

STOCHASTIC BIFURCATION MODELS¹

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We study an ordinary differential equation controlled by a stochastic process. We present results on existence and uniqueness of solutions, on associated local times (Trotter and Ray–Knight theorems) and on time and direction of bifurcation. A relationship with Lipschitz approximations to Brownian paths is also discussed.

1. Introduction. Let B_t be a continuous function of t , let $t_0, x_0, \beta_1, \beta_2 \in \mathbb{R}$, and consider the ordinary differential equation

$$(1.1) \quad \frac{dX_t}{dt} = \begin{cases} \beta_1, & \text{if } X_t < B_t, \\ \beta_2, & \text{if } X_t > B_t, \end{cases} \quad t \in \mathbb{R}, \quad X(t_0) = x_0.$$

Among the results we prove are the following:

1. Although in general there will not be a unique solution to (1.1), there will be a unique Lipschitz solution to (1.1) if B_t is a typical Brownian motion path.
2. Let B_t be a Brownian motion with $B_0 = 0$ and let $X_t^{x_0}$ denote the solution to (1.1) when $t_0 = 0$ and $X(t_0) = x_0$. The map $y \rightarrow X_t^y$ is a one-to-one map of \mathbb{R} onto \mathbb{R} . The smoothness of this map is controlled by the local time at 0 of $X_t^y - B_t$. If we call this local time L_t^y and β_1, β_2 satisfy suitable assumptions, then L_t^y is jointly continuous in y and t and $\{L_\infty^y, y \geq 0\}$ and $\{L_\infty^{-y}, y \geq 0\}$ are strong Markov processes. We show that this implies that for a fixed $t > 0$, the function $y \rightarrow X_t^y$ is of class $C^{1+\gamma}$ with $\gamma < 1/2$, but it is not $C^{3/2}$.
3. As we shall see below, (1.1) is an example of a bifurcation model; if B_t is a Brownian motion, $\beta_1 < 0$ and $\beta_2 > 0$, each of the events $\{\lim_{t \rightarrow \infty} X_t = +\infty\}$ and $\{\lim_{t \rightarrow \infty} X_t = -\infty\}$ has positive probability. The bifurcation time is defined by $T_* = \sup\{t: X_t = B_t\}$. We calculate both the probability of $\{\lim_{t \rightarrow \infty} X_t = +\infty\}$ and the expectation of the bifurcation time using excursion theory.
4. Equation (1.1) sheds light on the best Lipschitz approximation to Brownian paths. In particular we obtain an estimate on the lower bound on the best constant in the Komlós–Major–Tusnády result concerning strong approximations of Brownian motion by random walks.

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Equation (1.1) is similar to an equation that arose in the course of an economic study and its accompanying probabilistic model in Burdzy, Frankel and Pautzner (1997, 1998). These papers introduce and study an economics model whose technical side is based on the following equation:

$$(1.2) \quad \frac{dX_t}{dt} = \begin{cases} -\beta X_t, & \text{if } X_t < f(B_t), \\ \beta(1 - X_t), & \text{if } X_t > f(B_t), \end{cases} \quad t \geq 0, \quad X(0) = x_0 \in (0, 1),$$

where B_t is a Brownian motion starting from $B_0 = b_0$, $\beta > 0$ is a fixed constant and f is a nonincreasing Lipschitz function. The case when $x_0 = f(b_0)$ is of special interest. Results on the time and direction of the stochastic bifurcation were crucial elements of these two papers.

We also consider the following equation, more general than (1.1):

$$(1.3) \quad \frac{dX_t}{dt} = \begin{cases} \beta_1 |X_t - B_t|^{\alpha_1}, & \text{if } X_t < B_t, \\ \beta_2 |X_t - B_t|^{\alpha_2}, & \text{if } X_t > B_t, \end{cases} \quad t \in \mathbb{R}, \quad X(t_0) = x_0.$$

[If $\alpha_1 = \alpha_2 = 0$, then (1.3) reduces to (1.1).] Equation (1.3) was inspired by the following model. Consider a pendulum with rigid arm which is turned upside down (see Figure 1).

Let X_t denote the distance of the weight W from its unstable rest position at the top of the vertical arm. When $X_t = x$ and x is small, the weight is about $c_1 x^2$ units below its rest position, and therefore $c_2 x^2$ units of potential energy must have been converted to kinetic energy, given by $c_3 (dX/dt)^2$. Hence, we have the approximate relationship $dX/dt = c_4 X_t$, assuming infinitesimally small velocity at the rest position. Note that if the initial velocity at the rest position is close to zero, then the time it takes the pendulum to move any fixed nonzero distance from the rest position is very large. We now add stochastic oscillations to our pendulum model. We suppose that the base A of the pendulum vibrates according to a Brownian motion B_t . Then the position X_t of the weight W relative to A is $X_t - B_t$ and we have $dX/dt = c_4 (X_t - B_t)$, which is (1.3) with $\alpha_1 = \alpha_2 = 1$ and $-\beta_1 = \beta_2 = c_4$.

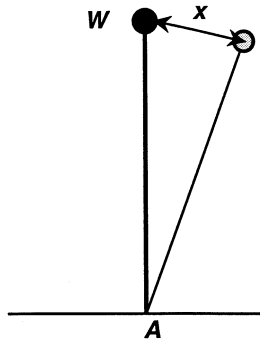


FIG. 1. An example of an unstable system—an “upside down” pendulum. Random oscillations of the base result in stochastic bifurcation.

The solutions to (1.1) exhibit fast switching between two kinds of excursions. See Karatzas and Shreve [(1988), Section 6.5] for a closely related model. Mandelbaum, Shepp and Vanderbei (1990) also consider a model with fast switching between two kinds of excursions, but we were not able to find a direct connection with our own model.

The rest of the paper consists of five sections. Section 2 contains results on existence and uniqueness of solutions to (1.1), (1.3) and related equations. The process B_t will generally be a Brownian motion, but Theorems 2.3 and 2.4 also apply to some fractional Brownian motions (see Examples 2.3 and 2.4).

Let X_t^y denote the solution to (1.1) with $X_0^y = y$. For a fixed $t \geq 0$, the function $y \rightarrow X_t^y$ is a transformation of \mathbb{R} onto itself. How smooth is this map? How many derivatives does the function $y \rightarrow X_t^y$ have and are they continuous? To answer these questions, one is led to study the local time of $X_t^y - B_t$. Section 3 is devoted to a number of results about local times related to (1.1), including analogues of the Trotter and Ray–Knight theorems. See Knight (1981), Leuridan (1998), Norris, Rogers and Williams (1987), Revuz and Yor (1991) and Yor (1997) for old and new variants of the Ray–Knight theorem. Our local times are defined as local times at points, but they may also be viewed as local times of Brownian motion on a random curve—see (5.15) in Föllmer, Protter and Shiryaev (1995) for a result on local times on nonrandom curves.

Section 4 gives explicit formulas for the probability of upward bifurcation for (1.3) and the expected bifurcation time for (1.1), with some indication how to proceed in the more general case (1.3). This extends results from Burdzy, Frankel and Pausner (1998). Section 5 takes a look at the solutions to (1.1) as Lipschitz approximations to the Brownian path. As a consequence, we obtain some lower bounds related to the Komlós–Major–Tusnády construction; see Theorem 5.6. Finally, Section 6 is a list of open problems.

In Sections 3–5, we consider Brownian motion defined on the whole real line \mathbb{R} , that is, the process $\{B_t, -\infty < t < \infty\}$, where $\{B_t, t \in (0, \infty)\}$ and $\{B_{-t}, t \in (0, \infty)\}$ are independent Brownian motions starting from 0 with variance $\mathbb{E}B_t^2 = \mathbb{E}B_{-t}^2 = \sigma^2 t$. Unless stated otherwise, we will assume that all Brownian motions (including those with drift and/or reflection) have infinitesimal variance σ^2 and that all constants are strictly positive and finite.

2. Existence and uniqueness of solutions. In this section we present several theorems on the existence and uniqueness of solutions to differential equations similar to (1.1). There is considerable overlap among the theorems, but each contains cases not covered by the other. We first present our main results. They are followed by some remarks and examples. The proofs are relegated to the end of the section.

We start with the equation

$$(2.1) \quad \frac{dX}{dt} = \begin{cases} \beta_1 |X_t - B_t|^{\alpha_1}, & \text{if } X_t < B_t, \\ \beta_2 |X_t - B_t|^{\alpha_2}, & \text{if } X_t > B_t, \end{cases} \quad t \in \mathbb{R}, \quad X(t_0) = x_0,$$

where B_t is a Brownian motion, $\alpha_1, \alpha_2 > -1$ and $\beta_1, \beta_2 \in \mathbb{R}$.

First note that the function $X_t = B_t$ is a solution to (2.1) with $t_0 = 0$ and $x_0 = 0$, because neither of the conditions on the right-hand side of (2.1) is ever satisfied. We would like to disregard such a solution for two reasons. First, the economics model behind (1.2) required that the solutions to (1.2) be Lipschitz. Second, the example $X_t = B_t$ is rather artificial. For $\alpha_1, \alpha_2 \geq 0$ it is natural to require that X_t be a Lipschitz function. We generalize this to all $\alpha_1, \alpha_2 > -1$ by writing an integrated version of (2.1), namely,

$$(2.2) \quad X_t = x_0 + \int_{t_0}^t [\beta_1 |X_s - B_s|^{\alpha_1} \mathbf{1}_{\{X_s - B_s \leq 0\}} + \beta_2 |X_s - B_s|^{\alpha_2} \mathbf{1}_{\{X_s - B_s > 0\}}] ds.$$

It is easy to see that solutions to (2.2) satisfy (2.1), but the example $X_t = B_t$ shows that the opposite statement is not true.

THEOREM 2.1. *For fixed $t_0, x_0, \beta_1, \beta_2 \in \mathbb{R}, \sigma^2 > 0$ and $\alpha_1, \alpha_2 > -1$, there exist a Brownian motion B_t and a process X_t which satisfy (2.2) with the initial condition as in (2.1). The solution X_t is unique in law. We may construct X_t in such a way that (X_t, B_t) is a strong Markov process relative to the appropriate filtration. If we assume in addition that $\alpha_1, \alpha_2 \geq 0$, then for a given Brownian motion B_t there exists a unique solution to (2.2), a.s.*

Our next theorem is a result on existence. We will state the result for the following generalization of the equation (1.1):

$$(2.3) \quad \frac{dX_t}{dt} = \begin{cases} F_1(X_t), & \text{if } X_t > B_t, \\ F_2(X_t), & \text{if } X_t < B_t, \end{cases} \quad t \in \mathbb{R}, \quad X(t_0) = x_0.$$

THEOREM 2.2. *Assume that F_1 and F_2 are continuous functions and that $|F_1|$ and $|F_2|$ are bounded by $\beta < \infty$. If B_t is a continuous process, then (2.3) has a Lipschitz solution, a.s. There exists a maximal Lipschitz solution $\{X_t^+, t \geq t_0\}$ to (2.3); it is adapted to the filtration $\mathcal{F}_t = \sigma(B_s, s \in [t_0, t])$.*

Haya Kaspı pointed out to us that measurability of a solution to (2.3) is the most delicate point of Theorem 2.2.

We will say that L_t^x is a local time for a process B_t if it is the occupation time density,

$$\int_{-\infty}^{\infty} h(x) L_t^x dx = \int_0^t h(B_t) dt \quad \text{a.s.}$$

for all h bounded and measurable. Note that if B_t is continuous and the local time L_t^x is jointly continuous, then $\sup_x L_t^x < \infty$, a.s. for each t .

We will use the traditional Markovian notation \mathbb{P}^x to denote the distribution of $\{B_t, t \geq t_0\}$ conditioned by $\{B_{t_0} = x\}$, even though we do not assume the Markov property for B_t in Theorems 2.3 and 2.4 below.

THEOREM 2.3. *Let $t_0 > 0, x_0, \beta_1, \beta_2 \in \mathbb{R}$. Assume that:*

(i) *The process B_t is continuous and has a jointly continuous local time L_t^x ;*

(ii) If A_t is an adapted process with $A_{t_0} = x_0$ whose paths are Lipschitz continuous with Lipschitz constant M , then for each x the law of $\{B_t + A_t, t_0 \leq t \leq t_0 + s\}$ under \mathbb{P}^x is mutually absolutely continuous with respect to the law of $\{B_t, t_0 \leq t \leq t_0 + s\}$ under \mathbb{P}^{x+x_0} , for every $s > 0$.

Then with probability one there exists a random $s_0 > 0$ and a unique Lipschitz solution to (1.1) on $[t_0, t_0 + s_0]$.

If in addition we assume that B_t is strong Markov then there is a unique Lipschitz solution to (1.1) for all $t \geq t_0$.

REMARK 2.4. If W_t is a Brownian motion and f is a strictly increasing function such that both f and f^{-1} are Lipschitz continuous, it is easy to check that $B_t = f(W_t)$ is a strong Markov process that satisfies the other assumptions of Theorem 2.3.

THEOREM 2.5. Let $t_0, x_0 \in \mathbb{R}$. Assume that F_1 and F_2 are bounded, Lipschitz functions. Suppose that both are bounded by M and that both have Lipschitz constant less than or equal to M . Let B_t be a continuous process such that:

(i) There exist $c_1 > 0$ and $\gamma \in (0, 1)$ such that whenever $s < t$,

$$(2.4) \quad \mathbb{P}(B_t \in dy \mid \mathcal{F}_s) \leq \frac{c_1}{(t-s)^\gamma} dy, \quad y \in \mathbb{R}.$$

(ii) If A_t is an adapted process with $A_{t_0} = x_0$ whose paths are Lipschitz continuous with Lipschitz constant M , then for each x the law of $\{B_t + A_t, t_0 \leq t \leq t_0 + s\}$ under \mathbb{P}^x is mutually absolutely continuous with respect to the law of $\{B_t, t_0 \leq t \leq t_0 + s\}$ under \mathbb{P}^{x+x_0} , for every $s > 0$.

Then with probability 1, there exists a unique solution to (2.3) for all $t \geq t_0$.

We will show in Example 2.10 below that Theorem 2.5 applies to some fractional Brownian motions. As in Remark 2.4, some functions of fractional Brownian motions also satisfy the hypotheses of Theorem 2.5.

Let $f(x, b) = \beta_1 \mathbf{1}_{\{x \leq b\}} + \beta_2 \mathbf{1}_{\{x > b\}}$ and suppose that $\alpha_1 = \alpha_2 = 0$. Then (2.2) may be written as

$$(2.5) \quad X_t = x_0 + \int_{t_0}^t f(X_s, B_s) ds.$$

The function $(x, b) \rightarrow f(x, b)$ is discontinuous. In applications such as that in Burdzy, Frankel and Puzner (1997), it may be argued that a model with continuous dX/dt might be more realistic. Let us replace f with a continuous approximation,

$$f_\varepsilon(x, b) = \beta_1 \mathbf{1}_{\{x < b - \varepsilon\}} + \beta_2 \mathbf{1}_{\{x > b + \varepsilon\}} + \left[\frac{\beta_2 - \beta_1}{2\varepsilon} (x - b + \varepsilon) + \beta_1 \right] \mathbf{1}_{\{b - \varepsilon \leq x \leq b + \varepsilon\}}$$

and consider the corresponding equation

$$(2.6) \quad X_t^\varepsilon = x_0 + \int_{t_0}^t f_\varepsilon(X_s^\varepsilon, B_s) ds.$$

We will show that the solutions to (2.6) converge to those of (2.5), and thus many results about solutions to (2.5) proved later in this article may be applied to give asymptotic results for the solutions to (2.6).

THEOREM 2.6. *Assume that (2.5) and (2.6) are defined relative to the same Brownian motion B_t . Equation (2.6) has a unique Lipschitz solution. As $\varepsilon \rightarrow 0$, the functions X_t^ε converge to the unique solution X_t of (2.5), a.s.*

Note that the convergence in Theorem 2.6 is uniform on compact sets as all functions X_t^ε are Lipschitz with constant $\max\{|\beta_1|, |\beta_2|\}$.

REMARK 2.7. For the economics model behind (1.2), one does not necessarily want to require the Markov property to hold. The proof of Theorem 2.3 uses the strong Markov property to do an induction argument. For Theorem 2.5 we have in mind examples where B_t is a Gaussian process; see Example 2.10 below. In general, $B_{T+t} - B_T$ will not be Gaussian when T is a stopping time.

EXAMPLE 2.8. We present an elementary example of a continuous deterministic function $t \rightarrow B_t$ for which there are multiple solutions to (1.1). Let $\beta_1 < 0$, $\beta_2 > 0$,

$$B_t = \begin{cases} (1 + \beta_2)t, & \text{for } t \in [0, 1], \\ 1 + \beta_2, & \text{for } t > 1, \\ 0, & \text{for } t < 0. \end{cases}$$

There are uncountably many solutions to (1.1) with this choice of B_t and the initial condition $X_0 = 1$. Here are two of them:

$$X_t^1 = \begin{cases} 0, & \text{for } t \leq -1/\beta_2, \\ 1 + \beta_2 t, & \text{for } t > -1/\beta_2; \end{cases}$$

$$X_t^2 = \begin{cases} 0, & \text{for } t \leq -1/\beta_2, \\ 1 + \beta_2 t, & \text{for } t \in (-1/\beta_2, 1], \\ 1 + \beta_2, & \text{for } t \in (1, 5], \\ 1 + \beta_2 + 5\beta_1 + \beta_1 t, & \text{for } t > 5. \end{cases}$$

EXAMPLE 2.9. As we noted earlier in this section, $X_t = B_t$ is a solution to (1.1) but a rather trivial one. In this example, we will show a less trivial and perhaps more interesting non-Lipschitz solution to (1.1). Take $\beta_1 = \beta_2 = 0$ in (1.1); in other words, consider the equation

$$\frac{dX_t}{dt} = 0 \quad \text{if } X_t \neq B_t, \quad t \in \mathbb{R}, \quad X(t_0) = x_0.$$

The function $X_t = 0$ is a solution to this equation and, moreover, it is the only Lipschitz solution, by Theorem 2.1. Let Y_t be a skew Brownian motion, that is, a process which may be constructed by flipping positive excursions of a standard Brownian motion \tilde{B}_t to the negative side with probability p_1 and negative excursions to the positive side with probability p_2 , independently of each other. Suppose that $p_1 \neq p_2$ so that the process Y_t is not a standard Brownian motion. Let L_t be the local time of Y_t at 0. By a result of Harrison and Shepp (1981) [see also Exercise X (2.24) in Revuz and Yor (1991)], for a suitable constant $c_1 \neq 0$, the process $Y_t - c_1 L_t$ is a standard Brownian motion. If we take $B_t = Y_t - c_1 L_t$ then $X_t = c_1 L_t$ is a non-Lipschitz solution to our equation.

EXAMPLE 2.10. We provide an example of a process satisfying the assumptions of Theorem 2.5 that is not strong Markov. Let B_t be fractional Brownian motion of index $H \in (0, 1/2]$. This means that B_t is a mean zero Gaussian process with

$$\text{Cov}(B_s, B_t) = c_1(s^{2H} + t^{2H} - |t - s|^{2H}).$$

Then B_t has a stochastic integral representation

$$B_t = \int_{-\infty}^t R(t, u) dZ_u,$$

where Z_u is a standard Brownian motion and

$$R(t, u) = c_2[((t - u)^+)^{H-1/2} - (u^-)^{H-1/2}];$$

see, for example, Rogers (1997). Conditioning on \mathcal{F}_s with $s > 0$, the law of B_t given \mathcal{F}_s is that of a Gaussian process with variance

$$c_2^2 \mathbb{E} \left[\left(\int_s^t (t - u)^{H-1/2} dZ_u \right)^2 \middle| \mathcal{F}_s \right] = c_2^2 \int_s^t (t - u)^{2H-1} du = c_3(t - s)^{2H}.$$

Assumption (i) of Theorem 2.5 is immediate from this.

We now show (ii). We give the argument for the case $t_0 = x_0 = 0$, $s = 1$; the extension to the general case is routine.

If $H = 1/2$, then B_t is standard Brownian motion, and (ii) follows from the Girsanov theorem; so we suppose $H < 1/2$. Let $\alpha = H + 1/2$. See Decreusefond and Üstünel (1997) for more details of some of the steps in the following argument. Let $F(a, b, c, z)$ be the standard Gauss hypergeometric function and define an operator K_H on functions on $[0, 1]$ by

$$\begin{aligned} (K_H f)(t) &= \frac{1}{\Gamma(H + 1/2)} \\ &\quad \times \int_0^t (t - x)^{H-1/2} F(H - 1/2, 1/2 - H, H + 1/2, 1 - t/x) f(x) dx. \end{aligned}$$

Let $\mathcal{H}_H = \{K_H h: h \in L^2([0, 1])\}$ and define

$$\|f\|_{\mathcal{H}_H} = \|K_H^{-1} f\|_{L^2}.$$

For $\beta \in (0, 1)$, define

$$(I^\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^x f(t)(x-t)^{\beta-1} dt$$

and

$$(D^\beta f)(x) = \frac{d}{dx}(I^{1-\beta} f)(x).$$

By Decreusefond and Üstünel [(1997), Theorem 2.1, Theorem 3.3 and the proof of Theorem 3.3] we have that \mathcal{H}_H is dense in the set of continuous functions on $[0, 1]$ that are null at 0 and that K_H is an isomorphism from $L^2([0, 1])$ onto $I^{H+1/2}(L^2([0, 1]))$. By Proposition 2.1 of that paper, D^β is the inverse to I^β .

Since K_H^{-1} is continuous from $I^{H+1/2}(L^2)$ into L^2 , then $K_H^{-1} \circ I^{H+1/2}$ is continuous from L^2 into itself, and so there exists a constant c_4 such that

$$\|K_H^{-1} I^{H+1/2} g\|_{L^2} \leq c_4 \|g\|_{L^2}.$$

Thus if $f \in \mathcal{H}_H$, then

$$\|K_H^{-1} f\|_{L^2} \leq c_4 \|D^{H+1/2} f\|_{L^2},$$

or

$$\|f\|_{\mathcal{H}_H} \leq c_4 \|D^\alpha f\|_{L^2}.$$

Let A_t be a uniformly Lipschitz process as in the statement of Theorem 2.5. By Theorem 4.9 of Decreusefond and Üstünel (1997) and the Novikov condition discussed just after that theorem, (ii) will hold if for each $T \in (0, 1)$ we have

$$\mathbb{E} \exp[\|A(\cdot)\|_{\mathcal{H}_H}^2 / 2] < \infty.$$

By the above paragraph, it is enough to show

$$(2.7) \quad \mathbb{E} \exp\left(\int_0^T |D^\alpha A_t|^2 dt / 2\right) < \infty.$$

To show (2.7), by an approximation argument it suffices to show that for each fixed $T > 0$ there exists c_5 (depending on T) such that if f is a C^∞ function on $[0, \infty)$ with $f(0) = 0$, then

$$(2.8) \quad \sup_{0 \leq t \leq T} |D^\alpha f(x)| \leq c_5 \|f'\|_\infty.$$

Then (2.7) will follow easily from (2.8) and our assumptions on A_t .

Note that by a change of variables,

$$I^{1-\alpha} f(x) = c_6 \int_0^x f(x-t)t^{-\alpha} dt$$

and by the Leibniz formula and the fact that $f(0) = 0$,

$$\frac{d}{dx} I^{1-\alpha} f(x) = c_6 \int_0^x f'(x-t)t^{-\alpha} dt = c_6 \int_0^x f'(t)(x-t)^{-\alpha} dt = I^{1-\alpha} f'(x).$$

Since $\alpha = H + 1/2 < 1$, then $|x - t|^{-\alpha}$ is integrable on $[0, x]$. So, for $u = f'$,

$$|D^\alpha f(x)| = |I^{1-\alpha} u(x)| \leq \|u\|_\infty \int_0^x |x - t|^{-\alpha} dt \leq c_7 \|u\|_\infty$$

for $x \leq T$. This gives (2.8), and thus a fractional Brownian motion with parameter $H \in (0, 1/2]$ satisfies the assumptions of Theorem 2.5.

EXAMPLE 2.11. The weaker version of Theorem 2.3, that is, the one without the assumption on the Markov character of B_t , applies to fractional Brownian motions with parameter $H \in (0, 1/2]$. Assumption (ii) of Theorem 2.3 is the same as (ii) of Theorem 2.5; we have verified that assumption in the previous example. As for assumption (i) of Theorem 2.3, the joint continuity of the local time for the fractional Brownian motion follows from Lemma 8.8.1, Theorem 8.8.2 and the proof of Theorem 8.8.4 in Adler (1981).

