UNIQUENESS FOR THE SKOROKHOD PROBLEM IN AN ORTHANT: CRITICAL CASES

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Abstract. Consider the Skorokhod problem in the closed non-negative orthant: find a solution $(g(t), m(t))$ to

$$g(t) = f(t) + Rm(t),$$

where f is a given continuous vector-valued function with $f(0)$ in the orthant, R is a given $d \times d$ matrix with 1's along the diagonal, g takes values in the orthant, and m is a vector-valued function that starts at 0, each component of m is non-decreasing and continuous, and for each i the i^{th} coordinate of m increases only when the i^{th} coordinate of g is 0. The stochastic version of the Skorokhod problem replaces f by the paths of Brownian motion. It is known that there exists a unique solution to the Skorokhod problem if the spectral radius of $|Q|$ is less than 1, where $Q = I - R$ and $|Q|$ is the matrix whose entries are the absolute values of the corresponding entries of Q. The first result of this paper shows pathwise uniqueness for the stochastic version of the Skorokhod problem holds if the spectral radius of $|Q|$ is equal to 1. The second result of this paper settles the remaining open cases for uniqueness for the deterministic version when the dimension d is two.

1. Introduction

sect-intro

We consider both the deterministic and stochastic versions of the Skorokhod problem in the d-dimensional orthant $D = \{(x_1, \dots, x_d) : x_i \geq 0, i = 1, \dots, d\}$ with oblique reflection on the boundary of D that is constant on each face of D.

First we define the deterministic Skorokhod problem. Let R be a fixed $d \times d$ matrix.

intro-SP **Definition 1.1.** A driving function f is a continuous function from $[0, \infty)$ to $\mathbb{R}^d$ with $f(0) \in D$. The deterministic Skorokhod problem is to find $g(t)$ and $m(t)$ so that
(1) g is a continuous function from $[0, \infty)$ to D;
(2) m is a continuous d dimensional function on $[0, \infty)$ with $m(0) = 0$ and each component of m is non-decreasing;
(3) the Skorokhod equation holds, namely, $g(t) = f(t) + Rm(t)$ for all $t \geq 0$; and
(4) m_j increases only at those time t when $g_j(t) = 0$, $j = 1, \dots, d$.

The stochastic Skorokhod problem is very similar, but with the driving function replaced by a Brownian motion. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space with $\{\mathcal{F}_t\}$ being a right continuous complete filtration.

skor-prob **Definition 1.2.** Let $B(t)$ be standard d-dimensional Brownian motion in $\mathbb{R}^d$ started at $x_0 \in D$ and adapted to $\{\mathcal{F}_t\}$. The stochastic Skorokhod problem is to find $X(t)$ and

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$M(t)$, adapted to $\{\mathcal{F}_t\}$, so that almost surely
(1) X is a continuous d-dimensional vector-valued process taking values in D;
(2) M is a continuous d dimensional vector-valued process with $M(0) = 0$ and each component is non-decreasing;
(3) the Skorokhod equation holds, namely, $X(t) = B(t) + RM(t)$ for all $t \geq 0$; and
(4) M_j increases only at those times t when $X_j(t) = 0$.

Nothing precludes $\{\mathcal{F}_t\}$ being strictly larger than the filtration generated by B. We refer to X as obliquely reflecting Brownian motion (ORBM) in D. The process M_i is referred to as the local time of X on the i^{th} face $D_i = \{x \in D : x_i = 0\}$. When $X(t) \in D_i$, the direction of reflection is given by the i^{th} row of R. An edge is a set of the form $D_j \cap D_k$ for some $j \neq k$. For the cases we are interested in, no M_i charges any edge (that is, if E is an edge, then $\int_0^t 1_E(X_s)\, dM_i(s) = 0$ for all t almost surely) and so the direction of reflection will be immaterial when $X(t)$ is at an edge as long as it is deterministic and points into the interior of D; see [RW88].

ORBM is the limit process in many queueing theory models, and there is a large literature concerning this topic. See [Wil95] for some of the references. The model has also been of independent interest in probability theory and is related to partial differential equations with oblique boundary conditions.

For a vector v, writing $v > 0$ means each component of v is positive, $v \geq 0$ that each component is non-negative. A matrix R is an $\mathcal{S}$-matrix if there exists $x \geq 0$ with $Rx > 0$. A principal submatrix of D is a square matrix obtained from R by deleting some of the rows and corresponding columns of R. A matrix R is called completely-$\mathcal{S}$ if R and all of its principal submatrices are $\mathcal{S}$-matrices. A necessary and sufficient condition for the deterministic Skorokhod problem to have a solution for all continuous driving functions f is that R be completely-$\mathcal{S}$; see [BeK91], [MVdH87], and [DW96]. In Remark 1.6 we show that, as far as questions of existence and uniqueness of solutions to the Skorokhod problem go, we may assume that all the diagonal elements of R are equal to 1. We make that assumption throughout the remainder of the paper.

There has been considerable interest in when the solutions to the deterministic and stochastic Skorokhod problems are unique. Harrison and Reiman [HR81] proved that uniqueness holds for both problems (pathwise uniqueness in the stochastic case) if $Q = I - R$ is the transition matrix of a class of transient Markov chains and Williams [Wil95] observed that the proof actually holds provided the spectral radius of $|Q|$ is strictly less than 1, where $|Q|$ is the matrix whose entries are the absolute values of the corresponding entries of Q.

The first main result of this paper concerns pathwise uniqueness for the stochastic Skorokhod equation, and can be considered a critical case since we look at when the spectral radius of $|Q|$ is exactly 1.

main-theorem **Theorem 1.3.** *Let B be a Brownian motion adapted to $\{\mathcal{F}_t\}$. Suppose R is completely-$\mathcal{S}$ and the spectral radius of $|Q|$ is less than or equal to 1. If (X, M) and (X', M') are two solutions to the Skorokhod problem given by Definition 1.2, then almost surely $(X(t), M(t)) = (X'(t), M'(t))$ for all t. Moreover there exists a solution (X, M) to the Skorokhod equation that is adapted to the filtration generated by B.*

The proofs in [HR81, Wil95] use the contraction mapping principle and require that the spectral radius of $|Q|$ be strictly less than 1. The proof of Theorem 1.3 is necessarily quite different. Our tools include a fixed point theorem of Ishikawa [Ish76] as well as a decomposition of Q analogous to that of a finite state Markov chain.

The current authors recently showed in [BB24] that there is a large class of 2×2 matrices which are completely-$\mathcal{S}$ but for which pathwise uniqueness for the corresponding ORBM does not hold.

The questions of uniqueness in law for the solution of the stochastic Skorokhod problem and the strong Markov property for the set of solutions as the starting point varies have been settled by [TW93].

