PATHWISE NON-UNIQUENESS FOR BROWNIAN MOTION IN A QUADRANT WITH OBLIQUE REFLECTION

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ABSTRACT. Consider the Skorokhod equation in the closed first quadrant:

$$X_t = x_0 + B_t + \int_0^t \mathbf{v}(X_s)\, dL_s,$$

where B_t is standard 2-dimensional Brownian motion, X_t takes values in the quadrant for all t, and L_t is a process that starts at 0, is non-decreasing and continuous, and increases only at those times when X_t is on the boundary of the quadrant. Suppose $\mathbf{v}$ equals $(-a_1, 1)$ on the positive x axis, equals $(1, -a_2)$ on the positive y axis, and $\mathbf{v}(0)$ points into the closed first quadrant. Let $\theta_i = \arctan a_i$, $i = 1, 2$. It is known that there exists a solution to the Skorokhod equation for all $t \geq 0$ if and only if $\theta_1 + \theta_2 < \pi/2$ and moreover the solution is unique if $|a_1 a_2| < 1$.

Suppose now that $\theta_1 + \theta_2 < \pi/2$, $\theta_2 < 0$, $\theta_1 > -\theta_2 > 0$ and $|a_1 a_2| > 1$. We prove that for a large class of (a_1, a_2), namely those for which

$$\frac{\log |a_1| + \log |a_2|}{a_1 + a_2} > \pi/2,$$

pathwise uniqueness for the Skorokhod equation fails to hold.

1. INTRODUCTION

We consider Brownian motion in the upper right closed quadrant D with oblique reflection on the boundaries. Let $\mathbf{n}(x)$ denote the unit inward pointing normal vector at $x \in \partial D$ and let $\mathbf{v}(x)$ denote the vector of reflection, normalized so that $\mathbf{n}(x) \cdot \mathbf{v}(x) = 1$ for each $x \in \partial D$. In this paper we restrict attention to the case where $\mathbf{v}$ is equal to a constant on each of the x and y axes. The value of $\mathbf{v}(0)$ turns out to be immaterial as long as it points into D; see Remark 2.1. Such processes have been extensively studied. They arise, for example, as the limit of certain queueing models. See [Wil95] for references.

Many properties of obliquely reflecting Brownian motion (ORBM) are governed by a parameter α which is defined as follows. Let θ_1, θ_2 be the angles of reflection on the x and y axes, resp., with positive θ pointing towards the origin, so that $\mathbf{v}(z) = (-\tan \theta_1, 1)$ at all points z on the x axis except the origin and $\mathbf{v}(z) = (1, -\tan \theta_2)$ for all z on the y axis except the origin. Let

$$\alpha = \frac{\theta_1 + \theta_2}{\pi/2}. \tag{1.1}$$

When $\alpha < 1$, ORBM will be a semimartingale (see [Wil85]) and can be represented by what is known as the Skorokhod equation. If $B_t = (B_t^1, B_t^2)$ is standard 2–dimensional

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Brownian motion adapted to some filtration $\{\mathcal{F}_t\}$, the Skorokhod equation is

$$X_t = x_0 + B_t + \int_0^t \mathbf{v}(X_s)\, dL_s, \qquad t \geq 0. \tag{1.2}$$

Here L_t, known as the local time on the boundary, is a non-decreasing continuous process starting at 0 that increases only at those times when $X_t \in \partial D$.

When does there exist a solution to the Skorokhod equation and when is that solution pathwise unique? If $\alpha < 1$, a solution to the Skorokhod equation exists ([Wil95]). It is important to note that this solution may not be adapted to the filtration generated by $\{B_t\}$.

Let

$$a_i = \tan \theta_i, \qquad i = 1, 2. \tag{1.3}$$

Set

$$\beta = |a_1 a_2|. \tag{1.4}$$

It is known (see Section 2) that if $\beta < 1$, then the solution to the Skorokhod equation is pathwise unique and the solution is adapted to the filtration generated by $\{B_t\}$.

Let

$$\psi = \frac{\log |a_1| + \log |a_2|}{(\theta_1 + \theta_2)(a_1 + a_2)}. \tag{1.5}$$

The main result of this paper is that if $0 < \alpha < 1$, $\beta > 1$, a_1 and a_2 are of opposite signs, and $\psi > 1/\alpha$, then for almost every ω there is more than one solution to the Skorokhod equation (1.2). Moreover there does not exist a solution that is adapted to the filtration generated by $\{B_t\}$. See Section 2 for a precise statement.

The reason for the restriction $\alpha < 1$ is that ORBM is a semimartingale if and only if $\alpha < 1$, and so the Skorokhod equation makes sense only in this case. When $\alpha \leq 0$ the ORBM is transient and upon leaving the origin never returns to the origin again. Our techniques do not work in this case. We also do not know what happens in the case $\beta > 1, \psi \leq 1/\alpha$.

There exist processes, in fact strong Markov processes, that are uniquely defined in terms of a submartingale problem when $\alpha \geq 1$ (see [VW85]), but they are not semimartingales. See [KR10] for a representation of such processes as Dirichlet processes.

The following historical review refers to the form of the problem given in (2.1)-(2.2) and later in Section 2. Mandelbaum [Man87] showed with

$$R = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$$

that there is non-uniqueness for the corresponding deterministic Skorokhod equation (where B is replaced by any continuous driving function χ), even though the R-matrix is a P-matrix, which implies uniqueness for a time-discretized version of the problem. Specifically, Mandelbaum introduced homogeneous differential inequalities such that, for an arbitrary R, uniqueness holds for the deterministic Skorokhod equation if and only if zero is the unique solution of the differential inequalities. His proof was constructive in that he gave a procedure for creating a driving path χ from a non-zero solution

Y to the differential inequalities. For the above 2×2 R-matrix, Mandelbaum depicted the corresponding Y in the form of a snail-shell converging to the origin in $\mathbb{R}^2$ (χ was not shown). Various renderings of the corresponding χ (which is intricate) and the resulting reflected path, have been given by various people over the years. One published source is the Ph.D. thesis [Whi03], which in a background chapter recounts a form of χ and the reflected path that emerged from discussions of Mandelbaum with Robert Vanderbei in the mid 1980s (private communication). Subsequent to Mandelbaum, but independently, Bernard and El Kharroubi [BeK91] demonstrated that a driving χ, for which non-uniqueness holds, can in fact be linear; their example, however, lives in $\mathbb{R}^3$ and its intricacy stems from the matrix R.

There is a considerable literature devoted to the construction of reflected Brownian motion in domains other than quadrants. That research includes theorems on existence and uniqueness of solutions to stochastic differential equations representing reflected Brownian motion. Techniques used in that area are considerably different from those in this and related papers. Some references include [BB08, DI93, DI08, BCMR17, LS84, VW85].

In Section 2 we summarize some results from the literature that we need, state the main theorem, and give some ideas how the proof goes. Section 3 shows how Brownian motion and certain Bessel processes transform under certain conformal mappings. In Section 4 we review some excursion theory and extend some of those results to ones we will need. A computation of the expected value of the displacement of $\widehat{X}$, a process related to our solution, is given in Section 5. Sections 6 and 7 discuss the existence of an invariant measure for a certain Markov chain and a Harnack inequality for this chain, resp. A strong law of large numbers for the number of excursions from one side of D to the other is given in Section 8. Section 9 provides the key estimate needed, and then in Section 10 we use this to prove our non-uniqueness results. Finally, in Section 11 we give an index of the notation we use.

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2. Preliminaries

Many of the previous results concerning uniqueness are stated for a slightly different but equivalent formulation of the Skorokhod equation.

We let $B_t = (B_t^1, B_t^2)$ be a standard Brownian motion started at $x_0 \in D$, let R be the matrix

$$R = \begin{pmatrix} 1 & -a_1 \\ -a_2 & 1 \end{pmatrix}, \tag{2.1}$$

and let

$$X_t = B_t + RM_t, \tag{2.2}$$

where M_t is a two dimensional process, each of whose coordinates are non-decreasing continuous processes started at 0 with M_t^1 increasing only when X_t is on Γ_d and M_t^2 increasing only when X_t is on Γ_u, where

$$\Gamma_u = \{(x,y) : x = 0, y \geq 0\}, \qquad \Gamma_d = \{(x,y) : y = 0, x \geq 0\}. \tag{2.3}$$

It is easy to see that if L_t is the local time on the boundary in (1.2), then $L_t = M_t^1 + M_t^2$ and

$$M_t^1 = \int_0^t 1_{\Gamma_d}(X_s)\, dL_s, \qquad M_t^2 = \int_0^t 1_{\Gamma_u}(X_s)\, dL_s$$

give the relationship between this formulation and that in (1.2).

Remark 2.1. In view of Reiman and Williams [RW88], neither M^1 nor M^2 charge the origin, and therefore the reflection vector at the origin is of little importance as long as it is a fixed vector pointing into D.

A solution to (1.2) is clearly a semimartingale, and by [TW93] the law of the solution is equal to the law of the corresponding solution to (2.2). In particular L from (1.2) does not charge the origin, and again the reflection vector at the origin can be any fixed vector pointing into D.

Harrison and Reiman [HR81] used the contraction mapping principle to show there exists a solution to (2.2) which is pathwise unique provided $Q = I - R$ is the transition matrix of a certain class of transient Markov chains. It was observed in [Wil95, Theorem 2.1] that the proof of [HR81] carries through provided the spectral radius of $|Q|$ is strictly less than 1, where $|Q|$ is the matrix whose entries are the absolute values of the corresponding entries of Q. In this case, as we vary the starting point x_0 we can obtain a strong Markov process. In fact the result of [HR81] and [Wil95] is valid in higher dimensional orthants.

Looking only at the 2–dimensional case, an elementary calculation shows that the eigenvalues of $|Q|$ are $\pm\sqrt{|a_1 a_2|}$, and therefore pathwise uniqueness of (2.2) holds if $\beta = |a_1 a_2| < 1$.

Suppose $a_1, a_2 \geq 0$. If $\alpha < 1$, then $\theta_1 + \theta_2 < \pi/2$, and hence

$$\begin{aligned} \beta = a_1 a_2 &= (\tan\theta_1)(\tan\theta_2) < (\tan\theta_1)(\tan(\tfrac{\pi}{2} - \theta_1)) \\ &= (\tan\theta_1)(\cot\theta_1) = 1. \end{aligned} \tag{2.4}$$

Therefore the question of pathwise existence and uniqueness for the Skorokhod equation is completely resolved when $a_1, a_2 \geq 0$ and $\alpha < 1$.

A 2×2 matrix R is said to be completely–S if the diagonal terms are strictly positive and there exist $z_1, z_2 > 0$ such that if Z is the 2×1 column matrix whose entries are z_1 and z_2, then the entries of RZ are strictly positive.

For the R given in (2.1), this translates to

$$z_1 - a_2 z_2 > 0, \qquad -a_1 z_1 + z_2 > 0. \tag{2.5}$$

When $a_1, a_2 > 0$, this is equivalent to $z_1/z_2 > a_2$ and $z_1/z_2 < 1/a_1$, and these two equations are solvable only if $\beta < 1$. If one or both of a_1, a_2 are negative, then the equations (2.5) are solvable and R is completely–S

Sometimes the definition of completely–S is stated to allow $z_1, z_2 \geq 0$ rather than $z_1, z_2 > 0$. The two definitions are equivalent: if the entries of RZ are strictly positive for some $z_1, z_2 \geq 0$, they will also be strictly positive for $z_1 + \varepsilon, z_2 + \varepsilon$ for small enough ε.

An important fact is that if R is completely–S, then the Skorokhod equation (2.2) will always have a solution, even when B_t^1 and B_t^2 are replaced by any continuous functions whatsoever. We have the following proposition; see [BeK91], [MVdH87], or [DW96] for the proof.

Proposition 2.2. *Let $t_0 > 0$. Suppose R is a completely-S matrix, $x_0 \in D$, and $f : [0, t_0] \to \mathbb{R}^2$ is continuous with $f(0) = 0$.*
(i) There exists a function $g : [0, t_0] \to \mathbb{R}^2$ that is continuous and is a solution to

$$g(t) = x_0 + f(t) + RM(t), \tag{2.6}$$

where $M(t)$ is a function with values in $\mathbb{R}^2$ each of whose components is non-decreasing and continuous and where M^i, $i = 1, 2$, increases only when g is in Γ_d, Γ_u, resp.
(ii) Let $g : [0, t_0] \to \mathbb{R}^2$ be any function that satisfies the conditions of (i). There exists a constant χ_1 depending only on a_1 and a_2 but not t_0, t_1 or t_2 such that for all $0 \leq t_1 \leq t_2 \leq t_0$

$$\sup_{u_1, u_2 \in [t_1, t_2]} \Big(|g(u_2) - g(u_1)| + |M(u_2) - M(u_1)| \Big) \leq \chi_1 \sup_{u_1, u_2 \in [t_1, t_2]} |f(u_2) - f(u_1)|. \tag{2.7}$$

Returning to (2.2) with B a 2–dimensional Brownian motion, Proposition 2.2(i) does not assert that the solution is adapted to the filtration generated by $\{B_t\}$. When the pair (X, B) satisfies (2.2) and X is adapted to the filtration generated by $\{B_t\}$, the pair (X, B) is called a strong solution. We will see later (Section 10) that pathwise uniqueness is essentially equivalent to the solution being a strong solution.

Suppose that again B_t is 2–dimensional Brownian motion. We say a pair (X, B) solving (2.2) is a weak solution if B is adapted to the filtration generated by $\{X_t\}$. For existence and uniqueness of weak solutions we refer the reader to Taylor and Williams [TW93].

The techniques we develop in this paper do not say anything about the case $\alpha \leq 0$, $\beta > 1$. Therefore in the remainder of this paper we consider only the case where $0 < \alpha < 1$, $\beta > 1$, and one of the a_i is positive, the other negative. To be specific, we take $\theta_1 > 0$ and $\theta_2 < 0$.

We can now state our main theorem. Recall the definition of ψ given in (1.5).

Theorem 2.3. *Suppose $a_1 > 0$, $a_2 < 0$, $\alpha > 0$, $\beta > 1$, and $\psi > 1/\alpha$. Then there exist two processes X and Y solving (1.2) such that with probability one*

$$\sup_{s \geq 0} |X_s - Y_s| > 0.$$

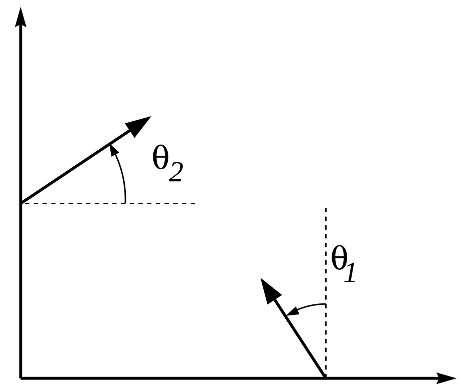

FIGURE 1. According to our conventions, $\theta_1 > 0$ and $\theta_2 < 0$.

Moreover no strong solution to the Skorokhod equation exists.

Remark 2.4. (i) The conditions $0 < \alpha < 1$, $\beta > 1$, and $\psi > 1/\alpha$ are satisfied by some θ_1 and θ_2. For example, take $\theta_1 = 3\pi/8 - \varepsilon$ and $\theta_2 = -3\pi/8 + 2\varepsilon$ for a very small $\varepsilon > 0$.

However, there exist θ_1 and θ_2 satisfying $0 < \alpha < 1$ and $\beta > 1$ but not satisfying $\psi > 1/\alpha$. For example, take $\theta_1 = 7\pi/16$ and $\theta_2 = -\pi/8$.

(ii) The region $(\theta_1, \theta_2) \in [\pi/4, \pi/2] \times [-\pi/2, 0]$ is depicted in Fig. 2. The conditions $0 < \alpha < 1$ and $\beta > 1$ are satisfied in the triangle to the right of the red and green lines. The yellow region is where the condition $\psi > 1/\alpha$ is satisfied. The picture was generated using Mathematica.

The proof is quite technical so we outline some of the steps.

Suppose we have two solutions $X = (X^1, X^2)$ and Y_n to (1.2) driven by the same Brownian motion B, with X started on the y axis a distance approximately 2^{-n} away from the origin and Y_n started a distance η_n to the right of X on the same horizontal line. As Y_n moves to the y axis, the local time on the y axis for X must increase by η_n in order for X to remain in D. Because the vector of reflection is $(1, -a_2)$, X has an additional displacement of $a_2\eta_n$ in addition to its motion due to B_t^2.

Once Y_n arrives at the y axis, we are now in the situation where X and Y_n are both on the y axis, but a distance $|a_2|\eta_n$ apart. We then watch until the pair X and Y_n, still vertically aligned, move so that the lower of the two is on the x axis and the upper is $|a_2|\eta_n$ above. By the argument in the preceding paragraph, after both have reached the x axis, the distance between X and Y_n is now $|a_1|\,|a_2|\eta_n = \beta\eta_n$, and the two processes are again on the same horizontal line. The net result is that the distance between X and Y_n has increased by a factor of β. This is why we need $\beta > 1$.

A major part of the proof consists of estimating the number of times the distance grows by a factor of β. We need to ensure that the number of times is sufficiently large so that when $|X|$ is approximately 1, the distance between X and Y_n is not negligible, independently of n. This is where the condition $\psi > 1/\alpha$ appears. In the limit we get two processes started at 0 which are not identical.