EXAMPLE 2.12. Fabes and Kenig (1981) gave an example of a process B_t satisfying

$$dB_t = \sigma(B_t, t) dW_t,$$

where W_t is a standard Brownian motion, σ is Hölder continuous in the first variable, σ is bounded above and below by positive constants and the distribution of B_1 does not have a density with respect to Lebesgue measure. B_t is a space-time strong Markov process. Because σ is bounded below, it is not hard to see that B_t has a jointly continuous local time [cf. Revuz and Yor (1991), Chapter 6] and that hypothesis (ii) of Theorem 2.3 holds. Thus this process B_t is an example where the assumptions of Theorem 2.3 hold, but those of Theorem 2.5 do not.

The rest of the section contains proofs of our main results. The following lemma is immediate.

LEMMA 2.13. *Let $\tilde{B}_t = B_{-t}$ and $\tilde{X}_t = X_{-t}$. If X_t is a solution to (2.1) then \tilde{X}_t is a solution to*

$$\frac{d\tilde{X}}{dt} = \begin{cases} -\beta_1 |\tilde{X}_t - \tilde{B}_t|^{\alpha_1}, & \text{if } \tilde{X}_t < \tilde{B}_t, \\ -\beta_2 |\tilde{X}_t - \tilde{B}_t|^{\alpha_2}, & \text{if } \tilde{X}_t > \tilde{B}_t, \end{cases} \quad t \in \mathbb{R}, \quad \tilde{X}(-t_0) = x_0.$$

PROOF OF THEOREM 2.1. For simplicity, assume that $t_0 = 0$. The equation

$$Y_t = x_0 + \int_0^t [\beta_1 |Y_s|^{\alpha_1} \mathbf{1}_{\{Y_s \leq 0\}} + \beta_2 |Y_s|^{\alpha_2} \mathbf{1}_{\{Y_s > 0\}}] ds - \int_0^t dB_s, \quad t \geq 0,$$

has a weak solution which is unique in law by Theorem 5.15 in Karatzas and Shreve (1988). For $X_t = Y_t + B_t$, the last equation is equivalent to (2.2) for $t \geq 0$. This proves the first assertion of the theorem. The strong uniqueness in the case $\alpha_1, \alpha_2 \geq 0$ follows from Proposition 5.17 of Karatzas and Shreve (1988). We note that although the function $y \rightarrow y^\alpha$ is not bounded, that proposition clearly applies by using a truncation argument. The part of the solution

to (2.1) for $t < t_0 = 0$ can be obtained in a similar way using Lemma 2.13. That X_t may be constructed so that (X_t, B_t) is a strong Markov process follows from the weak uniqueness in a standard manner; see Bass (1997), Section I.5 or Stroock and Varadhan (1979), Chapter 6. \square

PROOF OF THEOREM 2.2. We start by showing that for each ω and for any u_1 and z_1 there exists a maximal solution $\tilde{X}_t^{u_1, z_1}$ to the equation

$$dX_t/dt = F_1(X_t), \quad t \in \mathbb{R}, \quad X(u_1) = z_1.$$

First, it is well known that there exists at least one solution to the equation since F_1 is continuous. Since $|F_1|$ is bounded by β , all solutions are Lipschitz with constant β and so their supremum $\tilde{X}_t^{u_1, z_1}$ is also a Lipschitz function with constant β . Next note that the maximum of any two solutions is also a solution to the equation. This and the Lipschitz property of solutions easily imply that there exists a sequence of solutions converging to $\tilde{X}_t^{u_1, z_1}$, uniformly on compact intervals. Now a standard argument can be used to show that $\tilde{X}_t^{u_1, z_1}$ is a solution to the equation.

The analogous maximal solution to $dX_t/dt = F_2(X_t)$ with the initial condition $X(u_1) = z_1$ will be denoted $\tilde{X}_t^{u_1, z_1}$.

The following properties of solutions $\tilde{X}_t^{u_1, z_1}$ follow easily from the definition of $\tilde{X}_t^{u_1, z_1}$ and from the continuity and boundedness of F_1 and F_2 . First, if $\tilde{X}_{u_2}^{u_1, z_1} = z_2$ then the functions $\tilde{X}_t^{u_1, z_1}$ and $\tilde{X}_t^{u_2, z_2}$ are identical. Second, if $z_n \rightarrow z_\infty$ as $n \rightarrow \infty$ then the sequence of functions $\tilde{X}_t^{u_1, z_n}$ converges to $\tilde{X}_t^{u_1, z_\infty}$ as $n \rightarrow \infty$, uniformly on compact sets. Finally, if $z_1 < z_2$ then $\tilde{X}_t^{u_1, z_1} \leq \tilde{X}_t^{u_1, z_2}$ for all t .

We start by proving the existence of a solution to (2.3) for $t \geq t_0$. Consider a small $\delta > 0$. We proceed to define a δ -approximate solution X_t^δ to (2.3). First suppose that $B_{t_0} < x_0$. By the continuity of the paths of B_t , for almost every path of B_t , there exist a unique time $t_1 \in (t_0, \infty]$ and a function X_t^δ defined for $t \in (t_0, t_1)$, such that $X_{t_0}^\delta = x_0$, $X_{t_1}^\delta = B_{t_1}$ if $t_1 < \infty$, and $X_t^\delta = \tilde{X}_t^{t_0, x_0}$ for all $t \in (t_0, t_1)$. We then let $X_t^\delta = X_{t_1}^\delta + \beta(t - t_1)$ for all $t \in [t_1, t_1 + \delta]$, if $t_1 < \infty$. If $B_{t_0} > x_0$ we use the same procedure to define X_t^δ for $t \in [t_0, t_1 + \delta]$ except that we use the function $\tilde{X}_t^{t_0, x_0}$ in place of $\tilde{X}_t^{t_0, x_0}$. If $B_{t_0} = x_0$, we let $t_1 = t_0$ and $X_t^\delta = X_{t_1}^\delta + \beta(t - t_1)$ for $t \in [t_1, t_1 + \delta]$.

We have defined X_t^δ on an interval $[t_0, t_1 + \delta]$. Let $x_1 = X_{t_1 + \delta}^\delta$. Let us replace the initial condition in (2.3) by $X(t_1 + \delta) = x_1$ and define an approximate solution X_t^δ to (2.3) on an interval $[t_1 + \delta, t_2 + \delta]$ using the same method as above. By induction, we can construct a (possibly infinite) sequence of times $\{t_k\}$ and a continuous function X_t^δ which satisfies (2.3) on every interval $(t_k + \delta, t_{k+1})$ and which is linear on every interval $[t_k, t_k + \delta]$, for $k \geq 1$. Note that the function X_t^δ is defined for all $t \geq t_0$ because $t_{k+1} \geq t_k + \delta$ for every k .

By construction, the δ -approximate solution X_t^δ is a Lipschitz function with Lipschitz constant β .

For every integer $m \geq 1$, consider a $1/m$ -approximate solution $X_t^{1/m}$. All of these functions are Lipschitz with the same constant β , and they all satisfy

$X_{t_0}^{1/m} = x_0$. Let X_t be defined by

$$X_t = \limsup_{m \rightarrow \infty} X_t^{1/m} = \lim_{n \rightarrow \infty} \sup_{m > n} X_t^{1/m}.$$

The supremum of an arbitrary family of Lipschitz functions with constant β is a Lipschitz function with the same constant, and the same remark applies to the limit of a sequence of such functions. Hence, for every n , the function $Y_t^n = \sup_{m > n} X_t^{1/m}$ is Lipschitz with constant β , and the same is true of X_t . Note that Y_t^n converge in a monotone way to X_t , uniformly on compact intervals, because all these functions are Lipschitz with the same constant β .

We will show that X_t is a solution to (2.3). Let

$$W(\delta) = \bigcup_{\{(s,x): s \geq t_0, B_s = x\}} \{(t, y): y = x + (t - s)\beta, t \in [s, s + \delta]\}.$$

For $\delta \leq \delta_1$, the portion of the graph of X_t^δ which lies outside $W(\delta_1)$ satisfies (2.3), by construction.

The set of t such that $B_t = X_t$ is closed because both functions B_t and X_t are continuous. Consider any interval (s_1, s_2) such that $B_t \neq X_t$ for all $t \in (s_1, s_2)$. Suppose without loss of generality that $B_t < X_t$ for all $t \in (s_1, s_2)$. Choose an arbitrarily small $\delta_1 > 0$. Note that as $\delta \rightarrow 0$, the open sets $W^c(\delta)$ converge to the complement of $\{(s, x): s \geq t_0, B_s = x\}$. Let $\delta_2 > 0$ be so small that the (closed) portion of the graph of X_t between $s_1 + \delta_1$ and $s_2 - \delta_1$ does not intersect $W(\delta_2)$. Let $s_0 = s_1 + \delta_1$. Since the Y_t^n converge to X_t , there exists a sequence m_j such that $X_{s_0}^{1/m_j} \rightarrow X_{s_0}$. For sufficiently large j , the point $(s_0, X_{s_0}^{1/m_j})$ lies outside $W(\delta_2)$ and we also have $1/m_j < \delta_2$. Then, for t in a neighborhood of s_0 , the function X_t^{1/m_j} must be given by $X_t^{1/m_j} = \tilde{X}_t^{s_0, X_{s_0}^{1/m_j}}$. We will show that $X_t = \tilde{X}_t^{s_0, X_{s_0}}$ for $t \in (s_0, s_2 - \delta_1)$.

Suppose that this is not true and let $s_3 = \inf\{t \in [s_0, s_2 - \delta_1]: X_t \neq \tilde{X}_t^{s_0, X_{s_0}}\}$. Note that the functions $\tilde{X}_t^{s_0, X_{s_0}}$ and $\tilde{X}_t^{s_3, X_{s_3}}$ are identical. Since (s_3, X_{s_3}) lies outside $W(\delta_2)$, an argument similar to the one given above shows that for some $\delta_3, \delta_4 > 0$ and all $m > 1/\delta_2$, the functions $X_t^{1/m}$ must satisfy $X_t^{1/m} = \tilde{X}_t^{s_3, X_{s_3}^{1/m}}$ for $t \in [s_3, s_3 + \delta_3]$, if $|X_{s_3}^{1/m} - X_{s_3}| \leq \delta_4$. If for some $k, m > 1/\delta_2$ we have $|X_{s_3}^{1/m} - X_{s_3}| \leq \delta_4$ and $X_{s_3}^{1/k} \leq X_{s_3}^{1/m}$ (note that we are not assuming that $|X_{s_3}^{1/k} - X_{s_3}| \leq \delta_4$) then $X_t^{1/k} \leq X_t^{1/m}$ for $t \in [s_3, s_3 + \delta_3]$. This is because $X_t^{1/m} = \tilde{X}_t^{s_3, X_{s_3}^{1/m}}$ for $t \in [s_3, s_3 + \delta_3]$, and if $X_v^{1/k} = X_v^{1/m}$ for some v in this interval, we must have $X_t^{1/k} = \tilde{X}_t^{s_3, X_{s_3}^{1/k}}$ for all $t \geq v$ in the interval $[s_3, s_3 + \delta_3]$, recalling the construction of the approximate solutions. Consider an $n > 1/\delta_2$ such that $Y_{s_3}^n \leq X_{s_3} + \delta_4/2$. If $Y_{s_3}^n = X_{s_3}^{1/m}$ for some $m > n$ then we have $X_{s_3}^{1/k} \leq X_{s_3}^{1/m}$ for all $k > n$. This implies that $\tilde{X}_t^{s_3, X_{s_3}^{1/k}} \leq \tilde{X}_t^{s_3, X_{s_3}^{1/m}}$ for all $k > n$ and t . Hence, for $t \in [s_3, s_3 + \delta_3]$,

$$Y_t^n = \sup_{k > n} X_t^{1/k} = \sup_{k > n} \tilde{X}_t^{s_3, X_{s_3}^{1/k}} = \tilde{X}_t^{s_3, X_{s_3}^{1/m}} = \tilde{X}_t^{s_3, Y_{s_3}^n}.$$

Suppose that there is no such m but then, necessarily, $Y_{s_3}^n = \lim_{j \rightarrow \infty} X_{s_3}^{1/m_j}$, for some increasing sequence $\{m_j\}$ with the property that $X_{s_3}^{1/m_j}$ is also increasing. Then, using the monotonicity properties of $\tilde{X}_t^{u_1, z_1}$ discussed at the beginning of the proof and earlier in this paragraph, we obtain for $t \in [s_3, s_3 + \delta_3]$,

$$Y_t^n = \sup_{k > n} X_t^{1/k} = \sup_{k > n} \tilde{X}_t^{s_3, X_{s_3}^{1/k}} = \lim_{j \rightarrow \infty} \tilde{X}_t^{s_3, X_{s_3}^{1/m_j}} = \tilde{X}_t^{s_3, Y_{s_3}^n}.$$

We have shown that $Y_t^n = \tilde{X}_t^{s_3, Y_{s_3}^n}$ for $t \in [s_3, s_3 + \delta_3]$. Recall that X_t is a monotone limit of Y_t^n . In view of the monotonicity and continuity of $z_1 \rightarrow \tilde{X}_t^{u_1, z_1}$, we obtain

$$X_t = \lim_{n \rightarrow \infty} Y_t^n = \lim_{n \rightarrow \infty} \tilde{X}_t^{s_3, Y_{s_3}^n} = \tilde{X}_t^{s_3, X_{s_3}} = \tilde{X}_t^{s_0, X_{s_0}},$$

for $t \in [s_3, s_3 + \delta_3]$. This contradicts the definition of s_3 and proves our claim.

Thus X_t satisfies (2.3) on $(s_1 + \delta_1, s_2 - \delta_1)$ and, in view of the arbitrary nature of δ_1 , the same claim extends to the whole interval (s_1, s_2) . The argument applies to all intervals (s_1, s_2) such that $B_t \neq X_t$ for all $t \in (s_1, s_2)$. This implies that X_t is a Lipschitz solution to (2.3). The proof of the existence of a Lipschitz solution is complete.

The existence of the solution to (2.3) for $t < t_0$ may be proved in a completely analogous way. The two solutions can be combined into one function X_t in an obvious way. It remains to check if the differential equation (2.3) is satisfied at $t = t_0$. It is easy to see that if $B_{t_0} < x_0$ then $dX_t/dt = F_1(X_t)$ for all t in some intervals $(t_0 - \delta, t_0)$ and $(t_0, t_0 + \delta)$ with $\delta > 0$. This and the continuity of X_t at $t = t_0$ evidently imply that $dX_t/dt = F_1(X_t)$ for $t = t_0$ and so (2.3) is satisfied for $t = t_0$. The case when $B_{t_0} > x_0$ is analogous. When $B_{t_0} = x_0$, (2.3) is trivially satisfied by X_t for $t = t_0$.

Since the functions $\{X_t^{1/m}, t \geq t_0\}$ are adapted to the filtration $\mathcal{F}_t^B = \sigma(B_s, s \in [t_0, t])$, so is their lim sup, X_t . If B_t is strong Markov then the process $\{(B_t, X_t), t \geq t_0\}$ is strong Markov with respect to the filtration $\{\mathcal{F}_t^B, t \geq t_0\}$.

We will show that the function $\{X_t, t \geq t_0\}$ constructed above is the largest of all Lipschitz solutions to (2.3); that is, if X_t^* is another Lipschitz solution, then $X_t \geq X_t^*$ for all $t \geq t_0$. Consider any Lipschitz solution X_t^* to (2.3) and suppose that $X_t^* > X_t$ for some $t \geq t_0$. Then there must exist $\delta = 1/m_j$ such that $X_t^* > X_t^\delta$ for some $t \geq t_0$. Fix such δ and let S be the infimum of those t such that $X_t^* > X_t^\delta$. If $S \in [t_j + \delta, t_{j+1})$ for some j , then $X_S^* = X_S^\delta \neq B_S$ a.s., and, by continuity, we must have $X_s^* \neq B_s$ and $X_s^\delta \neq B_s$ for all s in some nondegenerate interval $[S, S + \delta_1)$. On this interval one of the conditions in (2.3) is satisfied by both X_t^* and X_t^δ , so $X_s^* = X_s^\delta = \tilde{X}^{S, X_s^*}$ for all $s \in [S, S + \delta_1)$ or $X_s^* = X_s^\delta = \tilde{X}^{S, X_s^*}$ for all $s \in [S, S + \delta_1)$. This contradicts the definition of S . Next, suppose that $S \in [t_j, t_j + \delta)$ for some j . On this interval, the derivative of X_t^δ is equal to β . It is easy to see that a Lipschitz solution X_t^* to (2.3) cannot grow faster than that on this interval, and so $S \geq t_j + \delta$, a contradiction which completes the proof of our claim.

A similar construction gives a solution $\{X_t, t \leq t_0\}$ to (2.3) which is maximal among all Lipschitz solutions on the interval $(-\infty, t_0]$ with constant β . Note that X_t is measurable with respect to the σ -field $\sigma(B_s, s \in [t, t_0])$ for $t < t_0$.

The maximal solution X_t of (2.3) is consistent in the following sense. Consider a fixed path $\{B_t, t \in \mathbb{R}\}$ and the corresponding maximal solution X_t . Now choose any $s > 0$ and suppose that $X_s = z$. Let $\{X_u^*, u \geq s\}$ be the largest Lipschitz solution with constant β for (2.3) on the interval $[s, \infty)$ with the initial condition $X_s^* = z$ and the path $\{B_t, t \in \mathbb{R}\}$ truncated to $\{B_t, t \geq s\}$. Then it is easy to see that $X_u^* = X_u$ for all $u \geq s$. It follows that for $s \geq 0$, the portion $\{X_t, t \in [s, u]\}$ of the solution to (2.3) may be defined only in terms of X_s and $\{B_t, t \in [s, u]\}$.

In a similar fashion we can construct a minimal solution to (2.3); this minimal solution is also adapted to the filtration of B_t . Uniqueness would follow once we prove the maximal and minimal solutions are equal for all s a.s. \square

PROOF OF THEOREM 2.3. Without loss of generality, we assume that $t_0 = 0$. Let X^+ and X^- be the maximal and minimal solutions to (1.1). By (2.3) the \mathbb{P}^x law of $B_t - X^-(t)$ is mutually absolutely continuous with respect to the \mathbb{P}^x law of B_t , so under \mathbb{P}^x , $B_t - X^-(t)$ has a jointly continuous local time \tilde{L}_t^z such that $\sup_z \tilde{L}_t^z < \infty$, a.s. for each t .

Let $\beta = \max(|\beta_1|, |\beta_2|)$ and

$$U(1) = \inf \left\{ t > 0: \sup_z \tilde{L}_t^z \geq 1/(4\beta) \right\},$$

with the convention that $\inf \emptyset = \infty$. If $t \leq U(1)$ and $a > 0$, then

$$\int_0^t \mathbf{1}_{(B_s - X^-(s) \in [0, a])} ds = \int_0^a \tilde{L}_t^z dz \leq a/(4\beta).$$

Let $a > 0$ and

$$S = \inf \{ t > 0: X^+(t) - X^-(t) \geq a \}.$$

Since both X^+ and X^- satisfy (1.1), if $V = U(1) \wedge S$,

$$\begin{aligned} X^+(V) - X^-(V) &\leq 2\beta \int_0^V \mathbf{1}_{(X^-(u) \leq B_u \leq X^+(u))} du \\ &\leq 2\beta \int_0^V \mathbf{1}_{(0 \leq B_u - X^-(u) \leq X^+(u) - X^-(u))} du \\ &\leq 2\beta \int_0^V \mathbf{1}_{(0 \leq B_u - X^-(u) \leq a)} du \\ &\leq 2a\beta/(4\beta) = a/2. \end{aligned}$$

Since $X^+(V) - X^-(V) = a$ if $U(1) > S$, we must have $V = U(1)$. This is true for all $a > 0$, so $X^+(t) = X^-(t)$ for $t \leq U(1)$.

Now assume that B_t is strong Markov and let $U(j+1) = U(j) + U(1) \circ \theta_{U(j)}$, $j = 1, 2, \dots$, where θ is the shift operator associated with the process B_t . An

induction argument using the strong Markov property at $U(j)$ shows that $X^+(t) = X^-(t)$ for $t \leq U(j+1)$ for $j = 1, 2, \dots$. The continuity of B_t and \tilde{L}_t^z easily implies $U(j) \rightarrow \infty$, a.s., so $X^+(t) = X^-(t)$ for all $t \geq 0$. \square

The proof of Theorem 2.5 will be split into several lemmas.

For the remainder of the section, let $\delta = (1 - \gamma)/4$. Note that $\delta \in (0, 1/4)$ since $\gamma \in (0, 1)$. The constants c_1, c_2, \dots , in the proofs in this section may depend on γ and δ .

LEMMA 2.14. *Let $\alpha \geq 1$, $t \leq 1$, $A > 0$, $C_t = \int_0^t \mathbf{1}_{(0 < B_s < As^\alpha)} ds$. Assume that condition (i) of Theorem 2.5 holds. There exist c_1 and c_2 independent of α and A such that for $\lambda > 0$,*

$$\mathbb{P}(C_t > \lambda) \leq c_1 \exp(-c_2 \lambda \alpha^\delta / (At^{\alpha+2\delta})).$$

PROOF. First let us compute $\mathbb{E}(C_t - C_u | \mathcal{F}_u)$ for $u \in [0, t]$. Let $R = R(\alpha) = \alpha^{1/\alpha}$. Note that $R \geq 1$, $R = \exp(\alpha^{-1} \log \alpha) \leq c_3$, and

$$1 - R^{-1} = 1 - \exp(-\log \alpha / \alpha) \leq \log \alpha / \alpha \leq c_4 \alpha^{-1/2},$$

where c_3 and c_4 do not depend on α as long as $\alpha \geq 1$.

By condition (i) of Theorem 2.5,

$$\begin{aligned} \mathbb{E}(C_t - C_u | \mathcal{F}_u) &= \int_u^t \mathbb{P}(B_s \in (0, As^\alpha) | \mathcal{F}_u) ds \\ &\leq \int_u^t \frac{c_5 As^\alpha}{(s-u)^\gamma} ds. \end{aligned}$$

Let us examine

$$I = \int_u^t \frac{s^\alpha}{(s-u)^\gamma} ds.$$

Suppose first that $u < t/R$. We observe, using the fact that $R \geq 1$,

$$\begin{aligned} (2.9) \quad \int_u^{t/R} \frac{s^\alpha}{(s-u)^\gamma} ds &\leq \left(\frac{t}{R}\right)^\alpha \int_u^{t/R} \frac{ds}{(s-u)^\gamma} \leq \frac{t^\alpha}{\alpha} \int_u^t \frac{ds}{(s-u)^\gamma} \\ &= \frac{t^\alpha}{\alpha} \int_0^{t-u} \frac{ds}{s^\gamma} \leq c_6 \frac{t^\alpha}{\alpha} t^{1-\gamma}. \end{aligned}$$

On the other hand, in view of the inequality $1 - R^{-1} \leq c_4 \alpha^{-1/2}$,

$$\begin{aligned} \int_{t/R}^t \frac{s^\alpha}{(s-u)^\gamma} ds &\leq t^\alpha \int_{t/R}^t \frac{ds}{(s-u)^\gamma} \\ &\leq t^\alpha \int_{t/R}^t \frac{ds}{(s-t/R)^\gamma} \\ &= t^\alpha \int_0^{t(1-1/R)} \frac{ds}{s^\gamma} \\ &= c_7 t^\alpha t^{1-\gamma} (1 - R^{-1})^{1-\gamma} \\ &\leq c_8 t^{\alpha+1-\gamma} / \alpha^{(1-\gamma)/2}. \end{aligned}$$

Recalling that $\alpha \geq 1$ and combining with (2.9),

$$I \leq \frac{c_9 t^{\alpha-1-\gamma}}{\alpha^{(1-\gamma)/2}}.$$

Now suppose $u \geq t/R$. Then

$$\begin{aligned} \int_u^t \frac{s^\alpha}{(s-u)^\gamma} ds &\leq t^\alpha \int_u^t \frac{ds}{(s-u)^\gamma} = t^\alpha \int_0^{t-u} \frac{ds}{s^\gamma} \\ &= c_9 t^\alpha (t-u)^{1-\gamma} \leq c_9 t^\alpha (t-t/R)^{1-\gamma} \\ &= c_9 t^{\alpha+1-\gamma} (1-R^{-1})^{1-\gamma}. \end{aligned}$$

As before, this is less than or equal to $c_{10} t^{\alpha+1-\gamma}/\alpha^{(1-\gamma)/2}$.

