Correction to “Stability of parabolic Harnack inequalities on metric measure spaces”

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Dr. N. Kajino pointed us out that the proof of Lemma 3.3 in the paper [BBK] is inadequate, since there is no easy way to control the Green function $g_\lambda(x,y)$ near the boundary of D. Since there are also some other minor errors in Section 3, we have made a revision from page 499, line 6 to the end of Section 3. We thank Dr. Kajino for pointing out the error and for his comments on the revision.

Let Y be the process associated with the Dirichlet form $(\mathcal{E},\mathcal{F})$. Let G_λ be the λ-resolvent associated with the process Y; that is,

$$G_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(Y_t)dt,$$

for bounded measurable f. Let $p_t(\cdot,\cdot)$ be the heat kernel of Y. Then the Green kernel of G_λ is given by

$$g_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p_t(x,y)dt.$$

We will use the Green kernel to build a cut-off function φ.

Lemma 3.2. *Let $x_0 \in X$. Then there exist $\delta \in (0,1)$ and $C_1 = C_1(\delta) > 0$ such that if $\lambda = c_0\Psi(\delta R)^{-1}$, then*

$$g_\lambda(x,y) \le C_1 \frac{\Psi(R)}{V(x_0,R)}, \qquad x \in B(x_0,R)^c,\ y \in B(x_0,\delta R), \tag{3.1}$$

$$g_\lambda(x,y) \ge 2C_1 \frac{\Psi(R)}{V(x_0,R)}, \qquad x,y \in B(x_0,\delta R). \tag{3.2}$$

Proof. This follows easily from HK(Ψ) by integration. □

Lemma 3.3. *There exists $\theta > 0$ such that the following holds. Let $x_0 \in X$, $R > 0$, $x_1 \in B(x_0,R)$, and $\lambda \ge c\Psi(R)^{-1}$. Then*

$$|g_\lambda(x_1,y) - g_\lambda(x_1,y')| \le c_1 \Big(\frac{d(y,y')}{R}\Big)^\theta \frac{\Psi(R)}{V(x_0,R)} \qquad \text{for } y,y' \in B(x_0,2R)^c. \tag{3.3}$$

(ii) If $d(y,y') \geq R/4$ then (3.3) follows immediately from (3.1). Otherwise we use the Hölder continuity of $p_t(x_1,\cdot)$, which follows from PHI(Ψ) by a standard argument; see [BGK], Corollary 4.2. (Note that to handle small values of t we need to extend the function $p_\cdot(x_1,\cdot)$ from $(0,\infty)\times B(x_0,R)^c$ to $\mathbb{R}\times B(x_0,R)^c$, by setting $p_s(x_1,y)=0$ for $s<0$.) Once we have the Hölder continuity of $p_t(x_1,\cdot)$, integrating gives (3.3). □

The following lemma is given in [BH] Chapter I, Proposition I.4.1.1 when $u,f\in\mathcal{F}$ are non-negative and bounded. By a standard approximation argument, it can be proved for the unbounded case as well.

Lemma 3.A. *For $u\in\mathcal{F}$, let $\Phi(u)=(u\vee 0)\wedge 1$. Then $\Phi(u)\in\mathcal{F}$ and the following holds.*

$$\int_X f d\Gamma(\Phi(u),\Phi(u)) \leq \int_X f d\Gamma(u,u) \qquad \forall f\in\mathcal{F} \ \ \textit{with} \ \ f\geq 0.$$

Let δ be as in Lemma 3.2, fix $x_0\in X$ and let $B'=B(x_0,\delta R), B''=B(x_0,\delta R/8)$, $B=B(x_0,R)$, $2B=B(x_0,2R)$. By Remark 2.6(2-3) it is enough to prove CS(Ψ) with a scale factor of δ^{-1} rather than 2.