There are some complications. We need to ensure that X does not hit 0, so we argue that it suffices to look at excursions. It is possible that Y_n goes only part ways towards

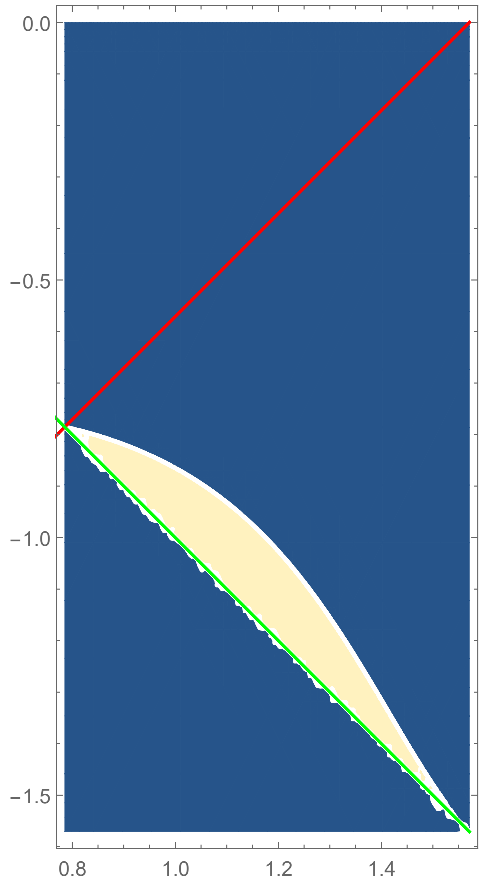

FIGURE 2. The region $(\theta_1, \theta_2) \in [\pi/4, \pi/2] \times [-\pi/2, 0]$.

the y axis in the argument above, before moving to the x axis. We also need to bound the probability of this happening.

We could not find a construction of the approximating processes in which the initial distance between them was deterministic because the variance of the distance between them at time 1 would be too large for our purposes. To overcome this challenge, we "look into the future" and choose the initial distance so that it makes the two processes have a distance of order 1 at a time of order 1.

Throughout this paper the letter c with subscripts will denote constants whose exact value is unimportant and which may change from occurrence to occurrence.

3. Conformal mappings

In order to facilitate the computations of the expected number of excursions, we will use two conformal mappings. The first is

$$\mathbb{F}(z) = e^{i(\pi/2-\theta_1)} z^{\alpha}, \tag{3.1}$$

which maps D into a wedge $\widetilde{D} = \mathbb{F}(D)$ so that the angles of reflection are horizontal and point at each other.

Fig. 3 shows $\widetilde{D}$ with the vectors of reflection horizontal and pointing at each other. The angles in the picture are $\chi = \theta_1 + \theta_2$ and $\gamma = \pi/2 - \theta_1$.

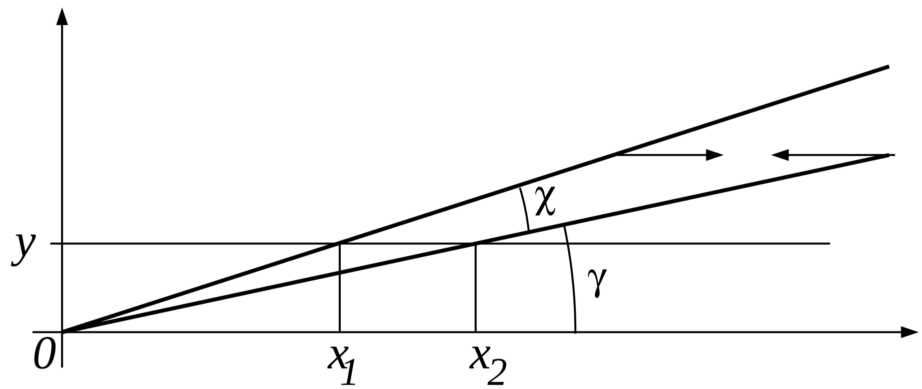

FIGURE 3.

The second conformal mapping,

$$g(z) = \log z, \tag{3.2}$$

maps the wedge $\widetilde{D}$ into a strip $\widehat{D} = \log(\widetilde{D})$.

In this section we determine the law of the time change of the image of X under $\mathbb{F}$, and then the image of a closely related process under g.

Since $\mathbb{F}$ maps the vectors of reflection into horizontal vectors, there is no push for the vertical component of $\mathbb{F}(X_t)$ due to hitting the boundary, and so we expect the vertical component to be a time change of Brownian motion.

Here is the precise statement.

Proposition 3.1. *Let X be a solution to* (1.2) *stopped on hitting the origin. Let $A_t = \int_0^t |\mathbb{F}'(X_s)|^2 \, ds$ and $C_t = \inf\{s : A_s \geq t\}$. Let $\widetilde{X}_t = \mathbb{F}(X_{C_t})$. Then $\widetilde{X}$ solves the following Skorokhod equation in $\widetilde{D}$:*

$$\widetilde{X}_t = \mathbb{F}(x_0) + \widetilde{B}_t + \int_0^t \widetilde{\mathbf{v}}(\widetilde{X}_s) \, d\widetilde{L}_s, \tag{3.3}$$

where $\widetilde{B}_t$ is a standard Brownian motion in $\mathbb{R}^2$, $\widetilde{\mathbf{v}}(z)$ points horizontally into the interior of $\widetilde{D}$, $\widetilde{\mathbf{v}}(z)$ is normalized so that the component normal to the boundary of $\partial\widetilde{D}$ is equal to 1, $\widetilde{L}$ is a continuous non-decreasing process starting at 0 that increases only when $\widetilde{X}_t$ is in $\partial\widetilde{D}$, and $\widetilde{X}$ is stopped upon hitting the origin.

When we work with $\widetilde{X}$ we only start the process away from the origin and we will subsequently condition it to never hit 0, hence we do not need to worry what happens if $\widetilde{X}$ were to continue after hitting the origin.

Proof. If $\mathbb{F}(z) = U(z) + iV(x)$ (we use upper case U and V to avoid any confusion with the vectors of reflection), recall that U and V are harmonic, hence $U_{xx} + U_{yy} = 0$ in the interior of D and the same for V. Here U_x is the partial derivative with respect to x

and the notations U_y, U_{xx}, U_{xy}, U_{yy} are the respective partials. If $B_t = (B_t^1, B_t^2)$ is the planar Brownian motion in (1.2), the quadratic variations are $d\langle B^1\rangle_t = d\langle B^2\rangle_t = dt$ and $d\langle B^1, B^2\rangle_t = 0$. Let $\mathcal{U}_t = U(X_t)$, $\mathcal{V}_t = V(X_t)$.

Using Ito's formula,

$$\begin{aligned}
d\mathcal{U}_t &= U_x(X_t)\,dB_t^1 + U_y(X_t)\,dB_t^2 + U_x(X_t)\mathbf{v}_1(X_t)\,dL_t + U_y(X_t)\mathbf{v}_2(X_t)\,dL_t \\
&\quad + \tfrac{1}{2}U_{xx}(X_t)\,d\langle B^1\rangle_t + U_{xy}(X_t)\,d\langle B^1, B^2\rangle_t + \tfrac{1}{2}U_{yy}(X_t)\,d\langle B^2\rangle_t \\
&= U_x(X_t)\,dB_t^1 + U_y(X_t)\,dB_t^2 + U_x(X_t)\mathbf{v}_1(X_t)\,dL_t + U_y(X_t)\mathbf{v}_2(X_t)\,dL_t,
\end{aligned}$$

and similarly for $\mathcal{V}_t$.

It is standard that $(\mathcal{U}_{C_t}, \mathcal{V}_{C_t})$ is a Brownian motion except for the local time terms (see [Bas95, pp. 310–311] for an exposition), so we focus on the local time terms. These terms can be written as $\mathbf{q}(\mathcal{U}_t, \mathcal{V}_t)\,dM_t$, where

$$\begin{aligned}
\mathbf{q}_0(w_1, w_2) &= b(z)\Big(U_x(z)\mathbf{v}_1(z) + U_y(z)\mathbf{v}_2(z), V_x(z)\mathbf{v}_1(z) + V_y(z)\mathbf{v}_2(z)\Big), \\
z &= \mathbb{F}^{-1}(w_1, w_2), \\
dM_t &= (1/b(\mathcal{U}_t, \mathcal{V}_t))\,dL_t.
\end{aligned}$$

Here b is chosen so the normal component of $\mathbf{q}$ is equal to 1.

We let $\widetilde{L}_t = M_{C_t}$ and note that since C_t is strictly increasing and continuous, then $\widetilde{L}_t$ is non-decreasing and continuous.

It remains to show that $\mathbf{q}$ has the desired direction. At any point $z \in \partial D$ except for the origin, $\mathbb{F}$ is analytic in a neighborhood of z and hence conformal. If $z \in \Gamma_d$, the angle between the vector $(1, 0)$ and $\mathbf{v}$ is $\frac{\pi}{2} + \theta_1$. It is easy to check that the angle between Γ_d and $\mathbb{F}(\Gamma_d)$ is $\frac{\pi}{2} - \theta_1$. Therefore the angle between $\mathbf{q}(\mathbb{F}(z))$ and the vector $(1, 0)$ is equal to π, as claimed. The argument for $z \in \Gamma_u$ is similar. □

For our second result, we suppose $\widetilde{X}$ is as in Proposition 3.1, except that $\widetilde{X}^2$ is now a 3-dimensional Bessel process. More precisely, we suppose

$$\widetilde{X}_t = \widetilde{x}_0 + \widetilde{B}_t + \widetilde{E}_t + \int_0^t \widetilde{\mathbf{v}}(\widetilde{X}_s)\,d\widetilde{L}_s, \tag{3.4}$$

where $\widetilde{B}_t$ is a standard Brownian motion in $\mathbb{R}^2$, $\widetilde{\mathbf{v}}$ and $\widetilde{L}$ are as in Proposition 3.1, $\widetilde{x}_0 = f(x_0)$, $\widetilde{E}_t^1 = 0$ for all t, and

$$\widetilde{E}_t^2 = \int_0^t \frac{1}{\widetilde{X}_s^2}\,ds.$$

.

Let $\widehat{D} = \log(\widetilde{D})$, $\widehat{\Gamma}_u = \log(\widetilde{\Gamma}_u)$, and $\widehat{\Gamma}_d = \log(\widetilde{\Gamma}_d)$.

Proposition 3.2. *Let $\widetilde{X}_t$ be as in* (3.4)*. Let $\widetilde{A}_t = \int_0^t |1/\widetilde{X}_s|^2\,ds$ and $\widetilde{C}_t = \inf\{s : \widetilde{A}_s \geq t\}$. Let $\widehat{X}_t = \log(\widetilde{X}_{\widetilde{C}_t})$. Then $\widehat{X}_t$ solves the following Skorokhod equation in $\widehat{D}$:*

$$\widehat{X}_t = \log(\widetilde{x}_0) + \widehat{B}_t + \widehat{E}_t + \int_0^t \widehat{\mathbf{v}}(\widehat{X}_s)\,d\widehat{L}_s, \tag{3.5}$$

where $\widehat{B}_t$ is a standard Brownian motion in $\mathbb{R}^2$, $\widehat{\mathbf{v}}$ is at an angle θ_1, θ_2 with $\widehat{\Gamma}_d, \widehat{\Gamma}_u$, resp. and has normal component 1, $\widehat{L}$ is a continuous non-decreasing process starting at 0 that increases only when $\widehat{X}_t$ is in $\partial\widehat{D}$, and

$$\widehat{E}_t = \int_0^t (1, \cot(\widehat{X}^2_s))\, ds. \tag{3.6}$$

Proof. The local time terms and martingale terms are handled just as in Proposition 3.1, so we focus only on the drift terms.

Let $z = x + iy = re^{i\theta}$ and let $w = U + iV = \log z$. For the partial derivatives we have

$$U_x = \frac{x}{x^2+y^2}, \qquad U_y = \frac{y}{x^2+y^2}, \qquad V_x = -U_y, \qquad V_y = U_x.$$

When we apply Ito's formula as in Proposition 3.1, we have what we obtained there plus an additional drift term: if $(\mathcal{U}_t, \mathcal{V}_t) = \log(\widetilde{X}_t)$, then

$$d\mathcal{U}_t = \text{ martingale} + \text{local time term } + U_y(\widetilde{X}_t)\, dt$$

and a similar expression for $d\mathcal{V}_t$ with the U's replaced by V's. Therefore

$$d\mathcal{U}_t = \text{ martingale} + \text{local time term } + \frac{\widetilde{X}^2_t}{(\widetilde{X}^1_t)^2 + (\widetilde{X}^2_t)^2} \cdot \frac{1}{\widetilde{X}^2_t} dt,$$

$$d\mathcal{V}_t = \text{ martingale} + \text{local time term } + \frac{\widetilde{X}^1_t}{(\widetilde{X}^1_t)^2 + (\widetilde{X}^2_t)^2} \cdot \frac{1}{\widetilde{X}^2_t} dt.$$

Since $(\mathcal{U}_t, \mathcal{V}_t) = \log \widetilde{X}_t$, then $\widetilde{X}_t = (e^{\mathcal{U}_t} \cos \mathcal{V}_t, e^{\mathcal{U}_t} \sin \mathcal{V}_t)$. We therefore have

$$d\mathcal{U}_t = \text{ martingale} + \text{local time term } + \frac{1}{e^{2\mathcal{U}_t}}\, dt,$$

$$d\mathcal{V}_t = \text{ martingale} + \text{local time term } + \frac{\cot \mathcal{V}_t}{e^{2\mathcal{U}_t}}\, dt.$$

Since $\left|\widetilde{X}_t\right|^{-2} = e^{-2\mathcal{U}_t}$, a time change argument using $\widetilde{A}_t = \int_0^t |1/\widetilde{X}_s|^2\, ds$ shows that the drift term is given by (3.6). □

4. Some excursion theory

The following review of excursion theory is taken from [BCJ06] because a crucial element of our argument, just as in [BCJ06], is the proper normalization of several quantities. See, e.g., [Mai75] for the foundations of excursion theory in the abstract setting and [Bur87] for the special case of excursions of Brownian motion. Although [Bur87] does not discuss reflecting Brownian motion, all the results we need from that book readily apply in the present context.

An "exit system" for excursions of reflecting Brownian motion X from ∂D is a pair (L^*_t, H^x) consisting of a positive continuous additive functional L^*_t and a family of "excursion laws" $\{H^x\}_{x\in\partial D}$. We will explain below that $L^*_t = L^X_t$. Let Δ denote the "cemetery" point outside $\mathbb{R}^2$ and let $\mathcal{C}$ be the space of all functions $f : [0, \infty) \to \mathbb{R}^2 \cup \{\Delta\}$ which are continuous and take values in $\mathbb{R}^2$ on some interval $[0, \zeta)$, and are equal to

Δ on $[\zeta, \infty)$. For $x \in \partial D$, the excursion law H^x is a σ-finite (positive) measure on $\mathcal{C}$, such that the canonical process is strong Markov on (t_0, ∞), for every $t_0 > 0$, with the transition probabilities of Brownian motion killed upon hitting ∂D. Moreover, H^x gives zero mass to paths which do not start from x. We will be concerned only with "standard" excursion laws; see Definition 3.2 of [Bur87]. For every $x \in \partial D$ there exists a unique standard excursion law H^x in D, up to a multiplicative constant.

Excursions of X from ∂D will be denoted e or e_s. If $s < u$, $X_s, X_u \in \partial D$, and $X_t \notin \partial D$ for $t \in (s, u)$ then $\mathrm{e}_s = \{\mathrm{e}_s(t) = X_{t+s}, t \in [0, u-s)\}$ and $\zeta(\mathrm{e}_s) = u - s$. By convention, $\mathrm{e}_s(t) = \Delta$ for $t \geq \zeta$, so $\mathrm{e}_t \equiv \Delta$ if $\inf\{s > t : X_s \in \partial D\} = t$. Let $\mathcal{E}_u = \{\mathrm{e}_s : s \leq u\}$.

Let $\sigma_t = \inf\{s \geq 0 : L^*_s \geq t\}$ and let I be the set of left endpoints of all connected components of $(0, \infty) \setminus \{t \geq 0 : X_t \in \partial D\}$. The following is a special case of the exit system formula of [Mai75],

$$\mathbb{E}\left[\sum_{t \in I} V_t \cdot f(\mathrm{e}_t)\right] = \mathbb{E}\int_0^\infty V_{\sigma_s} H^{X(\sigma_s)}(f)\, ds = \mathbb{E}\int_0^\infty V_t H^{X_t}(f)\, dL^*_t, \tag{4.1}$$

where V_t is a predictable process, either non-negative or such that at least one of the terms of (4.1) with V replaced by $|V|$ is finite, and $f : \mathcal{C} \to [0, \infty)$ is a universally measurable function which vanishes on excursions e_t identically equal to Δ. Here and elsewhere $H^x(f) = \int_{\mathcal{C}} f\, dH^x$.

The normalization of the exit system is somewhat arbitrary. For example, if (L^*_t, H^x) is an exit system and $c \in (0, \infty)$ is a constant then $(cL^*_t, (1/c)H^x)$ is also an exit system. One can even make c dependent on $x \in \partial D$. Let $\mathbb{P}^y_D$ denote the distribution of Brownian motion starting from y and killed upon exiting D. Theorem 7.2 of [Bur87] shows how to choose a "canonical" exit system; that theorem is stated for the usual planar Brownian motion but it is easy to check that both the statement and the proof apply to reflecting Brownian motion. According to that result, we can take L^*_t to be the continuous additive functional whose Revuz measure is a constant multiple of the arc length measure on ∂D and the H^x's to be standard excursion laws normalized so that

$$H^x(A) = \lim_{\delta \downarrow 0} \frac{1}{\delta}\, \mathbb{P}_D^{x+\delta \mathbf{n}(x)}(A), \tag{4.2}$$

for any event A in the σ-field generated by the process on an interval $[t_0, \infty)$, for any $t_0 > 0$.

The Revuz measure of L^X is the measure $dx/(2|D|)$ on ∂D: if the initial distribution of X is the uniform probability measure μ in D then

$$\mathbb{E}^\mu \int_0^1 \mathbf{1}_A(X_s)\, dL^X_s = \int_A dx/(2|D|) \tag{4.3}$$

for any Borel set $A \subset \partial D$; see [FOT94, Ex. 5.2.2].