Since $\delta = (1-\gamma)/4$, $t \leq 1$ and $\alpha \geq 1$,

$$\mathbb{E}(C_t - C_u \mid \mathcal{F}_u) \leq c_{11} A t^{\alpha+4\delta}/\alpha^{2\delta} \leq c_{11} A t^{\alpha+2\delta}/\alpha^\delta.$$

This says that almost surely the process $\mathbb{E}(C_t \mid \mathcal{F}_u)$ does not exceed C_u by more than $c_{11} A t^{\alpha+2\delta}/\alpha^\delta$ for any $u \leq t$. In particular,

$$\mathbb{E}(C_t - C_T \mid \mathcal{F}_T) \leq c_{11} A t^{\alpha+2\delta}/\alpha^\delta$$

for every stopping time T bounded by t . We apply Theorem I.6.11 of Bass (1995) to deduce that there exist c_{12} and c_{13} such that

$$\mathbb{E} \exp(c_{12} C_t \alpha^\delta / (A t^{\alpha+2\delta})) \leq c_{13}.$$

Our result easily follows from this estimate. \square

LEMMA 2.15. *Given $\xi > 0$, there exist c_1, c_2 such that if $\alpha \geq 1$, $A, B > 0$, $B/A > \xi$, and $\beta = \alpha + \delta$, then*

$$\mathbb{P}(C_t \geq B t^\beta \text{ for some } t \leq 1/2) \leq c_1 \exp(-c_2 B \alpha^\delta / A).$$

PROOF. Let $t_k = 2^{-1-k/\beta}$, $k = 0, 1, \dots$. The process C_t is increasing. So if $C_t \geq B t^\beta$ for some $t \leq 1/2$, then for some $k \geq 1$ we must have $C_{t_{k-1}} \geq B (t_k)^\beta$. Hence

$$\begin{aligned} \mathbb{P}(C_t \geq B t^\beta \text{ for some } t \leq 1/2) &\leq \mathbb{P}(C_{t_{k-1}} \geq B (t_k)^\beta \text{ for some } k \geq 1) \\ (2.10) \qquad \qquad \qquad &\leq \sum_{k=1}^{\infty} \mathbb{P}(C_{t_{k-1}} \geq B (t_k)^\beta). \end{aligned}$$

Using Lemma 2.14, this is bounded by

$$\begin{aligned} &\sum_{k=1}^{\infty} c_3 \exp(-c_4 B (t_k)^\beta \alpha^\delta / (A t_{k-1}^{\alpha+2\delta})) \\ &= \sum_{k=1}^{\infty} c_3 \exp\left(-c_4 \frac{B \alpha^\delta}{A} 2^{-\beta-k-(1-(k-1)/\beta)(\alpha+2\delta)}\right) \\ &= \sum_{k=1}^{\infty} c_3 \exp\left(-c_4 \frac{B \alpha^\delta}{A} 2^{k\delta/\beta+\delta-(\alpha+2\delta)/(\alpha+\delta)}\right). \end{aligned}$$

Since $\alpha \geq 1$ and $\delta \in (0, 1/4)$, the quantity $2^{\delta - (\alpha + 2\delta)/(\alpha + \delta)}$ is bounded below and above by absolute constants, so the last displayed formula admits a bound

$$(2.11) \quad \begin{aligned} & \sum_{k=1}^{\infty} c_3 \exp\left(-c_5 \frac{B\alpha^\delta}{A} 2^{k\delta/\beta}\right) \\ &= c_3 \exp\left(-c_5 \frac{B\alpha^\delta}{A}\right) \sum_{k=1}^{\infty} \exp\left(-c_5 \frac{B}{A} \alpha^\delta (2^{k\delta/\beta} - 1)\right). \end{aligned}$$

The infinite sum in the last expression is bounded by

$$\begin{aligned} \sum_{k=1}^{\infty} \exp\left(-c_5 \frac{B}{A} (2^{k\delta/\beta} - 1)\right) &\leq \sum_{k=1}^{\infty} \exp\left(-\frac{c_5 B k \delta \log 2}{A\beta}\right) \\ &\leq \frac{1}{1 - \exp(-c_5 B \delta \log 2 / (A\beta))} \\ &\leq c_6 A\beta / B. \end{aligned}$$

Combining this with (2.10) and (2.11), we obtain

$$\begin{aligned} & \mathbb{P}\left(C_t \geq Bt^\beta \text{ for some } t \leq \frac{1}{2}\right) \\ & \leq c_3 \exp\left(-c_5 \frac{B\alpha^\delta}{A}\right) c_6 \frac{A\beta}{B} \\ & = c_3 \exp\left(-c_5 \frac{B\alpha^\delta}{A} + \log c_6 + \log\left(\frac{A}{B}\right) + \log(\alpha + \delta)\right) \\ & \leq c_3 \exp\left(-c_5 \frac{B\alpha^\delta}{A} + \log c_6 - \log \xi + \log 2 + \log \alpha\right). \end{aligned}$$

The last expression is less than

$$c_7 \exp\left(-c_8 \frac{B\alpha^\delta}{A}\right)$$

for suitable c_7 and c_8 (depending on ξ and δ) and all $\alpha \geq 1$. \square

Let X_t^+ and X_t^- be the maximal and minimal solutions to (2.3) constructed in the proof of Theorem 2.2. Let $Y_t = X_t^+ - X_t^-$. We will show $Y_t = 0$, a.s. for $t \leq 1/2$.

LEMMA 2.16. *For each s ,*

$$\mathbb{P}(X_s^+ = B_s) = 0 \quad \text{a.s.}$$

and similarly with X_s^+ replaced by X_s^- .

PROOF. We know X_s^+ is a process whose paths are Lipschitz continuous. By assumption (ii) of Theorem 2.5, there exists a probability measure \mathbb{Q} which is equivalent to \mathbb{P} and such that the \mathbb{Q} law of $B_s - X_s^+$ is the same as the \mathbb{P} law of B_s . Then

$$\mathbb{Q}(X_s^+ = B_s) = \mathbb{Q}(B_s - X_s^+ = 0) = \mathbb{P}(B_s = 0).$$

This is equal to zero by (2.4). Since \mathbb{P} and \mathbb{Q} are equivalent, the lemma is proved. \square

LEMMA 2.17. $Y_t = 0$, a.s. if $t \leq 1/2$.

PROOF. The process X_t^+ satisfies

$$X_t^+ = x + \int_0^t [F_1(X_s^+)1_{(X_s^+ > B_s)} + F_2(X_s^+)1_{(X_s^+ < B_s)}] ds$$

and X_t^- satisfies a similar equation. Then, noting Lemma 2.16,

$$\begin{aligned} Y_t &= \int_0^t [F_1(X_s^+) - F_1(X_s^-)]1_{(B_s < X_s^- \leq X_s^+)} ds \\ &\quad + \int_0^t [F_2(X_s^+) - F_2(X_s^-)]1_{(X_s^- \leq X_s^+ < B_s)} ds \\ &\quad + \int_0^t [F_1(X_s^+) - F_2(X_s^-)]1_{(X_s^- < B_s < X_s^+)} ds. \end{aligned}$$

Therefore

$$\begin{aligned} (2.12) \quad Y_t &\leq M \int_0^t (X_s^+ - X_s^-) ds + 2M \int_0^t 1_{(X_s^- < B_s < X_s^+)} ds \\ &= M \int_0^t Y_s ds + 2M \int_0^t 1_{(0 < B_s - X_s^- < X_s^+ - X_s^-)} ds \\ &= M \int_0^t Y_s ds + 2M \int_0^t 1_{(0 < B_s - X_s^- < Y_s)} ds. \end{aligned}$$

Recall that we have assumed that F_j is bounded by M . Hence, the process Y_t is Lipschitz with constant $2M$. Since X_s^- has Lipschitz paths, there exists, by assumption (ii) of Theorem 2.5, a probability measure \mathbb{Q} equivalent to \mathbb{P} such that under \mathbb{Q} , $\{B_s - X_s^-, 0 \leq s \leq 1/2\}$ has the same law as $\{B_s, 0 \leq s \leq 1/2\}$ does under \mathbb{P} . So it suffices to show that for any Lipschitz process Y_s with constant M satisfying

$$(2.13) \quad Y_t \leq M \int_0^t Y_s ds + 2M \int_0^t 1_{(0 < B_s < Y_s)} ds,$$

we have

$$\mathbb{P}(Y_t \neq 0 \text{ for some } t \leq 1/2) = 0.$$

Let

$$D(A, \alpha) = \{Y_s \geq As^\alpha \text{ for some } s \leq 1/2\}.$$

As Y is Lipschitz with $|Y_t| \leq 2Mt$, then $D(3M, 1) = \emptyset$. Let $\varepsilon > 0$ and let $\eta = 1/4$. We will choose $N \geq 1$ and $j_0 \geq 0$ in a moment. Let $A_j = N^j$ if $j \leq j_0$ and $A_j = (1 + \eta)^j N^{j_0}$ for $j > j_0$. Let $\alpha_j = 1 + j\delta$. We want an estimate on the probability of $D(A_{j+1}, \alpha_{j+1}) - D(A_j, \alpha_j)$. If $\omega \notin D(A_j, \alpha_j)$, then $Y_s \leq A_j s^{\alpha_j}$ for all $s \leq 1/2$, and so from (2.13), for $t \leq 1/2$,

$$(2.14) \quad \begin{aligned} Y_t &\leq M \int_0^t A_j s^{\alpha_j} ds + 2M \int_0^t \mathbf{1}_{(0 < B_s < A_j s^{\alpha_j})} ds \\ &= \frac{MA_j t^{\alpha_j+1}}{\alpha_j + 1} + 2M \int_0^t \mathbf{1}_{(0 < B_s < A_j s^{\alpha_j})} ds. \end{aligned}$$

Let $\xi = (1 - \eta)/2M$ and let c_1 and c_2 be constants chosen as in Lemma 2.15 (depending on ξ). Find large j_0 so that

$$(2.15) \quad (1 + j_0\delta)^{\delta/2}/2M \geq 1,$$

$$(2.16) \quad \frac{M}{1 + j_0\delta} \leq \eta(1 + \eta)$$

and

$$(2.17) \quad c_1 \sum_{j=j_0}^{\infty} \exp(-c_2(1 - \eta^2)(1 + j\delta)^{\delta/2}) < \varepsilon/2.$$

Next choose N large so that

$$(2.18) \quad N \geq M/\eta$$

and

$$(2.19) \quad 2j_0 c_1 \exp(-c_2(1 - \eta)N/2M) < \varepsilon/2.$$

For $j \geq j_0$, we have

$$(2.20) \quad \frac{MA_j}{(\alpha_j + 1)} \leq \eta A_{j+1},$$

using (2.16). The same inequality holds for $j < j_0$ in view of (2.18).

In view of (2.14) and (2.20), for ω to be in $D(A_{j+1}, \alpha_{j+1}) - D(A_j, \alpha_j)$, we must have

$$(2.21) \quad \begin{aligned} \int_0^t \mathbf{1}_{(0 < B_s < A_j s^{\alpha_j})} ds &\geq \frac{Y_t}{2M} - \frac{A_j t^{\alpha_j+1}}{2(\alpha_j + 1)} \\ &\geq \frac{A_{j+1} t^{\alpha_{j+1}}}{2M} - \frac{A_j t^{\alpha_j+1}}{2(\alpha_j + 1)} \\ &\geq (1 - \eta) \frac{A_{j+1} t^{\alpha_{j+1}}}{2M} \end{aligned}$$

for some $t < 1/2$. Recall that we set $\xi = (1 - \eta)/2M$ and note that for all j we have $(1 - \eta)A_{j+1}/(2MA_j) \geq \xi$. By Lemma 2.15, the probability that the inequality (2.21) holds is less than or equal to

$$c_1 \exp\left(-c_2 \frac{(1 - \eta)A_{j+1}}{2MA_j} \alpha_j^\delta\right).$$

Using (2.15) and (2.17) for $j \geq j_0$, we obtain

$$\begin{aligned} c_1 \sum_{j=j_0}^{\infty} \exp\left(-c_2 \frac{(1 - \eta)A_{j+1}}{2MA_j} \alpha_j^\delta\right) \\ \leq c_1 \sum_{j=j_0}^{\infty} \exp\left(-c_2 \frac{1 - \eta^2}{2M} (1 + j_0\delta)^{\delta/2} (1 + j\delta)^{\delta/2}\right) < \varepsilon/2. \end{aligned}$$

From (2.19),

$$\begin{aligned} c_1 \sum_{j=0}^{j_0-1} \exp\left(-c_2 \frac{(1 - \eta)A_{j+1}}{2MA_j} \alpha_j^\delta\right) &\leq c_1 \sum_{j=0}^{j_0-1} \exp\left(-c_2 \frac{(1 - \eta)N}{2M}\right) \\ &\leq 2j_0 c_1 \exp\left(-c_2 \frac{(1 - \eta)N}{2M}\right) < \varepsilon/2. \end{aligned}$$

Hence,

$$c_1 \sum_{j=0}^{\infty} \exp\left(-c_2 \frac{(1 - \eta)A_{j+1}}{2MA_j} \alpha_j^\delta\right) \leq \varepsilon,$$

and so

$$\mathbb{P}\left(\bigcup_{j=0}^{\infty} D(A_j, \alpha_j)\right) \leq \varepsilon.$$

If $\omega \notin \bigcup_{j=0}^{\infty} D(A_j, \alpha_j)$, then $Y_t(\omega) \leq A_j t^{\alpha_j} \leq (1 + \eta)^j N^{j_0} (1/2)^{1+j\delta}$ for all $j \geq j_0$ and all $t \leq 1/2$. Since $(1 + \eta)(1/2) < 1$, letting $j \rightarrow \infty$ shows $Y_t(\omega) = 0$. Therefore

$$\mathbb{P}(Y_t \neq 0 \text{ for some } t \leq 1/2) \leq \varepsilon.$$

Since ε is arbitrary, this proves the lemma. \square

PROOF OF THEOREM 2.5. By Lemma 2.17 we have $Y_t = 0$ a.s. for $t \leq 1/2$. If we consider the law of $B_{t+1/2}$ given $\mathcal{F}_{1/2}$, it is not hard to see that assumptions (i) and (ii) of Theorem 2.5 apply to this process as well. So we apply the same argument to $X_{t+1/2}^+$ and $X_{t+1/2}^-$, and we obtain $Y_{t+1/2} = 0$ for $t \leq 1/2$ or $Y_t = 0$ for $t \leq 2(1/2)$. By an induction argument, we then have $Y_t = 0$ for all t , which proves uniqueness. \square

PROOF OF THEOREM 2.6. The existence and strong uniqueness of solutions X_t^ε to (2.6) can be proved in the same way as in Theorem 2.1.

Consider any sequence $\varepsilon_n \downarrow 0$ and with a slight abuse of notation let $X_t^n = X_t^{\varepsilon_n}$. Since all functions $t \rightarrow X_t^n$ are Lipschitz with constant β , we may suppose, passing to a subsequence, if necessary, that X_t^n converge to a function X_t^∞ . In order to finish the proof, it will suffice to show that $X_t^\infty = X_t$. Since (2.5) has a unique solution a.s., it will be enough to show that if ω is not in the null set where uniqueness does not hold, then $X_t^\infty(\omega)$ is a solution to (2.5). The functions X_t^n are Lipschitz with constant β , so the same is true of X_t^∞ . Let A be the set of times t such that $X_t^\infty = B_t$. The complement of the set A consists of a countable number of open intervals. Let $I = (t_1, t_2)$ be one of the intervals in the complement of A . Fix any $t_3 \in I$ and suppose without loss of generality that $X_{t_3}^\infty > B_{t_3}$. Choose some $t_4 \in (t_1, t_3)$ and $t_5 \in (t_3, t_2)$ and let a be the infimum of $X_t^\infty - B_t$ over $t \in (t_4, t_5)$. For sufficiently large n , we have $\varepsilon_n < a/3$ and $|X_t^n - X_t^\infty| < a/3$ for all $t \in (t_4, t_5)$. It follows that for large n and $t \in (t_4, t_5)$, we have $X_t^n - B_t > a/3 > \varepsilon_n$. Hence, for such n and t , $dX_t^n/dt = \beta_2$. This shows that $dX_t^\infty/dt = \beta_2$ for all $t \in I$. The same argument works for all other intervals in the complement of A . There is nothing to check for $t \in A$, so X_t^∞ is a solution to (2.5). \square

3. Local time. In the remaining part of the article we assume that B_t is a Brownian motion. In this section we will exclusively deal with solutions to (1.1). We will find several explicit formulas for the local time spent by B_t on the paths of the process X_t . Moreover, we will prove analogues of the Trotter and Ray–Knight theorems. The results on local times provide information about the behavior of the function $y \rightarrow X_t^y$ for fixed t ; see Remark 3.10.

Our first lemma is supposed to help develop a mental picture of the solutions of (1.1). So far, we have proved existence and uniqueness of a solution to (1.1) for fixed t_0 and x_0 (see Theorem 2.1).

LEMMA 3.1. (i) *With probability 1, for every $t_0, x_0 \in \mathbb{R}$, there exists a unique Lipschitz solution to (1.1). Let X_t^x denote the solution of (1.1) with $X_0^x = x$.*

(ii) *For a fixed t , the function $x \rightarrow X_t^x$ is continuous a.s.*

(iii) *If $x < y$, then $X_t^x < X_t^y$ for all $t \in \mathbb{R}$, a.s.*

PROOF. According to Theorem 2.1 applied with $\alpha_1 = \alpha_2 = 0$, for every fixed t_0 and x_0 there exists a unique Lipschitz solution to (1.1), a.s. Hence, the existence and uniqueness assertion holds for all rational t_0 and x_0 simultaneously, a.s.

Suppose Lipschitz functions X_t and \tilde{X}_t satisfy (1.1), not necessarily for the same values of t_0 and x_0 (we make no assumptions about rationality of the initial conditions t_0 and x_0 in this paragraph). We will show that $\hat{X}_t = \max(X_t, \tilde{X}_t)$ also satisfies the condition in (1.1). Consider any $s \in \mathbb{R}$ and assume without loss of generality that $X_s \geq \tilde{X}_s$. If $X_s = B_s$, then $\hat{X}_s = B_s$ and (1.1) is satisfied in a trivial way. If $X_s \neq B_s$, then the graph of X_t is a line

segment in a neighborhood of (s, X_s) . Since $\widetilde{X}_s \leq X_s$, then either the graph of \widetilde{X}_t lies strictly below that of X_t in a neighborhood of (s, X_s) , or it agrees with the graph of X_t in a neighborhood of the same point. In either case, the graph of \widehat{X}_t is a line segment in a neighborhood of (s, X_s) , with the slope satisfying (1.1).

Recall that X_t^x denotes the solution of (1.1) with $X_0^x = x$. We will show that $X_t^x \leq X_t^y$ for all $t \in \mathbb{R}$, a.s., assuming that $x < y$, and x and y are rational. Indeed, if $X_t^x > X_t^y$ for some t then $\widehat{X}_t^y = \max(X_t^x, X_t^y)$ is a solution to (1.1) with $\widehat{X}_0^y = y$, but this solution is different from X_t^y . This contradicts the uniqueness of solutions for rational t_0 and x_0 .

Next we will prove that if x and y are rational and $x < y$ then $X_t^x < X_t^y$ for all $t \in \mathbb{R}$. By Lemma 2.13, it is enough to consider $t > 0$. We will use the same method as in the proof of Theorem 2.3. Fix some rational x . By the Girsanov theorem, the law of $B_t - X_t^x$ is mutually absolutely continuous with the law of Brownian motion starting from $-x$, so $B_t - X_t^x$ has a jointly continuous local time \widetilde{L}_t^z such that $\sup_z \widetilde{L}_t^z < \infty$, a.s. for each t . Let $\beta = \max(|\beta_1|, |\beta_2|)$ and

$$U = \inf \left\{ t > 0: \sup_z \widetilde{L}_t^z \geq 1/(8\beta) \right\},$$

with the convention that $\inf \emptyset = \infty$. If $t \leq U$ and $a > 0$, then

$$\int_0^t \mathbf{1}_{(B_s - X_s^x \in [0, a])} ds = \int_0^a \widetilde{L}_t^z dz \leq a/(8\beta).$$

Let $a > 0$ be a rational number and $y = x + a$. Let

$$S = S(y) = \inf \left\{ t > 0: X_t^y - X_t^x \leq a/2 \right\}.$$

Let $V = U \wedge S$ and $W = W(y) = \sup \{ t \leq V: X_t^y - X_t^x \geq a \}$. Since both X_t^x and X_t^y satisfy (1.1),

$$\begin{aligned} X_V^y - X_V^x &\geq a - 2\beta \int_W^V \mathbf{1}_{(X_u^x \leq B_u \leq X_u^y)} du \\ &\geq a - 2\beta \int_W^V \mathbf{1}_{(0 \leq B_u - X_u^x \leq X_u^y - X_u^x)} du \\ &\geq a - 2\beta \int_W^V \mathbf{1}_{(0 \leq B_u - X_u^x \leq a)} du \\ &\geq a - 2a\beta/(8\beta) = 3a/4. \end{aligned}$$

Since $X_V^y - X_V^x = a/2$ if $U > S$, we must have $V = U$. This is true for all $a > 0$, so $X_U^y - X_U^x \geq (X_0^y - X_0^x)/2$ for all rational $y > x$.

Let $U(1) = U$ and $U(j+1) = U(j) + U(1) \circ \theta_{U(j)}$, $j = 1, 2, \dots$, where θ is the shift operator associated with the process B_t . An induction argument using the strong Markov property at $U(j)$ shows that $X_{U(j)}^y - X_{U(j)}^x \geq (X_0^y - X_0^x)/2^j$. It is elementary to see that $U(j) \rightarrow \infty$ as $j \rightarrow \infty$, so for a fixed rational x , all solutions X_t^y with rational $y > x$ will stay strictly above X_t^x for all $t > 0$, a.s.

The claim can be extended as usual to all rational t_0 and x_0 ; that is, if X and \tilde{X} are Lipschitz solutions to (1.1) with $X(t_0) = x < y = \tilde{X}(t_0)$ then we have $X(t) < \tilde{X}(t)$ for all $t \in \mathbb{R}$, for all rational t_0 , x and y simultaneously, a.s.

Now consider any real y and define \overline{X}_t^y as the supremum of X_t^x over rational $x < y$. We can use the argument presented in Theorem 2.2 to show that \overline{X}_t^y is a Lipschitz solution to (1.1) with $\overline{X}_0^y = y$. Similarly, let \underline{X}_t^y be a Lipschitz solution obtained as the infimum of X_t^x over rational $x > y$.

Let us show that $\overline{X}_t^y = \underline{X}_t^y$ and that this is the unique Lipschitz solution of (1.1) with $X_0^y = y$ for all real y simultaneously. Suppose that $\overline{X}_s^y < \underline{X}_s^y$ for some real y and s . By the continuity of the two solutions, there are rational t_0 , v and z such that $\overline{X}_{t_0}^y < v < z < \underline{X}_{t_0}^y$. Let \tilde{X}_t and \hat{X}_t be the solutions of (1.1) satisfying $\tilde{X}(t_0) = v$ and $\hat{X}(t_0) = z$. It follows from the definitions of \overline{X}_t^y and \underline{X}_t^y and the monotonicity of $x \rightarrow X_t^x$ for rational x that we must have $\hat{X}(0) = y = \tilde{X}(0)$. This, however, contradicts the fact that solutions with different rational starting points v and z at a rational time t_0 never meet.

Next suppose that for some real y , there is a solution X_t^y to (1.1) with $X_0^y = y$, but different from \underline{X}_t^y . Hence for some s we have $X_s^y \neq \underline{X}_s^y$, and we can assume without loss of generality that $X_s^y > \underline{X}_s^y$. Recalling the definition of \underline{X}_s^y , we can find a rational number x such that $x > y$ and $X_s^x \in (\underline{X}_s^y, X_s^y)$. Then $\tilde{X}_t^x = \max(X_t^x, \underline{X}_s^y)$ is a Lipschitz solution to (1.1) with $\tilde{X}_0^x = x$, different from the function X_t^x , thus contradicting the uniqueness of solutions for rational initial data. This completes the proof that there exists a unique Lipschitz solution to (1.1) for $t_0 = 0$ and all $x_0 \in \mathbb{R}$ simultaneously, a.s. This can be immediately extended to an assertion that applies to all rational t_0 simultaneously, with probability 1. Finally, this last assertion easily implies the claim that existence and uniqueness hold for all real t_0 and x_0 simultaneously.

The continuity of the function $x \rightarrow X_t^x$ follows from the fact that $\overline{X}_t^x = \underline{X}_t^x$ for all x and t .

Finally, we show the strict monotonicity of $x \rightarrow X_t^x$. Suppose that we have $x < y$ and $X_s^x = X_s^y$ for some $s \in \mathbb{R}$. The two functions X_t^x and X_t^y are not identical since $X_0^x = x \neq y = X_0^y$, but they are both solutions to (1.1) with $t_0 = s$ and $x_0 = X_s^x$. This contradicts the uniqueness of solutions to (1.1). \square

In the rest of this section, we will assume that $t_0 = 0$ and study the portion of the solution X_t to (1.1) for $t \geq 0$ only.