One can equivalently write the Skorokhod equation in terms of a real-valued local time; see [BB24, Section 2]. Also for matrices R that are not completely-$\mathcal{S}$, one can characterize ORBMs that correspond to R, but they will not be solutions to the Skorokhod equation; see [VW85].

The proof of Theorem 1.3 for the case where $|Q|$ is what is known as an irreducible matrix is given in Section 2. The general case is given in Section 3.

Our second main result concerns uniqueness for the deterministic Skorokhod problem equation in two dimensions and consists of two theorems. Suppose R has the form $R=\begin{pmatrix}1 & a_1\\ a_2 & 1\end{pmatrix}$.

There are five cases to consider:
(1) $|a_1a_2|<1$;
(2) $|a_1a_2|=1$, a_1,a_2 are of opposite signs;
(3) $|a_1a_2|=1$, a_1,a_2 are both positive;
(4) $|a_1a_2|>1$, a_1,a_2 are of opposite signs;
(5) $|a_1a_2|>1$, a_1,a_2 are both positive.
(It is easy to see that R will not be completely-$\mathcal{S}$ if $|a_1a_2|\geq 1$ and a_1,a_2 are both negative.)

Uniqueness holds in Case (1) by [HR81, Wil95]. Mandelbaum [Man87] gave a brief sketch to show uniqueness holds for Case (2). He also provided in [Man87] a detailed proof for a counterexample to show uniqueness fails in Case (4).

Mandelbaum's paper [Man87] has never been published. However an exposition of his results for Cases (2) and (4) may be found in [Bas24]. The counterexample (Case (4)) has been also presented in the Ph.D. thesis of Whitley [Whi03]. Stewart [Ste09] used similar methods to prove a counterexample in a related problem. See also [BeK91] for a counterexample to uniqueness when the dimension is 3.

Our first theorem covers Case (3), which may be considered a critical case.

T1-uniq **Theorem 1.4.** *Suppose $a_1>0, a_2>0, a_1a_2=1$, f is a continuous driving function, and (g,f,m) and $(\overline{g},f,\overline{m})$ are two solutions to the deterministic Skorokhod problem with matrix $R=\begin{pmatrix}1 & a_1\\ a_2 & 1\end{pmatrix}$. Then $g(t)=\overline{g}(t)$ for all t.*

Although g is uniquely determined, m is not: see Remark 4.2. We point out that the argument we use to prove Theorem 1.4 is well known to experts in the field. If we allow discontinuous driving functions f, in particular piecewise constant ones, then there are cases where there will be non-uniqueness for some driving functions ([Man87]).

Our second theorem settles Case (5). This result and its proof are new. Our method was motivated by the proofs in [BB24].

Theorem 1.5. *Suppose* $R = \begin{pmatrix} 1 & a_1 \\ a_2 & 1 \end{pmatrix}$ *and* $a_1 > 0, a_2 > 0, a_1 a_2 > 1$. *Then there is a continuous driving function* f *for which there is more than one solution to the deterministic Skorokhod problem.*

The proofs of Theorems 1.4 and 1.5 are given in Section 4.

Remark 1.6. If R is completely-$\mathcal{S}$, then all the diagonal elements must be positive. If we let $\widetilde{m}_i(t) = R_{ii} m_i(t)$ and $\widetilde{R}_{ij} = R_{ij}/R_{jj}$ for $i, j = 1, \dots, d$, then the Skorokhod equation can be rewritten as $g(t) = f(t) + \widetilde{R}\widetilde{m}(t)$. Note $\widetilde{R}_{ii} = 1$ for each i. Therefore there is no loss of generality in assuming that the diagonal elements of R are all equal to 1. The same remark holds for the stochastic Skorokhod problem.

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2. The irreducible case

A $d \times d$ matrix A is non-negative if each entry of A is non-negative, and we write $A \geq B$ if $A - B$ is non-negative. We denote the spectral radius of A by $\rho(A)$; this is the maximum of the moduli of the eigenvalues of A. Given a matrix A, we denote by $|A|$ the matrix whose entries are the absolute values of the corresponding entries of A, and similarly A^+ for the matrix whose entries are the positive parts of the corresponding entries of A. We write $v > 0$ for a d-dimensional vector v if all the coordinates are positive, $v \geq 0$ if all the coordinates are non-negative. We use $|v|$ for $(\sum_{i=1}^d |v_i|^2)^{1/2}$ and $\|A\| = \sup\{|Av| : |v| \leq 1\}$. Finally we use A^T for the transpose of a matrix A.

A non-negative matrix A is irreducible if for each i, j there exist $k_0, k_1, \dots, k_n$ with $k_0 = i$ and $k_n = j$ such that $A_{k_m, k_{m+1}} > 0$ for $0 \leq m \leq n-1$. The k's and n can depend on i and j. A number of equivalent definitions of irreducibility can be found in [HJ13, Theorem 6.2.24]. If the row sums of A are each equal to 1, then A can be viewed as the transition matrix of a Markov chain, and irreducibility of A is the same as irreducibility in the Markov chain context.

We need a few well-known facts about non-negative matrices.

(1) If $A \geq B \geq 0$, then ([HJ13, Theorem 8.1.18])

$$\rho(A) \geq \rho(B). \tag{2.1}$$

(2) The Gelfand formula ([HJ13, Corollary 5.6.14]):

$$\rho(A) = \lim_{m\to\infty} \|A^m\|^{1/m}. \tag{2.2}$$

(3) The Perron-Frobenius theorem ([HJ13, Theorem 8.4.4]): If $A \geq 0$ is irreducible, then there exists a positive eigenvalue $r = \rho(A)$ and left and right positive eigenvectors y, x such that $y^T A = r y^T$ and $Ax = rx$.
(4) A corollary to the Perron-Frobenius theorem ([HJ13, Theorem 8.3.1]): If $A \geq 0$, then there exists a non-negative non-zero x such that $Ax = \rho(A)x$.

Throughout the remainder of this section we assume $Q = I - R$ and $|Q|$ is irreducible with $\rho(|Q|) \leq 1$. Note that this implies that the diagonal elements of Q are zero.

We need the following elementary lemma.

add-identity **Lemma 2.1.** *If $C \geq 0$ and $\beta \geq 0$, then $\rho(C + \beta I) = \rho(C) + \beta$.*

Proof. By the corollary to the Perron-Frobenius theorem described in (4) above, there exists an eigenvector x for C such that $x \geq 0$, $x \neq 0$, and $Cx = \rho(C)x$. Then $(C + \beta I)x = \rho(C)x + \beta x$, which implies that $C + \beta I$ has an eigenvalue of size $\rho(C) + \beta$, and consequently $\rho(C + \beta I) \geq \rho(C) + \beta$.