The crucial fact for us, proved in [BCJ06], is that $L^*_t = L^X_t$. That is, the normalization of the local time L^X_t contained implicitly in (1.2) and the normalization of excursion laws H^x given in (4.2) match so that (dL^X_t, H^x) is an exit system for X_t from ∂D.

All of the above applies to the processes $\widetilde{X}$ in $\widetilde{D}$ and $\widehat{X}$ in $\widehat{D}$. We will put tildes or hats above all the corresponding objects.

5. Moving along the strip

Define

$$\begin{aligned} S_0^u &= 0, \qquad S_0^d = 0, \\ S_k^d &= \inf\{t \ge S_{k-1}^u : X_t \in \Gamma_d\}, \qquad k \ge 1, \\ S_k^u &= \inf\{t \ge S_k^d : X_t \in \Gamma_u\}, \qquad k \ge 1. \end{aligned} \tag{5.1}$$

These are the successive hits of X to Γ_u and Γ_d. Define $\widetilde{S}_k^u, \widetilde{S}_k^d$ and $\widehat{S}_k^u, \widehat{S}_k^d$ analogously, but with X, Γ_u, Γ_d replaced by $\widetilde{X}, \widetilde{\Gamma}_u, \widetilde{\Gamma}_d$ and $\widehat{X}, \widehat{\Gamma}_u, \widehat{\Gamma}_d$, resp.

Recall that $\widehat{D}$ is a horizontal strip with upper boundary $\widehat{\Gamma}_u$ and lower boundary $\widehat{\Gamma}_d$. In this section we compute the expected value of the difference in $\widehat{X}^1$ between $\widehat{S}_{k+1}^d$ and $\widehat{S}_k^d$.

Remark 5.1. We begin by recalling several formulas from the theory of one-dimensional diffusions; see, e.g., [KT81, Ch. 15, Sec. 3]. If the generator is given by

$$\frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx},$$

the scale function S is defined up to an additive constant by

$$S'(x) = \exp\left(-\int^x (2\mu(y)/\sigma^2(y))\, dy\right)$$

and the speed measure has a density given by

$$m(x) = \frac{1}{\sigma^2(x)S'(x)}.$$

If T_c denotes the hitting time of $c \in \mathbb{R}$ then for $a < x < b$,

$$p(x) = \mathbb{P}_x(T_b < T_a) = \frac{S(x) - S(a)}{S(b) - S(a)},$$

$$\begin{aligned} \mathbb{E}_x(T_a \wedge T_b) = 2p(x) &\int_x^b (S(b) - S(y))m(y)\, dy \\ &+ 2(1 - p(x)) \int_a^x (S(y) - S(a))m(y)\, dy. \end{aligned} \tag{5.2}$$

Let $\widehat{\tau}_u$ and $\widehat{\tau}_d$ denote the hitting times of $\widehat{\Gamma}_u$ and $\widehat{\Gamma}_d$, resp. Let $\{\widehat{H}^\cdot\}$ denote the excursion laws corresponding to the process $\widehat{X}$.

Lemma 5.2. *We have*

$$\widehat{H}^{(x,y)}(\widehat{\tau}_u < \infty) = \frac{1}{\cos^2\theta_1(\tan\theta_1 + \tan\theta_2)}, \qquad (x, y) \in \widehat{\Gamma}_d, \tag{5.3}$$

$$\widehat{H}^{(x,y)}(\widehat{\tau}_d < \infty) = \frac{1}{\cos^2\theta_2(\tan\theta_1 + \tan\theta_2)}, \qquad (x, y) \in \widehat{\Gamma}_u. \tag{5.4}$$

Proof. When $\widehat{X}$ is in the interior of the strip $\widehat{D}$, $\widehat{X}^2$ is a one-dimensional diffusion with drift coefficient $\cot y$ when $\widehat{X}^2_t = y$ by Proposition 3.2. We apply Remark 5.1 to $\widehat{X}^2$ with $a = \pi/2 - \theta_1$, $b = \pi/2 + \theta_2$, $\sigma \equiv 1$, and $\mu(y) = \cot y$. Note that

$$\sin(a) = \sin(\pi/2 - \theta_1) = \cos(\theta_1), \qquad \sin(b) = \sin(\pi/2 + \theta_2) = \cos(\theta_2), \tag{5.5}$$

$$\cot(a) = \cot(\pi/2 - \theta_1) = \tan(\theta_1), \qquad \cot(b) = \cot(\pi/2 + \theta_2) = -\tan(\theta_2). \tag{5.6}$$

By Remark 5.1,

$$S'(x) = \exp\left(-\int^x 2\cot y\, dy\right) = 1/\sin^2 x, \tag{5.7}$$

hence

$$S(x) = \int^x \sin^{-2} z\, dz = -\cot x$$

and

$$p(x) = \frac{S(x) - S(a)}{S(b) - S(a)} = \frac{-\cot x + \cot a}{-\cot b + \cot a}.$$

Consider $x = a + \delta$ where $\delta > 0$ is small. Then

$$-\cot x + \cot a = -\delta \frac{d}{dy}(\cot a)\Big|_{y=a} + o(\delta) = \delta/\sin^2 a + o(\delta).$$

Hence

$$p(x) = p(a+\delta) = \frac{-\cot x + \cot a}{-\cot b + \cot a} = \frac{\delta}{\sin^2 a(\cot a - \cot b)} + o(\delta).$$

We apply (4.2) and use (5.5)-(5.6) to see that

$$\widehat{H}^{(x,y)}(\widehat{\tau}_u < \infty) = \lim_{\delta \downarrow 0} \frac{1}{\delta} p(a+\delta) = \frac{1}{\sin^2 a(\cot a - \cot b)} = \frac{1}{\cos^2\theta_1(\tan\theta_1 + \tan\theta_2)}.$$

This completes the proof of (5.3). The proof of (5.4) is analogous. □

Set

$$\kappa = \frac{1}{(\theta_1 + \theta_2)(\tan\theta_1 + \tan\theta_2)}. \tag{5.8}$$

Proposition 5.3. *For $k \geq 1$,*

$$\widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\widehat{X}^1\left(\widehat{S}^d_{k+1}\right) - \widehat{X}^1\left(\widehat{S}^d_k\right)\right) = 1/\kappa.$$

Proof. Step 1. According to the exit system formula (4.1), the distribution of the amount of local time $\widehat{L}\left(\widehat{S}^u_k\right) - \widehat{L}\left(\widehat{S}^d_k\right)$ accumulated on $\widehat{\Gamma}_d$ between a time when $\widehat{X}$ hits $\widehat{\Gamma}_d$ and the first time after that when it hits $\widehat{\Gamma}_u$ is exponential with mean equal to $\left(\widehat{H}^{(x,y)}(\widehat{\tau}_u < \infty)\right)^{-1}$. In view of (5.3),

$$\widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\widehat{L}\left(\widehat{S}^u_k\right) - \widehat{L}\left(\widehat{S}^d_k\right)\right) = \cos^2\theta_1(\tan\theta_1 + \tan\theta_2).$$

The reflection vector on $\widehat{\Gamma}_d$ is $\widehat{\mathbf{v}} = (\widehat{\mathbf{v}}^1, \widehat{\mathbf{v}}^2) = (-\tan\theta_1, 1)$ according to our sign convention, so the last formula implies that

$$\widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\int_{\widehat{S}^d_k}^{\widehat{S}^u_k} \widehat{\mathbf{v}}^1(\widehat{X}_t)\, d\widehat{L}_t\right) = -\tan\theta_1\cos^2\theta_1(\tan\theta_1+\tan\theta_2).$$

Similarly,

$$\widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\int_{\widehat{S}^u_k}^{\widehat{S}^d_{k+1}} \widehat{\mathbf{v}}^1(\widehat{X}_t)\, d\widehat{L}_t\right) = -\tan\theta_2\cos^2\theta_2(\tan\theta_1+\tan\theta_2),$$

and so

$$\widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\int_{\widehat{S}^d_k}^{\widehat{S}^d_{k+1}} \widehat{\mathbf{v}}^1(\widehat{X}_t)\, d\widehat{L}_t\right) = -(\tan\theta_1\cos^2\theta_1+\tan\theta_2\cos^2\theta_2)(\tan\theta_1+\tan\theta_2).$$

Step 2. In this step, we will find $\widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\widehat{S}^d_{k+1}-\widehat{S}^d_k\right)$.

Formula (5.2) is concerned with the expected amount of time that it takes for the diffusion to go from an interior point in the interval (a,b) to one of the boundary points. We would like to apply this formula to the reflected process $\widehat{X}^2$. To achieve that, we will "unfold" the reflection. More precisely, we will consider a diffusion on the interval $(\pi/2-2\theta_1-\theta_2, \pi/2+\theta_2)$ with the same parameters $\sigma\equiv 1$ and $\mu(y)=1/y$ for y in $(\pi/2-\theta_1,\pi/2+\theta_2)$, the upper half of the interval. For y in $(\pi/2-2\theta_1-\theta_2,\pi/2-\theta_1)$, the lower half of the interval, we take $\sigma\equiv 1$ and make the drift μ symmetric, that is, $\mu(y) = -\mu(\pi/2-\theta_1+(\pi/2-\theta_1-y))$. The "unfolded" diffusion has no reflection at $\pi/2-\theta_1$. It is clear that the expected time for the new diffusion starting from $\pi/2-\theta_1$ to exit $(\pi/2-2\theta_1-\theta_2,\pi/2+\theta_2)$ is the same as for $\widehat{X}^2$ starting from $\pi/2-\theta_1$ to exit $(\pi/2-\theta_1,\pi/2+\theta_2)$. We will write $\mathbb{E}^*$ to denote the expectation corresponding to the unfolded diffusion.

In the following calculation we take $p(x)=1/2$ because by symmetry it represents the probability that the unfolded diffusion will exit the interval at the upper end. By (5.2),

$$\begin{aligned}
&\widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\widehat{S}^u_k-\widehat{S}^d_k\right) = \mathbb{E}^*_{\pi/2-\theta_1}(T_{\pi/2+\theta_2}\wedge T_{\pi/2-2\theta_1-\theta_2})\\
&= 2p(x)\int_x^b (S(b)-S(y))m(y)\,dy + 2(1-p(x))\int_a^x (S(y)-S(a))m(y)\,dy\\
&= \int_x^b (S(b)-S(y))m(y)\,dy + \int_a^x (S(y)-S(a))m(y)\,dy,\\
&= 2\int_x^b (S(b)-S(y))m(y)\,dy.
\end{aligned}$$

The last equality holds by symmetry. Hence,

$$\widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\widehat{S}^u_k-\widehat{S}^d_k\right) = 2\int_x^b (S(b)-S(y))m(y)\,dy = 2\int_{\pi/2-\theta_1}^{\pi/2+\theta_2} (\tan\theta_2+\cot y)\sin^2 y\,dy$$

$$= (\theta_1 + \theta_2)\tan(\theta_2) + \sin^2(\theta_1) + \sin(\theta_1)\cos\theta_1\tan\theta_2.$$

A calculation, based on similar unfolding arguments, yields

$$\begin{aligned}\widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\widehat{S}^d_{k+1} - \widehat{S}^u_k\right) &= \mathbb{E}^*_{\pi/2+\theta_2}(T_{\pi/2-\theta_1} \wedge T_{\pi/2+2\theta_2+\theta_1}) \\ &= (\theta_1+\theta_2)\tan(\theta_1) + \sin^2(\theta_2) + \tan(\theta_1)\sin(\theta_2)\cos\theta_2.\end{aligned}$$

Adding the last two formulas, we obtain

$$\begin{aligned}\widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\widehat{S}^d_{k+1} - \widehat{S}^d_k\right) = (\theta_1+\theta_2)\tan(\theta_2) + \sin^2(\theta_1) + \sin(\theta_1)\cos\theta_1\tan\theta_2 \\ + (\theta_1+\theta_2)\tan(\theta_1) + \sin^2(\theta_2) + \tan(\theta_1)\sin(\theta_2)\cos\theta_2.\end{aligned}$$

Step 3. The process $\widehat{X}$ satisfies the stochastic differential equation

$$\widehat{X}_t = x_0 + \widehat{B}_t + \int_0^t \left(1, \cot\left(\widehat{X}^2_s\right)\right) ds + \int_0^t \widehat{\mathbf{v}}(\widehat{X}_s)\, d\widehat{L}_s, \qquad \text{for } t \geq 0, \tag{5.9}$$

where $\widehat{B}$ is two-dimensional Brownian motion. Hence, the formulas derived in Steps 1 and 2 yield

$$\begin{aligned}&\widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\widehat{X}^1\left(\widehat{S}^d_{k+1}\right) - \widehat{X}^1\left(\widehat{S}^d_k\right)\right) = \widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\int_{\widehat{S}^d_k}^{\widehat{S}^d_{k+1}} \widehat{\mathbf{v}}_1(\widehat{X}_t)\, d\widehat{L}_t\right) + \widehat{\mathbb{E}}^{\widehat{h}}_{\cdot}\left(\widehat{S}^d_{k+1} - \widehat{S}^d_k\right) \\ &= -(\tan\theta_1\cos^2\theta_1 + \tan\theta_2\cos^2\theta_2)(\tan\theta_1 + \tan\theta_2) \\ &\quad + (\theta_1+\theta_2)\tan(\theta_2) + \sin^2(\theta_1) + \sin(\theta_1)\cos\theta_1\tan\theta_2 \\ &\quad + (\theta_1+\theta_2)\tan(\theta_1) + \sin^2(\theta_2) + \tan(\theta_1)\sin(\theta_2)\cos\theta_2 \\ &= (\theta_1+\theta_2)(\tan\theta_1 + \tan\theta_2).\end{aligned}$$

□

6. Invariant measures

This section is devoted to establishing the existence of an invariant measure, or equivalently, a stationary probability distribution, for a Markov chain defined in terms of $\widetilde{X}$.

Define

$$\widetilde{H}_b = \{(x,y) \in \widetilde{D} : y = b\}, \qquad H_b = \mathbb{F}^{-1}(\widetilde{H}_b), \qquad \widehat{H}_b = \log(\widetilde{H}_b), \tag{6.1}$$

where $\mathbb{F}$ is the function given in (3.1). Define

$$T_b = \inf\{t : X_t \in H_b\} \tag{6.2}$$

and define $\widetilde{T}_b$ and $\widehat{T}_b$ analogously, but with H_b replaced by $\widetilde{H}_b$ and $\widehat{H}_b$ and X replaced by $\widetilde{X}$ and $\widehat{X}$, resp.

We start with X_t, an ORBM in D satisfying (1.2). We define $\widetilde{X}$ by means of Proposition 3.1. Since $\widetilde{X}^2$ is a one-dimensional Brownian motion, the function

$$\widetilde{h}(x,y) = y, \qquad (x,y) \in \widetilde{D} \tag{6.3}$$

is harmonic. We then condition $\widetilde{X}$ by using the h-path transform of Doob with the function (6.3). This gives rise to new probability measures defined by

$$\widetilde{P}_z^{\widetilde{h}}(A) = \widehat{\mathbb{E}}_z[\widetilde{h}(\widetilde{X}_S); A]/\widetilde{h}(z), \tag{6.4}$$

where $\widetilde{X}$ is adapted to a filtration $\{\mathcal{F}_t\}$, S is a stopping time with respect to $\{\mathcal{F}_t\}$, and $A \in \mathcal{F}_S$. Note $\partial\widetilde{h}(x,y)/\partial x = 0$ and $\partial\widetilde{h}(x,y)/\partial y = 1$. Therefore under $\widetilde{P}_z^{\widetilde{h}}$ the process $\widetilde{X}$ is a 3-dimensional Bessel process in the second coordinate; see [Bas95, pp. 61–62]. Finally we apply Proposition 3.2 to obtain $\widehat{X}$.

Recall that if we have a strong Markov process Z_t, a point z is regular for a Borel set A if $\mathbb{P}^x(\tau_A = 0) = 1$, where $\tau_A = \inf\{t > 0 : Z_t \in A\}$.

We establish this for $\widetilde{X}$, for points $z \in \widetilde{H}_1$, and

$$\widetilde{A} = \{z = (x,y) : y > 1\}. \tag{6.5}$$

Lemma 6.1. *Let $(\widetilde{X}, \mathbb{P}_z^{\widetilde{h}})$ be the strong Markov process given by* (6.4)*. Let $\widetilde{A}$ be the set defined by* (6.5)*. Then every point of $\widetilde{H}_1$ is regular for $\widetilde{A}$.*

Proof. The process $\widetilde{X}^2$ is a 3 dimensional Bessel process defined in (3.4), and so solves the stochastic differential equation

$$X_t^2 = X_0^2 + B_t^2 + \int_0^t \frac{1}{\widetilde{X}_s^2}\, ds,$$

where B_t^2 is a Brownian motion independent of $\widetilde{X}_t^1$. Therefore $X_t^2 \geq X_0^2 + B_t^2$. It is well known that for one-dimensional Brownian motion the origin is regular with respect to the interval $(0,\infty)$, and using translation invariance it follows that for $\widetilde{X}_t^2$ the point 1 is regular for $(1,\infty)$. Consequently for $\widetilde{X}_t$ the point $(x,1)$ is regular for $\widetilde{A}$. $\square$

We now define a Markov chain on $\widetilde{H}_1$ by using the kernel

$$\widetilde{Q}(x, dy) = \widetilde{P}_{e^{-1}x}^{\widetilde{h}}(\widetilde{X}(\widetilde{T}_1) \in dy), \qquad x, y \in \widetilde{H}_1. \tag{6.6}$$

Our main result in this section is that there exists an invariant measure for this chain.