We continue with a discussion of some exit systems. Some of our results on exit systems may be of independent interest. We refer the reader to Blumenthal (1992), Burdzy (1987), Maisonneuve (1975) or Sharpe (1989) concerning the fundamentals of excursion theory.

Let $D = \{(b, x) \in \mathbb{R}^2: b = x\}$. We will construct an exit system (H^x, dL) for the process of excursions of (B_t, X_t) from the set D . The first element of an exit system is a family of excursion laws H^x . An excursion law H^x is an infinite σ -finite measure on the space C^* of functions (e_t^1, e_t^2) defined on $(0, \infty)$ (note that 0 is excluded) which take values in $\mathbb{R}^2 \cup \{\Delta\}$. Here Δ is the coffin (absorbing)

state. Let ν be the lifetime of an excursion, that is, $\nu = \inf\{t > 0: (e_t^1, e_t^2) = \Delta\}$. Then H^x -a.e., we have $(e_t^1, e_t^2) \in \mathbb{R}^2$ for $t \in (0, \nu)$ and $(e_t^1, e_t^2) = \Delta$ for $t \in [\nu, \infty)$. The measure H^x is strong Markov with respect to the transition probabilities of the process $\{(B_t, X_t), t \geq 0\}$ killed at the hitting time of D . Moreover, the H^x -measure of the set of paths for which $\lim_{t \downarrow 0} (e_t^1, e_t^2) \neq (x, x)$ is equal to 0. The second element of the exit system, dL , denotes the measure defined by a nondecreasing process L_t . The process L_t is a continuous additive functional, also known as a local time, for (B_t, X_t) on D . The process L_t does not increase on any interval (s, u) such that $(B_t, X_t) \notin D$ for $t \in (s, u)$; that is, $L_s = L_u$ for such intervals. Consider a maximal interval (s, u) such that $B_t \neq X_t$ for $t \in (s, u)$. Suppose $L_s = r$. Let $(e_t^1, e_t^2)_r = (B_{s+t}, X_{s+t})$ for $t \in (0, u - s)$ and $(e_t^1, e_t^2)_r = \Delta$ for $t \geq u - s$. Let $\mu(r) = \inf\{t > 0: L_t = r\}$. The collection of all "excursions" $\{(r, (e_t^1, e_t^2)_r)\}$ is a Poisson point process which, roughly speaking, has random mean measure $(r_2 - r_1) \int_{r_1}^{r_2} H^{\mu(r)}(A) dr$ on the set $(r_1, r_2) \times A$.

Next we apply some transformations to the excursions and excursion laws in order to simplify our description of the exit system. First, we note that by the translation invariance of the Brownian motion B_t and (1.1), the distribution of $(e_t^1 - x, e_t^2 - x)$ under H^x is the same for every $x \in \mathbb{R}$. Let this distribution be called H_1 . For H_1 -almost all excursions, the second component e_t^2 is a linear function of t until the excursion lifetime ν , with the slope equal to β_1 or β_2 . In the first case, $e_t^1 > e_t^2$ for $t \in (0, \nu)$, while the inequality goes the other way in the second case. Let H_{1+} denote the part of the measure H_1 which is supported on excursions with $e_t^1 > e_t^2$ and let H_{1-} be the part supported on the set where $e_t^1 < e_t^2$. Let H_{2+} be the distribution of $\{e_t^1 - e_t^2, t \in (0, \nu)\}$ under H_{1+} and let H_{2-} have the same definition relative to H_{1-} . Note that, by definition, the excursion laws H_{2+} and H_{2-} are supported on paths in $\mathbb{R} \cup \{\Delta\}$ rather than $\mathbb{R}^2 \cup \{\Delta\}$, since the second component becomes irrelevant after our last transformation.

Our transformations preserve the strong Markov property, but the last transformation creates a drift so that the measure H_{2+} has the transition probabilities of Brownian motion with drift $-\beta_1$, killed upon hitting 0. It is standard to show [see, e.g., Theorem 4.1 of Burdzy (1987)] that for any event A defined in terms of the process after some fixed time $s_0 > 0$, we have, up to a multiplicative constant,

$$(3.1) \quad H_{2+}(A) = \lim_{x \downarrow 0} \frac{1}{|x|} Q_{-\beta_1}^x(A),$$

where $Q_{-\beta_1}^x$ stands for the distribution of Brownian motion with drift $-\beta_1$, killed at the hitting time of 0. The normalization of the excursion laws is arbitrary as long as it matches the normalization of the local time, so we can use the normalization in (3.1). We next choose the normalization of the local time so that it matches that of H_{2+} . Given the normalization for H_{2+} , the normalization for H_{2-} is no longer arbitrary and we will have to prove that

$$(3.2) \quad H_{2-}(A) = \lim_{x \uparrow 0} \frac{1}{|x|} Q_{-\beta_2}^x(A).$$

Unless specified otherwise, all excursion laws in this paper will be normalized as in (3.1) or (3.2).

Let H_3 denote the excursion law for excursions of Brownian motion without drift away from 0. Let us split H_3 into positive and negative parts H_{3+} and H_{3-} , as in the case of H_2 . We normalize H_3 using a formula analogous to (3.1). Recall that ν denotes the lifetime of an excursion e , and that (3.1) defines the normalization of H_{2+} .

LEMMA 3.2. (i) *On the set where $\nu < \infty$,*

$$\frac{dH_{2+}}{dH_{3+}}(e) = \exp\left(-\frac{\beta_1^2 \nu}{2\sigma^2}\right).$$

(ii) *For a fixed time $s \in (0, \infty)$, the conditional distributions of H_{2+} and H_{3+} given $\{\nu = s\}$ are identical.*

(iii) *If $\beta_1 < 0$ then $H_{2+}(\nu = \infty) = 2|\beta_1|/\sigma^2$.*

(iv) *Formula (3.2) is the correct normalization for H_{2-} .*

Parts (i)–(iii) of Lemma 3.2 have obvious analogues for H_{2-} .

PROOF. Fix arbitrary $0 < s_0 < s_1 < \infty$ and let A be an event measurable with respect to $\sigma\{e_t, t \in (s_0, s_1)\}$. Since H_{3+} is assumed to be normalized using a formula analogous to (3.1), we have

$$\frac{H_{2+}(A \cap \{\nu = s_1\})}{H_{3+}(A \cap \{\nu = s_1\})} = \lim_{x \downarrow 0} \frac{Q_{-\beta_1}^x(A \cap \{\nu = s_1\})}{Q_0^x(A \cap \{\nu = s_1\})}.$$

An application of Girsanov's theorem, as in Karatzas and Shreve [(1988), (5.11), page 196], shows that

$$\frac{Q_{-\beta_1}^x(A \cap \{\nu = s_1\})}{Q_0^x(A \cap \{\nu = s_1\})} = \exp\left(\frac{x\beta_1}{\sigma^2} - \frac{\beta_1^2 s_1}{2\sigma^2}\right).$$

This and the previous formula imply

$$\frac{H_{2+}(A \cap \{\nu = s_1\})}{H_{3+}(A \cap \{\nu = s_1\})} = \exp\left(-\frac{\beta_1^2 s_1}{2\sigma^2}\right),$$

which then easily implies (i) and (ii).

As for (iii), we start with the formula

$$Q_{-\beta_1}^x(\nu = \infty) = 1 - \exp(2x\beta_1/\sigma^2),$$

with $\beta_1 < 0$ [Karlin and Taylor (1975), page 362]. Then (3.1) yields

$$H_{2+}(\nu = \infty) = \lim_{x \downarrow 0} \frac{1}{|x|} Q_{-\beta_1}^x(\nu = \infty) = \frac{2|\beta_1|}{\sigma^2},$$

as desired.

It remains to prove (iv). Fix arbitrarily small $\gamma > 0$ and let

$$A_1 = A_1(t) = \left\{ \max_{s \leq t} |X_s| > t^{1/2+\gamma} \right\}.$$

Note that $|X_t| \leq \beta t < t^{1/2+\gamma}$ for small $t > 0$ so we have $\mathbb{P}(A_1(t)) = 0$ if t is small. However, we will prove the result using only the property that $\lim_{t \rightarrow 0} \mathbb{P}(A_1(t)) = 0$ because we will need this version of the proof later in the paper. Let us take $t_0 = 0$ and $x_0 = 0$ so that $X_0 = 0$. Note that the excursion law normalization does not depend on t_0 and x_0 . Let $A_+ = A_+(s)$ be the event that the first excursion (e_t^1, e_t^2) of (B_t, X_t) from D with the property that $|e_t^1 - e_t^2| > s^{1/2+\gamma/2}$ for some $t \in (0, \nu)$ also has the property that $e_t^1 > e_t^2$ for $t \in (0, \nu)$. Let A_- be the analogous event with $e_t^1 < e_t^2$. Let $T(a)$ be the hitting time of a by B_t . For small $s > 0$,

$$\{T(s^{1/2+\gamma/2} + s^{1/2+\gamma}) < T(-s^{1/2+\gamma/2} + s^{1/2+\gamma}) < s\} \subset A_+(s) \cup A_1(s)$$

and

$$\{T(-s^{1/2+\gamma/2} - s^{1/2+\gamma}) < T(s^{1/2+\gamma/2} - s^{1/2+\gamma}) < s\} \subset A_-(s) \cup A_1(s).$$

It is elementary to check that

$$\begin{aligned} \lim_{s \rightarrow 0} \mathbb{P}(T(s^{1/2+\gamma/2} + s^{1/2+\gamma}) < T(-s^{1/2+\gamma/2} + s^{1/2+\gamma}) < s) \\ = \lim_{s \rightarrow 0} \mathbb{P}(T(-s^{1/2+\gamma/2} - s^{1/2+\gamma}) < T(s^{1/2+\gamma/2} - s^{1/2+\gamma}) < s) = 1/2. \end{aligned}$$

This and the fact that $\lim_{t \rightarrow 0} \mathbb{P}(A_1(t)) = 0$ imply that

$$(3.3) \quad \lim_{s \rightarrow 0} \mathbb{P}(A_+(s)) = \lim_{s \rightarrow 0} \mathbb{P}(A_-(s)) = 1/2.$$

The scale function $S(y)$ for Brownian motion with drift $-\beta_1$ is given by $S(y) = \exp(2\beta_1 y / \sigma^2)$ [Karlin and Taylor (1981), Chapter 15.4]. Let F_h be the event that the difference between the maximum and the minimum of an excursion exceeds h . Then, by (3.1),

$$\begin{aligned} H_{2+}(F_h) &= \lim_{x \downarrow 0} \frac{1}{x} Q_{-\beta_1}^x(T_h < T_0) = \lim_{x \downarrow 0} \frac{1}{x} \frac{S(x) - S(0)}{S(h) - S(0)} \\ &= \lim_{x \downarrow 0} \frac{1}{x} \frac{\exp(2\beta_1 x / \sigma^2) - 1}{\exp(2\beta_1 h / \sigma^2) - 1} = \frac{2\beta_1}{\sigma^2} \frac{1}{\exp(2\beta_1 h / \sigma^2) - 1}. \end{aligned}$$

An analogous formula holds for $H_{2-}(F_h)$, but we will write it with an additional multiplicative constant c_1 , since we have not proved that (3.2) is the right normalization yet:

$$H_{2-}(F_h) = c_1 \frac{2\beta_2}{\sigma^2} \frac{1}{\exp(2\beta_2 h / \sigma^2) - 1}.$$

Our goal is to show that $c_1 = 1$ is the correct choice for the constant.

Excursion theory tells us that the arrival times for excursions (e_t^1, e_t^2) of (B_t, X_t) from D with the property that $|e_t^1 - e_t^2| > s^{1/2+\gamma/2}$ for some $t \in (0, \nu)$, and with $e_t^1 > e_t^2$ for $t \in (0, \nu)$, form a Poisson point process on the local time scale with intensity $H_{2+}(F_{s^{1/2+\gamma/2}})$. This process is independent from the analogous process of excursions with $e_t^1 < e_t^2$. Formula (3.3) tells us that for

small s , the probability that the first arrival for the first process is earlier than the first arrival for the second process is close to $1/2$. Hence, the ratio of the intensities for the two Poisson point processes must converge to 1 as $s \rightarrow 0$. Therefore, we must have

$$\lim_{s \rightarrow 0} \frac{2\beta_1}{\sigma^2} \frac{1}{\exp(2\beta_1 s^{1/2+\gamma/2}/\sigma^2) - 1} \left(c_1 \frac{2\beta_2}{\sigma^2} \frac{1}{\exp(2\beta_2 s^{1/2+\gamma/2}/\sigma^2) - 1} \right)^{-1} = 1.$$

However, this is possible only if $c_1 = 1$. This completes the proof of (iv). \square

REMARK 3.3. (i) Lemma 3.2(iv) can be used to prove uniqueness for (1.1). In order to do so, one would have to consider an exit system for the maximal Lipschitz solution X_t to (1.1), constructed as in the proof of Theorem 2.2, and the analogous exit system for the minimal Lipschitz solution. Lemma 3.2(iv) shows that both exit systems are identical but this can be true only if the maximal and minimal solutions are the same. We will not formalize this argument, as it cannot be easily generalized to non-Markov processes. The delicate part of the argument would be to show that the maximal solution X_t is the sum of the excursions and it contains no component corresponding to a “push” proportional to local time.

(ii) According to Lemma 3.2(i)–(iii), if $\beta_1 < 0$, the excursion laws $H_{2+}^{\beta_1}$ and $H_{2+}^{-\beta_1}$ agree on the set of excursions with finite lifetime and the only difference is that $H_{2+}^{\beta_1}$ gives some mass to excursions with infinite lifetime, while $H_{2+}^{-\beta_1}$ does not.

For every $x \in \mathbb{R}$, consider the solution X_t^x to (1.1) with $X_0^x = x$. Let L_t^x denote the local time of $Y_t^x =_{\text{def}} B_t - X_t^x$ at 0, defined earlier in this section as the local time of (B_t, X_t^x) on the diagonal, accumulated between times 0 and t . Note that this is not the local time of a one-dimensional diffusion at level x .

PROPOSITION 3.4. *If $\beta_1 > 0$ and $\beta_2 < 0$, then for every $x \in \mathbb{R}$,*

$$\lim_{t \rightarrow \infty} \frac{L_t^x}{t} = \left(\frac{1}{|\beta_1|} + \frac{1}{|\beta_2|} \right)^{-1} \quad a.s.$$

PROOF. Fix some $x \in \mathbb{R}$. Our assumptions that $\beta_1 > 0$ and $\beta_2 < 0$ imply that there will never be an excursion of B_t from X_t^x with infinite lifetime, since the drift will always push the excursions of Y_t^x towards 0. This in turn implies that L_t^x will grow to infinity a.s.

Recall that we used $Q_{-\beta_1}^y$ to denote the distribution of Brownian motion with drift $-\beta_1$, killed at the hitting time of 0. By Theorem 7.5.3 of Karlin and Taylor (1975), we have for $y > 0$,

$$Q_{-\beta_1}^y(\nu \in dt) = \frac{|y|}{\sigma t^{3/2} \sqrt{2\pi}} \exp\left(-\frac{(|y| - |\beta_1|t)^2}{2\sigma^2 t}\right) dt,$$

where ν denotes the lifetime of the process. The same formula holds for $y < 0$, with β_1 replaced by β_2 . Using (3.1) and (3.2), we obtain

$$\begin{aligned}
H_2(\nu \in dt) &= \lim_{y \downarrow 0} \frac{1}{|y|} \mathbf{Q}_{-\beta_1}^y(\nu \in dt) + \lim_{y \uparrow 0} \frac{1}{|y|} \mathbf{Q}_{-\beta_2}^y(\nu \in dt) \\
&= \lim_{y \downarrow 0} \frac{1}{\sigma t^{3/2} \sqrt{2\pi}} \exp\left(-\frac{(|y| - |\beta_1|t)^2}{2\sigma^2 t}\right) dt \\
&\quad + \lim_{y \uparrow 0} \frac{1}{\sigma t^{3/2} \sqrt{2\pi}} \exp\left(-\frac{(|y| - |\beta_2|t)^2}{2\sigma^2 t}\right) dt \\
&= \frac{1}{\sigma t^{3/2} \sqrt{2\pi}} \exp\left[-\left(\frac{\beta_1^2}{2\sigma^2}\right)t\right] dt \\
&\quad + \frac{1}{\sigma t^{3/2} \sqrt{2\pi}} \exp\left[-\left(\frac{\beta_2^2}{2\sigma^2}\right)t\right] dt.
\end{aligned}$$

Let V_s^x be the inverse local time, that is, $V_s^x = \inf\{t > 0: L_t^x > s\}$. The process V_s^x is the sum of lifetimes of excursions which start before the local time reaches the level s . The Poisson character of the excursion process easily implies that

$$\begin{aligned}
\mathbb{E}V_s^x &= s \int_0^\infty t H_2(\nu \in dt) \\
&= s \int_0^\infty t \frac{1}{\sigma t^{3/2} \sqrt{2\pi}} \exp[-(\beta_1^2/2\sigma^2)t] dt \\
(3.4) \quad &\quad + s \int_0^\infty t \frac{1}{\sigma t^{3/2} \sqrt{2\pi}} \exp[-(\beta_2^2/2\sigma^2)t] dt \\
&= \left(\frac{1}{|\beta_1|} + \frac{1}{|\beta_2|}\right)s.
\end{aligned}$$

This and the strong law of large numbers for the Lévy process $s \rightarrow V_s^x$ [see page 92 of Bertoin (1996)] imply that

$$\frac{V_s^x}{s} \rightarrow \frac{1}{|\beta_1|} + \frac{1}{|\beta_2|},$$

a.s., as $s \rightarrow \infty$. This can be easily translated to the statement of the proposition. \square

We note that if $\beta_1, \beta_2 > 0$, then we will eventually have $X_t^x > B_t$, for every x . Hence, in this case, $L_\infty^x < \infty$ for every $x \in \mathbb{R}$, a.s. We will prove the next lemma under the assumption that $\beta_1 - \beta_2 > 0$. We believe that similar statements hold when $\beta_1 - \beta_2 < 0$, but technical difficulties prevent us from giving a formal proof in that case.

The following lemma contains the most complicated and technical argument in the whole article.

LEMMA 3.5. (i) Fix $x, a, \beta_1, \beta_2 \geq 0$ and assume that $\beta_1 - \beta_2 > 0$. Then

$$\mathbb{E}(L_\infty^{x+\delta} \mid L_\infty^x = a) = a - \delta \frac{\beta_1}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2}\right) \right) + o(\delta),$$

for $\delta \downarrow 0$.

(ii) If $a, \beta_1, \beta_2 \geq 0, x \leq 0$ and $\beta_1 - \beta_2 > 0$, then

$$\mathbb{E}(L_\infty^{x+\delta} \mid L_\infty^x = a) = a + \delta \left[\frac{\beta_2}{\beta_1 - \beta_2} - \frac{\beta_1}{\beta_1 - \beta_2} \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2}\right) \right] + o(\delta),$$

for $\delta \uparrow 0$.

PROOF. (i) Recall that $B_0 = 0$, that we have fixed $x, a, \beta_1, \beta_2 \geq 0$ and assumed that $\beta_1 - \beta_2 > 0$. Since the proof of the lemma is quite long, we will split it into several steps.

Step 1. We start with some transformations of the processes X_t^x and B_t , which will enable us to look at L_∞^x from a slightly different perspective. It is perhaps not necessary to make these transformations, but we find the transformed problem much easier to comprehend than the original one from an intuitive point of view.

We first offer a rough guide to our notation (whose validity is limited to this proof). Different Brownian motions with different drifts and reflected barriers will be denoted B_t^j , for $j = 1, 2, \dots$. The notation L_t^{j-} and L_t^{j+} will refer to the local time of B_t^j on the lower and upper reflected barriers (if any). We will write $v_j(a) = \inf\{t: L_t^{j-} = a\}$.

It is well known that the set $\{t: B_t = 0\}$ has zero Lebesgue measure, so the Girsanov theorem implies that the same is true of the set $\{t: X_t^x = B_t\}$. We will excise all intervals where $X_t^x > B_t$. First, we define a clock $\mathcal{C}_1(t) = \int_0^t \mathbf{1}_{\{X_s^x \leq B_s\}} ds$ and its inverse $b_1(t) = \inf\{s: \mathcal{C}_1(s) \geq t\}$. Since $\beta_1, \beta_2 > 0$, we will eventually have $X_t^x > B_t$, so we let $u_1 = \sup\{\mathcal{C}_1(t): t \geq 0\}$. Then we define new processes on the random interval $[0, u_1]$ by

$$\begin{aligned} B_t^1 &= B_{b_1(t)} - \beta_2(b_1(t) - t), \\ X_t^{1,x} &= X_{b_1(t)}^x - \beta_2(b_1(t) - t), \\ X_t^{1,x+\delta} &= X_{b_1(t)}^{x+\delta} - \beta_2(b_1(t) - t). \end{aligned}$$

For $t \in [0, u_1]$, we have $X_t^{1,x} = x + \beta_1 t$, the process B_t^1 is a Brownian motion staying above and reflected on the line $t \rightarrow x + \beta_1 t$, and the process $X_t^{1,x+\delta}$ is a solution to (1.1) with B_t replaced by B_t^1 .

Next we similarly excise the intervals where $X_t^{1,x+\delta} < B_t^1$. Let us define a new clock $\mathcal{C}_2(t) = \int_0^t \mathbf{1}_{\{X_s^{1,x+\delta} \geq B_s^1\}} ds$, its inverse $b_2(t) = \inf\{s: \mathcal{C}_2(s) > t\}$, a

random time $u_2 = \sup\{\mathcal{L}_2(t) : t \geq 0\}$, and processes

$$\begin{aligned} B_t^2 &= B_{b_2(t)}^1 - \beta_1(b_2(t) - t), \\ X_t^{2,x} &= X_{b_2(t)}^{1,x} - \beta_1(b_2(t) - t), \\ X_t^{2,x+\delta} &= X_{b_2(t)}^{1,x+\delta} - \beta_1(b_2(t) - t). \end{aligned}$$

For $t \in [0, u_2]$, we have $X_t^{2,x} = x + \beta_1 t$ and $X_t^{2,x+\delta} = x + \delta + \beta_2 t$. The process $\{B_t^2, t \in [0, u_2]\}$ is a Brownian motion reflected on the lines $t \rightarrow x + \beta_1 t$ and $t \rightarrow x + \delta + \beta_2 t$ and confined to the region between them. Note that $B_0^2 = x$ a.s. and that the lines $t \rightarrow x + \beta_1 t$ and $t \rightarrow x + \delta + \beta_2 t$ intersect at $t = \delta/(\beta_1 - \beta_2)$, so necessarily $u_2 \leq \delta/(\beta_1 - \beta_2)$.

The time u_2 corresponds to the start of the infinite excursion of B_t below the graph of X_t^x . By excursion theory and Lemma 3.2(iii), the distribution of $L_{u_2}^x$ is exponential with mean $\sigma^2/(2\beta_2)$. Hence, we may assume that the process B_t^2 is generated in the following way. Suppose that B_t^3 is a Brownian motion starting from $B_0^3 = x$, reflected on the lines $t \rightarrow x + \beta_1 t$ and $t \rightarrow x + \delta + \beta_2 t$ and confined to the region between them, but defined for all $t \in [0, \delta/(\beta_1 - \beta_2))$ rather than confined to some random time interval. Let L_t^{3-} be the local time of B_t^3 on the line $t \rightarrow x + \beta_1 t$ and let Z be an exponential random variable with mean $\sigma^2/(2\beta_2)$, independent of B_t^3 . If $v_3(s) = \inf\{t : L_t^{3-} = s\}$, then the distributions of the processes $\{B_t^2, t \in [0, u_2]\}$ and $\{B_t^3, t \in [0, v_3(Z)]\}$ are the same.

Let L_t^{3+} be the local time of B_t^3 accumulated on the line $t \rightarrow x + \delta + \beta_2 t$. The distribution of $L_\infty^{x+\delta}$ given $\{L_\infty^x = a\}$ is the same as the distribution of $L_{v_3(a)}^{3+}$, so we will try to find an approximate formula for $\mathbb{E}L_{v_3(a)}^{3+}$.

We continue our transformations. Let $B_t^4 = B_t^3 - x - \beta_1 t$. The process B_t^4 is a Brownian motion starting from 0, with drift $-\beta_1$, reflected on the horizontal axis and the line $t \rightarrow \delta - (\beta_1 - \beta_2)t$. The processes L_t^{3-} and L_t^{3+} can be identified with the local times L_t^{4-} and L_t^{4+} of B_t^4 on the horizontal axis and the line $t \rightarrow \delta - (\beta_1 - \beta_2)t$, respectively. Hence, it will suffice to show that the estimate given in part (i) of the lemma holds for $\mathbb{E}L_{v_4(a)}^{4+}$.