On the other hand, $C + \beta I$ will also be non-negative, hence there exists an eigenvector $\widehat{x}$ for $C + \beta I$ such that $\widehat{x} \geq 0$, $\widehat{x} \neq 0$, and $(C + \beta I)\widehat{x} = \rho(C + \beta I)\widehat{x}$. Then

$$C\widehat{x} = (C + \beta I)\widehat{x} - \beta\widehat{x} = \rho(C + \beta I)\widehat{x} - \beta\widehat{x},$$

which implies that C has an eigenvalue of size $\rho(C + \beta I) - \beta$, hence $\rho(C) \geq \rho(C + \beta I) - \beta$. □

In the case $Q \geq 0$ we have the following.

irred-p2 **Proposition 2.2.** *Suppose $Q \geq 0$, R is completely-$\mathcal{S}$, and $\rho(Q) \leq 1$. Then $\rho(Q) < 1$.*

Proof. Recall that we are assuming that $|Q|$ is irreducible in this section. Suppose $\rho(Q) = 1$. By the Perron-Frobenius theorem there exists a left eigenvector $y > 0$ such that $y^T Q = y^T$. Then $y^T R = 0$. Since R is completely-$\mathcal{S}$, there exists $x \geq 0$ such that $Rx > 0$. Now consider $y^T Rx$. On the one hand this equals 0. On the other, because $y > 0$ and $Rx > 0$, then $y^T Rx > 0$, a contradiction. □

irred-L1 **Lemma 2.3.** *Suppose $A \geq B \geq 0$, $A \neq B$, and A is irreducible. Then $\rho(A) > \rho(B)$.*

Proof. Since $A \geq B$ but $A \neq B$, there exist i_0, j_0 such that $A_{i_0 j_0} > B_{i_0 j_0}$. Fix $i \in \{1, \ldots, d\}$ for the moment. There exists $n_1(i)$ such that $A^{n_1(i)}_{i i_0} > 0$ and $n_2(i)$ such that $A^{n_2(i)}_{j_0 i} > 0$. Set $n(i) = n_1(i) + n_2(i) + 1$. Therefore

irr-E1 (2.3)
$$A^{n_1(i)}_{i i_0} A_{i_0 j_0} A^{n_2(i)}_{j_0 i} > B^{n_1(i)}_{i i_0} B_{i_0 j_0} B^{n_2(i)}_{j_0 i}.$$

Since

$$A^{n(i)}_{ii} = \sum A^{n_1(i)}_{i k_1} A_{k_1 k_2} A^{n_2(i)}_{k_2 i},$$

where the sum is over all indices $k_1, k_2 \in \{1, \ldots, d\}$ and a similar expression holds for $B^{n(i)}_{ii}$, using (2.3) we conclude $A^{n(i)}_{ii} > B^{n(i)}_{ii}$.

Suppose K is a positive integer. Then $(A^{n(i)}_{ii})^K > (B^{n(i)}_{ii})^K$. Observe that

j16.1 (2.4)
$$A^{Kn(i)}_{ii} = \sum A^{n(i)}_{i k_1} A^{n(i)}_{k_1 k_2} \cdots A^{n(i)}_{k_{K-2}, k_{K-1}} A^{n(i)}_{k_{K-1} i}$$

where the sum is over all $k_1, \dots, k_{K-1} \in \{1, \dots, d\}$ and a similar expression holds for $B_{ii}^{Kn(i)}$. Each summand in (2.4) is greater than or equal to the corresponding summand for B. However the summand where we have $k_1 = k_2 = \dots = k_{K-1} = i$ is equal to $(A_{ii}^{(n(i))})^K$, and this is strictly larger than $(B_{ii}^{(n(i))})^K$. It follows that $A_{ii}^{Kn(i)} > B_{ii}^{Kn(i)}$.

Let N be the least common multiple of $\{n(1), \dots, n(d)\}$. Then $A_{ii}^N > B_{ii}^N$ for each i, and hence there exists $\varepsilon > 0$ such that

$$A^N \geq B^N + \varepsilon I.$$

Therefore $\rho(A^N) \geq \rho(B^N + \varepsilon I) = \rho(B^N) + \varepsilon$ by (2.1) and Lemma 2.1.

Finally, by (2.2),

$$\rho(A^N) = \lim_{m\to\infty} \|A^{Nm}\|^{1/m} = \lim_{m\to\infty} (\|A^{Nm}\|^{1/Nm})^N = \rho(A)^N,$$

and similarly for B. Our result now follows. □

Fix $t_0 > 0$. Let $\mathcal{C}_d[0, t_0]$ be the set of functions f mapping $[0, t_0]$ to $\mathbb{R}^d$ such that each component of f is continuous. Let $\mathcal{C}_d^*[0, t_0]$ be the set of functions f in $\mathcal{C}_d[0, t_0]$ such that each component of f is non-decreasing.

Let f be a fixed element of $\mathcal{C}_d[0, t_0]$ with $f(0) \geq 0$. We define the non-linear operator T on $\mathcal{C}_d[0, t_0]$ by

def-T (2.5)
$$Tg(t) = \sup_{s\leq t} \Big(Qg(s) - f(s) \Big)^+ .$$

We define the non-linear operator U on $\mathcal{C}_d[0, t_0]$ by

def-U (2.6)
$$Ug(t) = \tfrac{1}{2} g(t) + \tfrac{1}{2} Tg(t).$$

We denote the iterates of U by $U^2 g = U(Ug)$, etc.

irred-T1 **Theorem 2.4.** *Suppose R is completely-$\mathcal{S}$. Let $Q = I - R$. Suppose either (a) $Q = 0$ or (b) $|Q|$ is irreducible and $\rho(|Q|) \leq 1$.*

(i) There exists a constant C_1 such that if $h \in \mathcal{C}_d^*[0, t_0]$ and $h(0) = 0$, then

j23.1 (2.7)
$$|Th(t) - Th(s)| \leq C_1 \Big(|h(t) - h(s)| + \sup_{s\leq u_1<u_2\leq t} |f(u_2) - f(u_1)| \Big).$$

(ii) Suppose $g(0) = 0$ and $g \in \mathcal{C}_d^*[0, t_0]$. For all $n \geq 1$ we have $U^n g \in \mathcal{C}_d^*[0, t_0]$ and $U^n g(0) = 0$. Moreover there exists a constant C_2 not depending on n such that for all $0 \leq s < t \leq t_0$ and all $n \geq 1$ we have

j4.1 (2.8)
$$|U^n g(t) - U^n g(s)| \leq C_2 \Big(|g(t) - g(s)| + \sup_{s\leq u_1<u_2\leq t} |f(u_2) - f(u_1)| \Big).$$