Proposition 6.2. *There exists an invariant probability $\widetilde{\nu}$ for the chain with kernel $\widetilde{Q}$.*

Proof. By page 316 of [TW93] and [SV79, Corollary 4.6], the law of $\widetilde{P}_{e^{-1}x_n}^{\widetilde{h}}$ converges weakly to the law of $\widetilde{P}_{e^{-1}x}^{\widetilde{h}}$ whenever $x_n, x \in D$ and $x_n \to x$. By the Skorokhod representation theorem (see, e.g., [Bas11, Chapter 31]), we can find a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and processes X_n' and X' such that the law of X_n' under $\mathbb{P}'$ is the same as that of $\widetilde{X}$ under $\widetilde{P}_{e^{-1}x_n}^{\widetilde{h}}$, the law of X' under $\mathbb{P}'$ is the same as that of $\widetilde{X}$ under $\widetilde{P}_{e^{-1}x}^{\widetilde{h}}$, and $X_n'(t)$ converges uniformly on compacts to $X'(t)$ almost surely with respect to $\mathbb{P}'$.

Let $T_n' = \inf\{t : X_n'(t) \in \widetilde{H}_1\}$ and define T' similarly. If F is a continuous function on $\widetilde{H}_1$, if $x_n, x \in \widetilde{H}_1$, and if $x_n \to x$, it may not be true that $F(X_n'(T_n'))$ converges to $F(X'(T'))$ for every ω. But the only ω's for which it is not true are those for which the path of X' just hits $\widetilde{H}_1$ from below but does not subsequently immediately enter the set $\widetilde{A}$ defined in (6.5). By the strong Markov property and Lemma 6.1, the probability of such paths is 0.

Therefore, using that F is bounded because $\widetilde{H}_1$ is compact,

$$\mathbb{E}'\, F(X_n'(T_n')) \to \mathbb{E}'\, F(X'(T')),$$

or

$$\widetilde{\mathbb{E}}^{\widetilde{h}}_{e^{-1}x_n} F(X_{T_1}) \to \widetilde{\mathbb{E}}^{\widetilde{h}}_{e^{-1}x} F(X_{T_1}).$$

This is the same as saying

$$\widetilde{Q}F(x_n) = \int F(y)\, \widetilde{Q}(x_n, dy) \to \int F(y)\, \widetilde{Q}(x, dy) = \widetilde{Q}F(x).$$

Therefore $\widetilde{Q}$ maps continuous functions to continuous functions.

Since $\widetilde{H}_1$ is compact, $\{\widetilde{Q}^n(x,\cdot)\}$ is tight for each x, where $\widetilde{Q}^n$ is the n^{th} iterate of $\widetilde{Q}$. The existence of an invariant measure now follows by the Krylov-Bogolyubov theorem ([Hai21, Theorem 1.10]). □

As a consequence to Proposition 7.1 we will show that the invariant measure is unique and that $\widetilde{Q}^n(x, dy)$ converges to $\widetilde{\nu}(dy)$ for every x.

Remark 6.3. If W_t is a one-dimensional Brownian motion or a 3-dimensional Bessel process started at w_0 and $b > 0$, then scaling tells us that $\sqrt{b}W_{t/b}$ is again a one-dimensional Brownian motion or a 3-dimensional Bessel process, but now started at $\sqrt{b}w_0$. If $\widetilde{X}$ is a solution to (3.3), then $\sqrt{b}\widetilde{L}_{t/b}$ is again a non-decreasing continuous process that increases only when $\widetilde{X}_{t/b}$ is in $\partial\widetilde{D}$. Therefore $\sqrt{b}\widetilde{X}_{t/b}$ is a solution to (3.3), started at $x_0/\sqrt{b}$. We have uniqueness in law for (3.3) by [TW93], hence except for the starting point, $\sqrt{b}\widetilde{X}_{t/b}$ has the same law as $\widetilde{X}_t$. We refer to this property as scaling.

Let $\widetilde{\nu}(1)$ be the probability distribution on $\widetilde{H}_1$ given in Proposition 6.2. Define a probability $\widetilde{\nu}(e^m)$ on $\widetilde{H}_{e^m}$ for integer m by

$$\widetilde{\nu}(e^m)(A) = \widetilde{\nu}(1)(e^{-m}A) \tag{6.7}$$

for Borel subsets $A \subset \widetilde{H}_{e^m}$, where $bA = \{by : y \in A\}$.

Remark 6.4. By scaling and the fact that $\widetilde{\nu}(1)$ is a stationary distribution for the kernel $\widetilde{Q}$, we see that if $n \le m$, then the distribution of $\widetilde{X}(\widetilde{T}(e^m))$ under $\widetilde{P}^{\widetilde{h}}_{\widetilde{\nu}(e^n)}$ is $\widetilde{\nu}(e^m)$.

7. Harnack inequality

In this section we prove a Harnack inequality by means of what is essentially a coupling argument.

Proposition 7.1. *There exists $q \in (0,1)$ such that for all integers i and n with $i \ge n+2$, $x, y \in \widetilde{H}_{e^n}$, and A a Borel subset of $\widetilde{H}_{e^i}$,*

$$1 - q^{i-n-1} \le \frac{\widetilde{\mathbb{P}}^{\widetilde{h}}_x(\widetilde{X}(\widetilde{T}_{e^i}) \in A)}{\widetilde{\mathbb{P}}^{\widetilde{h}}_y(\widetilde{X}(\widetilde{T}_{e^i}) \in A)} \le \left(1 - q^{i-n-1}\right)^{-1}. \tag{7.1}$$

Proof. By scaling it suffices to take $i = 0$. This implies that $n \le -2$. Define

$$\Phi(z) = \widetilde{P}^{\widetilde{h}}_z(\widetilde{X}(\widetilde{T}_1) \in A), \qquad z \in \widetilde{D} \setminus \{0\},$$

where $\widetilde{X}$ satisfies (3.3) with starting point z and $\widetilde{P}^{\widetilde{h}}_z$ is the law of $\widetilde{X}$ with starting point z. Let $b \in (0,1)$ and $x \in \widetilde{H}_{e^n}$. We first show there exists a continuous curve γ in $\widetilde{D}$ connecting x and $\widetilde{H}_1$ that avoids a neighborhood of 0 and such that $\Phi(z) \ge b\Phi(x)$ for all z lying on γ.

By [TW93] the solution to (1.2) is unique in law, in particular, unique up until first hitting 0. It follows readily that the solution to (3.3) is also unique in law up until first hitting 0, since the conformal map f is one-to-one and the time change is non-degenerate up until hitting 0. It then follows that the law of $\widetilde{X}$ under the h-path transform is unique. Since $\widetilde{X}$ transformed by (6.3) never hits 0, the law of $\widetilde{X}$ is unique without the restriction of not hitting 0. As in [TW93, page 316], this implies $(X, \widetilde{P}^{\widetilde{h}}_\cdot)$ is a strong Markov process and in addition is a Feller process.

If $\mathcal{F}_t$ is the filtration generated by $\widetilde{X}$, it is right continuous; see [Bas11, Proposition 20.7]. Letting

$$M_t = \widetilde{\mathbb{E}}_z[1_A(\widetilde{X}(\widetilde{T}_1)) \mid \mathcal{F}_{t\wedge\widetilde{T}_1}],$$

we see that not only is M_t a martingale, but it is also right continuous. By the strong Markov property,

$$M_t = \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{X}_{t\wedge\widetilde{T}_1}}[1_A(\widetilde{X}(\widetilde{T}_1))] = \Phi(\widetilde{X}_{t\wedge\widetilde{T}_1}).$$

If $S_b = \inf\{t \ge 0 : \Phi(\widetilde{X}_t) \le b\Phi(x)\}$, Doob's optional stopping theorem tells us that

$$\Phi(x) = \widetilde{\mathbb{E}}^{\widetilde{h}}_x[\Phi(\widetilde{X}_{S_b\wedge\widetilde{T}_1}].$$

If $S_b < \widetilde{T}_1$ almost surely, the right hand side will be less than or equal to $b\Phi(x)$, a contradiction. Therefore with positive probability there exists an ω such that $S_b \ge \widetilde{T}_1$. The graph of $\widetilde{X}_t(\omega)$ will be our desired γ.

We now want to apply the support theorem for diffusions (see [Bas98, pp. 25–26]) to $\widetilde{B}$. The domain in which we apply the support theorem is a positive distance from the origin, and hence the drift coefficients of $\widetilde{B}$ are bounded there. Let d be the length of $\widetilde{H}_1$. Let φ be the curve that moves horizontally to the left a distance $d+1$ and then horizontally to the right a distance $d+1$ and then vertically a distance $e+2$, all at unit speed. We start $\widetilde{X}$ at a point $z \in \widetilde{H}_1$. Let $\varepsilon = 1/4$. The support theorem says there is probability $p > 0$ that $\widetilde{B}$ stays within ε of this curve until time $2d + e + 4$. Since the local time term only contributes a push in the horizontal directions and not at all in the vertical directions, we see that with probability at least p the process $\widetilde{X}$ reflects off $\widetilde{\Gamma}_u$, eventually moves right until it reflects off $\widetilde{\Gamma}_d$, and finally exits the strip between $\widetilde{H}_{e^{-1}}$ and $\widetilde{H}_e$ through the upper boundary.

Let F_k be the event where the process $\widetilde{X}_t$ starts at a point in $\widetilde{H}_{e^k}$, there is an excursion from $\widetilde{\Gamma}_u$ to $\widetilde{\Gamma}_d$, and the process exits the strip between $\widetilde{H}_{e^{k-1}}$ and $\widetilde{H}_{e^{k+1}}$ for the first time after the excursion is completed, and the exit is through the top of the strip. By scaling, $\widetilde{\mathbb{P}}^{\widetilde{h}}_\cdot(F_k) \ge p$.

By the strong Markov property applied $n-1$ times,

$$\widetilde{\mathbb{P}}^{\widehat{h}}_z(\cup_{k=n+1}^{-1} F_k) = 1 - \widetilde{P}^{\widehat{h}}_z(\cap_{k=n+1}^{-1} F_k^c) \geq 1 - q^{-n-1},$$

where $q = 1-p$. This implies that except for an event of probability at most q^{-n-1}, the curve γ must intersect the path of $\widetilde{X}_t$.

Let $S_\gamma = \inf\{t : \widehat{X}_t \in \gamma\}$. Then by optional stopping,

$$\begin{aligned}\Phi(y) &= \widetilde{P}^{\widetilde{h}}_y(\widetilde{X}(\widetilde{T}_1) \in A) = \widetilde{E}^{\widetilde{h}}_y[M_{\widetilde{T}_1}] = \widetilde{\mathbb{E}}^{\widetilde{h}}_y[M_{\widetilde{T}_1 \wedge S_\gamma}] \\ &\geq \Big[b\Phi(x)\Big] \cdot \widehat{P}^{\widehat{h}}_y(S_\gamma < \widetilde{T}_1) \\ &\geq b(1-q^{-n-1})\Phi(x).\end{aligned}$$

Therefore

$$\mathbb{P}^{\widetilde{h}}_y(\widetilde{X}(\widetilde{T}_1 \in A) \geq b(1-q^{-n-1})\,\mathbb{P}^{\widetilde{h}}_x(\widetilde{X}(\widetilde{T}_1 \in A).$$

Letting $b \uparrow 1$ yields the right hand inequality in (7.1), and reversing the roles of x and y yields the left hand inequality. □

Corollary 7.2. *Suppose that $i \geq n+2$.*
(i) If μ_1 and μ_2 are two probability measures on $\widetilde{H}_{e^n}$ and A is a Borel subset of $\widetilde{H}_{e^i}$, then

$$-q^{i-n-1} \leq \widetilde{P}^{\widetilde{h}}_{\mu_1}(\widetilde{X}(\widetilde{T}_{e^i}) \in A) - \widetilde{P}^{\widetilde{h}}_{\mu_2}(\widetilde{X}(\widetilde{T}_{e^i}) \in A) \leq (1-q^{i-n-1})^{-1} - 1. \tag{7.2}$$

(ii) If μ_1 and μ_2 are two probability measures on $\widetilde{H}_{e^n}$, then the total variation distance between $\widetilde{\mathbb{P}}^{\widetilde{h}}_{\mu_1}(e^{-i}\widetilde{X}(\widetilde{T}_{e^i}) \in dz)$ and $\widetilde{\mathbb{P}}^{\widetilde{h}}_{\mu_2}(e^{-i}\widetilde{X}(\widetilde{T}_{e^i}) \in dz)$ tends to 0 as $i \to \infty$.
(iii) There exists a unique invariant probability measure for the Markov chain whose transition densities are given by the kernel $\widetilde{Q}$.

Proof. (i) Let A be a Borel subset of $\widetilde{H}_{e^i}$. Multiply all three terms in (7.1) by $\widetilde{P}^{\widetilde{h}}_y(\widetilde{X}(\widetilde{T}_{e^i}) \in A)$. Then integrate all three terms with respect to $\mu_1(dx) \times \mu_2(dy)$. Since μ_1 and μ_2 are probability measures we obtain

$$(1-q^{i-n-1})\widetilde{P}^{\widetilde{h}}_{\mu_2}(\widetilde{X}(\widetilde{T}_{e^i}) \in A) \leq \widetilde{P}^{\widetilde{h}}_{\mu_1}(\widetilde{X}(\widetilde{T}_{e^i}) \in A) \leq (1-q^{i-n-1})^{-1}\widetilde{P}^{\widetilde{h}}_{\mu_2}(\widetilde{X}(\widetilde{T}_{e^i}) \in A).$$

From this we deduce (7.2).

(ii) follows by replacing A by e^iA in (7.2) and then letting $i \to \infty$.

(iii) If μ_1, μ_2 are invariant probability measures and A is a Borel subset of $\widetilde{H}_1$, then from (7.2) with A replaced by e^iA and using the invariance of μ_1, μ_2, we obtain

$$-q^{i-n-1} \leq \mu_1(A) - \mu_2(A) \leq (1-q^{i-n-1})^{-1} - 1.$$

Letting $i \to \infty$, we see that $\mu_1(A) = \mu_2(A)$ for all Borel sets A. □

8. Strong law of large numbers for excursions

We need a strong law for the number of excursions between $\widetilde{\Gamma}_d$ and $\widetilde{\Gamma}_u$. Much of this section is devoted to showing our sequence satisfies the hypotheses of a version of the strong law for dependent random variables.

Define

$$S_k^{d-} = \sup\{t \le S_k^d : X_t \in \Gamma_u\}, \qquad k \ge 2, \tag{8.1}$$
$$S_k^{u-} = \sup\{t \le S_k^u : X_t \in \Gamma_d\}, \qquad k \ge 1,$$

and define $\widetilde{S}_k^{d-}, \widetilde{S}_k^{u-}, \widehat{S}_k^{d-}, \widehat{S}_k^{u-}$ analogously. Define

$$\widetilde{N}^d_{k_1,k_2} = \#\{j : \widetilde{T}_{e^{k_1}} \le \widetilde{S}_j^{u-} \le \widetilde{S}_j^u < \widetilde{T}_{e^{k_2}}\}, \qquad k_1 < k_2, \tag{8.2}$$
$$\widetilde{N}^u_{k_1,k_2} = \#\{j : \widetilde{T}_{e^{k_1}} \le \widetilde{S}_j^{d-} \le \widetilde{S}_j^d < \widetilde{T}_{e^{k_2}}\}, \qquad k_1 < k_2, \tag{8.3}$$

and define $\widehat{N}^d_{k_1,k_2}$ and $\widehat{N}^u_{k_1,k_2}$ analogously. The quantity $\widetilde{N}^d_{k_1,k_2}$ counts the number of times that $\widetilde{S}^u_j$ is between $\widetilde{T}_{e^{k_1}}$ and $\widetilde{T}_{e^{k_2}}$ and where an additional constraint holds: the excursion from $\widetilde{\Gamma}_d$ to $\widetilde{\Gamma}_u$ that ends at time $\widetilde{S}^u_j$ starts after $\widetilde{T}_{e^{k_1}}$.

Lemma 8.1. *(i) For any $m \ge 1$, the random variables $\widetilde{N}^d_{k,k+m}$, $k \ge n$, are identically distributed under $\widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(e^n)}$. Moreover, the distribution of the sequence $(\widetilde{N}^d_{k,k+m}, k \ge i)$ does not depend on i.*

(ii) For any $m \ge 1$, the random variables $\widehat{N}^d_{k,k+m}$, $k \ge n$, are identically distributed under $\widehat{\mathbb{P}}^{\widehat{h}}_{\widehat{\nu}(e^n)}$. Moreover, the distribution of the sequence $(\widehat{N}^d_{k,k+m}, k \ge i)$ does not depend on i.

Proof. Part (i) follows from the strong Markov property applied at time $\widetilde{T}(e^k)$ and Remark 6.4. Part (ii) follows from conformal invariance and the fact that time changing a process does not affect the values of $\widehat{N}^u_{k_1,k_2}$ and $\widehat{N}^d_{k_1,k_2}$. □

Lemma 8.2. *(i) Suppose that $n \le kj$ and let $\lambda = \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(e^n)} \widetilde{N}^d_{kj,(k+1)j}$. The mean λ depends on j but does not depend on k or n.*

(ii) For $j \ge 1$ there exists $\sigma^2 < \infty$ such that for all k and n such that $kj \ge n$,

$$\widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(e^n)} \left(\left(\widetilde{N}^d_{kj,(k+1)j} \right)^2 \right) < \sigma^2.$$

(iii) Fix $j \ge 1$ and for $n \le kj$ and $m \ge 1$ let

$$\varphi(m) = \left| \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(e^n)} \left[\left(\widetilde{N}^d_{kj,(k+1)j} - \lambda \right) \left(\widetilde{N}^d_{(k+m)j,(k+m+1)j} - \lambda \right) \right] \right|.$$

Then

$$\sum_{m \ge 1} \varphi(m)/m < \infty. \tag{8.4}$$

Proof. (i) follows by Lemma 8.1 (i). By the same lemma, σ^2 and $\varphi(m)$ do not depend on n (as long as $n \le kj$) and we can take $k = 0$ in the proofs of (ii)-(iii). By scaling, the strong Markov property, and the fact that $\widetilde{\nu}$ is a stationary probability distribution, it suffices to take $n = 0$.