Step 2. In this step we will obtain some estimates for reflected Brownian motions using excursion theory. Let B_t^5 be a Brownian motion with drift $-\beta_1$, confined to positive values by reflection on the horizontal axis. The Green function $G(z, y)$ for Brownian motion with drift $-\beta_1$, killed upon hitting 0 is given by

$$G(z, y) = \frac{1}{\beta_1} \left[\exp\left(\frac{2\beta_1 z}{\sigma^2}\right) - 1 \right] \exp\left(-\frac{2\beta_1 y}{\sigma^2}\right),$$

for $0 < z < y < \infty$, by (3.15) in Section 15.3 and Section 15.4.B of Karlin and Taylor (1981). Let $G_H^5(y)$ denote the Green function for the excursion law H_5 of B_t^5 from 0, that is, the function defined by

$$H_5\left(\int_0^\infty \mathbf{1}_{\{e(t) \in [z_1, z_2]\}} dt\right) = \int_{z_1}^{z_2} G_H^5(y) dy.$$

A formula analogous to (3.1) yields

$$G_H^5(y) = \lim_{z \downarrow 0} \frac{1}{z} G(z, y) = \frac{2}{\sigma^2} \exp\left(-\frac{2\beta_1 y}{\sigma^2}\right),$$

for $y > 0$.

Consider some $\delta_1 > 0$ and excise excursions of B_t^5 above the level δ_1 , just as we did with the excursions of B_t and B_t^1 . Let $\mathcal{C}_3(t) = \int_0^t \mathbf{1}_{\{B_s^5 \leq \delta_1\}} ds$, $b_3(t) = \inf\{s: \mathcal{C}_3(s) > t\}$ and $B_t^6 = B_{b_3(t)}^5$. The process B_t^6 is a reflected Brownian motion in $[0, \delta_1]$. Let $G_H^6(y)$ be the Green function for the excursion law H_6 of B_t^6 from 0. It is clear from the nature of the transformation which generates B_t^6 from the paths of B_t^5 that $G_H^6(y) = G_H^5(y)$ for $y \in (0, \delta_1)$. Hence,

$$H_6(\nu) = \int_0^{\delta_1} G_H^6(y) dy = \int_0^{\delta_1} \frac{2}{\sigma^2} \exp\left(-\frac{2\beta_1 y}{\sigma^2}\right) dy = \frac{1}{\beta_1} \left[1 - \exp\left(-\frac{2\beta_1 \delta_1}{\sigma^2}\right)\right].$$

Let L_t^{6-} and L_t^{6+} denote the local time of B_t^6 at 0 and δ_1 , respectively. Let $v_6(s) = \inf\{t: L_t^{6-} = s\}$. The random variable $v_6(s)$ is the sum of the lifetimes of excursions of B_t^6 from 0 which occur before L_t^{6-} reaches the level s . The last formula and excursion theory give

$$(3.5) \quad \mathbb{E}v_6(s) = s \frac{1}{\beta_1} \left[1 - \exp\left(-\frac{2\beta_1 \delta_1}{\sigma^2}\right)\right] =_{\text{df}} s\eta(\delta_1).$$

Next we will derive an estimate for $H_6(\nu > t)$. Recall that $Q_{-\beta_1}^z$ denotes the distribution of Brownian motion with drift $-\beta_1$, killed upon hitting 0. Let $\widehat{Q}_{-\beta_1}^z$ denote the distribution of Brownian motion starting from $z \in (0, \delta_1)$, with drift $-\beta_1$, reflected at δ_1 , and killed upon hitting 0. It is easy to see that

$$\widehat{Q}_{-\beta_1}^z(\nu > t) \leq Q_{-\beta_1}^z(\nu > t),$$

for all $t > 0$ and $z \in (0, \delta_1)$. By Lemma 3.2(i), Theorem 5.1(iii) of Burdzy (1987) and scaling,

$$(3.6) \quad \begin{aligned} H_6(\nu > t) &\leq \lim_{z \downarrow 0} \frac{1}{z} \widehat{Q}_{-\beta_1}^z(\nu > t) \\ &\leq \lim_{z \downarrow 0} \frac{1}{z} Q_{-\beta_1}^z(\nu > t) = H_5(\nu > t) \leq \int_t^\infty \frac{1}{\sigma} (2\pi s^3)^{-1/2} ds. \end{aligned}$$

A simple argument based on scaling and the Markov property applied at times $t = k\delta_1^2$, $k = 1, 2, \dots$, shows that there exists a constant $c_1 > 0$, such that

$$(3.7) \quad \widehat{Q}_{-\beta_1}^z(\nu > t) \leq \exp(-c_1 t \sigma^2 / \delta_1^2),$$

for all $t > \delta_1^2 / \sigma^2$ and $z \in (0, \delta_1)$. Another standard estimate is

$$\widehat{Q}_{-\beta_1}^z(\nu \geq \delta_1^2 / \sigma^2) \leq z c_2.$$

This combined with the previous estimate gives (with possibly new values for the constants),

$$\widehat{Q}_{-\beta_1}^z(\nu > t) \leq z c_2 \exp(-c_1 t \sigma^2 / \delta_1^2),$$

for all $t > \delta_1^2/\sigma^2$ and $z \in (0, \delta_1)$. We obtain from this an estimate analogous to (3.6) but applicable for $t > \delta_1^2/\sigma^2$,

$$(3.8) \quad H_6(\nu > t) \leq c_2 \exp(-c_1 t \sigma^2 / \delta_1^2).$$

Since the excursion process is a Poisson point process, we have from (3.6) and (3.8),

$$(3.9) \quad \begin{aligned} \text{Var } v_6(s) &= s \int_0^\infty t^2 H_6(\nu \in dt) \\ &\leq s \int_0^{\delta_1^2/\sigma^2} \frac{1}{\sigma} t^2 (2\pi t^3)^{-1/2} dt + s \int_{\delta_1^2/\sigma^2}^\infty \frac{1}{\sigma} \left(\frac{\delta_1^2}{\sigma^2}\right)^2 (2\pi t^3)^{-1/2} dt \\ &\quad + s \int_{\delta_1^2/\sigma^2}^\infty t^2 c_2 \frac{\delta_1^2}{c_1 \sigma^2} \exp\left(-c_1 \frac{t \sigma^2}{\delta_1^2}\right) dt \\ &\leq c_3 \frac{s \delta_1^3}{\sigma^4} + c_4 \frac{s \delta_1^3}{\sigma^4} + c_5 \frac{s \delta_1^8}{\sigma^8} \leq c_6 s \delta_1^3. \end{aligned}$$

Step 3. We will find a link between processes reflected on sloped lines (in space–time) and within an interval. We will need to define some more variables. First, $s_0 > 0$ should be considered a small constant whose value will be chosen later in the proof and which does not change with δ . Recall η defined in (3.5). Let $u_0 > 0$ and $\delta_1 \in (0, \delta)$ be defined by the following two equations, $u_0 = (\delta - \delta_1)/(\beta_1 - \beta_2)$ and $s_0 = u_0/\eta(\beta_1, \delta, \sigma^2)$.

Recall that B_t^6 is a reflected Brownian motion in $[0, \delta_1]$ and note that now δ_1 is defined relative to δ . Let B_t^7 be the analogous reflected Brownian motion in $[0, \delta]$.

Note that $\delta - (\beta_1 - \beta_2)t > \delta_1$ for $t \in (0, u_0)$. Hence, on the interval $(0, u_0)$, the upper reflecting boundary for B_t^6 lies below that for B_t^4 . This relationship between the upper reflecting boundaries implies that the excursion measure distribution of the lifetime of an excursion from 0 of the process B_t^4 is stochastically larger than that for an excursion of B_t^6 , for excursions within the interval $(0, u_0)$. It follows that one can construct B_t^4 and B_t^6 on a common probability space so that $v_6(s) \wedge t \leq v_4(s) \wedge t$ for all $t \leq u_0$, where $v_4(s) = \inf\{t: L_t^{4-} = s\}$. On the other hand, $\delta - (\beta_1 - \beta_2)t < \delta$ for $t > 0$, so the analogous relationship for B^7 goes in the opposite way, that is, $v_7(s) \wedge t \geq v_4(s) \wedge t$ for all $t \geq 0$.

Although the process B_t^4 starts from 0, by construction, it will be necessary to consider the case when it starts from some other value; in other words, we will now consider a process with the same transition probabilities but a different starting point. The starting point y will be reflected in the notation by writing \mathbb{P}^y or \mathbb{E}^y , as usual.

Let T_0^i be the hitting time of 0 for the process B_t^i for $i = 4, 6, 7$. By the previous remarks, we can construct versions of B_t^4 and B_t^7 on the same probability space so that they start from the same point y and $T_0^4 \wedge t \leq T_0^7 \wedge t$ for $t \leq u_0$.

By the strong Markov property applied at T_0^7 , we have $\mathbb{E}^y v_7(s) = \mathbb{E}^y T_0^7 + \mathbb{E}^0 v_7(s)$. It follows easily from (3.7), applied to δ rather than δ_1 , that

$$(3.10) \quad \mathbb{E}^y T_0^7 \leq c_7 \delta^2 / \sigma^2.$$

Consider arbitrarily small $\varepsilon \in (0, 1/4)$. We obtain, using (3.5) (applied with δ_1 replaced by δ) and (3.10),

$$\mathbb{E}^y(v_4(s_0)) \leq \mathbb{E}^y(v_7(s_0)) \leq \mathbb{E}^y T_0^7 + \mathbb{E}^0(v_7(s_0)) \leq c_7 \delta^2 / \sigma^2 + u_0.$$

For small $\delta > 0$, (3.5) shows that $\eta(\delta)$ is approximately $2\delta/\sigma^2$. Hence, $u_0 = s_0 \eta(\delta)$ is approximately equal to $2s_0 \delta / \sigma^2$. This shows that for small δ , the last displayed inequality yields

$$(3.11) \quad \mathbb{E}^y(v_4(s_0)) \leq u_0(1 + \varepsilon) = s_0 \eta(\delta)(1 + \varepsilon) \leq 2s_0 \delta (1 + \varepsilon)^2 / \sigma^2.$$

Next we will find a lower bound for the same quantity.

By the strong Markov property applied at T_0^6 , we have $\mathbb{E}^y v_6(s) = \mathbb{E}^y T_0^6 + \mathbb{E}^0 v_6(s)$ and $\text{Var}(v_6(s) | B_0^6 = y) = \text{Var}(T_0^6 | B_0^6 = y) + \text{Var}(v_6(s) | B_0^6 = 0)$. We have an estimate analogous to (3.10),

$$(3.12) \quad \mathbb{E}^y T_0^6 \leq c_7 \delta_1^2 / \sigma^2$$

and another estimate following from (3.7),

$$(3.13) \quad \text{Var}(T_0^6 | B_0^6 = y) \leq c_8 \delta_1^4 / \sigma^4,$$

for any $y \in [0, \delta_1]$.

We obtain, using (3.5) and (3.12),

$$\mathbb{E}^y(v_6(s_0(1 - \varepsilon))) \leq \mathbb{E}^y T_0^6 + \mathbb{E}^0(v_6(s_0(1 - \varepsilon))) \leq c_7 \delta_1^2 / \sigma^2 + s_0(1 - \varepsilon)\eta(\delta_1).$$

For small $\delta > 0$, δ_1 is also small and (3.5) shows that $\eta(\delta_1)$ is about $2\delta_1/\sigma^2$. Hence, $s_0(1 - \varepsilon)\eta(\delta_1)$ is approximately equal to $2s_0(1 - \varepsilon)\delta_1/\sigma^2$. This shows that for small δ , the last displayed inequality yields

$$(3.14) \quad \begin{aligned} \mathbb{E}^y(v_6(s_0(1 - \varepsilon))) &\leq s_0(1 - \varepsilon/2)\eta(\delta_1) \\ &\leq s_0(1 - \varepsilon/2)\eta(\delta) = u_0(1 - \varepsilon/2). \end{aligned}$$

A similar estimate for the variance follows from (3.9) and (3.13), for small δ ,

$$(3.15) \quad \begin{aligned} \text{Var}(v_6(s_0(1 - \varepsilon)) | B_0^6 = y) &\leq c_8 \delta_1^4 / \sigma^4 + c_6 s_0(1 - \varepsilon) \delta_1^3 / \sigma^4 \\ &\leq c_9 s_0(1 - \varepsilon) \delta_1^3 / \sigma^4. \end{aligned}$$

This estimate, (3.14) and the Chebyshev inequality yield,

$$\mathbb{P}^y(v_6(s_0(1 - \varepsilon)) \geq u_0) \leq \frac{c_9 s_0(1 - \varepsilon) \delta_1^3 / \sigma^4}{(\varepsilon u_0 / 2)^2} \leq \frac{c_{10} \delta_1^3}{\eta(\delta) \varepsilon^2 u_0 \sigma^4} = \frac{c_{10} \delta_1^3}{\eta^2(\delta) \varepsilon^2 s_0 \sigma^4}.$$

For small δ we have

$$(3.16) \quad \delta/\sigma^2 < \eta(\delta) < 4\delta/\sigma^2.$$

Hence,

$$(3.17) \quad \mathbb{P}^y(v_6(s_0(1-\varepsilon)) \geq u_0) \leq \frac{c_{11}\delta_1^3}{(\delta/\sigma^2)^2\varepsilon^2s_0\sigma^4} \leq \frac{c_{11}\delta_1}{\varepsilon^2s_0}.$$

We have from (3.14)–(3.17), for small δ ,

$$\begin{aligned} \mathbb{E}^y(v_6(s_0(1-\varepsilon)))^2 &= \text{Var}(v_6(s_0(1-\varepsilon)) | B_0^6 = y) + (\mathbb{E}^y v_6(s_0(1-\varepsilon)))^2 \\ &\leq c_9s_0(1-\varepsilon)\delta_1^3/\sigma^4 + (s_0(1-\varepsilon/2)\eta(\delta))^2 \\ &\leq 32s_0^2(1-\varepsilon/2)^2\delta^2/\sigma^4. \end{aligned}$$

We use this estimate, (3.5) and (3.16)–(3.17) to obtain, for sufficiently small δ ,

$$\begin{aligned} \mathbb{E}^y(v_4(s_0)) &\geq \mathbb{E}^y(v_4(s_0) \wedge u_0) \\ &\geq \mathbb{E}^y(v_6(s_0) \wedge u_0) \\ &\geq \mathbb{E}^y(v_6(s_0(1-\varepsilon)) \wedge u_0) \\ &\geq \mathbb{E}^y(v_6(s_0(1-\varepsilon))) - \mathbb{E}^y[v_6(s_0(1-\varepsilon))\mathbf{1}_{\{v_6(s_0(1-\varepsilon)) \geq u_0\}}] \\ (3.18) \quad &\geq s_0(1-\varepsilon)\eta(\delta_1) - (\mathbb{E}^y(v_6(s_0(1-\varepsilon)))^2\mathbb{E}^y(\mathbf{1}_{\{v_6(s_0(1-\varepsilon)) \geq u_0\}})^2)^{1/2} \\ &= s_0(1-\varepsilon)\eta(\delta_1) - (\mathbb{E}^y(v_6(s_0(1-\varepsilon)))^2\mathbb{P}^y(v_6(s_0(1-\varepsilon)) \geq u_0))^{1/2} \\ &\geq 2s_0(1-\varepsilon)^2\delta_1/\sigma^2 - \left(32s_0^2(1-\varepsilon/2)^2\delta^2/\sigma^4 \cdot \frac{c_{11}\delta_1}{\varepsilon^2s_0}\right)^{1/2}. \end{aligned}$$

It follows from (3.16) and the definition of δ_1 and u_0 that for small $\delta > 0$,

$$\delta_1 = \delta - s_0\eta(\delta)(\beta_1 - \beta_2) \geq \delta(1 - 4s_0(\beta_1 - \beta_2)/\sigma^2).$$

We will choose sufficiently small $s_0 > 0$ (relative to σ, β_1, β_2 and ε) so that $\delta_1 > \delta(1 - \varepsilon/2)$. Then the last inequality and (3.18) yield for small δ ,

$$(3.19) \quad \mathbb{E}^y(v_4(s_0)) \geq 2s_0(1-\varepsilon)^3\delta/\sigma^2.$$

Step 4. We will apply induction in order to obtain estimates for $\mathbb{E}^0 v_4(js_0)$ with integer $j \geq 1$. At the time $v_4(s_0)$, the distance between the reflecting barriers for B_t^4 is equal to $\tilde{\delta} = \delta - (\beta_1 - \beta_2)v_4(s_0)$, which is less than δ , so we can use the estimates (3.11) and (3.19) with δ replaced by $\tilde{\delta}$, assuming that δ itself is sufficiently small for the estimates to hold. By the strong Markov property,

$$\begin{aligned} \mathbb{E}^0(v_4(2s_0) - v_4(s_0) | v_4(s_0)) &\leq 2\tilde{\delta}s_0(1+\varepsilon)^2/\sigma^2 \\ &= 2[\delta - (\beta_1 - \beta_2)v_4(s_0)]s_0(1+\varepsilon)^2/\sigma^2, \end{aligned}$$

and so

$$\begin{aligned}\mathbb{E}^0(v_4(2s_0) - v_4(s_0)) &\leq \mathbb{E}^0 2[\delta - (\beta_1 - \beta_2)v_4(s_0)]s_0(1 + \varepsilon)^2/\sigma^2 \\ &\leq 2[\delta - (\beta_1 - \beta_2)2\delta s_0(1 + \varepsilon)^2/\sigma^2]s_0(1 + \varepsilon)^2/\sigma^2 \\ &= 2\delta(s_0/\sigma^2)[(1 + \varepsilon)^2 - 2(\beta_1 - \beta_2)(s_0/\sigma^2)(1 + \varepsilon)^4].\end{aligned}$$

It follows that for small $\delta > 0$,

$$\begin{aligned}\mathbb{E}^0 v_4(2s_0) &= \mathbb{E}^0 v_4(s_0) + \mathbb{E}^0(v_4(2s_0) - v_4(s_0)) \\ &\leq 2\delta(s_0/\sigma^2)(1 + \varepsilon)^2 \\ &\quad + 2\delta(s_0/\sigma^2)[(1 + \varepsilon)^2 - 2(\beta_1 - \beta_2)(s_0/\sigma^2)(1 + \varepsilon)^4] \\ &= 2\delta(s_0/\sigma^2)[2(1 + \varepsilon)^2 - 2(\beta_1 - \beta_2)(s_0/\sigma^2)(1 + \varepsilon)^4].\end{aligned}$$

More generally,

$$\begin{aligned}\mathbb{E}^0 v_4((j+1)s_0) &= \mathbb{E}^0 v_4(js_0) + \mathbb{E}^0(v_4((j+1)s_0) - v_4(js_0)) \\ &\leq \mathbb{E}^0 v_4(js_0) + 2[\delta - (\beta_1 - \beta_2)\mathbb{E}^0 v_4(js_0)]s_0(1 + \varepsilon)^2/\sigma^2 \\ &= \mathbb{E}^0 v_4(js_0)[1 - 2(\beta_1 - \beta_2)(s_0/\sigma^2)(1 + \varepsilon)^2] \\ &\quad + 2\delta(s_0/\sigma^2)(1 + \varepsilon)^2.\end{aligned}$$

From this we obtain by induction

$$\begin{aligned}\mathbb{E}^0 v_4(js_0) &\leq \mathbb{E}^0 v_4(s_0) \left[1 - 2(\beta_1 - \beta_2) \left(\frac{s_0}{\sigma^2} \right) (1 + \varepsilon)^2 \right]^{j-1} \\ &\quad + 2\delta \left(\frac{s_0}{\sigma^2} \right) (1 + \varepsilon)^2 \sum_{k=0}^{j-2} \left[1 - 2(\beta_1 - \beta_2) \left(\frac{s_0}{\sigma^2} \right) (1 + \varepsilon)^2 \right]^k \\ &\leq 2\delta \left(\frac{s_0}{\sigma^2} \right) (1 + \varepsilon)^2 \left[1 - 2(\beta_1 - \beta_2) \left(\frac{s_0}{\sigma^2} \right) (1 + \varepsilon)^2 \right]^{j-1} \\ &\quad + 2\delta \left(\frac{s_0}{\sigma^2} \right) (1 + \varepsilon)^2 \sum_{k=0}^{j-2} \left[1 - 2(\beta_1 - \beta_2) \left(\frac{s_0}{\sigma^2} \right) (1 + \varepsilon)^2 \right]^k \\ &= 2\delta \left(\frac{s_0}{\sigma^2} \right) (1 + \varepsilon)^2 \frac{1 - [1 - 2(\beta_1 - \beta_2)(s_0/\sigma^2)(1 + \varepsilon)^2]^j}{1 - [1 - 2(\beta_1 - \beta_2)(s_0/\sigma^2)(1 + \varepsilon)^2]} \\ &= \frac{\delta}{\beta_1 - \beta_2} \left(1 - \left[1 - 2(\beta_1 - \beta_2) \left(\frac{s_0}{\sigma^2} \right) (1 + \varepsilon)^2 \right]^j \right).\end{aligned}$$

Now fix an arbitrary $a > 0$, an arbitrarily small $\varepsilon > 0$ and choose a sufficiently small $s_0 > 0$ so that $\delta_1 > \delta(1 - \varepsilon/2)$ and such that for some integer j

we have $js_0 = a$, and, moreover, j is sufficiently large to imply the following:

$$\begin{aligned} & \frac{\delta}{\beta_1 - \beta_2} \left(1 - \left[1 - 2(\beta_1 - \beta_2) \left(\frac{s_0}{\sigma^2} \right) (1 + \varepsilon)^2 \right]^j \right) \\ &= \frac{\delta}{\beta_1 - \beta_2} \left(1 - \left[1 - 2(\beta_1 - \beta_2) \left(\frac{s_0}{\sigma^2} \right) (1 + \varepsilon)^2 \right]^{a/s_0} \right) \\ &\leq \frac{\delta}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)(1 + \varepsilon)^3}{\sigma^2} \right) \right). \end{aligned}$$

Then for sufficiently small $\delta > 0$ we have

$$(3.20) \quad \mathbb{E}^0 v_4(a) = \mathbb{E}^0 v_4(js_0) \leq \frac{\delta}{\beta_1 - \beta_2} (1 - \exp(-2a(\beta_1 - \beta_2)(1 + \varepsilon)^3/\sigma^2)).$$

A completely analogous argument using (3.19) in place of (3.11) yields

$$(3.21) \quad \mathbb{E}^0 v_4(a) \geq \frac{\delta}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)(1 - \varepsilon)^4}{\sigma^2} \right) \right).$$

Since $\varepsilon > 0$ is arbitrarily small, a standard argument based on (3.20) and (3.21) gives for $\delta \downarrow 0$,

$$(3.22) \quad \mathbb{E}^0 v_4(a) = \frac{\delta}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2} \right) \right) + o(\delta).$$

Step 5. The last part of the proof exploits a relationship between local time and certain stopping times. Recall the local times L_t^{4-} and L_t^{4+} , introduced earlier in the proof. We have for some standard Brownian motion B_t^8 ,

$$B_t^4 = B_t^8 - \beta_1 t + L_t^{4-} - L_t^{4+}.$$

One has to check that the normalization of the local time, defined relative to the normalization of the excursion laws in (3.1), is the correct one for the above ‘‘Lévy formula.’’ This can be done, for example, by comparing our normalizations with those in Theorems 3.6.17 and 6.2.23 in Karatzas and Shreve (1988).

Note that the σ -fields generated by B_t^4 and B_t^8 are identical so $v_4(a)$ is a stopping time for B_t^8 . We have $B_{v_4(a)}^4 = 0$ and $L_{v_4(a)}^{4-} = a$, so

$$(3.23) \quad 0 = B_{v_4(a)}^8 - \beta_1 v_4(a) + a - L_{v_4(a)}^{4+}.$$

Since $v_4(a)$ is bounded by $\delta/(\beta_1 - \beta_2)$ the optional stopping theorem yields $\mathbb{E} B_{v_4(a)}^8 = 0$, and so, using (3.22),

$$(3.24) \quad \begin{aligned} & \mathbb{E} L_{v_4(a)}^{4+} = a - \mathbb{E} \beta_1 v_4(a) \\ &= a - \delta \frac{\beta_1}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2} \right) \right) + o(\delta). \end{aligned}$$

Now recall that $L_{v_4(a)}^{4+}$ has the same distribution as $L_\infty^{x+\delta}$ given $\{L_\infty^x = a\}$. This observation and the last formula complete the proof of part (i) of the lemma.

(ii) The proof of part (ii) of the lemma uses a formula analogous to (3.24), but requires some additional work.

Recall that δ was positive in part (i) of the proof; it will be negative in the present part.

