tau^Y_{1/2}>R)\\ &=\mathbb{P}^1(Z_t\le 1+\tfrac{1}{2}g(t),\,0\le t\le R,\tau^Z_{1/2}>R)\\ &\le \mathbb{P}^1(B),\end{aligned}$$

where

$$B=\{Z_t\le 1+\tfrac{1}{2}g(t),\,0\le t\le R\}$$

and $\tau^Z_{1/2}$ is defined by (2.1); we use the fact that $Z_t=Y_t$ for $t<\tau^Y_{1/2}$.

Let

$$M_t=\exp\Big(\int_0^t a(Z_s)\,dW_s-\tfrac{1}{2}\int_0^t a(Z_s)^2\,ds\Big).$$

Let $\mathbb{Q}$ be defined by $d\mathbb{Q}/d\mathbb{P}^1=M_t$ on $\mathcal{F}_t$. By the Girsanov theorem, $Z_t=W_t-\int_0^t a(Z_s)\,ds$ is a Brownian motion under $\mathbb{Q}$.

By Hölder's inequality,

$$\mathbb{P}^1(B) = \mathbb{E}_{\mathbb{Q}}[M_R^{-1}; B] \le (\mathbb{E}_{\mathbb{Q}} M_R^{-r})^{1/r} (\mathbb{Q}(B))^{1/p},$$

where $r = p/(p-1)$. We bound the second factor by (3.3).

It remains to bound

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[M_R^{-r}] &= \mathbb{E}_{\mathbb{P}}^1[M_R^{1-r}] \\
&= \mathbb{E}_{\mathbb{P}}^1\Big[\exp\Big((1-r)\int_0^R a(Z_s)\,dW_s - \tfrac{1-r}{2}\int_0^R a(Z_s)^2\,ds\Big)\Big] \\
&= \mathbb{E}_{\mathbb{P}}^1\Big[\exp\Big((1-r)\int_0^R a(Z_s)\,dW_s - \tfrac{(1-r)^2}{2}\int_0^R a(Z_s)^2\,ds\Big) \\
&\qquad \times \exp\Big(\frac{(1-r)^2-(1-r)}{2}\int_0^R a(Z_s)^2\,ds\Big)\Big] \\
&\le \exp\Big(\frac{r^2-r}{2}R\Big).
\end{aligned}$$

Combining our estimates yields the proposition. □

4 Growth of local times

Suppose $\varepsilon \in (0, \frac{1}{2})$ and $0 < \delta \le \frac{1}{2}$. Choose $p > 1$ close to 1 so that $1/p^2 \ge 1-\varepsilon$. Choose $\beta \in (0, \frac{1}{2})$ small so that $\beta^2/4p(p-1) < \varepsilon/2$. Let

$$U_t = L^t_{T_1} - 1. \tag{4.1}$$

Recall that here t is actually the space variable for local time. Set

$$g(t) = \begin{cases} 4\delta, & t \le 16\delta^2/\beta^2; \\ \beta\sqrt{t}, & t > 16\delta^2/\beta^2. \end{cases}$$

Let

$$A = \{\exists t \in [0, \delta^\varepsilon] : U_t \ge g(t)\}. \tag{4.2}$$

Proposition 4.1.

$$\mathbb{P}(A^c) \le c_1 \delta^{1-2\varepsilon}.$$

Proof. We estimate the right hand side of (3.2) with $R = \delta^\varepsilon$ and $g(0) = 4\delta$. Observe that $g'(t)$ is zero unless $t > 16\delta^2/\beta^2$, in which case $g'(t) = \beta/2\sqrt{t}$. Hence

$$\begin{aligned}
\frac{1}{2p(p-1)}\int_0^{\delta^\varepsilon} g'(t)^2\,dt &\le \frac{\beta^2}{8p(p-1)}\int_{16\delta^2/\beta^2}^1 \frac{1}{t}\,dt \\
&= \frac{\beta^2}{4p(p-1)}\log(1/\delta) + c(p,\beta),
\end{aligned}$$

where $c(p,\beta)$ depends on p and β, but not δ.