Proof. (i) Suppose $h(0) = 0$ and $h \in \mathcal{C}_d^*[0, t_0]$. Let

$$F_i(s, t) = \sup_{s\leq u_1<u_2\leq t} |f_i(u_2) - f_i(u_1)|.$$

Writing the formula for Th in terms of coordinates, we have

T-coord (2.9)
$$(Th)_j(t) = \sup_{r\leq t} \Big(\sum_{k=1}^{d} Q_{jk} h_k(r) - f_j(r) \Big)^+ .$$

If w is a non-negative function and $0 \le s \le t$, then

$$\sup_{r \le t} w(r) \le \sup_{r \le s} w(r) + \sup_{s \le r \le t} (w(r) - w(s)).$$

This, (2.9) and the inequality $(a+b)^+ \le a^+ + b^+$ imply that

$$(Th)_j(t) - (Th)_j(s) \le \sup_{s \le r \le t} (Q(h_k(r) - h_k(s)))_j + F_j(s,t).$$

All the diagonal entries of Q are 0. If $Q_{jk} < 0$, then $r \to Q_{jk} h_k(r)$ is non-increasing. If $Q_{jk} \ge 0$, then $\sup_{s \le r \le t}(Q(h_k(r) - h_k(s)))_j = (Q(h_k(t) - h_k(s)))_j$. Therefore

Th-bound
$$(Th)_j(t) - (Th)_j(s) \le \sum_{k=1}^{d} Q^+_{jk}(h_k(t) - h_k(s)) + F_j(s,t). \tag{2.10}$$

This implies (2.7) because Q^+ is a fixed matrix not depending on h.

(ii) It is easy to see that for each n, we have $U^n g(0) = 0$, $U^n g$ is non-decreasing, and $U^n g$ is continuous.

In matrix terms, (2.10) can be written

T-diff
$$Th(t) - Th(s) \le Q^+(h(t) - h(s)) + F(s,t). \tag{2.11}$$

Let

$$P = \tfrac{1}{2} Q^+ + \tfrac{1}{2} I.$$

Since $h \in \mathcal{C}^*_d[0, t_0]$,

U-diff
$$Uh(t) - Uh(s) \le P(h(t) - h(s)) + \tfrac{1}{2} F(s,t). \tag{2.12}$$

Applying this to $h = U^n g$ we obtain

$$U^{n+1} g(t) - U^{n+1} g(s) \le P(U^n g(t) - U^n g(s)) + \tfrac{1}{2} F(s,t).$$

An induction argument shows that

T-matrix
$$U^n g(t) - U^n g(s) \le P^n(g(t) - g(s)) + \tfrac{1}{2} \sum_{i=1}^{n} P^{i-1} F(s,t). \tag{2.13}$$

In the case that $Q \ge 0$, we have by Proposition 2.2 that $\rho(Q^+) = \rho(Q) < 1$. In the case where Q has at least one negative entry we use Lemma 2.3 with $A = |Q|$ and $B = Q^+$ to see that $\rho(Q^+) < \rho(|Q|) \le 1$. By Lemma 2.1 we have $\rho(P) < 1$. By (2.2) there exists M such that $\|P^M\| < 1$. For any n we can write $n = kM + j$, where $k \ge 0$ and $0 \le j < M$. Let $\kappa = \|P^M\|$ and $\lambda = \|P\|$. Then (2.13) yields

quicontinuity
$$U^n g(t) - U^n g(s) \le \kappa^k \lambda^j |g(t) - g(s)| + \tfrac{1}{2} \Big(\sum_{k=0}^{\infty} \kappa^k \Big) \Big(\sum_{j=0}^{M-1} \lambda^j \Big) |F(s,t)|. \tag{2.14}$$

The sum over k is finite because $\kappa < 1$. It may be that $\lambda > 1$, but the sum over j is a finite sum. The desired uniform bound on $U^n g(t) - U^n g(s)$ thus follows from (2.14). $\square$

Next we define another norm on $\mathcal{C}_d[0,t_0]$. Suppose first that $Q \neq 0$. Since $|Q|$ is non-negative and irreducible, by the Perron-Frobenius theorem there exists $y > 0$ such that $y^T|Q| = ry^T$, where $0 < r = \rho(|Q|) \leq 1$.

Define

def-norm (2.15)
$$|f|_* = \sup_{s\leq t_0} \sum_{i=1}^d y_i |f_i(s)|.$$

Since $y_i > 0$ for each i, this norm is equivalent to the norm

$$|f| = \sup_{s\leq t_0} \sum_{i=1}^d |f_i(s)|.$$

If $Q = 0$, let $|f|_* = \sup_{s\leq t_0} |f(s)|$.

We show T is a non-expansive map with respect to $|\cdot|_*$.

irred-P3 **Proposition 2.5.** *Suppose $g \in \mathcal{C}_d[0,t_0]$, either (i) $Q = 0$ or (ii) $|Q|$ is irreducible, and $r = \rho(|Q|) \leq 1$. Then T is a non-expansive map: if $g_1, g_2 \in \mathcal{C}_d[0,t_0]$, then*

$$|Tg_1 - Tg_2|_* \leq |g_1 - g_2|_*.$$

Proof. Suppose $Q \neq 0$. If $g \in \mathcal{C}_d[0,t_0]$, then

$$\begin{aligned}|Qg|_* &= \sup_{s\leq t_0} \sum_{i,j=1}^d y_i |Q_{ij}\, g_j(s)| \leq \sup_{s\leq t_0} \sum_{j=1}^d (y^T|Q|)_j |g_j(s)| \\ &= \sup_{s\leq t_0} \sum_{j=1}^d r y_j |g_j(s)| \leq |g|_*\end{aligned}$$

since $r \leq 1$. Then

$$|Qg_1 - Qg_2|_* = |Q(g_1 - g_2)|_* \leq |g_1 - g_2|_*.$$

It follows readily from this and the definition of T that T is non-expansive.

In the case $Q = 0$, we see that $|Qg_1 - Qg_2|_* = 0$ and again T is non-expansive. $\square$

A key result of this section is the following.

irred-T2 **Theorem 2.6.** *Suppose $Q = I - R$ and either (i) $Q = 0$ or (ii) $|Q|$ is irreducible with $\rho(|Q|) \leq 1$. Suppose $f \in \mathcal{C}_d[0,t_0]$ with $f(0) \geq 0$. Let $g^{(0)}$ be the identically 0 function. For $k \geq 0$ define*

$$g^{(k+1)} = \tfrac{1}{2}\Big((g^{(k)} + T(g^{(k)})\Big).$$

Then $g^{(k)}$ converges with respect to $|\cdot|_*$ to a function $g^{(\infty)} \in \mathcal{C}^*_d[0,t_0]$. Moreover $g^{(\infty)}$ is a fixed point for T, that is, $Tg^{(\infty)} = g^{(\infty)}$.