Let $\lambda=c_0\Psi(\delta R)^{-1}$ and define

$$h := C_1\Psi(R)\frac{V(x_0,\delta R/8)}{V(x_0,R)}.$$

Integrating Lemma 3.2, we have the following:

$$G_\lambda 1_{B''}(x)\leq h, \qquad x\in B(x_0,R)^c, \tag{3.a}$$
$$G_\lambda 1_{B''}(x)\geq 2h, \qquad x\in B(x_0,\delta R), \tag{3.b}$$

$$|G_\lambda 1_{B''}(x)-G_\lambda 1_{B''}(y)|\leq c_1\Big(\frac{d(x,y)}{R}\Big)^\theta h, \quad x,y\in B(x_0,R)\setminus B(x_0,\delta R/2). \tag{3.c}$$

Now define

$$\varphi(x)=\Big(2\wedge h^{-1}G_\lambda 1_{B''}(x)-1\Big)^+=\Big(1\wedge(h^{-1}G_\lambda 1_{B''}(x)-1)\Big)^+=\Phi(h^{-1}G_\lambda 1_{B''}(x)-1).$$

We need to make sure that $\varphi\in\mathcal{F}$. For the purpose, let $\hat{1}_{B(x_1,s)}\in\mathcal{F}\cap C_0$ be a function which is 1 inside $B(x_1,s)$, between 0 and 1 in $B(x_1,2s)\setminus B(x_1,s)$ and 0 outside $B(x_1,2s)$. Then $h^{-1}G_\lambda 1_{B''}(x)-1=h^{-1}G_\lambda 1_{B''}(x)-\hat{1}_{2B}(x)$ for $x\in 2B$, so

$$\varphi(x)=\Phi(h^{-1}G_\lambda 1_{B''}(x)-1)=\Phi(h^{-1}G_\lambda 1_{B''}(x)-\hat{1}_{2B}(x))\wedge\hat{1}_B\in\mathcal{F}.$$

Using (3.a)–(3.c), it is easy to check that φ is a cut-off function for $B'\subset B$ that satisfies Definition 2.5 (a)–(c). To complete the proof of CS(Ψ), we need to establish (2.5).

Proposition 3.4. *Let $x_1 \in X$ and $f \in \mathcal{F}$. Let δ be defined by Lemma 3.2 and let $I = B(x_1, \delta s)$ with $0 < s \le R$ and $I^* = B(x_1, s)$. There exist $c_1, c_2 > 0$ such that for all $f \in \mathcal{F}$,*

$$\int_I f^2 d\Gamma(\varphi,\varphi) \le c_1 (s/R)^{2\theta} \Big(\int_{I^*} d\Gamma(f,f) + c_2 \Psi(s)^{-1} \int_{I^*} f^2 d\mu \Big). \tag{3.4}$$

Proof. **Step 1.** We first prove that there exists a cutoff function ψ for $B' \subset B$, which we do not require to be continuous, such that

$$\int_B f^2 d\Gamma(\psi,\psi) \le c_1 \Big(\int_X d\Gamma(f,f) + \Psi(R)^{-1} \int_X f^2 d\mu \Big). \tag{3.d}$$

Let $D = B(x_0, R-\varepsilon)$ for some $\varepsilon > 0$ and define

$$\mathcal{F}_D = \{ f \in \mathcal{F} : \widetilde{f} = 0 \text{ q.e. on } X - D \}.$$

Set

$$\mathcal{E}_\lambda(f,g) = \mathcal{E}(f,g) + \lambda \int fg \, d\mu.$$

Let $v = G^D_\lambda 1_{B'} \in \mathcal{F}$. Note that

$$v(x) \le \int_{B'} g^D(x,y) d\mu(y) \le \mathbb{E}^x[\tau_D] \le c\Psi(R), \qquad x \in D, \tag{3.5}$$

by Theorem 2.15. By [FOT] Theorem 4.4.1, $v \in \mathcal{F}_D$ and is quasi-continuous. Further, since Y is continuous, $v = 0$ on $\overline{D}^c$. Let $f \in \mathcal{F}$. Then

$$\int_B f^2 d\Gamma(v,v) \le \int_X f^2 d\Gamma(v,v) = \int_X d\Gamma(f^2 v, v) - \int_X 2 f v d\Gamma(f,v).$$