Recall the transformations of B_t from the proof of (i). It is easy to see that analogous transformations in the current case do not lead to B_t^4 which is a Brownian motion starting from 0, with drift $-\beta_1$, reflected on the horizontal axis and the line $t \rightarrow \delta - (\beta_1 - \beta_2)t$ (with $\delta > 0$), but instead they give a Brownian motion \tilde{B}_t^4 starting from 0, with drift $-\beta_2$, reflected on the horizontal axis and the line $t \rightarrow \delta + (\beta_1 - \beta_2)t$ (with $\delta < 0$).

A subtle but significant difference from (i) is that the infinite excursion of B_t from the graph of X_t^x will go in the direction of the graph of $X_t^{x+\delta}$ and so it will generate some more local time. By Lemma 3.2(i) and Remark 3.3(ii), the excursions with finite lifetimes have the same intensities for Brownian motions with drifts β_2 and $-\beta_2$, so we can use estimate (3.24) for the portion of the local time generated before the last, infinite excursion of B_t from the graph of X_t^x . The estimate has to be modified as β_1 has to be replaced by β_2 , and so we obtain

$$(3.25) \quad a - |\delta| \frac{\beta_2}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2}\right) \right) + o(\delta).$$

To this we will have to add the local time spent by B_t on the graph of $X_t^{x+\delta}$ during its final, infinite excursion from the graph of X_t^x . The rest of the proof is devoted to that calculation.

Let U be the first time when the final, infinite excursion of B_t from the graph of X_t^x hits the graph of $X_t^{x+\delta}$. Let $\delta_1 = |X_U^x - X_U^{x+\delta}|$. First, we will condition on δ_1 . The process $\{B_t, t \geq U\}$ is a Brownian motion conditioned not to hit the line $t \rightarrow B_U + \delta_1 + \beta_2 t$. By subtracting the drift and flipping the process to the other side of the horizontal axis, we may consider a Brownian motion B_t^9 starting from δ_1 , with drift β_2 , conditioned not to hit 0. We will estimate the amount of the local time this process spends on the graph of a solution Y_t to (1.1) with β_2 replaced by $-(\beta_1 - \beta_2)$, β_1 replaced by 0, and B_t replaced by B_t^9 .

Let H^{δ_1} be the excursion law for excursions above the level δ_1 for Brownian motion with drift β_2 , conditioned not to hit 0. Let F_∞ denote the set of excursions with infinite lifetime. We will compute $H^{\delta_1}(F_\infty)$. Let $Q_{\beta_2}^z$ be the distribution of Brownian motion with drift β_2 . Then

$$\begin{aligned} H^{\delta_1}(F_\infty) &= \lim_{z \downarrow 0} \frac{1}{z} Q_{\beta_2}^{\delta_1+z}(T_\infty < T_{\delta_1} \mid T_\infty < T_0) \\ &= \lim_{z \downarrow 0} \frac{1}{z} \frac{Q_{\beta_2}^{\delta_1+z}(T_\infty < T_{\delta_1} \text{ and } T_\infty < T_0)}{Q_{\beta_2}^{\delta_1+z}(T_\infty < T_0)} \\ &= \lim_{z \downarrow 0} \frac{1}{z} \frac{Q_{\beta_2}^{\delta_1+z}(T_\infty < T_{\delta_1})}{Q_{\beta_2}^{\delta_1+z}(T_\infty < T_0)}. \end{aligned}$$

Recall that the scale function $S(y)$ for Brownian motion with drift β_2 is equal to $\exp(-2\beta_2 y/\sigma^2)$. This gives

$$\begin{aligned} H^{\delta_1}(F_\infty) &= \lim_{z \downarrow 0} \frac{1}{z} \frac{S(\delta_1 + z) - S(\delta_1)}{S(\infty) - S(\delta_1)} \frac{S(\infty) - S(0)}{S(\delta_1 + z) - S(0)} \\ &= \lim_{z \downarrow 0} \frac{1}{z} \frac{\exp(-2\beta_2(\delta_1 + z)/\sigma^2) - \exp(-2\beta_2\delta_1/\sigma^2)}{0 - \exp(-2\beta_2\delta_1/\sigma^2)} \\ &\quad \times \frac{0 - 1}{\exp(-2\beta_2(\delta_1 + z)/\sigma^2) - 1} \\ &= \frac{2\beta_2}{\sigma^2[1 - \exp(-2\beta_2\delta_1/\sigma^2)]}. \end{aligned}$$

If we fix arbitrarily small $\varepsilon > 0$ then for sufficiently small $\delta_1 > 0$ we have

$$(3.26) \quad \frac{1}{\delta_1} \leq H^{\delta_1}(F_\infty) \leq (1 + \varepsilon) \frac{1}{\delta_1}.$$

We proceed to calculate the expected time to hit δ_1 for Brownian motion with drift β_2 , starting from $\delta_1 - z$ and conditioned not to hit 0, where $z \in (0, \delta_1)$. If we take

$$s(z) = \exp\left[-\int_0^z \frac{2\beta_2}{\sigma^2} dy\right] = \exp\left(\frac{-2\beta_2 z}{\sigma^2}\right)$$

and

$$S(z) = \int_0^z s(y) dy = \frac{\sigma^2}{2\beta_2} \left[1 - \exp\left(\frac{-2\beta_2 z}{\sigma^2}\right)\right],$$

then formula (9.9) on page 264 of Karlin and Taylor (1981) yields

$$\begin{aligned} &\mathbb{E}^{\delta_1 - z}(T_{\delta_1} \mid T_{\delta_1} < T_0) \\ &= \frac{2[S(\delta_1) - S(\delta_1 - z)]}{S(\delta_1)S(\delta_1 - z)} \int_0^{\delta_1 - z} \frac{S^2(y)}{\sigma^2 s(y)} dy + 2 \int_{\delta_1 - z}^{\delta_1} \frac{S(y)[S(\delta_1) - S(y)]}{\sigma^2 s(y)S(\delta_1)} dy \\ &= \frac{2[(\sigma^2/2\beta_2)[1 - \exp(-2\beta_2\delta_1/\sigma^2)] - (\sigma^2/2\beta_2)[1 - \exp(-2\beta_2(\delta_1 - z)/\sigma^2)]}{(\sigma^2/2\beta_2)[1 - \exp(-2\beta_2\delta_1/\sigma^2)](\sigma^2/2\beta_2)[1 - \exp(-2\beta_2(\delta_1 - z)/\sigma^2)]} \\ &\quad \times \int_0^{\delta_1 - z} \frac{(\sigma^4/4\beta_2^2)[1 - \exp(-2\beta_2 y/\sigma^2)]^2}{\sigma^2 \exp(-2\beta_2 y/\sigma^2)} dy \\ &\quad + 2 \int_{\delta_1 - z}^{\delta_1} \left(\frac{(\sigma^2/2\beta_2)[1 - \exp(-2\beta_2 y/\sigma^2)]}{\sigma^2 \exp(-2\beta_2 y/\sigma^2)(\sigma^2/2\beta_2)[1 - \exp(-2\beta_2\delta_1/\sigma^2)]} \right. \\ &\quad \left. \times \left[\frac{\sigma^2}{2\beta_2} \left[1 - \exp\left(\frac{-2\beta_2\delta_1}{\sigma^2}\right)\right] - \frac{\sigma^2}{2\beta_2} \left[1 - \exp\left(\frac{-2\beta_2 y}{\sigma^2}\right)\right] \right] \right) dy \end{aligned}$$

$$\begin{aligned}
&= \frac{2[\exp(-2\beta_2(\delta_1 - z)/\sigma^2) - \exp(-2\beta_2\delta_1/\sigma^2)]}{(\sigma^2/2\beta_2)[1 - \exp(-2\beta_2\delta_1/\sigma^2)][1 - \exp(-2\beta_2(\delta_1 - z)/\sigma^2)]} \\
&\quad \times \int_0^{\delta_1 - z} \frac{(\sigma^4/4\beta_2^2)[1 - \exp(-2\beta_2 y/\sigma^2)]^2}{\sigma^2 \exp(-2\beta_2 y/\sigma^2)} dy \\
&\quad + 2 \int_{\delta_1 - z}^{\delta_1} \frac{\sigma^2}{2\beta_2} [1 - \exp(-2\beta_2 y/\sigma^2)] \\
&\quad \quad \times \frac{\exp(-2\beta_2 y/\sigma^2) - \exp(-2\beta_2\delta_1/\sigma^2)}{\sigma^2 \exp(-2\beta_2 y/\sigma^2)[1 - \exp(-2\beta_2\delta_1/\sigma^2)]} dy.
\end{aligned}$$

The expected lifetime of an excursion below δ_1 for the Brownian motion with drift β_2 , starting from δ_1 and conditioned not to hit 0, is therefore equal to

$$\begin{aligned}
(3.27) \quad &\lim_{z \downarrow 0} \frac{1}{z} \mathbb{E}^{\delta_1 - z}(T_{\delta_1} \mid T_{\delta_1} < T_0) \\
&= \frac{2(2\beta_2/\sigma^2) \exp(-2\beta_2\delta_1/\sigma^2)}{(\sigma^2/2\beta_2)[1 - \exp(-2\beta_2\delta_1/\sigma^2)][1 - \exp(-2\beta_2\delta_1/\sigma^2)]} \\
&\quad \times \int_0^{\delta_1} \frac{(\sigma^4/4\beta_2^2)[1 - \exp(-2\beta_2 y/\sigma^2)]^2}{\sigma^2 \exp(-2\beta_2 y/\sigma^2)} dy \\
&\leq c_1 \delta_1,
\end{aligned}$$

for small $\delta_1 > 0$ and some c_1 depending on β_2 and σ^2 but not on δ_1 .

Fix arbitrarily small $\varepsilon > 0$. We are ready to derive estimates for the total amount of local time, say L_∞^{9+} , that B_t^9 spends on the graph of Y_t .

On one hand, the estimate (3.26) shows that L_∞^{9+} is stochastically bounded by an exponential random variable with mean δ_1 , for sufficiently small $\delta_1 > 0$.

Let $v_9^+(a)$ be the time spent by B_t^9 between Y_t and the horizontal axis before the time when L_t^{9+} hits a . Since Y_t is nonincreasing, the estimate (3.27) can be used as an upper bound for the expected duration of an excursion below Y_t , for every $t \geq 0$. Fix arbitrarily large $b < \infty$ and arbitrarily small $\varepsilon > 0$. We have from excursion theory,

$$\mathbb{E}v_9^+(b\delta_1) \leq b\delta_1 c_1 \delta_1,$$

and so, for sufficiently small $\delta_1 > 0$,

$$\mathbb{P}(v_9^+(b\delta_1) \geq \delta_1 \varepsilon) \leq \frac{bc_1 \delta_1^2}{\delta_1 \varepsilon} = \frac{bc_1}{\varepsilon} \delta_1 < \varepsilon.$$

We see that with probability greater than $1 - \varepsilon$, the distance between Y_t and the horizontal axis remains greater than $\delta_1 - \delta_1 \varepsilon (\beta_1 - \beta_2)$, at least until the time when L_t^{9+} exceeds $b\delta_1$. On this time interval and given this event, the intensity for the arrival process of the infinite excursion is bounded above by $(1 + \varepsilon)/(\delta_1(1 - \varepsilon))$, by (3.26). Hence, $\mathbb{E}L_\infty^{9+}/\delta_1$ can be made arbitrarily close

to 1, by choosing large b , then small ε and finally small $|\delta| > 0$ (note that $\delta_1 \leq |\delta|$).

Finally, in order to obtain an unconditioned estimate for $\mathbb{E}L_\infty^{9+}$, we have to average over the possible values of δ_1 . Let $\tilde{v}_4(a)$ be the time when the local time of \tilde{B}_t^4 [defined earlier in the proof of part (ii)] reaches a . The same argument which gives (3.22) yields the following estimate:

$$\begin{aligned} \mathbb{E}L_\infty^{9+} &= \mathbb{E}\delta_1 + o(\delta) \\ &= |\delta| - \mathbb{E}(\beta_1 - \beta_2)\tilde{v}_4(a) + o(\delta) \\ &= |\delta| - |\delta| \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2}\right) \right) + o(\delta) \\ &= |\delta| \exp(-2a(\beta_1 - \beta_2)/\sigma^2) + o(\delta). \end{aligned}$$

Adding this quantity to (3.25) gives the formula in Lemma 3.5(ii). \square

LEMMA 3.6. *Fix $x, a, \beta_1, \beta_2 > 0$ and assume that $\beta_1 - \beta_2 > 0$. Then*

$$\text{Var}(L_\infty^{x+\delta} \mid L_\infty^x = a) = \frac{|\delta|}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2}\right) \right) + o(\delta),$$

for $\delta \downarrow 0$. The same formula holds if $x < 0$ and $\delta \uparrow 0$.

PROOF. First suppose that $x, \delta > 0$ and recall the notation and definitions from the proof of Lemma 3.5(i). It follows from (3.23) that

$$L_{v_4(a)}^{4+} - a = B_{v_4(a)}^8 - \beta_1 v_4(a).$$

We have, using (3.22),

$$\begin{aligned} \mathbb{E}(L_{v_4(a)}^{4+} - a)^2 &= \mathbb{E}(B_{v_4(a)}^8 - \beta_1 v_4(a))^2 \\ &= \mathbb{E}(B_{v_4(a)}^8)^2 - 2\beta_1 \mathbb{E}[B_{v_4(a)}^8 v_4(a)] + \mathbb{E}(v_4(a))^2 \\ (3.28) \quad &= \mathbb{E}v_4(a) - 2\beta_1 \mathbb{E}[B_{v_4(a)}^8 v_4(a)] + \mathbb{E}(v_4(a))^2 \\ &= \frac{\delta}{\beta_1 - \beta_2} \left(1 - \exp\left(-\frac{2a(\beta_1 - \beta_2)}{\sigma^2}\right) \right) \\ &\quad + o(\delta) - 2\beta_1 \mathbb{E}[B_{v_4(a)}^8 v_4(a)] + \mathbb{E}(v_4(a))^2. \end{aligned}$$

Recall that $v_4(a)$ is bounded by $\delta/(\beta_1 - \beta_2)$. Hence,

$$(3.29) \quad \mathbb{E}(v_4(a))^2 \leq \delta^2/(\beta_1 - \beta_2)^2,$$

and

$$\begin{aligned}
& \mathbb{E}B_{v_4(a)}^8 v_4(a) \\
& \leq (\mathbb{E}(B_{v_4(a)}^8)^2 \mathbb{E}(v_4(a))^2)^{1/2} \\
(3.30) \quad & = (\mathbb{E}v_4(a) \mathbb{E}(v_4(a))^2)^{1/2} \\
& \leq \left(\left[\frac{\delta}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2} \right) \right) + o(\delta) \right] \frac{\delta^2}{(\beta_1 - \beta_2)^2} \right)^{1/2} \\
& = o(\delta).
\end{aligned}$$

Combining (3.28)–(3.30) yields

$$\mathbb{E}(L_{v_4(a)}^{4+} - a)^2 = \frac{\delta}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2} \right) \right) + o(\delta).$$

This implies

$$\begin{aligned}
\text{Var } L_{v_4(a)}^{4+} &= \mathbb{E}(L_{v_4(a)}^{4+} - \mathbb{E}L_{v_4(a)}^{4+})^2 \\
&= \mathbb{E}(L_{v_4(a)}^{4+} - a)^2 - (\mathbb{E}L_{v_4(a)}^{4+} - a)^2 \\
&= \frac{\delta}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2} \right) \right) + o(\delta) \\
&\quad - \left[\frac{\delta \beta_1}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2} \right) \right) + o(\delta) \right]^2 \\
&= \frac{\delta}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2} \right) \right) + o(\delta).
\end{aligned}$$

Similarly to the proof of Lemma 3.5(i), we have $\text{Var}(L_\infty^{x+\delta} \mid L_\infty^x = a) = \text{Var } L_{v_4(a)}^{4+}$, which combined with the last formula proves the lemma in the case $x > 0$.

Now consider the case $x, \delta < 0$. We argue as in the proof of Lemma 3.5(ii) that we have to add a contribution from the local time on $X_t^{x+\delta}$ generated by the infinite excursion of B_t below X_t^x . We have shown in the proof of Lemma 3.5(ii) that the local time on $X_t^{x+\delta}$ generated by the infinite excursion is stochastically bounded by an exponential random variable with mean δ so its variance is bounded by $2\delta^2$, and, therefore, the contribution to the variance from the infinite excursion is negligible. The same formula holds in the case $x < 0$ as in the case $x > 0$. \square

Recall that L_t^x denotes the local time of $B_t - X_t^x$ at 0. The following result is analogous to Trotter's theorem on the joint continuity of local times for Brownian motion [see Karatzas and Shreve (1988) or Knight (1981)].

THEOREM 3.7. *Assume that $\beta_1, \beta_2 > 0$ and $\beta_1 - \beta_2 > 0$. There exists a version of the process $(x, t) \rightarrow L_t^x$ which is jointly continuous in both variables.*

PROOF. Note that X_t^x and X_t^y increase at the same rate when B_t does not lie between X_t^x and X_t^y , and by the assumptions on β_1 and β_2 , they grow closer together when B_t does lie between them. Therefore, for all x, y and $t \geq 0$,

$$(3.31) \quad |X_t^x - X_t^y| \leq |x - y|.$$

Define $G(x) = \mathbb{E}L_\infty^x$. The excursion law for Brownian motion below the line $t \rightarrow \beta_2 t$ gives mass $2\beta_2/\sigma^2$ to excursions with infinite lifetime, by Lemma 3.2(iii). By excursion theory, the waiting time (in terms of local time) for the first excursion with infinite lifetime is exponential with mean $\sigma^2/(2\beta_2)$. This says that the distribution of L_∞^0 is exponential with mean $\sigma^2/(2\beta_2)$. This and the strong Markov property applied at the first time when B_t intersects X_t^x imply that for some $c_1 < \infty$ and all x , we have $G(x) \leq c_1$. An easy conditioning argument that combines this observation with Lemma 3.5 shows that for all x and y ,

$$(3.32) \quad |G(x) - G(y)| \leq c_2|x - y|.$$

The process (X_t^x, B_t) is strong Markov, and L_t^x is an additive functional. So by the Markov property,

$$\begin{aligned} \mathbb{E}[L_\infty^x - L_t^x \mid \mathcal{F}_t] &= \mathbb{E}[L_\infty^x \circ \theta_t \mid \mathcal{F}_t] \\ &= G(X_t^x - B_t). \end{aligned}$$

Therefore

$$\mathbb{E}[L_\infty^x - L_t^x \mid \mathcal{F}_t] \leq c_1.$$

Also, using (3.31) and (3.32),

$$\begin{aligned} |\mathbb{E}[(L_\infty^x - L_\infty^y) - (L_t^x - L_t^y) \mid \mathcal{F}_t]| &= |G(X_t^x - B_t) - G(X_t^y - B_t)| \\ &\leq c_2|X_t^x - X_t^y| \\ &\leq c_2|x - y|. \end{aligned}$$

By Bass (1995), Proposition I.6.14,

$$(3.33) \quad \mathbb{E}\left[\sup_t |L_t^x - L_t^y|^4\right] \leq c_4|x - y|^2.$$

By Kolmogorov's criterion and standard arguments [cf. the proof of Proposition I.6.16 of Bass (1995)], we deduce that there exists a version of L_t^x that is jointly continuous in x and t . \square

A classical Ray–Knight theorem [see Knight (1981), Revuz and Yor (1991) or Yor (1997)] asserts, roughly speaking, that if L_t^x is the local time for the standard Brownian motion then $x \rightarrow L_T^x$ is a diffusion for certain stopping times T . As a part of that theorem, the infinitesimal parameters of the diffusion are also given. We prove a similar result for our family of local times, with $T \equiv \infty$. Recall that we assume that $B_0 = 0$.

THEOREM 3.8. *Suppose that $\beta_1, \beta_2 > 0$ and $\beta_1 - \beta_2 > 0$. The distribution of L_∞^0 is exponential with mean $\sigma^2/(2\beta_2)$. The process $\{L_\infty^x, x \geq 0\}$ is a diffusion with the infinitesimal drift*

$$\tilde{\mu}(a) = -\frac{\beta_1}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2}\right) \right)$$

and infinitesimal variance

$$\tilde{\sigma}^2(a) = \frac{1}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2}\right) \right).$$

The process $\{L_\infty^{-x}, x \geq 0\}$ is a diffusion with the infinitesimal drift

$$\hat{\mu}(a) = -\frac{\beta_2}{\beta_1 - \beta_2} + \frac{\beta_1}{\beta_1 - \beta_2} \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2}\right)$$

and the same infinitesimal variance

$$\hat{\sigma}^2(a) = \frac{1}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2}\right) \right).$$

PROOF. We have already shown in the proof of Theorem 3.7 that the distribution of L_∞^0 is exponential with mean $\sigma^2/(2\beta_2)$.

Recall from Lemma 3.1 that if $x < y$ then $X_t^x < X_t^y$ for all t . The Markovian character of the process $\{L_\infty^x, x \geq 0\}$ at any fixed “time” $x = y$ follows from the independence of the Poisson processes of excursions of B_t below and above X_t^y . The same remark applies to $\{L_\infty^{-x}, x \geq 0\}$. The infinitesimal parameters of the processes were calculated in Lemmas 3.5 and 3.6.

The process $x \rightarrow L_\infty^x$ is continuous, by Theorem 3.7. Since its infinitesimal drift is bounded and the infinitesimal variance is nondegenerate, there is a unique (in law) Markov process with this infinitesimal drift and variance [cf. Bass (1997), Section IV.3], and this Markov process is in fact a strong Markov process. \square

THEOREM 3.9. *Suppose $\beta_1, \beta_2 > 0$ and $\beta_1 - \beta_2 > 0$. For fixed $t > 0$, we have a.s., for all $x, x_1, x_2 \in \mathbb{R}$,*

$$\frac{d}{dy} X_t^y \Big|_{y=x} = \exp\left(\frac{-2L_t^x(\beta_1 - \beta_2)}{\sigma^2}\right)$$

and

$$X_t^{x_2} - X_t^{x_1} = \int_{x_1}^{x_2} \exp\left(\frac{-2L_t^x(\beta_1 - \beta_2)}{\sigma^2}\right) dx.$$

PROOF. First we will prove an estimate analogous to (3.22) except that it will hold for $v_4(a)$ itself rather than its expectation. Recall the notation and definitions from the proof of Lemma 3.5(i).

The following estimate is completely analogous to (3.17) except that we state it for the process B_t^7 rather than B_t^6 , so δ_1 is replaced by δ in the bound:

$$\mathbb{P}^y(v_7(s_0(1-\varepsilon)) \geq u_0) \leq \frac{c_{11}\delta}{\varepsilon^2 s_0 \sigma^2}.$$

We can further modify the estimate by replacing $s_0(1-\varepsilon)$ with s_0 , so that

$$\mathbb{P}^y(v_7(s_0) \geq u_0/(1-\varepsilon)) \leq \frac{c_{11}\delta(1-\varepsilon)}{\varepsilon^2 s_0 \sigma^2}.$$

This and (3.16) imply that for small ε and δ we have

$$(3.34) \quad v_7(s_0) \leq u_0/(1-\varepsilon) \leq 2s_0\delta(1+\varepsilon)^2/\sigma^2$$

with probability greater than or equal to $1 - c_{11}\delta(1-\varepsilon)/(\varepsilon^2 s_0 \sigma^2)$. The inequality (3.34) is analogous to (3.11) and can be used in the same way as in the argument between (3.19) and (3.20) to prove a formula analogous to (3.20),

$$(3.35) \quad v_4^\delta(a) = v_4(js_0) \leq \frac{\delta}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)(1+\varepsilon)^3}{\sigma^2}\right) \right),$$

where δ in $v_4^\delta(a)$ indicates the dependence of $v_4^\delta(a)$ on δ . The above argument requires that we can use an estimate analogous to (3.34) at every stage of the inductive procedure, that is, at every stopping time $v_4(ms_0)$ for $m = 1, 2, \dots, j-1$. All of these estimates hold simultaneously with probability greater than

$$1 - (j-1)c_{11}\delta(1-\varepsilon)/(\varepsilon^2 s_0 \sigma^2).$$

This shows that the probability that (3.35) fails to hold is smaller than

$$(j-1)c_{11}\delta(1-\varepsilon)/(\varepsilon^2 s_0 \sigma^2).$$

Now fix arbitrarily small $\varepsilon_1 > 0$ and let $\delta_k = (1-\varepsilon_1)^k$. Let A_k be the event in (3.35) with δ replaced by δ_k ; that is,

$$A_k = \left\{ v_4^{\delta_k}(a) \leq \frac{\delta_k}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)(1+\varepsilon)^3}{\sigma^2}\right) \right) \right\}.$$

We have

$$\sum_{k=0}^{\infty} (j-1)c_{11}\delta_k(1-\varepsilon)/(\varepsilon^2 s_0 \sigma^2) = \sum_{k=0}^{\infty} (j-1)c_{11}(1-\varepsilon_1)^k(1-\varepsilon)/(\varepsilon^2 s_0 \sigma^2) < \infty,$$

so only a finite number of events A_k may fail to hold. Consider an ω and k_0 such that all events A_k , $k \geq k_0$, hold for this ω . Suppose that $\delta \in (0, \delta_{k_0})$. Then $\delta \in [\delta_{k_1-1}, \delta_{k_1}]$ for some $k_1 \geq k_0$. Since A_{k_1} holds, we have

$$\begin{aligned} v_4^\delta(a) &\leq v_4^{\delta_{k_1}}(a) \leq \frac{\delta_{k_1}}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)(1+\varepsilon)^3}{\sigma^2}\right) \right) \\ &\leq \frac{\delta(1-\varepsilon_1)}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)(1+\varepsilon)^3}{\sigma^2}\right) \right). \end{aligned}$$

This inequality holds with probability one for all sufficiently small $\delta > 0$. Since $\varepsilon > 0$ and $\varepsilon_1 > 0$ are arbitrarily small, we see that a.s.,

$$\limsup_{\delta \rightarrow 0^+} \frac{v_4^\delta(a)}{\delta} \leq \frac{1}{\beta_1 - \beta_2} \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2}\right) \right).$$

The same lower bound can be obtained for \liminf in a completely analogous way, so with probability 1,

$$(3.36) \quad \lim_{\delta \rightarrow 0^+} \frac{v_4^\delta(a)}{\delta} = \frac{1 - \exp(-2a(\beta_1 - \beta_2)/\sigma^2)}{\beta_1 - \beta_2}.$$

Suppose $a > 0$ and let $v(a) = \inf\{t \geq 0: L_t^x = a\}$. Fix some $x \in \mathbb{R}$ and consider $\delta > 0$. We will first find the right-hand side derivative $(d^+/dy)X_{v(a)}^y|_{y=x}$. Let $T = \inf\{t: B_t = X_t^x\}$ and let U_1 be the amount of time spent by B_t between the graphs of X_t^x and $X_t^{x+\delta}$ on the time interval $[0, T]$. We will write U_2 to denote the amount of time spent by B_t between the graphs of X_t^x and $X_t^{x+\delta}$, between times T and $v(a)$.