Proof. Since the norms $|\cdot|$ and $|\cdot|_*$ are equivalent, sets that are compact with respect to $|\cdot|$ are compact with respect to $|\cdot|_*$ and vice versa. Let $\mathcal{D}$ be the set of functions $h \in \mathcal{C}^*_d[0,t_0]$ such that $h(0) = 0$ and

j4.2 (2.16)
$$|h(t) - h(s)| \leq C_2\Big(\sup_{s\leq u_1<u_2\leq t} |f(u_2) - f(u_1)|\Big),$$

whenever $0 \le s \le t \le t_0$, where C_2 is the constant in Theorem 2.4 (ii). By the Ascoli-Arzelà theorem, $\mathcal{D}$ is a compact subset of $\mathcal{C}_d[0,t_0]$.

Note that $g^{(k+1)} = Ug^{(k)} = U^k g^0$. Using that $g^{(0)}$ is the zero function, Theorem 2.4(ii) tells us that $\{g^{(k)} : k \ge 0\}$ is contained in $\mathcal{D}$. We observed in Proposition 2.5 that T is non-expansive with respect to the norm $|\cdot|_*$. By Theorem 2.4(i) and the Ascoli-Arzelà theorem, T maps $\mathcal{D}$ into a compact subset of $\mathcal{C}_d[0,t_0]$. We now apply a theorem of Ishikawa ([Ish76, Theorem 1]) to obtain our result: we take $t_n = \frac{1}{2}$ for each n in his theorem and his D is our $\mathcal{D}$. □

Here is the result we will need in the next section. We do not assume here that Y is a Brownian motion. We do require it to be a semimartingale so that X will be a semimartingale.

irred-T3 **Theorem 2.7.** *Let $Y(t)$ be a semimartingale with trajectories in $\mathcal{C}_d[0,t_0]$, adapted to a filtration $\{\mathcal{F}_t\}$, with $Y(0) \ge 0$. Suppose either (i) $Q = 0$ or (ii) $|Q|$ is irreducible and $\rho(|Q|) \le 1$. Then there exist processes $X(t)$ taking values in $\mathcal{C}_d[0,t_0]$ and $M(t)$ taking values in $\mathcal{C}^*_d[0,t_0]$ such that $X(t) \ge 0$ and $M(t) \ge 0$ for all t, both X and M are adapted to $\{\mathcal{F}_t\}$, $M(0) = 0$,*

$$X(t) = Y(t) + RM(t),$$

and for each i, $M_i(t)$ increases only at times when $X(t) \in D_i$, the i^{th} face of D.

Proof. Let $M^{(0)}(t)$ be identically 0 and for $k \ge 0$ define

$$M^{(k+1)}(t)(\omega) = \tfrac{1}{2}\Big(M^{(k)}(t)(\omega) + (TM^{(k)})(t)(\omega)\Big),$$

where in the definition of T given by (2.5) we replace $f(s)$ by $Y(s)(\omega)$. Induction shows that each $M^{(k)}$ is adapted to $\{\mathcal{F}_t\}$ and its components are non-decreasing.

Since $Y(s)(\omega)$ is continuous almost surely, we apply Theorem 2.6 for each ω. Let $M(t)(\omega)$ be the limit of $M^k(t)(\omega)$ as $k \to \infty$. Clearly $M(t)$ is also adapted to $\{\mathcal{F}_t\}$ and its components are non-decreasing.

Let

$$X(t) = Y(t) + RM(t).$$

The only parts of the theorem that are not easy consequences of Theorem 2.6 are that $X(t) \ge 0$ and that $M_i(t)$ increases only when $X(t) \in D_i$.

Since M is a fixed point for T, for each ω we have

irred-T3E1
$$M(t) = \sup_{s\le t}\Big[QM(s) - Y(s)\Big]^+ = \sup_{s \le t}\Big[-RM(s) + M(s) - Y_s\Big]^+ = \sup_{s\le t}\Big[M(s) - X(s)\Big]^+, \tag{2.17}$$

using the definition of X. Therefore $M(t) \ge M(t) - X(t)$, which implies $X(t) \ge 0$.

Suppose M_i has a point of increase at a time t_1 yet $X_i(t_1) > 0$; we will obtain a contradiction. Let us suppose $t_1 \in (0,t_0)$, the cases when $t_1 = 0$ or $t_1 = t_0$ being similar. By the continuity of X_i there exist $h, \varepsilon > 0$ such that $t_1 + h \le t_0$, $t_1 - h \ge 0$, and $X_i(s) \geq \varepsilon$ for $t_1 - h \leq s \leq t_1 + h$. Since t_1 is a point of increase for M_i, we can take ε smaller if necessary so that $M_i(t_1 - h) \leq M_i(t_1 + h) - \varepsilon$. If $s \leq t_1 - h$, then

$$M_i(s) - X_i(s) \leq M_i(s) \leq M_i(t_1 - h) \leq M_i(t_1 + h) - \varepsilon.$$

If $t_1 - h \leq s \leq t_1 + h$,

$$M_i(s) - X_i(s) \leq M_i(s) - \varepsilon \leq M_i(t_1 + h) - \varepsilon.$$

But then

$$\sup_{s \leq t_1 + h} (M_i(s) - X_i(s)) \leq M_i(t_1 + h) - \varepsilon,$$

which contradicts (2.17) with t replaced by $t_1 + h$. □

sect-general

3. Pathwise uniqueness

The reader who is only interested in the irreducible case may at this point jump to the proof of Theorem 1.3 at the end of this section.

In this section we consider the case where Q might not be irreducible, and reduce it to the irreducible case by a decomposition of R.

Let $1 \leq J \leq d$. Let I_J be the $J \times J$ identity matrix. We start with an elementary lemma.

general-L1 **Lemma 3.1.** *Suppose $A \geq 0$ and A_0 is a principal submatrix of A. Then $\rho(A_0) \leq \rho(A)$.*

Proof. A_0 is obtained from A be deleting certain rows and columns, say the $i_1, \ldots, i_k$ rows and columns. Let B be the matrix obtained from A by changing all entries in the $i_1, \ldots, i_k$ rows and columns to 0's. By (2.1), $\rho(A) \geq \rho(B)$. Now delete those rows and columns from B to obtain A_0. It is easy to check that the non-zero eigenvalues of A_0 and B are the same. □

We say that $\mathcal{P}(J)$ holds if the following occurs.