(ii) It is well known that the probability a 3-dimensional Bessel process started at 1 ever hits the level e^{-1} is equal to e^{-1}. Hence the probability that this process hits the level e before the level e^{-1} is greater than 1/2. We apply this to $\widetilde{X}^2$.

Let

$$\begin{aligned} U_0 &= 0, \qquad \widetilde{X}^2_0 = 1, \\ U_k &= \inf\{t \ge U_{k-1} : \widetilde{X}^2_t / \widetilde{X}^2_{U_{k-1}} = 1/2 \text{ or } 2\}. \end{aligned}$$

Using the strong Markov property repeatedly shows that $\log \widetilde{X}^2_{U_k}$ is an asymmetric simple random walk on the integers started at 0 with positive mean.

Recall Hoeffding's inequality: if S_n is the sum of i.i.d. random variables that are bounded between a and b, then

$$\mathbb{P}(|S_n - \mathbb{E}\, S_n| \ge t) \le 2\exp(-2t^2/n(b-a)^2).$$

Let μ denote the mean of $\log \widetilde{X}^2_{U_1}$ and let

$$J = \inf\{k \ge 0 : \log \widetilde{X}^2_{U_k} = j\}.$$

By Hoeffding's inequality with $a = -1, b = 1$, and scaling, we have

$$\begin{aligned} \widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(1)}(J \ge M) &\le \widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(1)}(\log \widetilde{X}^2_{U_M} \le j) = \mathbb{P}^{\widetilde{h}}_{\widetilde{\nu}(1)}(\log \widetilde{X}^2_{U_M} - M\mu \le j - M\mu) \\ &\le 2\exp(-2(M\mu - j)^2/4M) \end{aligned}$$

as long as $M\mu \ge j$.

Let Λ_k be the number of excursions from $\widetilde{\Gamma}_d$ to $\widetilde{\Gamma}_u$ that start in $[\widetilde{U}_{k-1}, \widetilde{U}_k)$. Look at the time when $\widetilde{X}$ hits the middle line between $\widetilde{\Gamma}_d$ and $\widetilde{\Gamma}_u$, say, at a point (x, y). By the support theorem for diffusions there is probability $p_1 > 0$, independent of (x, y), that $\widetilde{X}$ will hit $\widetilde{H}_{y/2} \cup \widetilde{H}_{2y}$ before hitting $\widetilde{\Gamma}_d \cup \widetilde{\Gamma}_u$. This and the strong Markov property implies that Λ_k has a geometric tail: for some $c_1, c_2 > 0$ and $K \ge 1$ and all k,

$$\widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(1)}(\Lambda_k \ge K) \le c_1 \exp(-c_2 K).$$

Consider I large enough so that $\sqrt{I} \ge 2j/\mu$. Let $M = [\sqrt{I}]$. If $\widetilde{N}^d_{0,j} \ge I$, then either $J \ge M$ or for some $k \le M$ we have $\Lambda_k \ge I/M$. Using the above estimates on J and Λ_k we obtain

$$\widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(1)}(\widetilde{N}^d_{0,j} \ge I) \le 2e^{-c_3\sqrt{I}} + c_4\sqrt{I}e^{-c_5\sqrt{I}}.$$

This estimate for the tail probabilities implies that all moments of $\widetilde{N}^d_{0,j}$ under $\widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(1)}$ are finite.

(iii) We have

$$\varphi(m) = \left| \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(1)} \left[\left(\widetilde{N}^d_{0,j} - \lambda \right) \left(\widetilde{N}^d_{mj,(m+1)j} - \lambda \right) \right] \right|$$

$$= \left| \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(1)} \left[\left(\widetilde{N}^d_{0,j} - \lambda \right) \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{X}(\widetilde{T}_{e^j})} \left(\widetilde{N}^d_{mj,(m+1)j} - \lambda \right) \right] \right|$$

$$\le \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(1)} \left[\left| \widetilde{N}^d_{0,j} - \lambda \right| \cdot \left| \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{X}(\widetilde{T}_{e^j})} \left(\widetilde{N}^d_{mj,(m+1)j} \right) - \lambda \right| \right]. \tag{8.5}$$

By Proposition 7.1 and the strong Markov property applied at stopping times $\widetilde{T}_{e^j}$ and $\widetilde{T}_{e^{mj}}$, for $m \ge 2$,

$$\frac{\widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{X}(\widetilde{T}_{e^j})} \left(\widetilde{N}^d_{mj,(m+1)j} \right)}{\lambda} = \frac{\widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{X}(\widetilde{T}_{e^j})} \left(\widetilde{N}^d_{mj,(m+1)j} \right)}{\widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(1)} \left(\widetilde{N}^d_{mj,(m+1)j} \right)}$$

$$\le (1 - q^{(m-1)j-1})^{-1} \frac{\widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(1)} \left(\widetilde{N}^d_{mj,(m+1)j} \right)}{\widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(1)} \left(\widetilde{N}^d_{mj,(m+1)j} \right)} = (1 - q^{(m-1)j-1})^{-1}.$$

Similarly,

$$\frac{\widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{X}(\widetilde{T}_{e^j})} \left(\widetilde{N}^d_{mj,(m+1)j} \right)}{\lambda} \ge 1 - q^{(m-1)j-1},$$

so

$$\left| \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{X}(\widetilde{T}_{e^j})} \left(\widetilde{N}^d_{mj,(m+1)j} \right) - \lambda \right| \le \lambda c_5 q^{(m-1)j-1}.$$

We combine this with (8.5) to see that

$$\varphi(m) \le \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(1)} \left[\left| \widetilde{N}^d_{0,j} - \lambda \right| \cdot \lambda c_5 q^{(m-1)j-1} \right] \le c_6 q^{(m-1)j-1}.$$

This implies (8.4). □

Lemma 8.3. *For each $\varepsilon > 0$ there exists j_0 such that if $j \ge j_0$ and $n \le k$ then*

$$\kappa j < \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(e^n)} \widetilde{N}^d_{k,k+j} < (\kappa + \varepsilon) j. \tag{8.6}$$

Proof. Let d be the length of $\widehat{H}_b$. Since $\widehat{D}$ is a horizontal strip, d does not depend on b. By the construction of $\widehat{D}$ we note that d is finite. Let

$$J = J(j) = \min\{m \ge 1 : \widehat{X}^1(\widehat{S}^d_m) - \widehat{X}^1(0) \ge j + 2d\}.$$

By scaling, the strong Markov property, and the fact that $\widehat{\nu}$ is an invariant measure for $\widetilde{X}$, we may take $n = 0$. We will show that for some $c_1 < \infty$ and all $j \ge 1$,

$$0 \le \widehat{\mathbb{E}}^{\widehat{h}}_{\widetilde{\nu}(1)} (J) - \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(1)} \widetilde{N}^d_{k,k+j} \le c_1. \tag{8.7}$$

Note that $\widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(1)} \widetilde{N}^d_{k,k+j} = \widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)} \widehat{N}^d_{k,k+j}$ so the above inequality is equivalent to

$$0 \le \widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)} (J) - \widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)} \widehat{N}^d_{k,k+j} \le c_1. \tag{8.8}$$

Since $\widehat{X}_0 \in \widehat{H}_1$ under $\widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}$ and $\widehat{X}^1(\widehat{S}^d_J) \geq j + 2d$, we must have $\widehat{S}^d_J \geq \widehat{T}_{e^j}$, and therefore $J \geq \widehat{N}^d_{k,k+j}$. This implies the lower bound in (8.8).

By the support theorem for diffusions there exists a $p_1 > 0$ such that for every m, if $\widehat{X}_0 \in \widehat{H}_{e^m}$ then $\widehat{X}$ will hit $\widehat{\Gamma}_d$ before hitting $\widehat{H}_{e^{m+1}}$ with probability greater than p_1. Hence, by Lemmas 8.1 (ii) and 8.2 (i),

$$\begin{aligned}\widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}(J) - \widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}\widehat{N}^d_{k,k+j} &\leq \sum_{m\geq 0}(1-p_1)^m \widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}\left(\widehat{N}^d_{k+j+m,k+j+m+1}+1\right)\\ &= \sum_{m\geq 0}(1-p_1)^m \widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}\left(\widehat{N}^d_{k,k+1}+1\right)\\ &= \sum_{m\geq 0}(1-p_1)^m \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(1)}\left(\widetilde{N}^d_{k,k+1}+1\right) < c_1 < \infty.\end{aligned}$$

This completes the proof of (8.8) and, hence, of (8.7).

In view of Lemma 8.2, (8.7) implies that

$$\widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}(J) < \infty. \tag{8.9}$$

.

Note that $\left\{\widehat{X}^1\left(\widehat{S}^d_{k+1}\right) - \widehat{X}^1\left(\widehat{S}^d_k\right), k \geq 1\right\}$ is an i.i.d. sequence under $\widehat{\mathbb{P}}^{\widehat{h}}_{\widehat{\nu}(1)}$, and

$$\widehat{X}^1(\widehat{S}^d_J) = \sum_{k=0}^{J}\left(\widehat{X}^1\left(\widehat{S}^d_{k+1}\right) - \widehat{X}^1\left(\widehat{S}^d_k\right)\right).$$

This, (8.9) and Wald's identity imply that

$$\widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}\left(\widehat{X}^1(\widehat{S}^d_J) - \widehat{X}^1(0)\right) = \widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}(J)\,\widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}\left(\widehat{X}^1\left(\widehat{S}^d_{k+1}\right) - \widehat{X}^1\left(\widehat{S}^d_k\right)\right). \tag{8.10}$$

It follows from (8.10) and Proposition 5.3 that

$$\widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}(J) = \kappa\widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}\left(\widehat{X}^1(\widehat{S}^d_J) - \widehat{X}^1(0)\right). \tag{8.11}$$

Recall that J depends on j. We will show that for every $\varepsilon > 0$ there exist j_1 and c_2 such that for $j > j_1$,

$$j + 2d \leq \widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}\left(\widehat{X}^1(\widehat{S}^d_J) - \widehat{X}^1(0)\right) \leq j + c_2. \tag{8.12}$$

The lower bound follows from the definition of J.

Using the support theorem for diffusions it is easy to see that there exists a $p_2 > 0$ such that if $\widehat{X}$ starts from a point in $\widehat{H}_b$ then it will hit $\widehat{\Gamma}_d$ before hitting $\widehat{H}_{b+1}$ with probability greater than p_2. By the strong Markov property applied at the hitting times of $\widehat{H}_{b+k}$, if $\widehat{X}$ starts from a point in $\widehat{H}_b$, it will hit $\widehat{H}_{b+k}$ before hitting $\widehat{\Gamma}_d$ with probability smaller than $(1-p_2)^{k-1}$. Hence,

$$\widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}\left(\widehat{X}^1(\widehat{S}^d_J) - \widehat{X}^1(\widehat{T}_{e^j})\right) \leq \sum_{m=0}^{\infty}(m+1)\,\mathbb{P}^{\widehat{h}}_{\widehat{\nu}(1)}\left(\widehat{X}^1(\widehat{S}^d_J) - \widehat{X}^1(\widehat{T}_{e^j}) \geq m\right)$$

$$\leq \sum_{m=0}^{\infty}(m+1)(1-p_2)^{m-1} \leq c_2.$$

This implies (8.12).

Combining (8.11) and (8.12) yields for every $\varepsilon > 0$ and sufficiently large j,

$$\kappa j < \kappa(j+2d) \leq \widehat{\mathbb{E}}^{\widehat{h}}_{\widehat{\nu}(1)}(J) \leq \kappa(j+c_2) < (\kappa+\varepsilon)j.$$

The lemma follows from this and (8.7). □

Proposition 8.4. *For each $\varepsilon > 0$ and $p < 1$ there exists $j_1 > 0$ such that for all $n \geq j_1$ with n/j_1 an integer,*

$$\widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(e^{-n})}(I_1) \geq p,$$

where

$$I_1 = \bigcap_{k=1}^{n/j_1} \left\{ (\kappa-\varepsilon)kj_1 \leq \widetilde{N}^d_{-kj_1,0} \leq (\kappa+\varepsilon)kj_1 \right\}. \tag{8.13}$$

Proof. Let $\varepsilon \in (0,1)$ and choose $j > 4/\varepsilon > 4$ so that the conclusion of Lemma 8.3 holds with ε replaced by $\varepsilon/4$. Let $p \in (0,1)$.

Recall that $\lambda = \widetilde{\mathbb{E}}^{\widetilde{h}}_{\widetilde{\nu}(e^{-n})}\widetilde{N}^d_{kj,(k+1)j}$ for $-n \leq kj$. The estimate in Lemma 8.3 can be written as

$$\kappa j < \lambda < (\kappa + \varepsilon/4)j. \tag{8.14}$$

Recall that $\widetilde{N}^d_{k_1,k_2} = \#\{i : \widetilde{T}_{e^{k_1}} \leq \widetilde{S}^{u-}_i \leq \widetilde{S}^u_i < \widetilde{T}_{e^{k_2}}\}$ for $k_1 < k_2$. We also need

$$\widetilde{N}^D_{k_1,k_2} = \#\{i : \widetilde{T}_{e^{k_1}} \leq \widetilde{S}^u_i < \widetilde{T}_{e^{k_2}}\}.$$

Of course $\widetilde{N}^d_{k_1,k_2} \leq \widetilde{N}^D_{k_1,k_2}$. They need not be equal since it is possible that $\widetilde{S}^{u-}_j < \widetilde{T}_{e^{k_1}}$. However if $i_0, i_0+1, \ldots, i_0+n$ are those i for which $\widetilde{T}_{e^{k_1}} \leq \widetilde{S}^u_i < \widetilde{T}_{e^{k_2}}$, then $\widetilde{S}^{u-}_{i_0+\ell} \geq \widetilde{S}^u_{i_0} \geq \widetilde{T}_{e^{k_1}}$ if $\ell \geq 1$. Hence $\widetilde{N}^d_{k_1,k_2}$ and $\widetilde{N}^D_{k_1,k_2}$ can differ by at most 1. Therefore, for $k > 0$,

$$\sum_{m=1}^{kj_1/j} \widetilde{N}^d_{-mj,(-m+1)j} \leq \widetilde{N}^d_{-kj_1,0} \leq \sum_{m=1}^{kj_1/j} \widetilde{N}^D_{-mj,(-m+1)j} \leq \sum_{m=1}^{kj_1/j} \widetilde{N}^d_{-mj,(-m+1)j} + kj_1/j. \tag{8.15}$$

Lemma 8.2 shows that the sequence $\left\{\widetilde{N}^d_{mj,(m+1)j} - \lambda, m \geq 0\right\}$ satisfies the assumptions of [Lyo88, Cor. 11], a version of the strong law of large numbers for dependent random variables. The SLLN and (8.14) imply that

$$\frac{\sum_{m=1}^{M}\left[\widetilde{N}^d_{(M-m)j,(M-m+1)j} - \lambda\right]}{M} = \frac{\sum_{m=1}^{M}\left[\widetilde{N}^d_{(m-1)j,mj} - \lambda\right]}{M} \to 0 \tag{8.16}$$

almost surely with respect to $\widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(e^{-n})}$ provided $n \geq 1$. Thus there exists $m_0 \geq 1$ such that the left hand side will be less than $\varepsilon/2$ in absolute value for all $M \geq m_0$ with probability at least p. This can be written as

$$\widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(e^{-n})}\Big(\bigcap_{M=m_0}^{\infty}\Big\{(\lambda-\varepsilon/2)M \le \sum_{m=1}^{M}\widetilde{N}^d_{(M-m)j,(M-m+1)j} \le (\lambda+\varepsilon/2)M\Big\}\Big) \ge p.$$

Take $j_1 = m_0 j$. Letting $M = kj_1/j$ for $k = 1,\dots,n/j_1$, we have

$$\widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(e^{-n})}\Big(\bigcap_{k=1}^{n/j_1}\Big\{(\lambda-\varepsilon/2)kj_1/j \le \sum_{m=1}^{kj_1/j}\widetilde{N}^d_{(M-m)j,(M-m+1)j} \le (\lambda+\varepsilon/2)kj_1/j\Big\}\Big) \ge p.$$

Now use stationarity, scaling, and the strong Markov property to obtain

$$\widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(e^{-n})}\Big(\bigcap_{k=1}^{n/j_1}\Big\{(\lambda-\varepsilon/2)kj_1/j \le \sum_{m=1}^{kj_1/j}\widetilde{N}^d_{-mj,(-m+1)j} \le (\lambda+\varepsilon/2)kj_1/j\Big\}\Big) \ge p \tag{8.17}$$

if n/j_1 is an integer larger than 1.