If $x \geq 0$, then $U_1 = 0$. If $x < 0$ then U_1 is not greater than the amount of time U_3 spent by B_t between the lines $t \rightarrow x + \beta_1 t$ and $t \rightarrow x + \delta + \beta_1 t$, until the hitting time T . Standard arguments show that for any arbitrarily small $\varepsilon > 0$, we have $U_3/\delta^{2-\varepsilon} \rightarrow 0$ as $\delta \rightarrow 0$, a.s. Note that the distance between $X_t^{x+\delta}$ and X_t^x decreases by $(\beta_1 - \beta_2)u$ on any interval where the Brownian motion B_t spends u units between these functions. Hence,

$$X_{v(a)}^{x+\delta} - X_{v(a)}^x = \delta - (\beta_1 - \beta_2)(U_1 + U_2).$$

The random variable U_2 may be identified with $v_4^\delta(a)$, so (3.36) gives for any fixed a , a.s.,

$$\begin{aligned} \frac{d^+}{dy} X_{v(a)}^y \Big|_{y=x} &= \lim_{\delta \rightarrow 0} \frac{X_{v(a)}^{x+\delta} - X_{v(a)}^x}{\delta} \\ &= \lim_{\delta \rightarrow 0} \frac{\delta - (\beta_1 - \beta_2)(U_1 + U_2)}{\delta} \\ &= 1 - \lim_{\delta \rightarrow 0} \frac{(\beta_1 - \beta_2)U_1}{\delta} - \lim_{\delta \rightarrow 0} \frac{(\beta_1 - \beta_2)v_4^\delta(a)}{\delta} \\ &= 1 - 0 - \left(1 - \exp\left(\frac{-2a(\beta_1 - \beta_2)}{\sigma^2}\right) \right) \\ &= \exp\left(\frac{-2L_{v(a)}^x(\beta_1 - \beta_2)}{\sigma^2}\right). \end{aligned}$$

The above holds simultaneously for all rational a , with probability 1. Since $t \rightarrow L_t^x$ and $t \rightarrow X_t^y - X_t^z$ are continuous monotone functions, an elementary argument can be used to extend the last formula to fixed times; that is,

$$(3.37) \quad \frac{d^+}{dy} X_t^y \Big|_{y=x} = \exp\left(-\frac{2L_t^x(\beta_1 - \beta_2)}{\sigma^2}\right),$$

simultaneously for all $t \geq 0$, a.s.

Fix some $t > 0$. By Fubini's theorem, (3.37) holds for almost all x , a.s. We have $|X_t^y - X_t^z| \leq |y - z|$ for all y and z . Since the function $y \rightarrow X_t^y$ is Lipschitz, it has a derivative almost everywhere and so for a fixed t , we may replace the right-hand derivative with the usual derivative in (3.37), for almost all x . The function $x \rightarrow L_t^x$ is continuous, so the derivative in (3.37) is equal almost everywhere to a continuous function. This implies that the derivative is equal to the function everywhere. This proves the first assertion of the theorem. The second one follows from the first one and from the Lipschitz character of $y \rightarrow X_t^y$. \square

REMARK 3.10. Suppose that X_t^y are solutions to (1.1) and assume that $\beta_1, \beta_2 > 0$ and $\beta_1 - \beta_2 > 0$. Fix some $t > 0$ and consider the function $y \rightarrow X_t^y$. We will sketch an argument showing that $y \rightarrow X_t^y$ is $C^{1+\gamma}$ for every $\gamma < 1/2$, that is, that the function has a derivative which is Hölder continuous with Hölder exponent γ .

Fix any $z \in \mathbb{R}$. With probability 1, $B_t \neq X_t^z$, and with strictly positive probability, there exists $\varepsilon > 0$ such that $B_s \neq X_s^y$ for all $y \in (z - \varepsilon, z + \varepsilon)$ and $s \geq t$. It follows that if a local property holds for the function $y \rightarrow L_\infty^y$ with probability 1, it must hold for $y \rightarrow L_t^y$, with probability 1. Since $y \rightarrow L_\infty^y$ is a diffusion, its paths are Hölder continuous with exponent γ for every $\gamma < 1/2$. It follows that the same is true of $y \rightarrow L_t^y$. Theorem 3.9 now implies that $y \rightarrow X_t^y$ is $C^{1+\gamma}$ for every $\gamma < 1/2$. The same argument shows that $y \rightarrow X_t^y$ is not $C^{3/2}$.

4. Time and direction of bifurcation. We will first address the question of the direction of bifurcation for (1.3). We will say that a positive bifurcation occurs if for some t_1 we have $X_t > B_t$ for all $t > t_1$. The definition of a negative bifurcation is analogous. If β_1 and β_2 have the same sign then it is easy to see that a bifurcation will occur with probability 1 and its direction will be the same as the sign of β_k 's. If $\beta_1 > 0$ and $\beta_2 < 0$ then there will be no bifurcation. The next theorem deals with the only remaining nontrivial case.

THEOREM 4.1. *Consider (1.3) with $t_0 = x_0 = 0$. Assume that $\beta_1 < 0$ and $\beta_2 > 0$. Let*

$$\lambda_j = \frac{2|\beta_j|^{1/(\alpha_j+1)}(\alpha_j+1)^{\alpha_j/(\alpha_j+1)}}{\sigma^{2/(\alpha_j+1)}\Gamma(1/(\alpha_j+1))},$$

for $j = 1, 2$. The probability of a negative bifurcation is equal to $\lambda_1/(\lambda_1 + \lambda_2)$. When $\alpha_1 = \alpha_2 = 0$, the formula simplifies to $|\beta_1|/(|\beta_1| + |\beta_2|)$.

Before proving Theorem 4.1 we present a lemma which may have some interest of its own.

LEMMA 4.2. *Assume that $\beta_1 < 0$ and $\beta_2 > 0$. Consider a solution X_t to (1.3) with $t_0 = x_0 = 0$. There exists $\gamma > 0$, depending on $\alpha_1, \alpha_2, \beta_1, \beta_2$, such that $X_t/t^{1/2+\gamma}$ converges in probability to 0 as $t \rightarrow 0$.*

PROOF. Let us assume that $-1 < \alpha_1 \leq 0 \leq \alpha_2$. The other cases may be treated in a similar way. Let $U = \sup\{s \leq t: X_s - B_s \geq -t^{1/2}\}$. For $s \in (U, t)$ we have $X_s - B_s < -t^{1/2}$ so for such s , $|dX_s/ds| \leq \beta_1 t^{(1/2)\alpha_1}$. It follows that

$$|X_t - X_U| \leq -(t - U)\beta_1 t^{(1/2)\alpha_1} \leq -\beta_1 t^{1+(1/2)\alpha_1}$$

and so

$$\begin{aligned} B_t - X_t &\leq (B_t - B_U) + (B_U - X_U) + (X_U - X_t) \\ &\leq \left(\max_{s \in (0, t)} B_s - \min_{s \in (0, t)} B_s \right) + t^{1/2} - \beta_1 t^{1+(1/2)\alpha_1}. \end{aligned}$$

This implies that

$$\begin{aligned} &\mathbb{E}|B_t - X_t|^{\alpha_1} \mathbf{1}_{\{B_t - X_t > 0\}} \\ &\leq \mathbb{E} \left| \left(\max_{s \in (0, t)} B_s - \min_{s \in (0, t)} B_s \right) + t^{1/2} - \beta_1 t^{1+(1/2)\alpha_1} \right|^{\alpha_1} \\ &\leq 3^{\alpha_1} \left[\mathbb{E} \left(\max_{s \in (0, t)} B_s - \min_{s \in (0, t)} B_s \right)^{\alpha_1} + t^{(1/2)\alpha_1} + |\beta_1|^{\alpha_1} t^{\alpha_1 + (1/2)\alpha_1^2} \right] \\ &\leq c_1 t^{(1/2)\alpha_1} + c_2 |\beta_1|^{\alpha_1} t^{\alpha_1 + (1/2)\alpha_1^2}. \end{aligned}$$

Recall from (2.2) that

$$X_t = \int_0^t [\beta_1 |X_s - B_s|^{\alpha_1} \mathbf{1}_{\{X_s - B_s \leq 0\}} + \beta_2 |X_s - B_s|^{\alpha_2} \mathbf{1}_{\{X_s - B_s > 0\}}] ds.$$

From this we have the following estimate:

$$\begin{aligned} \mathbb{E}X_t &\geq \mathbb{E} \int_0^t \beta_1 |X_s - B_s|^{\alpha_1} \mathbf{1}_{\{X_s - B_s \leq 0\}} ds \\ &= \int_0^t \mathbb{E}(\beta_1 |X_s - B_s|^{\alpha_1} \mathbf{1}_{\{X_s - B_s \leq 0\}}) ds \\ &\geq \beta_1 \int_0^t (c_1 s^{(1/2)\alpha_1} + c_2 |\beta_1|^{\alpha_1} s^{\alpha_1 + (1/2)\alpha_1^2}) ds \\ &= \beta_1 (c_3 t^{1+(1/2)\alpha_1} + c_4 |\beta_1|^{\alpha_1} t^{1+\alpha_1 + (1/2)\alpha_1^2}). \end{aligned}$$

Since $\alpha_1 > -1$, the exponents $1 + (1/2)\alpha_1$ and $1 + \alpha_1 + (1/2)\alpha_1^2$ are greater than $1/2$ and so for some $\gamma > 0$ and every $c_5 > 0$, $\liminf_{t \rightarrow 0} c_5 \mathbb{E}X_t/t^{1/2+\gamma} \geq 0$. It follows that

$$(4.1) \quad \lim_{t \rightarrow 0} \mathbb{P}(X_t/t^{1/2+\gamma} < -c_6) = 0,$$

for every $c_6 > 0$.

Recall that $\alpha_2 \geq 0$. Since

$$X_t \leq \int_0^t \beta_2 |X_s - B_s|^{\alpha_2} \mathbf{1}_{\{X_s - B_s > 0\}} ds,$$

an elementary argument shows that for small t we have $X_t \leq 2\beta_2 t$ if $B_s \leq 1$ for all $s \in (0, t)$. It is clear that $\mathbb{P}(\max_{s \in (0, t)} B_s > 1)$ goes to 0 as $t \rightarrow 0$ so

$$\lim_{t \rightarrow 0} \mathbb{P}(X_t/t > 2\beta_2) = 0.$$

This and (4.1) prove the lemma. \square

PROOF OF THEOREM 4.1. The assertion of the theorem deals only with probabilities, so we can use any solution to (1.3), as we have uniqueness in law by Theorem 2.1. The same theorem shows that a solution X_t may be constructed so that (X_t, B_t) is a strong Markov process, and hence we may apply excursion theory to it. Recall the discussion at the beginning of Section 3. The same analysis of excursion laws and the exit system applies to the solutions of (1.3) for arbitrary $\alpha_1, \alpha_2 > -1$. Let us briefly recall the facts that we will need in our present argument. Let $D = \{(b, x) \in \mathbb{R}^2: b = x\}$ and let (H^x, dL) be an exit system for the process of excursions of (B_t, X_t) from the set D . The generic excursion may be denoted (e_t^1, e_t^2) . By the translation invariance of the Brownian motion B_t and (1.3), the distribution of $(e_t^1 - x, e_t^2 - x)$ under H^x is the same for every $x \in \mathbb{R}$. Let this distribution be called H_1 . Let H_{1+} denote the part of the measure H_1 which is supported on excursions with $e_t^1 > e_t^2$ and let H_{1-} be the part supported on the set where $e_t^1 < e_t^2$. Let H_{2+} be the distribution of $\{e_t^1 - e_t^2, t \in (0, \nu)\}$ under H_{1+} and let H_{2-} have the same definition relative to H_{1-} . We have, up to a multiplicative constant,

$$(4.2) \quad H_{2+}(A) = \lim_{x \downarrow 0} \frac{1}{|x|} Q_+^x(A),$$

where Q_+^x stands for the distribution of the diffusion Y_t with the same infinitesimal variance as Brownian motion (i.e., σ^2) but with drift $-\beta_1|Y_t|^{\alpha_1}$, killed at the hitting time of 0. We will normalize H_{2+} as in (4.2). We will have to prove that the following formula gives the correct normalization for H_{2-} :

$$(4.3) \quad H_{2-}(A) = \lim_{x \uparrow 0} \frac{1}{|x|} Q_-^x(A).$$

Here Q_-^x denotes the distribution of diffusion Z_t with Brownian quadratic variation (namely, σ^2) and drift $-\beta_2|Z_t|^{\alpha_2}$, killed at the hitting time of 0.

The proof that (4.3) is the correct normalization for H_{2-} can proceed exactly as the proof of Lemma 3.2(iv), thanks to Lemma 4.2. It only remains to find and compare the formulas analogous to those for $H_{2+}(F_h)$ and $H_{2-}(F_h)$. Recall that F_h is the event that the difference between the maximum and the minimum of an excursion exceeds h . The scale function for a diffusion on $(0, \infty)$ with infinitesimal drift $\mu(x) = -\beta_1 x^{\alpha_1}$ and variance σ^2 is given by [see Karlin and Taylor (1981), page 194]

$$(4.4) \quad \begin{aligned} S(x) &= \int_1^x \exp\left(-\int_0^y \frac{2\mu(z)}{\sigma^2} dz\right) dy = \int_1^x \exp\left(-\int_0^y \frac{-2\beta_1 z^{\alpha_1}}{\sigma^2} dz\right) dy \\ &= \int_1^x \exp\left(\frac{2\beta_1 y^{\alpha_1+1}}{\sigma^2(\alpha_1+1)}\right) dy. \end{aligned}$$

By (4.2),

$$\begin{aligned} H_{2+}(F_h) &= \lim_{x \downarrow 0} \frac{1}{x} \mathbf{Q}_+^x(T_h < T_0) = \lim_{x \downarrow 0} \frac{1}{x} \frac{S(x) - S(0)}{S(h) - S(0)} \\ &= \lim_{x \downarrow 0} \frac{1}{x} \frac{\int_0^x \exp(2\beta_1 y^{\alpha_1+1}/\sigma^2(\alpha_1+1)) dy}{\int_0^h \exp(2\beta_1 y^{\alpha_1+1}/\sigma^2(\alpha_1+1)) dy} \\ &= \frac{1}{\int_0^h \exp(2\beta_1 y^{\alpha_1+1}/\sigma^2(\alpha_1+1)) dy}. \end{aligned}$$

If we use (4.3), we obtain in the same way

$$H_{2-}(F_h) = \frac{1}{\int_0^h \exp(-2\beta_2 y^{\alpha_2+1}/\sigma^2(\alpha_2+1)) dy},$$

which implies that

$$\lim_{h \rightarrow 0} H_{2+}(F_h)/H_{2-}(F_h) = 1,$$

and this confirms that the normalization in (4.3) is correct.

The probability for the process Y_t starting from δ never to hit 0 is equal to

$$\lim_{b \rightarrow \infty} \frac{S(\delta) - S(0)}{S(b) - S(0)} = \frac{\int_0^\delta \exp(2\beta_1 y^{\alpha_1+1}/\sigma^2(\alpha_1+1)) dy}{\int_0^\infty \exp(2\beta_1 y^{\alpha_1+1}/\sigma^2(\alpha_1+1)) dy}.$$

It follows that $H_{2+}(F_\infty)$, that is, the measure given to positive excursions which do not return to 0, is given by

$$\begin{aligned} (4.5) \quad & \lim_{\delta \rightarrow 0} \frac{1}{\delta} \frac{\int_0^\delta \exp(2\beta_1 y^{\alpha_1+1}/\sigma^2(\alpha_1+1)) dy}{\int_0^\infty \exp(2\beta_1 y^{\alpha_1+1}/\sigma^2(\alpha_1+1)) dy} \\ &= \left[\int_0^\infty \exp\left(\frac{2\beta_1 y^{\alpha_1+1}}{\sigma^2(\alpha_1+1)}\right) dy \right]^{-1} \\ &= \frac{(-2\beta_1/\sigma^2(\alpha_1+1))^{1/(\alpha_1+1)}(\alpha_1+1)}{\Gamma(1/(\alpha_1+1))} \\ &= \frac{(-2\beta_1)^{1/(\alpha_1+1)}(\alpha_1+1)^{\alpha_1/(\alpha_1+1)}}{\sigma^{2/(\alpha_1+1)}\Gamma(1/(\alpha_1+1))} =_{\text{df}} \lambda_1. \end{aligned}$$

An analogous formula holds for $\lambda_2 =_{\text{df}} H_{2-}(F_\infty)$. The processes of excursions on both sides are independent so the probability of the negative bifurcation is the same as the probability that the first arrival of an infinite excursion in the Poisson process on the negative side comes before the analogous event on the other side. The probability in question is the ratio of λ_1 and $\lambda_1 + \lambda_2$. \square

REMARK 4.3. Mike Harrison pointed out that Theorem 4.1 may be proved without using excursion theory. One can calculate the probability that the diffusion $X_t - B_t$ will go to infinity using an explicit formula for the scale function of this diffusion. The excursion theory approach has its advantages,

though. First, excursion theory seems to be the right tool for the proof of Theorem 4.4 below. Second, the excursion theory may be used to find the positive bifurcation probability when the vector process (X_t, B_t) is Markov but $X_t - B_t$ is not. The solution of (1.2), studied in Burdzy, Frankel and Pauzner (1998), is an example of such a situation.

Let T_* denote the bifurcation time; that is, let T_* be the supremum of t with $X_t = B_t$.

THEOREM 4.4. *Consider the solution to (1.1) with $t_0 = x_0 = 0$, $\beta_1 < 0$ and $\beta_2 > 0$. Then*

$$\mathbb{E}T_* = \frac{\sigma^2}{|2\beta_1\beta_2|}.$$

PROOF. By Remark 3.3(ii), the distribution of the excursion law on excursions with finite lifetime remains the same if we change β to $-\beta$. Hence, (3.1) applies in the case $\beta_1 < 0$ and $\beta_2 > 0$, and we have

$$\mathbb{E}V_s = \left(\frac{1}{|\beta_1|} + \frac{1}{|\beta_2|} \right) s,$$

for $V_s = \inf\{t \geq 0: L_t \geq s\}$. By excursion theory, the infinite excursion of B_t from X_t^0 occurs independently from finite excursions in the Poisson point process of excursions, so the expected bifurcation time is equal to

$$\mathbb{E}T_* = \int_0^\infty \lambda e^{-\lambda s} \mathbb{E}V_s ds,$$

where λ is the intensity of the Poisson process arrival for infinite excursions. We have

$$\lambda = \frac{2(|\beta_1| + |\beta_2|)}{\sigma^2},$$

from (4.5), taking into account infinite excursions on both sides. It follows that

$$\begin{aligned} \mathbb{E}T_* &= \int_0^\infty \lambda e^{-\lambda s} \mathbb{E}V_s ds = \int_0^\infty \lambda e^{-\lambda s} \left(\frac{1}{|\beta_1|} + \frac{1}{|\beta_2|} \right) s ds = \frac{1}{\lambda} \left(\frac{1}{|\beta_1|} + \frac{1}{|\beta_2|} \right) \\ &= \frac{\sigma^2}{2(|\beta_1| + |\beta_2|)} \left(\frac{1}{|\beta_1|} + \frac{1}{|\beta_2|} \right) = \frac{\sigma^2}{|2\beta_1\beta_2|}. \quad \square \end{aligned}$$

REMARK 4.5. (i) It is also the case that

$$\mathbb{E}T_* = \frac{\sigma^2(\beta_1 + \beta_2)}{2\beta_1\beta_2^2}$$

if $\beta_1, \beta_2 > 0$. We leave the proof to the reader.

(ii) A similar result can be obtained for any values of $\alpha_1, \alpha_2 > -1$, but the formula does not seem to have a compact form, so we only sketch how it can be obtained. The proof of Theorem 4.4 needs two ingredients. One of them is

the expected amount of local time before the infinite excursion occurs. This is equal to the expectation of the minimum of two independent exponential random variables whose expected values are inverses of the quantity in (4.5) [for (α_1, β_1) and (α_2, β_2)].

The second ingredient is the expectation of the inverse local time at s , for the process with finite excursions only. This is equal to s times the expected lifetime of a finite excursion under the excursion law. Here is how we can calculate this quantity. For arbitrary $\alpha_1 > -1$ we write as in (4.4),

$$s(x) = \exp\left(-\frac{2\beta_1 x^{\alpha_1+1}}{\sigma^2(\alpha_1+1)}\right),$$

$$S(x) = \int_1^x \exp\left(-\frac{2\beta_1 y^{\alpha_1+1}}{\sigma^2(\alpha_1+1)}\right) dy.$$

For $0 < x < y < \infty$, the Green function for Brownian motion Y_t with drift $-\beta_1$ (the negative sign is due to restriction of the excursion law to finite excursions) is given by [see Remark 3.3 on page 198 of Karlin and Taylor (1981)]

$$G(x, y) = \frac{2[S(x) - S(0)][S(\infty) - S(y)]}{\sigma^2 s(y)[S(\infty) - S(0)]}.$$

Hence, the expected lifetime of an excursion is equal to

$$\begin{aligned} \lim_{x \downarrow 0} \frac{1}{x} \int_0^\infty G(x, y) dy &= \lim_{x \downarrow 0} \frac{1}{x} \int_0^\infty \frac{2[S(x) - S(0)][S(\infty) - S(y)]}{\sigma^2 s(y)[S(\infty) - S(0)]} dy \\ &= \frac{2}{\sigma^2[S(\infty) - S(0)]} \int_0^\infty \frac{[S(\infty) - S(y)]}{s(y)} dy \\ &= \frac{2}{\sigma^2 \int_0^\infty \exp(-2\beta_1 z^{\alpha_1+1}/\sigma^2(\alpha_1+1)) dz} \\ &\quad \times \int_0^\infty \frac{\int_y^\infty \exp(-2\beta_1 z^{\alpha_1+1}/\sigma^2(\alpha_1+1)) dz}{\exp(-2\beta_1 y^{\alpha_1+1}/\sigma^2(\alpha_1+1))} dy. \end{aligned}$$

Adding this to the analogous quantity for α_2 gives the expected lifetime of a finite excursion under the excursion law.

5. Lipschitz approximations. In this section we will address the question of how well a Lipschitz function can approximate a Brownian path. Our analysis will be based on the fact, proved in Lemma 5.1 below, that a certain solution X_t^* to (1.1) may be looked upon as a Lipschitz approximation to B_t .