Prop-PJ **Definition 3.2.** $\mathcal{P}(J)$: Let R' be any principal submatrix of R of size $J \times J$. Let $Q' = I_J - R'$. For any semimartingale Y' taking values in $\mathcal{C}_J[0, t_0]$ with $Y_0 \geq 0$ and measurable with respect to a filtration $\{\mathcal{F}_t\}$ there exist J-dimensional processes X' and M' with trajectories in $\mathcal{C}_J[0, t_0]$ and adapted to $\{\mathcal{F}_t\}$, $X'(t) \geq 0$ for all t, $M'(0) = 0$, each component of M' is non-decreasing and increases only when the corresponding component of X' is 0, and

$$X'(t) = Y'(t) + R'M'(t).$$

Note that for $\mathcal{P}(J)$ to hold, the condition stated in Definition 3.2 has to hold for each continuous semimartingale Y', not just for a particular one.

We will prove that Property $\mathcal{P}(J)$ holds for each $J \leq d$ by induction, but first we must show how to decompose the set of indices into communicating classes. This decomposition is analogous to decomposing a Markov chain state space into communicating classes.

Let A be a $J \times J$ matrix. We say $i \to j$ if there exist $i_0, i_1, \ldots, i_k$ such that $i_0 = i$, $i_k = j$, and $A_{i_m i_{m+1}} \neq 0$ for $m = 0, \ldots, k-1$. We say i and j are equivalent if $i = j$ or if $i \to j$ and $j \to i$. If an equivalence class has more than one element, let i and j be any two elements; then $i \to j$ and $j \to i$, so $i \to i$ and $j \to j$. It follows from this that an equivalence class with at least two elements is irreducible in the matrix sense. Note also that a square matrix is irreducible if and only if its transpose is irreducible.

Let $E_1, \dots, E_m$ be the equivalence classes. Let us write $E_k \Rightarrow E_\ell$ if there exists $i \in E_k$ and $j \in E_\ell$ such that $i \to j$. We cannot have both $E_k \Rightarrow E_\ell$ and $E_\ell \Rightarrow E_k$ for $\ell \neq k$ or else E_k and E_ℓ would not be distinct equivalence classes.

general-P1 **Proposition 3.3.** *Suppose R is completely-$\mathcal{S}$, $Q = I_d - R$, and $\rho(|Q|) \le 1$. Then Property $\mathcal{P}(J)$ holds for $J = d$.*

Proof. We show Property $\mathcal{P}(J)$ holds for each $J \le d$ using induction. The case when $J = 1$ follows by Theorem 2.7. We suppose $J_0 \ge 2$ and that Property $\mathcal{P}(J)$ holds for $J < J_0$, and we will prove that Property $\mathcal{P}(J)$ holds for $J = J_0$.

Let R' be a $J_0 \times J_0$ principal submatrix of R, and let $Q' = I_{J_0} - R'$. Since R' is a principal submatrix of R, Q' is a principal submatrix of Q, and, therefore, $|Q'|$ is a principal submatrix of $|Q|$. We assumed that $\rho(|Q|) \le 1$ so Lemma 3.1 implies that $\rho(|Q'|) \le 1$. If $|Q'|$ is irreducible, then we apply Theorem 2.7 and we have that Property $\mathcal{P}(J)$ holds and we are done.

So we suppose $|Q'|$ is not irreducible. Let $A = |Q'|^T$. We will decompose A rather than $|Q'|$ because if $R_{ij} \neq 0$, then the local time M_j of the component X_j gives a push to the component X_i; we want the notation $i \to j$ to be consistent with this.

Decompose A into equivalence classes as described above. There must be at least one E_{i_0} such that $E_k \not\Rightarrow E_{i_0}$ for all $k \neq i_0$. By renumbering the axes we may suppose $E_{i_0} = \{1, \dots, j_1\}$ for some $1 \le j_1 < J_0$. If $j \in E_{i_0}$ and $k \notin E_{i_0}$, then $k \not\to j$, so

decompose
$$|Q'|_{jk} = A^T_{kj} = 0. \tag{3.1}$$

Let Y' be a J_0-dimensional semimartingale adapted to $\{\mathcal{F}_t\}$ with $Y'(0) \ge 0$. Let $R^{(a)}$ be the $j_1 \times j_1$ principal submatrix of R' obtained by deleting the $j_1+1, \dots, J_0$ rows and columns of R', and let $Q^{(a)} = I_{j_1} - R^{(a)}$. By our construction, either $Q^{(a)}$ will be a 1×1 matrix or $|Q^{(a)}|$ will be irreducible. Since R is completely-$\mathcal{S}$, so is $R^{(a)}$. By Lemma 3.1, $\rho(|Q^{(a)}|) \le 1$. Let $Y^{(a)}$ be the j_1-dimensional vector obtained from Y' by deleting the $j_1+1, \dots, J_0$ entries. Using Theorem 2.7 we can find $X^{(a)}$ and $M^{(a)}$ satisfying the Skorokhod problem for $R^{(a)}$ and $Y^{(a)}$. Also both $X^{(a)}$ and $M^{(a)}$ are adapted to $\{\mathcal{F}_t\}$. Our solution $(X^{(a)}, M^{(a)})$ satisfies

general-E10
$$X_j^{(a)}(t) = Y_j^{(a)}(t) + \sum_{k=1}^{j_1} R_{jk}^{(a)} M_k^{(a)}(t), \qquad j = 1, \dots, j_1. \tag{3.2}$$

Let $R^{(b)}$ be the principal submatrix of R' obtained by deleting the $1, \dots, j_1$ rows and columns, let $Q^{(b)} = I_{J_0 - j_1} - R^{(b)}$, and note as above $\rho(|Q^{(b)}|) \le 1$. Let $Y^{(b)}$ be the $(J_0 - j_1)$-dimensional vector-valued semimartingale defined by

general-E21
$$Y_j^{(b)}(t) = Y'_{j+j_1}(t) + \sum_{k=1}^{j_1} R'_{j+j_1,k} M_k^{(a)}(t), \qquad j = 1, \dots, J_0 - j_1. \tag{3.3}$$

The process $Y^{(b)}$ is adapted to $\{\mathcal{F}_t\}$. Using our induction hypothesis we can find $(X^{(b)}, M^{(b)})$ adapted to $\{\mathcal{F}_t\}$ satisfying the Skorokhod problem, and in particular,

general-E12

$$X_j^{(b)}(t) = Y_j^{(b)}(t) + \sum_{k=1}^{J_0-j_1} R_{jk}^{(b)} M_k^{(b)}(t), \qquad j = 1, \dots, J_0 - j_1. \tag{3.4}$$

We now set $X'_j(t)$ equal to $X_j^{(a)}(t)$ if $j \le j_1$ and equal to $X_{j-j_1}^{(b)}$ if $j > j_1$ and similarly for M'.