By (8.14), using $j > 4$,

$$\begin{aligned}(\kappa-\varepsilon/2)kj_1 &\le (\lambda/j-\varepsilon/2)kj_1 = (\lambda-j\varepsilon/2)kj_1/j \le (\lambda-\varepsilon/2)kj_1/j,\\ (\kappa+\varepsilon/2)kj_1 &\ge (\lambda/j-\varepsilon/4+\varepsilon/2)kj_1 = (\lambda+j\varepsilon/4)kj_1/j \ge (\lambda+\varepsilon/2)kj_1/j.\end{aligned}$$

Hence, (8.17) implies

$$\widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(e^{-n})}\Big(\bigcap_{k=1}^{n/j_1}\Big\{(\kappa-\varepsilon/2)kj_1 \le \sum_{m=1}^{kj_1/j}\widetilde{N}^d_{-mj,(-m+1)j} \le (\kappa+\varepsilon/2)kj_1\Big\}\Big) \ge p. \tag{8.18}$$

Since $1/j < \varepsilon/4$, we have for $k > 0$,

$$(\kappa+\varepsilon/2)kj_1 + kj_1/j \le (\kappa+3\varepsilon/4)kj_1.$$

We combine this, (8.15) and (8.18) to obtain

$$\widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(e^{-n})}\left(\bigcap_{k=1}^{n/j_1}\Big\{(\kappa-\varepsilon/2)kj_1 \le \widetilde{N}^d_{-kj_1,0} \le (\kappa+3\varepsilon/4)kj_1\Big\}\right) \ge p.$$

This completes the proof. □

9. Back to the quadrant

We have used $\widetilde{D}$ and $\widehat{D}$ to establish a number of results. In this section we convert these to the corresponding results for D.

Recall the definitions of $\widetilde{H}_b, H_b, \widetilde{T}_b$, and T_b given in (6.1) and (6.2). Define

$$h(z) = \widetilde{h}(\mathbb{F}(z)) \tag{9.1}$$

and

$$\nu(e^n)(A) = \widetilde{\nu}(e^n)(\mathbb{F}(A)), \tag{9.2}$$

where A is a Borel subset of H_{e^n} and $\mathbb{F}(A) = \{\mathbb{F}(z) : z \in A\}$.

An easy calculation shows that the imaginary part of the analytic function $\mathbb{F}$ is

$$\Phi(re^{i\theta}) = r^\alpha \cos(\alpha\theta - \theta_1).$$

By the definition of $\widetilde{h}$ we have that $\widetilde{h}(w)$ is the imaginary part of w. Therefore $h(z) = \widetilde{h}(\mathbb{F}(z))$ is the imaginary part of $\mathbb{F}(z)$, and hence

$$h(re^{i\theta}) = \Phi(re^{i\theta}) = r^\alpha \cos(\alpha\theta - \theta_1). \tag{9.3}$$

The function h is harmonic in the interior of D since it is the imaginary part of an analytic function. By the calculation of [VW85, (2.11)], $\nabla\Phi \cdot \mathbf{v} = 0$ on the boundary of D. Using Ito's formula, we see that $h(X_t)$ is a continuous martingale when X is a solution to (1.2).

Define $\mathbb{P}^h_z$ to be the h-path transform of $\mathbb{P}$, defined analogously to (6.4). Thus if S is a stopping time relative to a filtration $\{\mathcal{F}_t\}$ with respect to which X is adapted and $A \in \mathcal{F}_S$, then

$$\mathbb{P}^h_x(A) = \mathbb{E}^h_z[h(X_S); A]/h(z). \tag{9.4}$$

Since $h(X_t)$ is a time change of one-dimensional Brownian motion, $\mathbb{P}_z(T_a < \infty) = 1$ if $z \in H_b$ and $0 < a < b$. Then

$$\mathbb{P}^h_z(T_a < \infty) = \mathbb{E}^z[h(X(T_a)); T_a < \infty]/h(z) = a/b. \tag{9.5}$$

Lemma 9.1. *For each $\varepsilon_1 > 0$ there exist n_1 and j_0 such that for all $n \geq n_1$ and $j_1 \geq j_0$,*

$$\mathbb{P}^h_{\nu(e^{-n})}(G) \leq 2e^{-j_1\varepsilon_1},$$

where

$$G = \bigcup_{m=1}^{n/j_1} \left\{ \inf_{t \geq T(e^{-mj_1})} h(X_t) \leq e^{-mj_1(1+\varepsilon_1)} \right\}. \tag{9.6}$$

Proof. By (9.5), if $z \in H_b$ and $0 < a < b$ then $\mathbb{P}^h_z(T_a < \infty) = a/b$, so

$$\mathbb{P}^h_{\nu(e^{-n})} \left(\inf_{t \geq T(e^{-mj_1})} h(X_t) \leq e^{-mj_1(1+\varepsilon_1)} \right) \leq e^{-mj_1(1+\varepsilon_1)}/e^{-mj_1} = e^{-mj_1\varepsilon_1}. \tag{9.7}$$

This leads to

$$\mathbb{P}^h_{\nu(e^{-n})}(G) \leq \sum_{m=1}^{n/j_1} e^{-mj_1\varepsilon_1} \leq \frac{e^{-j_1\varepsilon_1}}{1 - e^{-j_1\varepsilon_1}} \leq 2e^{-j_1\varepsilon_1} \tag{9.8}$$

if j_1 is sufficiently large. □

Let T be a finite stopping time for X and let

$$U(T,t) = \inf_{T \leq s \leq T+t} (B_s^2 - B_T^2), \qquad V(T,t) = \sup_{T \leq s \leq T+t} |X_s - X_T|.$$

Let

$$F_1(T) = \left\{ U(T, e^{-2mj_1(1+3\varepsilon_1)/\alpha}) \leq -e^{-mj_1(1+4\varepsilon_1)/\alpha} \right\}, \tag{9.9}$$

$$F_2(T) = \left\{ V(T, e^{-2mj_1(1+3\varepsilon_1)/\alpha}) \leq \chi_1 e^{-mj_1(1+2\varepsilon_1)/\alpha} \right\}, \tag{9.10}$$

where χ_1 is the constant from Proposition 2.2.

Lemma 9.2. *There exists a positive constant* c_1 *such that if* $h(X_T) \geq e^{-mj_1}$ *and* $\varepsilon_1 > 0$, *then for all positive* m, n, j_1 *we have*

$$\mathbb{P}^h_{\nu(e^{-n})}(F_1(T)^c) \leq c_1 e^{-mj_1\varepsilon_1}, \tag{9.11}$$

$$\mathbb{P}^h_{\nu(e^{-n})}(F_2(T)^c) \leq c_1 e^{-mj_1\varepsilon_1}. \tag{9.12}$$

Proof. By the strong Markov property it suffices to prove our result if we take T to be identically equal to 0 and we replace $\mathbb{P}^h_{\nu(e^{-n})}$ by $\mathbb{P}^h_z$ in (9.11) and (9.12), with $h(z) \geq e^{-mj_1}$.

Let

$$\begin{aligned} A_1 &= e^{-mj_1(1+\varepsilon_1)}, \qquad t_1 = e^{-2mj_1(1+3\varepsilon_1)/\alpha}, \\ \lambda_1 &= e^{-mj_1(1+4\varepsilon_1)/\alpha}, \qquad \lambda_2 = e^{-mj_1(1+2\varepsilon_1)/\alpha}. \end{aligned} \tag{9.13}$$

Let $G_1 = \{\inf_{t\geq 0} h(X_t) \leq A_1\}$. By our choice of A_1 and the argument in (9.7),

$$\mathbb{P}^h_z(G_1) \leq A_1/h(z) \leq e^{-mj_1\varepsilon_1}. \tag{9.14}$$

We have $h(x) \leq |x|^\alpha$ by (9.3). A standard calculation shows that $|\nabla h(x)|/h(x) \leq c_2/|x|$. Hence on the event G_1^c, for all $t \geq 0$,

$$\frac{|\nabla h(X_t)|}{h(X_t)} \leq \frac{c_2}{|X_t|} \leq \frac{c_2}{h(X_t)^{1/\alpha}} \leq c_2 e^{mj_1(1+\varepsilon_1)/\alpha}. \tag{9.15}$$

The process X_t solves the Skorokhod equation, where B_t is a 2-dimensional Brownian motion under $\mathbb{P}_z$. By [Bas95, pp. 61–62], the process $W_t = B_t - D_t$ is a Brownian motion under $\mathbb{P}^h_z$, where

$$D_t = \int_0^t \frac{\nabla h(X_s)}{h(X_s)}\, ds.$$

Use (9.13) and (9.15) to see that on the event G_1^c we have

$$\sup_{s\leq t_1} |D_s| \leq c_2 e^{mj_1(1+\varepsilon_1)/\alpha} t_1 = c_2 e^{-mj_1(1+5\varepsilon_1)/\alpha}. \tag{9.16}$$

By Brownian scaling and standard estimates for Brownian motion,

$$\mathbb{P}^h_z\Big(\inf_{s\leq t_1}(W_s^2 - W_0^2) \geq -2\lambda_1\Big) \leq c_3\lambda_1/\sqrt{t_1} \leq c_4 e^{-mj_1\varepsilon_1} \tag{9.17}$$

and

$$\mathbb{P}^h_z\Big(\sup_{s\leq t_1}|W_s - W_0| \geq \tfrac{1}{2}\lambda_2\Big) \leq 2e^{-\lambda_2^2/8t_1} \leq 2\exp\Big(-e^{2\varepsilon_1 mj_1/\alpha}/8\Big). \tag{9.18}$$

Suppose $mj_1\varepsilon_1$ is sufficiently large that the right hand side of (9.16) is smaller than λ_1. We have $D_0 = 0$, hence $W_0 = B_0$, and thus on G_1^c we have

$$\Big|\inf_{s\leq t_1}(W_s^2 - W_0^2) - \inf_{s\leq t_1}(B_s^2 - B_0^2)\Big| \leq \sup_{s\leq t_1}|D_s| \leq c_2 e^{-mj_1(1+5\varepsilon_1)/\alpha} \leq \lambda_1.$$

If the event in (9.17) does not hold, that is, if $\inf_{s\leq t_1}(W_s^2 - W_0^2) < -2\lambda_1$, then the above inequality implies that

$$\inf_{s\leq t_1}(B_s^2 - B_0^2) \leq \inf_{s\leq t_1}(W_s^2 - W_0^2) + \lambda_1 < -\lambda_1.$$

This means that $F_1(0)$ holds. In view of (9.14) and (9.17),

$$\mathbb{P}^h_z(F_1(0)^c) \le \mathbb{P}^h_z(G_1) + \mathbb{P}^h_z(G_1^c \cap F_1(0)^c) \le c_5 e^{-mj_1\varepsilon_1/\alpha}$$

and (9.11) follows for $mj_1\varepsilon_1$ sufficiently large. If $mj_1\varepsilon_1$ is small, (9.11) follows by adjusting c_1.

Similarly suppose $mj_1\varepsilon_1$ is sufficiently large that the right hand side of (9.16) is smaller than $\lambda_2/2$. Then on G^1_c we have

$$\Big| \sup_{s\le t_1} |B_s - B_0| - \sup_{s\le t_1} |W_s - W_0| \Big| \le \sup_{s\le t_1} |D_s| \le c_2 e^{-mj_1(1+5\varepsilon_1)/\alpha} \le \lambda_2/2$$

if $mj_1\varepsilon_1$ is sufficiently large. If the event in (9.18) does not hold, that is, if $\inf_{s\le t_1} |W_s - W_0| < \lambda_2/2$, then the above inequality implies that

$$\inf_{s\le t_1} |B_s - B_0| \le \inf_{s\le t_1} |W_s - W_0| + \lambda_2/2 < \lambda_2.$$

Therefore using Proposition 2.2, (9.14) and (9.18),

$$\begin{aligned}
\mathbb{P}^h_z(F_2(0)^c) &\le \mathbb{P}^h_z(G_1) + \mathbb{P}^h_z(G_1^c \cap F_2(0)^c)\\
&= \mathbb{P}^h_z(G_1) + \mathbb{P}^h_z\left(G_1^c \cap \left\{ \sup_{s\le t_1} |X_s - X_0| > \chi_1\lambda_2 \right\}\right)\\
&\le \mathbb{P}^h_z(G_1) + \mathbb{P}^h_z\left(G_1^c \cap \left\{ \sup_{s\le t_1} |B_s - B_0| > \lambda_2 \right\}\right)\\
&\le e^{-mj_1\varepsilon_1} + 2\exp\Big(- e^{2\varepsilon_1 m j_1/\alpha}/8\Big),
\end{aligned}$$

and (9.12) follows. □

Remark 9.3. Suppose $t_0 > 0$, $f : [0, t_0] \to \mathbb{R}^2$ is continuous with $f(0) \in D \setminus \{0\}$ and $g : [0, t_0] \to D$ satisfies

$$g(t) = f(t) + \int_0^t \mathbf{v}(g(s))\, dL^g_s$$

for all t up until the first time g hits the origin, where L^g is a continuous non-decreasing function with $L^g_0 = 0$ and L^g increases only when $g \in \partial D$. Uniqueness holds for this problem.

To see this, let g be any such solution. Note that g will move in tandem with f when g is in the interior of D. If $g(s) \in \Gamma_d$, then it is easy to check that

$$\begin{aligned}
L^2_g(s+\varepsilon) - L^2_g(s) &= \sup_{s\le r\le s+\varepsilon} \Big[-(f^2(r) - f^2(s))^+ \Big],\\
g^2(s+\varepsilon) - g^2(s) &= f^2(s+\varepsilon) - f^2(s) + (L^2_g(s+\varepsilon) - L^2_g(s)),\\
g^1(s+\varepsilon) - g^1(s) &= f^1(s+\varepsilon) - f^1(s) - a_1(L^2_g(s+\varepsilon) - L^2_g(s)),
\end{aligned}$$

provided ε is small enough so that $g(r)$ does not hit Γ_u in the time interval $[s, s+\varepsilon]$. Similar formulas hold when $g(s) \in \Gamma_u$. These imply that if there is uniqueness up to time s and $g(s) \neq 0$, there will be uniqueness for all times $[s, s+\varepsilon]$ if ε is sufficiently small (depending on s). From these we deduce uniqueness for all s up until the first time g hits 0. Unlike the stochastic case, there is no difficulty with the "piecing together argument," because there is no randomness here.

For the corresponding result for stochastic processes, use the above result for each path separately.

For $t_1 > 0$ and f a continuous function from $[0, t_1]$ to $\mathbb{R}^2$, define

$$\operatorname{Osc}(f, t_1, y) = \sup_{0 \le s_1 \le s_2 \le s_1 + y \le t_1} |f(s_2) - f(s_1)|. \tag{9.19}$$

Lemma 9.4. *Suppose $t_1, c_1 > 0$, $f : [0, t_1] \to \mathbb{R}^2$ is continuous, and g_1 and g_2 are two solutions to* (2.6) *with $g_1(0), g_2(0) \neq 0$, $|g_1(0) - g_2(0)| \le c_1/4$, and $\sup_{s \le t_* \wedge t_1} |g_1(s) - g_2(s)| \ge c_1$. Here $t_* = \inf\{t > 0 : g_2(t) = 0\}$. Let χ_1 be defined by Proposition 2.2. Then*

$$t_* \ge \inf\{y : \operatorname{Osc}(f, t_1, y) \ge c_1/4\chi_1\}.$$

Proof. Let $u = \inf\{t : |g_1(t) - g_2(t)| \ge 3c_1/4\}$ and note that $u \le t_*$. Then

$$|(g_1(u) - g_1(0)) - (g_2(u) - g_2(0))| \ge c_1/2.$$

By Proposition 2.2(ii)

$$|g_i(u) - g_i(0)| \le \chi_1 \operatorname{Osc}(f, t_1, u), \qquad i = 1, 2.$$

Therefore $c_1/2 \le 2\chi_1 \operatorname{Osc}(f, t_1, u)$. We have $u \le t_*$ and $\operatorname{Osc}(f, t_1, u) \ge c_1/4\chi_1$. Our result follows. □

The following is the key estimate of the paper.

Theorem 9.5. *Suppose $a_1 > 0$, $a_2 < 0$, $\alpha > 0$, $\beta > 1$, and $\psi > 1/\alpha$. Let X be a solution to* (1.2) *with initial distribution $\mathbb{P}^h_{\nu(e^{-n})}$. Suppose $p_1 \in (0, 1)$. There exist $c_1 > 0$ and n_1 such that if $n \ge n_1$, then the following hold.*

(i) There exist a random variable $Y_0 \neq 0$ that is measurable with respect to the σ-field $\sigma\{X_t, t \ge 0\}$ and a continuous solution to

$$Y_t = Y_0 + B_t + \int_0^t \mathbf{v}(Y_s)\, dL^Y_s, \qquad 0 \le t \le T_*, \tag{9.20}$$

where $T_* = \inf\{t \ge 0 : Y_t = 0\}$.

(ii)

$$\mathbb{P}^h_{\nu(e^{-n})}(C_i) > p_1, \qquad i = 1, 2, 3,$$

where

$$\begin{aligned} C_1 &= \left\{|X_0 - Y_0| \le e^{-n}\right\}, \\ C_2 &= \{T_* \ge \inf\{y : \operatorname{Osc}(B, T_1, y) \ge c_1/4\chi_1\}\}, \\ C_3 &= \left\{ \sup_{0 \le t \le T_1 \wedge T_*} |X_t - Y_t| \ge c_1 \right\}. \end{aligned} \tag{9.21}$$

Proof. The random variable Y_0 will be defined using information from the future of X, but this does not pose a problem with the solution to (9.20). By Proposition 2.2(i) this equation has a solution in the deterministic sense and therefore for every ω such that B_t is continuous. By Remark 9.3, the solution will be unique at least up until time T_*.

Step 1. We begin by setting some parameters and making some definitions. By the formula for h in (9.3) there exists $d > 0$ not depending on b such that $|x| \geq db^{1/\alpha}$ for $x \in T_b$. Let $\varepsilon > 0$ and set

$$\rho = \psi/\kappa = \log|\tan\theta_1| + \log|\tan\theta_2|. \tag{9.22}$$

Write $\varepsilon_1 = \varepsilon\rho$. We can take ε small enough so that $(1+4\varepsilon_1)/\alpha < \psi - \varepsilon_1$ and $(1+\varepsilon)/\alpha < \psi - \varepsilon_1$. We choose $\varepsilon > 0$ smaller if necessary so that $\rho(\kappa - \varepsilon) > 1$. We take m_1 large enough that for $j_1 \geq 1$ and $m \geq m_1$,

$$e^{-(\psi-\varepsilon_1)mj_1} \leq e^{-(m+1)j_1(1+4\varepsilon_1)/\alpha}, \tag{9.23}$$

$$e^{-(\psi-\varepsilon_1)mj_1} < de^{-(m+1)j_1(1+\varepsilon)/\alpha}. \tag{9.24}$$

We assume that n is so large that $n/j_1 > m_1$.