Therefore

$$\mathbb{P}(A^c) \le c_1(\delta^{1-\varepsilon/2})^{1/p^2}(1/\delta)^{\beta^2/4p(p-1)} + c_2 e^{-c_3\delta^{-\varepsilon}} \le c_4\delta^{1-2\varepsilon}.$$

□

For $s \in [0,1]$ let

$$X^s_t = L^t_{T(1+s)} - L^t_{T(1)} - s. \tag{4.3}$$

Let

$$B_s = \{\exists t \in [0, \delta^\varepsilon] : X_t^s \le -\tfrac{1}{4} g(t)\}. \tag{4.4}$$

For U, an estimate involving a power of δ close to 1 is the best we can expect. However the exponential estimate we obtain in the next proposition allows us to take the supremum over a large number of values of s.

Proposition 4.2. *For $s \in [0, \delta^\varepsilon]$*

$$\mathbb{P}(B_s) \le c_1 \log(1/\delta) e^{-c_2/\delta^\varepsilon}.$$

Proof. Let $I_0 = [0, 16\delta^2/\beta^2]$. Let M be the smallest positive integer such that $2^M(16\delta^2/\beta^2)$ is larger than δ^ε. For $1 \le m \le M$ let

$$I_m = [2^{m-1}(16\delta^2/\beta^2), 2^m(16\delta^2/\beta^2)].$$

For $0 \le m \le M$ let

$$C_m = \{\exists t \in I_m : X_t^s \le -\tfrac{1}{4} g(t)\}.$$

By Proposition 3.3, for $1 \le m \le M$,

$$\mathbb{P}(C_m) \le c_1 \exp\Big(-c_2 \frac{2^{m-1}\delta^2}{s 2^m \delta^2}\Big).$$

Because $s \le \delta^\varepsilon$, this is bounded by $c_1 e^{-c_2 \delta^{-\varepsilon}}$. Similarly

$$\mathbb{P}(C_0) \le c_1 \exp\Big(-c_2 \frac{\delta^2}{s\delta^2}\Big) \le c_3 e^{-c_4 \delta^{-\varepsilon}}.$$

Since $M \le c \log(\delta^{\varepsilon - 2})$,

$$\mathbb{P}(\cup_{m=0}^M C_m) \le c_1 \log(1/\delta) e^{-c_2 \delta^{-\varepsilon}}.$$

Observing that $B_s \subset \cup_{m=0}^M C_m$ completes the proof. □

Proposition 4.3. *There exists c such that*

$$\mathbb{P}(\exists u \in [1, 1+\delta^\varepsilon] : (L^*_{T_u} - u) \le \delta) \le c\delta^{2-4\varepsilon}.$$

c depends on ε but not δ.

Proof. Let $J = [\delta^{\varepsilon-1}] + 1$ and let $0 = s_0 < s_1 < \cdots < s_J = \delta^\varepsilon$ be points of the interval $[0, \delta^\varepsilon]$ such that $s_{j+1} - s_j \le \delta$ for all j. Let

$$D_j = \{\sup_{t \ge 0} (U_t + X_t^{s_j}) \le 2\delta\}.$$

We know $\mathbb{P}(D_0) \le 2\delta$ by Proposition 3.2.

Suppose $1 \le j \le J$. If $\omega \in A \cap B_{s_j}^c$, then there exists $t \in [0, \delta^\varepsilon]$ such that $U_t(\omega) \ge g(t)$ but $X_t^{s_j}(\omega) \ge -\frac{1}{4} g(t)$. But then

$$U_t(\omega) + X_t^{s_j}(\omega) \ge g(t) - \tfrac{1}{4} g(t) \ge 3\delta,$$

which implies $\omega \notin D_j$. Therefore $D_j \subset A^c \cup B_{s_j}$. It follows that

$$\cup_{j=1}^J D_j \subset A^c \cup (\cup_{j=1}^J B_{s_j}).$$

Using Propositions 4.1 and 4.2 and the fact that $J \le c\delta^{\varepsilon-1}$, we then have

$$\begin{aligned}\mathbb{P}(\exists j \le J : \sup_{t \ge 0} (U_t + X_t^{s_j}) \le 2\delta) &\le 2\delta + c_1 \delta^{1-2\varepsilon} + c_2 \delta^{\varepsilon - 1} \log(1/\delta) e^{-c_3 \delta^{-\varepsilon}} \\ &\le c_4 \delta^{1-2\varepsilon}.\end{aligned}$$