If $j \le j_1$ and $k > j_1$, then $j \in E_{i_0}$ and $k \notin E_{i_0}$ and by (3.1), $R'_{jk} = Q_{jk} = 0$. Using this and (3.2) we obtain

general-E31

$$X'_j(t) = Y'_j(t) + \sum_{k=1}^{J_0} R'_{jk} M'_k(t). \tag{3.5}$$

If $j > j_1$, using (3.3) and (3.4), we see that $X^{(b)}$ solves

general-E13

$$X_{j-j_1}^{(b)}(t) = Y'_j(t) + \sum_{k=1}^{j_1} R_{jk}^{(a)} M_k^{(a)}(t) + \sum_{k=1}^{J_0-j_1} R_{j-j_1,k}^{(b)} M_k^{(b)}(t). \tag{3.6}$$

Then

$$X'_j(t) = X_{j-j_1}^{(b)}(t) = Y'_j(t) + \sum_{k=1}^{j_1} R'_{jk} M'_k(t) + \sum_{k=j_1+1}^{J_0} R'_{jk} M'_k(t). \tag{3.7}$$

Therefore (3.5) is also valid if $j > j_1$.

Thus Property $\mathcal{P}(J_0)$ holds. By induction, Property $\mathcal{P}(J)$ holds for all J, and in particular, for $J = d$. $\square$

If we now let $Y = B$, a d-dimensional Brownian motion started at $x_0 \in D$, and let $\{\mathcal{F}_t\}$ be the filtration generated by B, we have constructed a solution X to the Skorokhod problem which is adapted to the filtration $\{\mathcal{F}_t\}$. A solution adapted to the filtration generated by B is called a strong solution. The solution whose existence is guaranteed by [BeK91] is not in general a strong solution.

Proof of Theorem 1.3. Let B be a d-dimensional Brownian motion and let (X, M) be the solution constructed using Proposition 3.3. Note that our construction shows that (X, M) is adapted to the filtration generated by B. Let (Z, N) be any other solution satisfying Definition 1.2. By [TW93, Thm. 1.3] there is uniqueness in law for the solution to the Skorokhod problem for a given starting point and R (this is also referred to as weak uniqueness), and consequently the law of (B, X, M) is equal to the law of (B, Z, N).

Let r be a non-negative rational. Since $X(r)$ is adapted to the filtration generated by $\{B(s) : s \le r\}$, there is a Borel measurable map φ from $\mathcal{C}_d[0, r]$ to D such that $X(r) = \varphi(B)$ a.s. Because the laws of (X, B) and (Z, B) are equal, we must have that $Z(r)$ also equals $\varphi(B)$ a.s. Therefore $X(r) = Z(r)$ a.s. An analogous argument shows that $M(r) = N(r)$, a.s.

This holds for every non-negative rational. Since X and Z are both continuous a.s., then X and Z are identical with probability one. Similarly, M and N are indistinguishable. □

se-and-wu **Remark 3.4.** The preceding proof that strong existence plus weak uniqueness implies pathwise uniqueness is the same as a well-known proof for the analogous result for stochastic differential equations.

4. The two dimensional case

sect-2d

In this section we consider only the deterministic Skorokhod problem in the case $d = 2$.

L1-uniq **Lemma 4.1.** *Suppose $C > 0$. There is a unique solution for every continuous driving function for the deterministic Skorokhod problem with matrix $R = \begin{pmatrix} 1 & a_1 \\ a_2 & 1 \end{pmatrix}$ if and only if there is a unique solution for every continuous driving function for the Skorokhod problem with matrix $S = \begin{pmatrix} 1 & Ca_1 \\ a_2/C & 1 \end{pmatrix}$.*

Proof. If we write out $g = f + Rm$ in coordinates and multiply the second equation by $1/C$, we get

$$g_1 = f_1 + m_1 + a_1 m_2,$$
$$\frac{1}{C} g_2 = \frac{1}{C} f_2 + \frac{1}{C} a_2 m_1 + \frac{1}{C} m_2.$$

Let $\widetilde{m}_1 = m_1$, $\widetilde{g}_1 = g_1$, $\widetilde{f}_1 = f_1$, and

$$\widetilde{m}_2 = \frac{1}{C} m_2, \quad \widetilde{g}_2 = \frac{1}{C} g_2, \quad \widetilde{f}_2 = \frac{1}{C} f_2.$$

We then have

$$\widetilde{g} = \widetilde{f} + S\widetilde{m}.$$

It follows that there will be two distinct solutions to the Skorokhod problem for a driving function f with respect to the matrix R if and only if there are two distinct solutions to the Skorokod problem with driving function $\widetilde{f}$ with respect to the matrix S. □

Proof of Theorem 1.4. In view of Lemma 4.1, if $a_1 > 0, a_2 > 0$ and $a_1 a_2 = 1$, we may reduce the question of uniqueness for the Skorokhod problem to the case where $a_1 = a_2 = 1$.

Let $u_1 = \frac{1}{\sqrt{2}}(1, 1)$ and $u_2 = \frac{1}{\sqrt{2}}(1, -1)$. Let (f_{u_1}, f_{u_2}) be the coordinates of f with respect to the orthonormal basis (u_1, u_2). Thus f_{u_j} is the inner product of (f_1, f_2) with u_j, $j = 1, 2$. Define g_{u_j} and $\overline{g}_{u_j}$, $j = 1, 2$, similarly. We use (f_1, f_2), (g_1, g_2), $(\overline{g}_1, \overline{g}_2)$, (m_1, m_2), and $(\overline{m}_1, \overline{m}_2)$ for the coordinates with respect to the usual basis $\{(1, 0), (0, 1)\}$.

Notice $Rm = (m_1 + m_2, m_1 + m_2)$. A calculation shows that $g = f + Rm$ implies

E1-uniq (4.1)
$$g_{u_1}(t) = f_{u_1}(t) + \sqrt{2}(m_1(t) + m_2(t)),$$
$$g_{u_2}(t) = f_{u_2}(t),$$

and similarly with g replaced by $\overline{g}$ and m replaced by $\overline{m}$. Therefore $g_{u_2}(t) = f_{u_2}(t) = \overline{g}_{u_2}(t)$ for all t.