Let $\Theta = \sup\{k : S^d_k < T_1\}$, $K = \inf\{k : S^d_k > T_{e^{-m_1 j_1}}\}$ and $Y_0 = X_0 + (0, e^{-\rho\Theta})$.

We will write $S^d_{X,k} = S^d_k$ and $S^u_{X,k} = S^u_k$ to emphasize the dependence on X, where S^d_k and S^u_k are defined in (5.1). We will use the analogous notation for Y:

$$\begin{aligned} &S^u_{Y,0} = 0, \qquad S^d_{Y,0} = 0,\\ &S^d_{Y,k} = \inf\{t \geq S^u_{Y,k-1} : Y_t \in \Gamma_d\}, \qquad k \geq 1,\\ &S^u_{Y,k} = \inf\{t \geq S^d_{Y,k} : Y_t \in \Gamma_u\}, \qquad k \geq 1. \end{aligned}$$

Step 2. Let

(9.25)

$$\begin{aligned} A_k &= \{S^d_{X,k} < S^d_{Y,k} < S^u_{X,k} < S^u_{Y,k} < S^d_{Y,k+1} < S^d_{X,k+1} < S^u_{Y,k+1} < S^u_{X,k+1} < S^d_{X,k+2}\},\\ E^k &= \left\{Y_t \neq 0, 0 \leq t \leq S^d_{X,k+2}\right\}. \end{aligned}$$

Equations (1.2) and (9.20) show that X and Y move in tandem until one of the two processes hits ∂D. Our assumptions that $Y_0 = X_0 + (0, e^{-\rho\Theta})$, $\theta_1 > 0$, and $\theta_2 < 0$ imply that if $A_1 \cap E^1$ holds, then the following is true for $k = 1$: (The reader may wish to consult the last few paragraphs of Section 2 at this point.)

$$\begin{aligned} &X^1(S^d_{X,k}) = Y^1(S^d_{X,k}) \text{ and } X^2(S^d_{X,k}) < Y^2(S^d_{X,k}), && (9.26)\\ &X^1(S^d_{Y,k}) < Y^1(S^d_{Y,k}) \text{ and } X^2(S^d_{Y,k}) = Y^2(S^d_{Y,k}),\\ &X^1(S^u_{X,k}) < Y^1(S^u_{X,k}) \text{ and } X^2(S^u_{X,k}) = Y^2(S^u_{X,k}),\\ &X^1(S^u_{Y,k}) = Y^1(S^u_{Y,k}) \text{ and } X^2(S^u_{Y,k}) > Y^2(S^u_{Y,k}),\\ &X^1(S^d_{Y,k+1}) = Y^1(S^d_{Y,k+1}) \text{ and } X^2(S^d_{Y,k+1}) > Y^2(S^d_{Y,k+1}),\\ &X^1(S^d_{X,k+1}) > Y^1(S^d_{X,k+1}) \text{ and } X^2(S^d_{X,k+1}) = Y^2(S^d_{X,k+1}),\\ &X^1(S^u_{Y,k+1}) > Y^1(S^u_{Y,k+1}) \text{ and } X^2(S^u_{Y,k+1}) = Y^2(S^u_{Y,k+1}),\\ &X^1(S^u_{X,k+1}) = Y^1(S^u_{X,k+1}) \text{ and } X^2(S^u_{X,k+1}) < Y^2(S^u_{X,k+1}),\\ &X^1(S^d_{X,k+2}) = Y^1(S^d_{X,k+2}) \text{ and } X^2(S^d_{X,k+2}) < Y^2(S^d_{X,k+2}). && (9.27) \end{aligned}$$

Moreover,

$$X(S^d_{Y,k}) - Y(S^d_{Y,k}) = X(S^u_{X,k}) - Y(S^u_{X,k}), \tag{9.28}$$

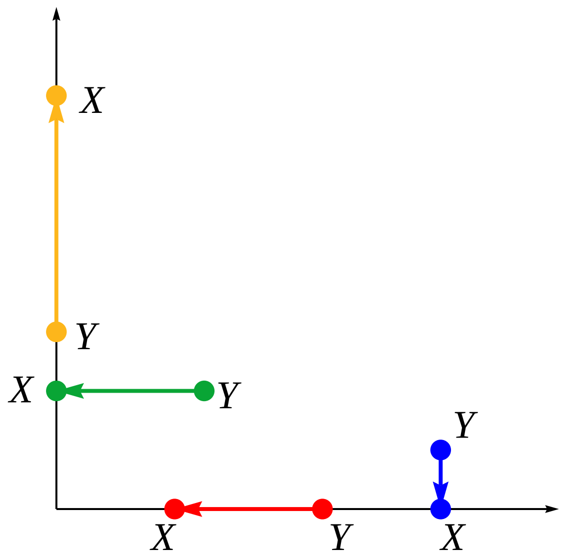

FIGURE 4. Positions of X and Y at different times t. (a) blue, $t = S^d_{X,k}$; (b) red, $t = S^d_{Y,k}$; (c) green, $t = S^u_{X,k}$; (d) orange, $t = S^u_{Y,k}$.

$$
\begin{aligned}
X(S^u_{Y,k}) - Y(S^u_{Y,k}) &= X(S^d_{Y,k+1}) - Y(S^d_{Y,k+1}),\\
X(S^d_{X,k+1}) - Y(S^d_{X,k+1}) &= X(S^u_{Y,k+1}) - Y(S^u_{Y,k+1}),\\
X(S^u_{X,k+1}) - Y(S^u_{X,k+1}) &= X(S^d_{X,k+2}) - Y(S^d_{X,k+2}). \qquad (9.29)
\end{aligned}
$$

We also have

$$
\begin{aligned}
|X(S^d_{Y,k}) - Y(S^d_{Y,k})| &= |X(S^d_{X,k}) - Y(S^d_{X,k})| \cdot |\tan\theta_1|, \qquad (9.30)\\
|X(S^u_{Y,k}) - Y(S^u_{Y,k})| &= |X(S^u_{X,k}) - Y(S^u_{X,k})| \cdot |\tan\theta_2|,\\
|X(S^d_{X,k+1}) - Y(S^d_{X,k+1})| &= |X(S^d_{Y,k+1}) - Y(S^d_{Y,k+1})| \cdot |\tan\theta_1|,\\
|X(S^u_{X,k+1}) - Y(S^u_{X,k+1})| &= |X(S^u_{Y,k+1}) - Y(S^u_{Y,k+1})| \cdot |\tan\theta_2|,\\
|X(S^d_{X,k+2}) - Y(S^d_{X,k+2})| &= |X(S^d_{X,k}) - Y(S^d_{X,k})| e^{2\rho}. \qquad (9.31)
\end{aligned}
$$

The conditions in (9.27) are the same as in (9.26) except that k is replaced by $k+2$. The cycle depicted in Fig. 4 is repeated twice, once as shown, and then with the roles of X and Y interchanged. At the end of the two cycles, the configuration of X and Y is similar to the one at the beginning of the cycle. We have already noticed that if $A_1 \cap E^i$ holds, then (9.26)-(9.31) are true for $k = 1$. An induction argument shows that if $A_i \cap E^i$ holds for all odd $i \le m$, then (9.26)-(9.31) hold for all odd $k \le m$. It then follows from (9.31) and an induction argument that

$$
|X(S^d_{X,m}) - Y(S^d_{X,m})| = e^{-\rho\Theta + m\rho}. \qquad (9.32)
$$

Step 3. Recall the definitions of I_1 from (8.13) and G from (9.6). Assume that $I_1 \cap G^c$ holds.

Consider an odd $k \le K-1$ and let m be such that

$$(\kappa-\varepsilon)mj_1 \le \Theta - k \le (\kappa-\varepsilon)(m+1)j_1. \tag{9.33}$$

Assume that $A_i \cap E^i$ holds for all odd $i < k$, and therefore (9.31) holds (with i in place of k) for all odd $i < k$. We use (9.32) and (9.33) to see that

$$\begin{aligned} |X(S^d_{X,k}) - Y(S^d_{X,k})| &= |X(0)-Y(0)|e^{2\rho k/2} \le |X(0)-Y(0)|e^{\rho(\Theta-(\kappa-\varepsilon)mj_1)} \\ &= e^{-\rho\Theta}e^{\rho(\Theta-(\kappa-\varepsilon)mj_1)} = e^{-(\psi-\varepsilon_1)mj_1}. \end{aligned} \tag{9.34}$$

Since I_1 holds, for $i \le n/j_1$,

$$(\kappa-\varepsilon)ij_1 \le \widetilde{N}^d_{-ij_1,0} \le (\kappa+\varepsilon)ij_1.$$

Neither a time change nor a conformal mapping change the validity of these inequalities, so we have

$$(\kappa-\varepsilon)ij_1 \le N^d_{-ij_1,0} \le (\kappa+\varepsilon)ij_1. \tag{9.35}$$

We apply the left hand side inequality with $i = m+1$ and (9.33) to see that $S^d_{X,k} \ge T_{e^{-(m+1)j_1}}$. By Lemma 9.1, (9.24) and (9.34),

$$|X(S^d_{X,k})| > d\, e^{-(m+1)j_1(1+\varepsilon)/\alpha} > e^{-(\psi-\varepsilon_1)mj_1} \ge |X(S^d_{X,k}) - Y(S^d_{X,k})|,$$

where d was defined at the beginning of the proof. We will show below that the inequality extends to $t \in [S^d_{X,k}, S^d_{X,k+1}]$, that is, $|X_t| > |X_t - Y_t|$. It follows that Y does not visit the origin, showing the event E^k holds.

We will argue that $S^d_{Y,k} < S^u_{X,k} \wedge S^u_{Y,k}$ if the events $F_1(T)$ and $F_2(T)$ defined in (9.9)-(9.10) hold with $T = S^d_{X,k}$. We will analyze the motions of X and Y on the interval $[S^d_{X,k}, S^d_{Y,k}]$.

First, recall that $\theta_1 > 0$ and $X^1(S^d_{X,k}) = Y^1(S^d_{X,k})$. It follows that $X^1_t \le Y^1_t$ for $t \in [S^d_{X,k}, S^u_{Y,k}]$ and therefore $S^u_{X,k} \le S^u_{Y,k}$.

Next we show that $S^d_{Y,k} < S^u_{X,k}$. Since $S^d_{X,k} \ge T_{e^{-(m+1)j_1}}$ and $F_2(S^d_{X,k})$ holds, we have

$$\sup\left\{\left|X_t - X(S^d_{X,k})\right| : S^d_{X,k} \le t \le S^d_{X,k} + e^{-2(m+1)j_1(1+3\varepsilon_1)/\alpha}\right\} \le e^{-(m+1)j_1(1+2\varepsilon_1)/\alpha}. \tag{9.36}$$

Recall that $S^d_{X,k} \ge T_{e^{-(m+1)j_1}}$ and G^c holds. Thus $|X(S^d_{X,k})| \ge de^{-(m+1)j_1(1+\varepsilon_1)/\alpha}$. This and (9.36) imply that for $m \ge 1$ and sufficiently large j_1,

$$\begin{aligned} &\inf\left\{X^1_t : S^d_{X,k} \le t \le S^d_{X,k} + e^{-2(m+1)j_1(1+3\varepsilon_1)/\alpha}\right\} \\ &\qquad \ge de^{-(m+1)j_1(1+\varepsilon_1)/\alpha} - e^{-(m+1)j_1(1+2\varepsilon_1)/\alpha} > 0. \end{aligned}$$

Therefore

$$S^u_{X,k} > S^d_{X,k} + e^{-2(m+1)j_1(1+3\varepsilon_1)\psi/\alpha}. \tag{9.37}$$

Since $X(S^d_{X,k}) \in \Gamma_d$, (9.34) implies that $Y^2(S^d_{X,k}) \le e^{-(\psi-\varepsilon_1)mj_1}$. In view of (9.23), $Y^2(S^d_{X,k}) \le e^{-(m+1)j_1(1+4\varepsilon_1)/\alpha}$. This and the event $F_1(S^d_{X,k})$ holding imply that

$$\begin{aligned} -Y^2(S^d_{X,k}) &= Y^2(S^d_{Y,k}) - Y^2(S^d_{X,k}) \\ &\le \inf\left\{Y^2_t - Y^2(S^d_{X,k}) : S^d_{X,k} \le t \le \min(S^d_{X,k} + e^{-2(m+1)j_1(1+3\varepsilon_1)/\alpha}, S^d_{Y,k})\right\} \end{aligned}$$

$$= \inf \left\{ B_t^2 - B^2(S^d_{X,k}) : S^d_{X,k} \le t \le \min(S^d_{X,k} + e^{-2(m+1)j_1(1+3\varepsilon_1)/\alpha}, S^d_{Y,k}) \right\}$$
$$= \max(-e^{-(m+1)j_1(1+4\varepsilon_1)/\alpha}, B^2(S^d_{Y,k}) - B^2(S^d_{X,k}))$$
$$= \max(-e^{-(m+1)j_1(1+4\varepsilon_1)/\alpha}, Y^2(S^d_{Y,k}) - Y^2(S^d_{X,k}))$$
$$= \max(-e^{-(m+1)j_1(1+4\varepsilon_1)/\alpha}, -Y^2(S^d_{X,k})) = -Y^2(S^d_{X,k}).$$

Thus the inequality is in fact an equality and, therefore,

$$S^d_{Y,k} \le S^d_{X,k} + e^{-2(m+1)j_1(1+3\varepsilon_1)/\alpha}.$$

This and (9.37) imply that $S^d_{Y,k} < S^u_{X,k}$, as claimed. We conclude that E^k holds.

Step 4. Suppose that $I_1 \cap G^c$ holds and also A_k holds for each odd $k \le K-1$. We apply (9.35) with $i = m_1$ to obtain,

$$(\kappa - \varepsilon)m_1 j_1 \le N^d_{-m_1 j_1, 0} \le (\kappa + \varepsilon) m_1 j_1.$$

Then $\Theta - K \le (\kappa+\varepsilon)m_1 j_1$, and in view of (9.32),

$$|X(S^d_{X,K}) - Y(S^d_{X,K})| = e^{-\rho\Theta + K\rho} \ge e^{-\rho(\kappa+\varepsilon)m_1 j_1}.$$

We conclude that C_3 in (9.21) holds if we set $c_1 = e^{-\rho(\kappa+\varepsilon)m_1 j_1}$.

Let m_2 be such that

$$(\kappa - \varepsilon) m_2 j_1 \le \Theta \le (\kappa - \varepsilon)(m_2+1) j_1. \tag{9.38}$$

We are assuming that I_1 holds, so $\{N^d_{-(m_2+1)j_1,0} \ge (\kappa-\varepsilon)(m_2+1)j_1\}$ holds. If we take $n = (m_2+1)j_1$ then $\Theta \ge (\kappa - \varepsilon)n$, and therefore

$$|X(0) - Y(0)| = e^{-\rho\Theta} \le e^{-\rho(\kappa-\varepsilon)n} \le e^{-n}.$$

This proves that C_1 holds.

Step 5. It remains to bound the probability that $A_k \cap E^k$ fails. We have proved the following three inequalities:

$$S^d_{X,k} < S^d_{Y,k} < S^u_{X,k} < S^u_{Y,k}, \tag{9.39}$$

illustrated by (a), (b) and (c) (blue, red and green) in Fig. 4. To complete one "cycle" in the definition of A_k one has to prove that

$$S^u_{X,k} < S^u_{Y,k} < S^d_{Y,k+1} < S^d_{X,k+1},$$

illustrated (partly) by (c) and (d) (green and orange) in Fig. 4. The event A_k also contains another "cycle," with the roles of X and Y interchanged. Each of the four sets of inequalities (two per "cycle") are proved similarly to the proof of (9.39). This explains the factor of 1/4 in the first line of the estimate below.

An event $A_k \cap C_2^k$ fails only if $I_1 \cap I_2$ fails or $F_1(T) \cap F_2(T)$ fails with $T = S^d_{X,k}$. Let $p = (1-p_1)/12$. By Proposition 8.4 and Lemmas 9.1 and 9.2,

$$\tfrac{1}{4}\,\mathbb{P}^h_{\nu(e^{-n})}\left(\bigcup_{k=1,\dots K-1,\ k \text{ odd}} A_k^c \right) \le \mathbb{P}^h_{\nu(e^{-n})}(I_1^c)) + \mathbb{P}^h_{\nu(e^{-n})}(I_2^c))$$
$$+ \sum_{m \ge m_1} \mathbb{P}^h_{\nu(e^{-n})}(F_1(T_{e^{mj_1}}) \cap I_1 \cap I_2) + \sum_{m \ge m_1} \mathbb{P}^h_{\nu(e^{-n})}(F_2(T_{e^{mj_1}}) \cap I_1 \cap I_2)$$

$$\leq p+p$$
$$+\sum_{m=1}^{\infty}(\kappa+\varepsilon)mj_1\left(c_1e^{-2mj_1\varepsilon/\alpha}+c_3e^{-mj_1\varepsilon}\exp\left(-c_4e^{-2mj_1\varepsilon}\right)\right)$$
$$\leq 3p=\tfrac{1}{4}(1-p_1).$$

The last inequality holds provided we choose j_1 sufficiently large. The factor $(\kappa+\varepsilon)mj_1$ on the second to last line is there to account for multiple k's corresponding to each m; being on the event I_1 controls the number of such k. As long as n is large enough that $e^{-n}\leq c_1/4$, an application of Lemma 9.4 establishes C_2, and the proof is complete.