LEMMA 5.1. *Assume that $\beta_1 < 0 < \beta_2$. For almost every ω there exists a unique $\bar{x} = \bar{x}(\omega)$ such that the solution $X_0^{\bar{x}}$ to (1.1), that is, the solution satisfying $X_0^{\bar{x}}(\omega) = \bar{x}(\omega)$, has the property that there exist arbitrarily large t with $X_t^{\bar{x}}(\omega) = B_t(\omega)$.*

It is easy to see that if $\beta_1 < 0 < \beta_2$ then with probability 1, all solutions X_t^x have the property that there exist arbitrarily small $t > -\infty$ such that $X_t^x = B_t$. Lemma 2.13 shows that a result analogous to Lemma 5.1 holds when $\beta_2 < 0 < \beta_1$, and we require that the solution intersects the Brownian path for arbitrarily small $t > -\infty$.

PROOF. We will first prove the existence. The law of the iterated logarithm easily implies that for some random $x > 0$, the functions $t \rightarrow x + \beta_2 t$ and $t \rightarrow -x + \beta_1 t$ stay above and below the trajectory of B_t for $t \geq 0$, respectively. This shows that there exist both large and small (random) x such that X_t^x does not intersect the trajectory of B_t for $t > 0$.

Let A be the set of all x such that $X_t^x > B_t$ for all t greater than some $t_1 = t_1(x)$. By Lemma 3.1(i) and the above remarks, the set A is a nonempty semiinfinite interval. We will show that it is open. Consider an x such that $X_t^x > B_t$ for all t greater than some t_1 . Then $X_t^x = X_{t_1}^x + \beta_2(t - t_1)$ for $t > t_1$. Let $c_1 = \inf\{X_t^x - B_t; t > t_1 + 1\}$ and note that $c_1 > 0$, by the continuity of $X_t^x - B_t$. By Lemma 3.1(ii), the function $y \rightarrow X_{t_1+1}^y$ is continuous so we can find $\varepsilon > 0$ such that $X_{t_1+1}^y > X_{t_1+1}^x - c_1/2$ for all $y > x - \varepsilon$. It follows easily that for such y , we have $X_t^y = X_{t_1+1}^y + \beta_2(t - t_1 - 1)$ and so $X_t^y > B_t$ for $t > t_1 + 1$. This proves that A is open. The same is true of the set A' of x 's with the property that $X_t^x < B_t$ for all t greater than some $t_1 = t_1(x)$. Hence, $(A \cup A')^c$ is nonempty and so we must have at least one x for which $X_t^x = B_t$ for arbitrarily large t .

We turn to the proof of uniqueness. Since $\beta_1 < 0 < \beta_2$, for every fixed x , there will occur a bifurcation at a finite time for the solution X_t^x . This holds for all rational x simultaneously, a.s. If it were true that for two distinct y and z , the solutions X_t^y and X_t^z intersected B_t for arbitrary large t , then the same would be true for a solution X_t^x , for some rational x between y and z . Such an event has probability 0. \square

Consider (1.1) with $-\beta_1 = \beta_2 = \beta > 0$. Let X_t^* denote the solution of (1.1) constructed in Lemma 5.1. That is, $X_t^* = X_t^{\bar{x}}$.

LEMMA 5.2. *We have with probability 1,*

$$\begin{aligned} \limsup_{t \rightarrow -\infty} \frac{X_t^* - B_t}{\log |t|} &= \limsup_{t \rightarrow \infty} \frac{X_t^* - B_t}{\log t} \\ &= \limsup_{t \rightarrow -\infty} \frac{B_t - X_t^*}{\log |t|} \\ &= \limsup_{t \rightarrow \infty} \frac{B_t - X_t^*}{\log t} \geq \frac{\sigma^2}{2\beta}. \end{aligned}$$

Note that $\liminf_{t \rightarrow -\infty} |X_t^* - B_t|/\log |t| = 0$ as X_t^* crosses B_t for arbitrarily large t , and the same is true for the other analogous \liminf 's.

PROOF. Let \tilde{X}_t be a solution to (1.1) with $t_0 = x_0 = 0$ and $-\beta_1 = \beta_2 = -\beta$, and let $Y_t = B_t - \tilde{X}_t$. The process Y_t is a diffusion which spends zero time on the real axis, which behaves like Brownian motion with drift β when $Y_t < 0$, and it is a Brownian motion with drift $-\beta$ when $Y_t > 0$. By Karlin and Taylor [(1981), Chapter 15.5, (5.34)], the process Y_t has a stationary probability distribution with a density

$$\psi(y) = \frac{\beta}{\sigma^2} \exp\left(-\frac{2\beta|y|}{\sigma^2}\right).$$

Let $\{\hat{Y}_t, t \in \mathbb{R}\}$ be the process which has density $\psi(y)$ for every fixed t and which has the transition probabilities of Y_t . The initial distributions of processes \hat{Y}_t and Y_t are different because $Y_0 = 0$. Let

$$\hat{X}_t = -\hat{Y}_0 + \int_0^t \text{sgn}(\hat{Y}_s) \beta ds$$

and

$$\hat{B}_t = \hat{Y}_t + \hat{X}_t.$$

It is easy to check that \hat{B}_t is a Brownian motion with $\hat{B}_0 = 0$ and that \hat{X}_t solves (1.1) with B_t replaced by \hat{B}_t and $\beta_1 = -\beta_2 = \beta$. Moreover, \hat{X}_t has the property that $\hat{X}_t = \hat{B}_t$ for infinitely many arbitrarily large negative and arbitrarily large positive t . This follows for large positive t from the triviality of the tail σ -field. The same argument applies to large negative t by time reversal. If we now time-reverse \hat{B}_t and \hat{X}_t , we will obtain a Brownian motion and a corresponding solution to (1.1) which satisfies the defining properties of X_t^* . Hence, we may construct B_t and the corresponding process X_t^* by letting $B_t = \hat{B}_{-t}$ and $X_t^* = \hat{X}_{-t}$.

The scale function $S(y)$ for Brownian motion with drift $-\beta$ is given by $S(y) = \exp(2\beta y/\sigma^2)$ [Karlin and Taylor (1981), Chapter 15.4]. Let T_a be the hitting time of a by the process Y . The mass $H(F_h)$ given by the excursion law for the process \hat{Y}_t to positive excursions whose height exceeds h is equal to

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{P}^\varepsilon(T_h < T_0) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \frac{S(\varepsilon) - S(0)}{S(h) - S(0)} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \frac{\exp(2\beta\varepsilon/\sigma^2) - 1}{\exp(2\beta h/\sigma^2) - 1} \\ &= \frac{2\beta}{\sigma^2} \frac{1}{\exp(2\beta h/\sigma^2) - 1}. \end{aligned}$$

Fix some small $\varepsilon > 0$ and let $h_k = k \log 2(1 - \varepsilon)\sigma^2/(2\beta)$. Let L_t denote the local time of \hat{Y}_t at 0 with $L_0 = 0$, and let A_k denote the event that there exists a positive excursion of \hat{Y}_t whose height exceeds h_k and which starts at a time t such that $2^k \leq L_t < 2^{k+1}$. The probability of A_k is the probability that a Poisson random variable with mean $\lambda_k = 2^k H(F_{h_k})$ takes a nonzero value.

Thus, $\mathbb{P}(A_k^c) = e^{-\lambda_k}$. For large k ,

$$\begin{aligned}\lambda_k &= 2^k \frac{2\beta}{\sigma^2} \frac{1}{\exp(2\beta h_k/\sigma^2) - 1} \geq 2^k \frac{2\beta}{\sigma^2} \exp\left(\frac{-2\beta h_k}{\sigma^2}\right) \\ &= 2^k \frac{2\beta}{\sigma^2} \exp\left(-2 \frac{\beta}{\sigma^2} \frac{k \log 2(1-\varepsilon)\sigma^2}{2\beta}\right) = 2^k \frac{2\beta}{\sigma^2} 2^{-k(1-\varepsilon)} = \frac{2\beta}{\sigma^2} 2^{k\varepsilon}.\end{aligned}$$

This implies that $\sum_k \mathbb{P}(A_k^c) = \sum_k e^{-\lambda_k} < \infty$. By the Borel–Cantelli lemma, only a finite number of the events A_k^c occur. Hence,

$$\begin{aligned}\limsup_{t \rightarrow \infty} \frac{\widehat{Y}_t}{\log L_t} &\geq \limsup_{k \rightarrow \infty} \sup \left\{ \frac{\widehat{Y}_t}{\log L_t} : L_t \in [2^k, 2^{k+1}] \right\} \\ &\geq \limsup_{k \rightarrow \infty} \sup \left\{ \frac{\widehat{Y}_t}{\log 2^{k+1}} : L_t \in [2^k, 2^{k+1}] \right\} \\ &\geq \limsup_{k \rightarrow \infty} \frac{h_k}{(k+1) \log 2} \\ &= \limsup_{k \rightarrow \infty} \frac{k \log 2(1-\varepsilon)\sigma^2}{2\beta(k+1) \log 2} \\ &= \frac{(1-\varepsilon)\sigma^2}{2\beta}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily small and, by Proposition 3.4, $\lim_{t \rightarrow \infty} L_t/t = \beta/2$, a.s., we obtain, with probability 1,

$$\limsup_{t \rightarrow \infty} \frac{\widehat{Y}_t}{\log t} = \frac{\sigma^2}{2\beta}.$$

A similar argument yields

$$-\liminf_{t \rightarrow \infty} \frac{\widehat{Y}_t}{\log t} = \limsup_{t \rightarrow -\infty} \frac{\widehat{Y}_t}{\log |t|} = -\liminf_{t \rightarrow -\infty} \frac{\widehat{Y}_t}{\log |t|} = \frac{\sigma^2}{2\beta}.$$

Recall from the first part of the proof that $\widehat{Y}_t = B_{-t} - X_{-t}^* - \widehat{Y}_0$. This combined with the results for \widehat{Y}_t implies the proposition. \square

The function $t \rightarrow a + \beta|t|$ is Lipschitz with constant β . For some random a , this function is greater than B_t for every t , by the law of the iterated logarithm. Since the infimum of an arbitrary family of Lipschitz functions with constant β is again a Lipschitz function with constant β , there exists a smallest Lipschitz function Z_t^+ with constant β with the property that $Z_t^+ \geq B_t$ for all t . Let Z_t^- be the largest Lipschitz function with constant β such that $Z_t^- \leq B_t$ for all t . Note that Z_t^+ and Z_t^- are not measurable with respect to $\sigma\{B_s, s \leq t\}$.

LEMMA 5.3. *Assume that $B_0 = 0$. We have with probability 1,*

$$\limsup_{t \rightarrow -\infty} \frac{Z_t^+ - Z_t^-}{\log t} = \limsup_{t \rightarrow \infty} \frac{Z_t^+ - Z_t^-}{\log t} \leq \frac{\sigma^2}{2\beta}.$$

PROOF. Consider $a_1, a_2 > 0$. Let

$$\begin{aligned} A_{++} &= \{\exists t > 0: B_t = a_1 + \beta t\}, & A_{+-} &= \{\exists t > 0: B_t = -a_2 - \beta t\}, \\ A_{-+} &= \{\exists t < 0: B_t = a_1 - \beta t\}, & A_{--} &= \{\exists t < 0: B_t = -a_2 + \beta t\}. \end{aligned}$$

The probability that B_t ever hits the line $t \rightarrow a_1 + \beta t$ is equal to $\exp(-2a_1\beta/\sigma^2)$ [Karlin and Taylor (1975), page 362]. The probability that B_t crosses the line $a_1 + \beta t$ at some $t_1 > 0$ and then crosses the line $-a_2 - \beta t$ for some $t > t_1$ is bounded by $\exp(-2a_1\beta/\sigma^2) \exp(-2a_2\beta/\sigma^2)$, by the strong Markov property applied at t_1 . The probability of crossing first $-a_2 - \beta t$ and then $a_1 + \beta t$ is bounded by the same quantity. Hence,

$$\mathbb{P}(A_{++} \cap A_{+-}) \leq 2 \exp(-2(a_1 + a_2)\beta/\sigma^2).$$

The same estimate holds for $\mathbb{P}(A_{--} \cap A_{-+})$, by symmetry. We obtain

$$\mathbb{P}(A_{++} \cap A_{--}) = \mathbb{P}(A_{-+} \cap A_{+-}) = \exp(-2(a_1 + a_2)\beta/\sigma^2),$$

from the independence of the processes $\{B_t, t \geq 0\}$ and $\{B_t, t \leq 0\}$. It follows that

$$\begin{aligned} &\mathbb{P}(Z_0^+ - B_0 \geq a_1, B_0 - Z_0^- \geq a_2) \\ &= \mathbb{P}(Z_0^+ \geq a_1, Z_0^- \leq -a_2) \\ &\leq \mathbb{P}([A_{++} \cap A_{+-}] \cup [A_{--} \cap A_{-+}] \cup [A_{++} \cap A_{--}] \cup [A_{-+} \cap A_{+-}]) \\ &\leq 8 \exp(-2(a_1 + a_2)\beta/\sigma^2). \end{aligned}$$

Choose $\varepsilon \in (0, 1)$. Let $m > 8$ be an integer large enough that $(m-1)/(m(1-\varepsilon)) > 1$. We have for any $y > 0$,

$$\begin{aligned} \mathbb{P}(Z_0^+ - Z_0^- \geq y) &\leq \sum_{j=0}^m \mathbb{P}\left(Z_0^+ - B_0 \geq \frac{jy}{m}, B_0 - Z_0^- \geq \frac{(m-j-1)y}{m}\right) \\ &\leq \sum_{j=0}^m 8 \exp\left(-2\left(\frac{jy}{m} + \frac{(m-j-1)y}{m}\right) \frac{\beta}{\sigma^2}\right) \\ &\leq 9m \exp\left(\frac{-2(m-1)y\beta}{m\sigma^2}\right). \end{aligned}$$

Fix some large $b < \infty$. Consider an integer $k > 0$. Let n be the integer part of

$$\frac{2\beta(1-\varepsilon)b2^k}{\sigma^2 k \log 2},$$

and let $x_k = b2^k/n$, and $t_j^k = j2^k/n$. We have

$$\begin{aligned}
\mathbb{P}(Z_{t_j^k}^+ - Z_{t_j^k}^- \geq x_k) &= \mathbb{P}(Z_0^+ - Z_0^- \geq x_k) \\
&\leq 9m \exp\left(\frac{-2(m-1)x_k\beta}{m\sigma^2}\right) \\
&= 9m \exp\left(-\frac{(m-1)2\beta b2^k}{mn\sigma^2}\right) \\
&\leq 9m \exp\left(-\frac{(m-1)2\beta b2^k\sigma^2 k \log 2}{m2\beta(1-\varepsilon)b2^k\sigma^2}\right) \\
&= 9m \exp\left(\frac{-k(m-1)\log 2}{m(1-\varepsilon)}\right) = 9m \cdot 2^{-k(m-1)/(m(1-\varepsilon))}.
\end{aligned}$$

For some $c_1 < \infty$, using $(m-1)/(m(1-\varepsilon)) > 1$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty} \sum_{0 \leq t_j^k \leq 2^k} \mathbb{P}(Z_{t_j^k}^+ - Z_{t_j^k}^- \geq x_k) \\
&\leq c_1 \sum_{k=1}^{\infty} 2n \cdot 9m \cdot 2^{-k(m-1)/(m(1-\varepsilon))} \\
&\leq c_1 \sum_{k=1}^{\infty} \frac{2\beta(1-\varepsilon)b2^k}{\sigma^2 k \log 2} \cdot 36m \cdot 2^{-k(m-1)/(m(1-\varepsilon))} < \infty.
\end{aligned}$$

By the Borel–Cantelli lemma, for all sufficiently large k and all $t_j^k \in [0, 2^k]$, we have $Z_{t_j^k}^+ - Z_{t_j^k}^- \leq x_k$. If $Z_{t_j^k}^+ - Z_{t_j^k}^- \leq x_k$ then for $t \in [(t_j^k + t_{j-1}^k)/2, (t_j^k + t_{j+1}^k)/2]$,

$$Z_t^+ - Z_t^- \leq x_k + 2\beta|t - t_j| \leq b2^k/n + \beta2^k/n = (b2^k/n)(1 + \beta/b) = x_k(1 + \beta/b).$$

This implies that for large k , we have for all $t \in [0, 2^k]$,

$$Z_t^+ - Z_t^- \leq x_k(1 + \beta/b).$$

We obtain

$$\begin{aligned}
\limsup_{t \rightarrow \infty} \frac{Z_t^+ - Z_t^-}{\log t} &\leq \limsup_{k \rightarrow \infty} \sup_{t \in [2^{k-1}, 2^k]} \frac{Z_t^+ - Z_t^-}{\log t} \\
&\leq \limsup_{k \rightarrow \infty} \sup_{t \in [2^{k-1}, 2^k]} \frac{Z_t^+ - Z_t^-}{\log 2^{k-1}} \\
&\leq \limsup_{k \rightarrow \infty} \frac{x_k(1 + \beta/b)}{(k-1)\log 2} \\
&\leq \limsup_{k \rightarrow \infty} \frac{(b2^k/n)(1 + \beta/b)}{(k-1)\log 2}
\end{aligned}$$

$$\begin{aligned} &\leq \limsup_{k \rightarrow \infty} \frac{b2^k \sigma^2 k \log 2(1 + \beta/b)}{2\beta(1 - \varepsilon)b2^k(k - 1) \log 2} \\ &= \frac{\sigma^2(1 + \beta/b)}{2\beta(1 - \varepsilon)}. \end{aligned}$$

Since ε may be chosen arbitrarily small and b may be chosen arbitrarily large, with probability 1,

$$\limsup_{t \rightarrow \infty} \frac{Z_t^+ - Z_t^-}{\log t} \leq \frac{\sigma^2}{2\beta}.$$

The result for $t \rightarrow -\infty$ follows by symmetry. \square

THEOREM 5.4. (i) *With probability 1,*

$$\begin{aligned} \limsup_{t \rightarrow -\infty} \frac{X_t^* - B_t}{\log t} &= \limsup_{t \rightarrow \infty} \frac{X_t^* - B_t}{\log t} \\ &= \limsup_{t \rightarrow -\infty} \frac{B_t - X_t^*}{\log t} = \limsup_{t \rightarrow \infty} \frac{B_t - X_t^*}{\log t} \\ &= \limsup_{t \rightarrow -\infty} \frac{Z_t^+ - Z_t^-}{\log t} = \limsup_{t \rightarrow \infty} \frac{Z_t^+ - Z_t^-}{\log t} = \frac{\sigma^2}{2\beta}. \end{aligned}$$

(ii) $\mathbb{E}|B_t - X_t^*| = \frac{1}{2}\sigma^2/\beta$, for every $t \in \mathbb{R}$.

(iii) $\mathbb{E}(Z_t^+ - B_t) = \mathbb{E}(B_t - Z_t^-) = \frac{3}{4}\sigma^2/\beta$, for every $t \in \mathbb{R}$.

Theorem 5.4(i) shows, in a sense, that Z^+ and Z^- are as good Lipschitz approximations to B_t as X^* . However, the comparison comes out differently when we look at the averages presented in (ii) and (iii).

PROOF. It is elementary to check that we always have $Z_t^- \leq X_t^* \leq Z_t^+$. This and Lemmas 5.2 and 5.3 yield (i).

Recall the stationary density $\psi(y)$ for Y_t from the proof of Lemma 5.2. This is the same as the density for the distribution of $B_t - X_t^*$. Hence

$$\mathbb{E}|B_t - X_t^*| = \int_{-\infty}^{\infty} |y| \frac{\beta}{\sigma^2} \exp\left(-\frac{2\beta|y|}{\sigma^2}\right) dy = \frac{1}{2} \frac{\sigma^2}{\beta},$$

which proves (ii).

For $a > 0$, the probability that B_t crosses the line $a + \beta t$ for some $t > 0$ is equal to $\exp(-2a\beta/\sigma^2)$ [Karlin and Taylor (1975), page 362]. This is the same as the probability of crossing the line $a - \beta t$ for some $t < 0$. The probability that none of these events happen is $[1 - \exp(-2a\beta/\sigma^2)]^2$, and so

$$\mathbb{P}(Z_0^+ < a) = [1 - \exp(-2a\beta/\sigma^2)]^2.$$

This yields

$$\mathbb{E}Z_0^+ = \frac{3\sigma^2}{4\beta}.$$

We similarly have $\mathbb{E}Z_0^- = -3\sigma^2/(4\beta)$, and by translation invariance, for every t ,

$$\mathbb{E}(Z_t^+ - B_t) = \mathbb{E}(B_t - Z_t^-) = \frac{3\sigma^2}{4\beta}. \quad \square$$

If we let $\alpha_1 = \alpha_2 = 1$ and choose suitable β_1 and β_2 in (1.3), then $Y_t = X_t - B_t$ is an Ornstein–Uhlenbeck process. Results for such a process, closely related to Theorem 5.4(i), can be found in the paper of Darling and Erdős (1956).

COROLLARY 5.5. *For any random Lipschitz function $g(t)$ with constant β we have with probability 1,*

$$\limsup_{t \rightarrow \infty} \frac{g(t) - B_t}{\log t} \geq \frac{\sigma^2}{4\beta}.$$

PROOF. Suppose that $X_t^* - B_t = a$ for some t and $a > 0$. Let s be the largest time less than t such that $B_s = X_s^*$. Then we see that the quantity $\sup_{s \leq u \leq t} |g(u) - B_u|$ cannot be smaller than $a/2$ for any Lipschitz function $g(u)$ with constant β , by comparing $g(u)$ with the function $u \rightarrow B_s + a/2 + (u-s)\beta$. Since $\limsup_{t \rightarrow \infty} (X_t^* - B_t)/\log t = \sigma^2/(2\beta)$, for any Lipschitz function $g(t)$ with constant β , we must have

$$\limsup_{t \rightarrow \infty} \frac{g(t) - B_t}{\log t} \geq \frac{\sigma^2}{4\beta}. \quad \square$$

Corollary 5.5 sheds some new light on an old problem about strong approximations. Let us assume that $\sigma^2 = 1$, that is, we will consider now only standard Brownian motion. Suppose that $\{V_k\}_{k \geq 1}$ are i.i.d. random variables such that $|V_k| \leq \beta$, a.s. Let $S_n = \sum_{k=1}^n V_k$ and extend the function $n \rightarrow S_n$ to all positive real values by linear interpolation between S_n and S_{n+1} . Note that the random function S_t is Lipschitz with constant β .

The following is an immediate consequence of Corollary 5.5.

THEOREM 5.6. *Suppose that V_k and S_t are as above. If S_t and B_t are constructed on the same probability space (but not necessarily independent), then*

$$(5.1) \quad \limsup_{t \rightarrow \infty} \frac{S_t - B_t}{\log t} \geq \frac{1}{4\beta}.$$

Theorem 2.3.2 of Csörgő and Révész (1981) says that if the V_k have finite variance and

$$\limsup_{t \rightarrow \infty} \frac{|S_t - B_t|}{\log t} = 0,$$

then the V_k have a standard normal distribution. Our result (5.1) may be interpreted as a quantitative version of the same theorem, in the case when $|V_k|$ are bounded. A remarkable theorem of Komlós, Major and Tusnády [see

Csörgő and Révész (1981), Theorem 2.6.1] implies that if the V_k are bounded, then one may construct S_t and B_t on a common probability space so that

$$(5.2) \quad \limsup_{t \rightarrow \infty} \frac{|S_t - B_t|}{\log t} \leq C < \infty.$$

It is striking that one can achieve the same logarithmic order of approximation for a Lipschitz function S_t with independent increments $S_n - S_{n-1}$, as for an arbitrary Lipschitz function $g(t)$ with constant β . Rio (1991) proved that (5.2) holds with $C = 9/2$ if V_k are centered Poisson variables [the estimate had appeared in Section 5 of the preprint; that section was not included in the final version of the article, Rio (1994)]. No other estimates for C seem to be known so (5.1) is our own modest contribution to the field of strong approximations.

6. Open problems. We list a few questions we were not able to answer in this paper.

1. Can one prove pathwise uniqueness in Theorem 2.1 if one or both α_1 and α_2 belong to $(-1, 0)$?
2. Does a result analogous to Theorem 3.8 hold for $\beta_1, \beta_2 > 0$ with $\beta_1 - \beta_2 < 0$? A similar question can be asked about the case when $\beta_1 < 0 < \beta_2$; in the last case a special solution to (1.1), defined in Lemma 5.1, would have to play an important role. Can one generalize Theorem 3.8 to local times corresponding to solutions of (1.3) with α_1 and α_2 not necessarily equal to 0? Can one prove similar results in the case $\beta_1 < 0 < \beta_2$ but for the process $\{L_T^x, x \in \mathbb{R}\}$, with T a stopping time for B_t , for example the first hitting time of a given value?
3. Find the best $\gamma = \gamma(\alpha_1, \alpha_2, \beta_1, \beta_2) > 0$ in Lemma 4.2.
4. Find the best constants in (5.1) and (5.2).
5. Does there exist a unique Lipschitz solution to (2.3) if B_t is a fractional Brownian motion of index $H \in (1/2, 1)$?

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