If $\sup_{x\geq 0} L^x_{T(1+s_j)} - (1+s_j) \leq 2\delta$, then $\sup_{t\geq 0}(U_t + X^{s_j}_t) \leq 2\delta$, and so

$$\mathbb{P}(\exists j \leq J : \sup_{x\geq 0} L^x_{T(1+s_j)} - (1+s_j) \leq 2\delta) \leq c_4 \delta^{1-2\varepsilon}. \tag{4.5}$$

Let $L^+_t = \sup_{x>0} L^x_t$ and $L^-_t = \sup_{x<0} L^x_t$. If $L^*_{T(1+s_j)} - (1+s_j) \leq 2\delta$, then

$$L^+_{T(1+s_j)} - (1+s_j) \leq 2\delta \qquad \text{and} \qquad L^-_{T(1+s_j)} - (1+s_j) \leq 2\delta.$$

By independence, symmetry, and (4.5),

$$\mathbb{P}(E) \leq (c_1\delta^{1-2\varepsilon})^2 = c_2\delta^{2-4\varepsilon},$$

where

$$E = \{\exists j \leq J : L^*_{T(1+s_j)} - (1+s_j) \leq 2\delta\}.$$

If $u \leq \delta^\varepsilon$ and $u \in [s_j, s_{j+1}]$, then

$$\begin{aligned} L^*_{T(1+u)} - (1+u) &\geq L^*_{T(1+s_j)} - (1+s_j) + (s_j - u) \\ &\geq L^*_{T(1+s_j)} - (1+s_j) - \delta. \end{aligned}$$

We conclude that on the event E^c

$$L^*_{T(1+u)} - (1+u) > 2\delta - \delta = \delta.$$

Therefore

$$\mathbb{P}(\exists u \in [0, \delta^\varepsilon] : L^*_{T(1+u)} - (1+u) \leq \delta) \leq c\delta^{2-4\varepsilon}.$$

□

Theorem 4.4. *If $\gamma > 1/2$, then*

$$\liminf_{t\to\infty} \frac{L^*_{T_t} - t}{t/(\log t)^\gamma} = \infty, \qquad \textit{a.s.}$$

Proof. Let $r_K = 2^K$, $a > 0$, and

$$\delta_K = \frac{a}{(\log r_K)^\gamma}.$$

Divide $[r_K, r_{K+1}]$ into $[\delta_K^{-\varepsilon}] + 1$ equal subintervals. Each subinterval will have length less than or equal to $\delta_K^\varepsilon r_K$. Let

$$F_K = \{\exists t \in [r_K, r_{K+1}] : (L^*_{T_t} - t) \leq \delta_K r_K\}.$$

Then by scaling, Proposition 4.3, and our bound on the number of subintervals,

$$\mathbb{P}(F_K) \leq c_1 \delta_K^{-\varepsilon} \delta_K^{2-4\varepsilon} = c_1 \delta_K^{2-5\varepsilon}.$$

If $\gamma > \frac{1}{2}$, choose ε small enough so that $(2-5\varepsilon)\gamma > 1$. By the Borel-Cantelli lemma, $\mathbb{P}(F_K \text{ i.o.}) = 0$. This implies

$$\mathbb{P}\Big(L^*_{T_t} - t \leq \frac{at}{(\log t)^\gamma} \text{ i.o.}\Big) = 0.$$

Since a is arbitrary, the theorem follows. □

5 From random times to fixed times

Now we derive our results for fixed times from Theorem 4.4. For values r where T_r is approximately r^2, the argument is straightforward, but for other values of r a different argument is necessary to avoid an extraneous power of logarithm.

Let

$$I(t,h) = \sup_{|z|\le h} L_t^z.$$

Theorem 5.1. *Let $\gamma > 1$. There exists $\rho > 0$ such that with probability one,*

$$L_t^* > I(t, \sqrt{t}/(\log t)^\gamma) + \frac{c\sqrt{t}}{(\log t)^\rho}$$

for all t sufficiently large.