Note that the choice of p_1 affects j_1 and this in turn affects the value of the constant $c_1=e^{-\rho(\kappa+\varepsilon)m_1j_1}$ in the definition of C_3. □

We now obtain a suitable version of Corollary 7.2(i) for measures on D.

Lemma 9.6. *Fix $n\geq 1$. There exists a positive integer $j_1>n$ such that for each Borel subset A of $H_{e^{-n}}$, each $j\geq j_1$, and each $z\in H_{e^{-j}}$ we have*

$$|\,\mathbb{P}^h_z(X(T_{e^{-n}})\in A)-\nu(e^{-n})(A)|\leq 2q^{j-n-1},$$

where q is given in the statement of Corollary 7.2.

Proof. By Corollary 7.2(i) and scaling, we have

$$|\widetilde{\mathbb{P}}^{\widetilde{h}}_{\mathbb{F}(z)}(\widetilde{X}(T_{e^{-n}})\in A_1)-\widetilde{\mathbb{P}}^{\widetilde{h}}_{\widetilde{\nu}(e^{-j})}(\widetilde{X}(T_{e^{-n}})\in A_1)|\leq q^{j-n-1}/(1-q^{j-n-1}),$$

where $A_1=\mathbb{F}(A)$ is a Borel subset of $\widetilde{H}_{e^{-n}}$ and $\mathbb{F}(z)\in\widetilde{H}_{e^{-j}}$. The quantities above are not affected by time changes, so by scaling and the fact that $\widetilde{\nu}(e^{-n})$ is an invariant probability measure, we obtain

$$|\widetilde{\mathbb{P}}^{\widetilde{h}}_{\mathbb{F}(z)}(\mathbb{F}(X(T_{e^{-n}}))\in\mathbb{F}(A))-\widetilde{\nu}(e^{-n})(\mathbb{F}(A))|\leq 2q^{j-n-1}\tag{9.40}$$

provided $j\geq j_1$ and j_1 is chosen so that $q^{j_1-n-1}<1/2$, where A is a Borel subset of $H_{e^{-n}}$ and $z\in H_{e^{-j}}$.

From the definition of h-path transform, we have

$$\begin{aligned}\widetilde{\mathbb{P}}^{\widetilde{h}}_{\mathbb{F}(z)}(\mathbb{F}(X(T_{e^{-n}}))\in\mathbb{F}(A))&=\widetilde{\mathbb{E}}_{\mathbb{F}(z)}[\widetilde{h}(X(T_{e^{-n}}));\mathbb{F}(X(T_{e^{-n}}))\in\mathbb{F}(A)]/\widetilde{h}(z)\\&=\frac{e^{-n}}{e^{-j}}\widetilde{\mathbb{P}}_{\mathbb{F}(z)}(\mathbb{F}(X(T_{e^{-n}}))\in\mathbb{F}(A))=\frac{e^{-n}}{e^{-j}}\,\mathbb{P}_z(X(T_{e^{-n}})\in A)\\&=\mathbb{P}^h_z(X(T_{e^{-n}})\in A).\end{aligned}$$

Substituting in (9.40) yields our result. □

10. Non-uniqueness

First we argue that to prove non-uniqueness it is enough to look at excursions of a solution to (1.2).

Let B be 2-dimensional Brownian motion and let X be a solution to (1.2). Next kill (in the Markov sense) the process X at time T_1. Let

$$U_0=\sup\{t\leq T_1:X_t=0\}.$$

Following [MSW72] we see that $X(U_0+t)$ is a strong Markov process whose law is given by $\mathbb{P}^{h_0}_{\cdot}$, where $h_0(z) = \mathbb{P}_z(T_1 < T_0)$. (Strictly speaking, we should write $\mathbb{P}_z(\zeta < T_0)$, where ζ is the lifetime of X killed on hitting H_1.)

The function h defined in (9.1) is equal to 1 on H_1, 0 at 0, and as we observed, $h(X_t)$ is a martingale under $\mathbb{P}_{\cdot}$. Therefore by optional stopping, $h(z) = \mathbb{P}_z(T_1 < T_0) = h_0(z)$.

Let $U_b = \inf\{t \geq U_0 : X_t \in H_b\}$.

Lemma 10.1. *(i) Let $n \geq 1$. Then*

$$\lim_{j\to\infty} \sup_{z\in H_{e^{-j}}} \sup_{A} |\, \mathbb{P}_0(X(U_{e^{-n}}) \in A \mid X(U_{e^{-j}}) = z) - \nu(e^{-n})(A)| = 0,$$

where the supremum over A is the supremum over Borel subsets $A \subset H_{e^{-n}}$.
(ii) For $n \geq 1$,

$$\mathbb{P}_0(X(U_{e^{-n}}) \in A) = \nu(e^{-n})(A)$$

for all Borel subsets $A \subset H_{e^{-n}}$.

Proof. (i) Let $A \subset H_{e^{-n}}$, let $\varepsilon > 0$, and let $j \geq j_1$ so that $q^{j-n-1} \leq \varepsilon/2$, where q and j_1 are given by the statement of Lemma 9.6. By the conclusion of Lemma 9.6, if $z \in H_{e^{-j}}$,

$$|\, \mathbb{P}^h_z(X(T_{e^{-n}}) \in A) - \nu(e^{-n})(A)| < \varepsilon.$$

Our result follows because the distributions of $\{X(U_0+t), t \geq 0\}$ under $\mathbb{P}_{\cdot}$ and $\{X_t, t \geq 0\}$ under $\mathbb{P}^h_{\cdot}$ are identical and $\varepsilon > 0$ is arbitrary.

(ii) Let $j > n$. Our result follows because $U_{e^{-j}} < \infty$, a.s, so j can be taken arbitrary large in (i), and, therefore $\varepsilon > 0$ can be taken arbitrarily small. □

We now apply Theorem 9.5 with $p_1 = 1/2$ to construct a sequence of processes $\{Y^{(n)}\}$. For each $n \geq n_1$, let $Y^{(n)}_t = X_t$ for $t < U_{e^{-n}}$. At time $U_{e^{-n}}$ we introduce a jump. Let Θ_n be as in the proof of Theorem 9.5, but we write Θ_n instead of Θ to emphasize the dependence on n. We define

$$Y^{(n)}(U_{e^{-n}}) = \begin{cases} X(U_{e^{-n}}) + (0, e^{-\psi\Theta_n/\kappa}) & \text{if } e^{-\psi\Theta_n/\kappa} < e^{-n}, \\ X(U_{e^{-n}}) & \text{otherwise.} \end{cases} \tag{10.1}$$

We let $Y^{(n)}$ be the solution to

$$Y^{(n)}(U_{e^{-n}} + t) = Y^{(n)}(U_{e^{-n}}) + B(U_{e^{-n}} + t) - B(U_{e^{-n}}) + \int_{U_{e^{-n}}}^{U_{e^{-n}}+t} \mathbf{v}(Y^{(n)}_s)\, dL^{Y^{(n)}}_s,$$

for $0 \leq t \leq U_* := \inf\{s : Y^{(n)}_s = 0\}$, where B is the same as in (1.2) and Theorem 9.5. As noted in Remark 9.3, the solution to this equation is pathwise unique up to time U_*.

We now use Proposition 2.2 to find a pathwise solution to (1.2) for $t > U_*$:

$$Y^{(n)}(U_* + t) = Y^{(n)}(U_*) + B(U_* + t) - B(U_*) + \int_{U_*}^{U_*+t} \mathbf{v}(Y^{(n)}_s)\, dL^{Y^{(n)}}_s$$

for $t > 0$. We do not know whether or not the solution $Y^{(n)}$ is pathwise unique for times after U_*, but that will not matter, as it is the behavior before time U_* that will be used to show that the solution to (1.2) is not unique.

In view of (10.1), Theorem 9.5 shows that we have processes $Y^{(n)}$ such that

$$\mathbb{P}_0\left(\sup_{s\le U_*}|X_s-Y^{(n)}_s|\ge c_1\right)\ge 1/2.$$

The size of the jump of $Y^{(n)}$ at the time $U_{e^{-n}}$ is less than or equal to e^{-n}, a.s.

We next show tightness of the sequence $(X,B,Y^{(n)})$. Choose t_0 large so that $\mathbb{P}(T_1>t_0)\le 1/4$.

Lemma 10.2. *The triple* $(X,B,Y^{(n)})$ *is tight with respect to the topology of* $D[0,t_0]$.

Recall that each of X,B,Y^n is bivariate.

Proof. Since X and B are continuous processes, it suffices to show that $\{Y^{(n)}\}$ is tight. We use a criterion of Aldous, namely, tightness holds if whenever τ_n are stopping times and δ_n are positive reals tending to 0, then

$$|Y^{(n)}(\tau_n+\delta_n)-Y^{(n)}(\tau_n)|\to 0 \tag{10.2}$$

in probability (see [Bas11, page 264]). An examination of the proof of the Aldous criterion shows that we need (10.2) to hold for stopping times τ_n such that for each n, the time τ_n is a stopping time with respect to the filtration generated by $Y^{(n)}$. However we will establish (10.2) for arbitrary random times τ_n, not necessarily stopping times, so the choice of a filtration is a moot point.

The size of the jump of $Y^{(n)}$ at time $U_{e^{-n}}$ is less than or equal to e^{-n}, a.s., which tends to 0 as $n\to\infty$. By Proposition 2.2(ii) we have that if $U_{e^{-n}}\in[\tau_n,\tau_n+\delta_n]$, then

$$\begin{aligned}|Y^{(n)}(\tau_n+\delta_n)-Y^{(n)}(U_{e^{-n}})|&+|Y^{(n)}((U_{e^{-n}})-)-Y^{(n)}(\tau_n)|\\ &\le 2\chi_1\sup_{0\le s<t\le s+\delta_n\le t_0}|B_t-B_s|,\end{aligned}$$

which also goes to 0 as $\delta_n\to 0$. If $U_{e^{-n}}$ is not in that interval, we have the same bound for

$$|Y^{(n)}(\tau_n+\delta_n)-Y^{(n)}(\tau_n)|$$

by the same proposition. Therefore the Aldous criterion is satisfied. □

We use the Skorokhod representation theorem (cf. the proof of Proposition 6.2) to find a probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ and random processes X'_n,B'_n,Y'_n such that the law of (X'_n,B'_n,Y'_n) under $\mathbb{P}'$ is equal to the law of $(X,B,Y^{(n)})$ under $\mathbb{P}_0$ and (X'_n,B'_n,Y'_n) converges a.s. to a triple $(X'_\infty,B'_\infty,Y'_\infty)$ with respect to the topology of $D[0,t_0]$. Clearly B'_∞ is a Brownian motion. The processes X and $Y^{(n)}$ have oscillations bounded by a constant times those of Brownian motion B, by Proposition 2.2(ii). This implies that the same bounds for the oscillations apply to the limit $(X'_\infty,B'_\infty,Y'_\infty)$. Thus these processes are continuous, and it is well known (see, e.g. [Bas11, page 263]) that therefore the convergence of (X'_n,B'_n,Y'_n) is uniform, a.s.

To verify that (X'_∞,B'_∞) and (Y'_∞,B'_∞) are both solutions to (1.2), it is more convenient to look at the formulation given by (2.2).

Lemma 10.3. (X'_∞,B'_∞) *and* (Y'_∞,B'_∞) *are both solutions to* (2.2). *Moreover*

$$\mathbb{P}\left(\sup_{s\ge 0}|X_s-Y_s|>0\right)\ge 1/4.$$

Proof. Only a few steps need comment. The determinant of R is equal to $1 - a_1 a_2 > 0$, hence R is invertible. Therefore $(M^{Y'_n})^i$ converges uniformly on compacts to $(M^{Y'_\infty})^i$ for $i = 1, 2$.

The other step is to argue that $(M^{Y'_\infty})^1$ increases only when $Y'_\infty \in \Gamma_d$ and similarly for $(M^{Y'_\infty})^2$. If $Y'_\infty(t) \notin \Gamma_d$, then $Y'_\infty(s) \notin \Gamma_d$ for s in an open interval about t, and hence for n large, $Y'_n(s) \notin \Gamma_d$ for s in this interval. This means that $(M^{Y'_n})^1(s)$ is constant for s in an open interval containing t if n is sufficiently large. But then the same is true for $(M^{Y'_\infty})^1(s)$. □

The last step in proving that pathwise uniqueness fails almost surely is to show that the probability of non-uniqueness is 1 rather than only saying it is greater than 1/4.

Let $\mathcal{C}_2[0, t_0]$ be the set of functions from $[0, t_0]$ into $\mathbb{R}^2$ such that each component is continuous. Let $x_0 = 0$. Let $\mathcal{U}$ be the set of functions f in $\mathcal{C}_2[0, \infty)$ for which (2.6) has a unique solution for each $t_0 > 0$.

Theorem 10.4. *With $\mathbb{P}_0$ probability one,*

$$B_t(\omega) \notin \mathcal{U}.$$

Proof. Let $T^1 = T_1$, $U^0 = 0$,

$$U^k = \inf\{t > T^k : X(t) = 0\}, \qquad k \geq 1,$$

and

$$T^{k+1} = \inf\{t > U^k : X_t \in H_1\}, \qquad k \geq 1.$$

By the strong Markov property, the laws of $\{(X_t, B_t) : t \in [U^k, T^{k+1}]\}$ are i.i.d. Therefore the probability that $B \in \mathcal{C}_2[0, \infty) \setminus \mathcal{U}$ is less than $(3/4)^k$ for each k, hence is 0. □

We now turn to showing there exists no strong solution to (1.2) and equivalently, to (2.2). We first need the following lemma.

Recall that (X, B) solving (2.2) is called a weak solution if B is adapted to the filtration generated by $\{X_t\}$.

Lemma 10.5. *Suppose B_t is a 2–dimensional Brownian motion and (X, B) satisfies (2.2). Then (X, B) is a weak solution of (2.2).*

Proof. By Lemma 2.1 of [TW93]

$$\int_0^t 1_{\partial D}(X_s)\, ds = 0, \qquad \text{a.s.}$$

Therefore

$$\mathbb{E}\Big[\int_0^t 1_{\partial D}(X_s)\, dB^i_s\Big]^2 = 0, \qquad i = 1, 2,$$

and so

$$\int_0^t 1_{\partial D}(X_s)\, dB^i_s = 0, \qquad \text{a.s.,} \tag{10.3}$$

for $i=1,2$. Since M_t^i increases only when $X_t \in \partial D$, $i=1,2$, then

$$\int_0^t 1_{D^o}(X_s)\, dX_s^i = \int_0^t 1_{D^o}(X_s)\, dB_s^i, \qquad i=1,2, \tag{10.4}$$

where D^o is the interior of D. Combining (10.3) and (10.4)

$$B_t^i = \int_0^t 1_{D^o}(X_s)\, dB_s^i + \int_0^t 1_{\partial D}(X_s)\, dB_s^i = \int_0^t 1_{D^o}(X_s)\, dX_s^i,$$

which implies that B is adapted to the filtration generated by $\{X_t\}$. □

Theorem 10.6. *There does not exist a strong solution to* (2.2).

One could modify existing proofs of the analogue of this theorem for ordinary stochastic differential equations, but it is just as easy to use Girsanov's original proof, slightly modified.

Proof. Let B_t be a 2-dimensional Brownian motion and suppose (X,B) is a strong solution. We are not supposing here that this X is the same as any of the processes we have worked with up to now. We will prove that the solution to (2.2) is then pathwise unique, which contradicts Theorem 10.4.

Let (Y,B) be any other solution to (2.2). By Lemma 10.5 (Y,B) is a weak solution. By [TW93] the law of X is unique, hence the law of X equals the law of Y. By Lemma 10.5 strong solutions are also weak solutions, hence the law of (X,B) is equal to the law of (Y,B).

We know that with positive probability there is not uniqueness to (2.6) with $f(t)$ replaced by $B_t(\omega)$. Also X and Y are continuous processes. Therefore there exists a positive rational r such that $\mathbb{P}(X_r \neq Y_r) > 0$. Since X_r is adapted to the filtration generated by $\{B_s : s \le r\}$, there is a Borel measurable map φ from $\mathcal{C}_2[0,r]$ to D such that $X_r = \varphi(B)$ a.s. Because the laws of (X,B) and (Y,B) are equal, we must have that Y_r also equals $\varphi(B)$ a.s. But then $X_r = Y_r$ a.s., a contradiction. □

11. Index of notation

We present an index of notation that is used repeatedly.

Functions: $\widetilde{h}$ (6.3); h (9.1); $\mathbb{F}$ (3.1)

Matrices: R (2.1)

Measures and probabilities: H^x Section 4; $\mathbb{P}^h$ (9.4); $\widetilde{\mathbb{P}}^{\widetilde{h}}$ (6.4)

Parameters: a_1, a_2 (1.3); α (1.1); β (1.4); κ (5.8); θ_1, θ_2 Section 1; χ_1 Proposition 2.2; ψ (1.5)

Processes: L (1.2); X (1.2); $\widetilde{X}$ Proposition 3.1; $\widehat{X}$ Proposition 3.2

Random variables: $\widetilde{N}^d_{k_1,k_2}, \widetilde{N}^u_{k_1,k_2}$ (8.2); S^d_k, S^u_k (with and without tildes and hats) (5.1); $S^{d-}_k; S^{u-}_k$ (with and without tildes and hats) (8.1); T_b (with and without tildes and hats) (6.2)

Sets: D Section 1; $\widetilde{D}, \widehat{D}$ Section 3; $H_b, \widetilde{H}_b, \widehat{H}_b$ (6.1); Γ_d, Γ_d (2.3)

Vectors: $\mathbf{n}, \mathbf{v}$ Section 1

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