Let $v(t) = |g_{u_1}(t) - \overline{g}_{u_1}(t)|$. The function v is continuous and $v(0) = 0$. Let $t_0 > 0$ and suppose $g_{u_1}(t_0) < \overline{g}_{u_1}(t_0)$. With respect to the coordinate system (u_1, u_2) the quadrant D is a wedge symmetric about the u_1 axis. Therefore regardless of whether or not $(g_{u_1}(t_0), g_{u_2}(t_0))$ is on the boundary of the wedge, we see that $(\overline{g}_{u_1}(t_0), \overline{g}_{u_2}(t_0))$ is in the interior of D since $g_{u_2}(t_0) = \overline{g}_{u_2}(t_0)$. Hence there exists an $\varepsilon > 0$ such that $\overline{g}_{u_1}(t) - \overline{f}_{u_1}(t)$ does not increase for $t \in [t_0, t_0 + \varepsilon]$. The quantity $g_{u_1}(t) - f_{u_1}(t)$ is always non-decreasing and might possibly increase for some times in $[t_0, t_0 + \varepsilon]$, but in any case $v(t)$ is non-increasing for $t \in [t_0, t_0 + \varepsilon]$. The case where $\overline{g}_{u_1}(t_0) < g_{u_1}(t_0)$ is treated exactly the same. Therefore v is non-increasing on $\{t : v(t) \neq 0\}$. This implies that v is identically 0. □

R1-uniq **Remark 4.2.** Note that we do not assert that $m = \overline{m}$. This is because it need not be true in the case $a_1 = a_2 = 1$. Let $g(t)$ be identically 0 and $f(t) = (-t, -t)$. For $m = (m_1, m_2)$ we can take any choice of non-decreasing continuous functions $m_1(t)$ and $m_2(t)$ that start at 0 such that $m_1(t) + m_2(t) = t$.

For any matrix R that does not have determinant 0 we can invert R, and then m is uniquely determined once g is from the formula $m = R^{-1}(g - f)$.

We now turn to non-uniqueness.

Proof of Theorem 1.5. By virtue of Lemma 4.1 we may suppose $a_1 = \gamma > 1$ and $a_2 = 1$. Let $t_n = 2^{-n}$, $n = 0, 1, 2, \ldots$. We first define $f_n, g_n, \overline{g}_n, m_n$, and $\overline{m}_n$ on $[0, 2^{-n}]$; these will be the components that we use to construct our f.

Let $f_n(0) = m_n(0) = \overline{m}_n(0) = 0$ and let

non-E1
$$g_n(0) = (0, \gamma^{-n}), \quad \overline{g}_n(0) = (\gamma^{-n}, \gamma^{-n}). \tag{4.2}$$

Let $s_{n1} = \frac{1}{4}2^{-n}$, $s_{n2} = \frac{1}{2}2^{-n}$, and $s_{n3} = \frac{3}{4}2^{-n}$. The functions $f_n, g_n, \overline{g}_n, m_n$, and $\overline{m}_n$ will consist of four pieces.

(1) First f_n moves left a distance γ^{-n} at constant speed. To be precise, let $f_n(s_{n1}) = (-\gamma^{-n}, 0)$. Let

$$g_n(s_{n1}) = (0, 2\gamma^{-n}), \quad \overline{g}_n(s_{n1}) = (0, \gamma^{-n}), \quad m_n(s_{n1}) = (\gamma^{-n}, 0), \quad \overline{m}_n(s_{n1}) = 0.$$

Extend the definition of each of these to $[0, s_{n1}]$ by linear interpolation.

(2) Next f_n moves γ^{-n+1} to the right. That is, set

$$f_n(s_{n2}) = (\gamma^{-n+1} - \gamma^{-n}, 0), \quad g_n(s_{n2}) = (\gamma^{-n+1}, 2\gamma^{-n}), \quad \overline{g}_n(s_{n2}) = (\gamma^{-n+1}, \gamma^{-n}),$$
$$m_n(s_{n2}) = (\gamma^{-n}, 0), \quad \overline{m}_n(s_{n2}) = 0.$$

Define each of these on $[s_{n1}, s_{n2}]$ by linear interpolation.

(3) For the third piece, let f_n move down a distance $2\gamma^{-n}$, which leads to

$$f_n(s_{n3}) = (\gamma^{-n+1} - \gamma^{-n}, -2\gamma^{-n}), \quad g_n(s_{n3}) = (\gamma^{-n+1}, 0), \quad \overline{g}_n(s_{n3}) = (2\gamma^{-n+1}, 0),$$
$$m_n(s_{n3}) = (\gamma^{-n}, 0), \quad \overline{m}_n(s_{n3}) = (0, \gamma^{-n}).$$

Again, use linear interpolation between s_{n2} and s_{n3}.

(4) Finally f_n moves diagonally up and to the left as follows. Set

non-E2 (4.3)
$$f_n(2^{-n}) = (-\gamma^{-n}, \gamma^{-n+1} - 2\gamma^{-n}), \quad g_n(2^{-n}) = (0, \gamma^{-n+1}),$$
$$\overline{g}_n(2^{-n}) = (\gamma^{-n+1}, \gamma^{-n+1}), \quad m_n(s_{n3}) = (\gamma^{-n}, 0), \quad \overline{m}_n(s_{n3}) = (0, \gamma^{-n}).$$

Use linear interpolation once more for values between s_{n3} and 2^{-n}.

Observe that

non-E3 (4.4)
$$g_n(t) - g_n(0) = f_n(t) + Rm_n(t)$$

when $t = 0, s_{n1}, s_{n2}, s_{n3}, 2^{-n}$, and similarly when g_n and m_n are replaced by $\overline{g}_n$ and $\overline{m}_n$. Since all these quantities are defined by linear interpolation for times in between the above values of t, we have (4.4) holding for all $t \in [0, 2^{-n}]$ and the same when g_n and m_n are replaced by $\overline{g}_n$ and $\overline{m}_n$.

For $t \in (2^{-n}, 2^{-n+1}]$ define

$$f(t) = f_n(t - 2^{-n}) + \sum_{k=n+1}^{\infty} f_k(2^{-k}).$$

The series converges by (4.3). We use the same formula with f, f_k, f_n replaced by m, m_k, m_n and $\overline{m}, \overline{m}_k, \overline{m}_n$ to define m and $\overline{m}$. For $t \in (2^{-n}, 2^{-n+1}]$ define g by

$$g(t) = g_n(t - 2^{-n})$$

and define $\overline{g}$ similarly.

Given (4.4) it is straightforward to check that $g(t) = f(t) + Rm(t)$ and that the analogous equation holds for $\overline{g}$. Clearly g and $\overline{g}$ are distinct.

The only thing left to check is that m_j increases only when $g_j = 0$, $j = 1, 2$, and the analogous result for $\overline{m}, \overline{g}$. We do this for the first coordinate of m, the other cases being almost identical. Note that the first coordinate of m increases only when the first coordinate of one of the m_n increases. By our construction, at these times the first coordinate of g_n is 0, which implies that the first coordinate of g is 0 at these times. □

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