Proof. Without loss of generality assume $\gamma \le 2$. Choose $1/2 < b < \gamma/2$ and then choose $a < \gamma$ such that $\gamma/2 - a/2 > b$. Suppose

$$T_{r-} \le t \le T_r,$$

where $T_{r-} = \lim_{s\to r-} T_s$. Then $L_t^0 = r$.
Case 1. $t \le r^2(\log r)^a$. By [15], for t sufficiently large (depending on ω),

$$r = L_t^0 \le c\sqrt{t\log\log t},$$

so $\log r \le c\log t$. By Proposition 3.1 and symmetry, for sufficiently large t (also depending on ω),

$$\begin{aligned} I(t, \sqrt{t}/(\log t)^\gamma) - L_t^0 &\le c\frac{\sqrt{t}\log\log t}{(\log t)^{\gamma/2}} \\ &\le c\frac{r(\log r)^{a/2}\log\log r}{(\log r)^{\gamma/2}} \\ &= c\frac{r\log\log r}{(\log r)^{\gamma/2-a/2}}. \end{aligned}$$

For r sufficiently large, for all $s \in [r/2, r)$, by Theorem 4.4 we have

$$L_{T_s}^* - s \ge \frac{s}{2(\log s)^b}.$$

Letting s increase up to r,

$$\begin{aligned} L_t^* - r \ge L_{T_{r-}}^* - r &\ge \frac{r}{2(\log r)^b} \\ &\ge I(t, \sqrt{t}/(\log t)^\gamma) - r + c\frac{r}{(\log r)^b} \\ &\ge I(t, \sqrt{t}/(\log t)^\gamma) - r + c\frac{\sqrt{t}}{(\log t)^{b+a/2}} \end{aligned}$$

for t sufficiently large.
Case 2. $t > r^2(\log r)^a$. Then

$$L_t^0 = r \le c_1\frac{\sqrt{t}}{(\log t)^{a/2}}.$$

By this, Proposition 3.1, and symmetry, there exists $K > c_1$ such that

$$I(t, \sqrt{t}/(\log t)^\gamma) \le L_t^0 + K\frac{\sqrt{t}\log\log t}{(\log t)^{\gamma/2}} \le 2K\frac{\sqrt{t}}{(\log t)^{a/2}}$$

for t large. By Kesten's law of the iterated logarithm (see [15] and also [7]), there exists $\kappa > 0$ such that for t sufficiently large,

$$\begin{aligned} L_t^* &\geq \kappa\sqrt{t}/(\log\log t)^{1/2} \\ &\geq 3K\frac{\sqrt{t}}{(\log t)^{a/2}} \geq I(t, \sqrt{t}/(\log t)^\gamma) + K\frac{\sqrt{t}}{(\log t)^{a/2}}. \end{aligned}$$

In either case,

$$L_t^* \geq I(t, \sqrt{t}/(\log t)^\gamma) + c\frac{\sqrt{t}}{(\log t)^{b+a/2}}, \tag{5.1}$$

and we may take $\rho = b + a/2$. □

Proof of Theorem 2.1. Theorem 2.1(2) is already known; see [20]. For (1), let $\gamma > 1$. For large enough t,

$$L_t^* > I(t, \sqrt{t}/(\log t)^\gamma),$$

which means that L_t^z takes its maximum for z outside the interval

$$[-\sqrt{t}/(\log t)^\gamma, \sqrt{t}/(\log t)^\gamma].$$

Theorem 2.1(1) now follows. □

6 Random walks

Proof of Theorem 2.2. (2) follows from [20], so we only consider (1). By the invariance principle of [24] we can find a simple random walk S_n and a Brownian motion W_t such that for each $\varepsilon > 0$,

$$\sup_{k\in\mathbb{Z}} |L_n^k - N_n^k| = o(n^{1/4+\varepsilon}), \qquad \text{a.s.} \tag{6.1}$$

If $\gamma > 1$ and $K_n = \max_{k\in\mathbb{Z}, |k|\leq\sqrt{n}/(\log n)^\gamma} N_n^k$, by (6.1), Lemma 5.3 of [4], and Theorem 5.1, there exists $\rho > 0$ such that

$$\begin{aligned} N_n^* &\geq L_n^* - cn^{1/4+\varepsilon} \\ &\geq I(n, \sqrt{n}/(\log n)^\gamma) + c_1\frac{\sqrt{n}}{(\log n)^\rho} - c_2 n^{1/4+\varepsilon} \\ &\geq K_n + c_1\frac{\sqrt{n}}{(\log n)^\rho} - 2c_2 n^{1/4+\varepsilon} \\ &> K_n \end{aligned}$$

for n sufficiently large. We conclude the most visited site of S_n must be larger in absolute value than $\sqrt{n}/(\log n)^\gamma$ for n large. □

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