Since $v \in \mathcal{F}_D$ we have $f^2 v \in \mathcal{F}_D$, so by [FOT] Theorem 4.4.1,

$$\int_X d\Gamma(f^2 v, v) = \mathcal{E}(f^2 v, G^D_\lambda 1_{B'}) \le \mathcal{E}_\lambda(f^2 v, G^D_\lambda 1_{B'}) = \int_X f^2 v 1_{B'} d\mu \le c\Psi(R) \int_{B'} f^2 d\mu,$$

where we used (3.5) in the last inequality. Using Cauchy-Schwarz and (3.5), we obtain

$$\begin{aligned} \Big| \int_X 2 f v d\Gamma(f,v) \Big| &\le c \Big(\int_X v^2 d\Gamma(f,f) \Big)^{1/2} \Big(\int_X f^2 d\Gamma(v,v) \Big)^{1/2} \\ &\le c\Psi(R) \Big(\int_B d\Gamma(f,f) \Big)^{1/2} \Big(\int_X f^2 d\Gamma(v,v) \Big)^{1/2}. \end{aligned}$$

So, writing $H = \int_X f^2 d\Gamma(v,v)$, $J = \int_B d\Gamma(f,f)$, $K = \int_B f^2 d\mu$, we have

$$H \le c\Psi(R) K + c\Psi(R) J^{1/2} H^{1/2},$$

from which it follows that $H \le c\Psi(R)K + c\Psi(R)^2 J$. Let $\psi(x) = (v(x)/h) \wedge 1 = \Phi(v(x)/h)$. Computing similarly to Lemma 3.2 using [BGK] Theorem 3.1, $\psi(x) = 1$ for $x \in B(x_0, \delta R)$ so that ψ is a cut-off function for $I \subset I^*$. Further, using Lemma 3.A, we have $\int_X f^2 d\Gamma(\psi,\psi) \le h^{-2}H$. Thus (3.d) holds.

Step 2. In Step 2, we will consider the situation that either

$$I^* \subset B(x_0, \delta R) \tag{3.6}$$

or else

$$I^* \cap B(x_0, \delta R/2) = \emptyset. \tag{3.7}$$

Since $\varphi \equiv 1$ on $B(x_0, \delta R)$, (3.4) is clear if (3.6) holds. Thus, we consider when (3.7) holds. Let $\psi_s(x)$ be a cut-off function for $I \subset I^*$ given by Step 1. Let $\varphi_0(x) = h^{-1} G_\lambda 1_{B''}(x) \in \mathcal{F}$, $a_0 = \inf_{I^*} \varphi_0$ and $\varphi_1(x) = \varphi_0(x) - a_0 \hat{1}_{I^*}(x) \in \mathcal{F}$. Note that $\varphi = \Phi(\varphi_1 + a_0 - 1)$ on I^*. By (3.c) we have

$$\varphi_1(x) \le c(s/R)^\theta = L, \qquad x \in I^*.$$

Let

$$\begin{aligned} A &= \int_I f^2 d\Gamma(\varphi,\varphi), \\ D &= \int_{I^*} d\Gamma(f,f) + \Psi(s)^{-1} \int_{I^*} f^2 d\mu, \\ F &= \int_{I^*} f^2 \psi_s^2 d\Gamma(\varphi_1, \varphi_1). \end{aligned}$$

By Lemma 3.A, we have

$$\begin{aligned} A \le \int_I f^2 d\Gamma(\varphi_1,\varphi_1) \le F &= \int_{I^*} f^2 \psi_s^2 d\Gamma(\varphi_1, \varphi_0) \\ &= \int_{I^*} d\Gamma(f^2\psi_s^2\varphi_1, \varphi_0) - \int_{I^*} \varphi_1 d\Gamma(f^2\psi_s^2, \varphi_0). \end{aligned} \tag{3.8}$$

For the first term in (3.8)

$$\begin{aligned} \int_{I^*} d\Gamma(f^2\psi_s^2\varphi_1, \varphi_0) &= \int_X d\Gamma(f^2\psi_s^2\varphi_1, \varphi_0) \\ &= \mathcal{E}_\lambda(f^2\psi_s^2\varphi_1, h^{-1}G_\lambda 1_{B''}) - \lambda \int_X f^2\psi_s^2\varphi_1\varphi_0 d\mu \\ &\le \mathcal{E}_\lambda(f^2\psi_s^2\varphi_1, h^{-1}G_\lambda 1_{B''}) = h^{-1} \int_{B''} f^2\psi_s^2\varphi_1 d\mu = 0. \end{aligned}$$

Here we used the fact that $\varphi_1 \ge 0$ on I^* and that the support of ψ_s is in I^*, hence outside B'' (due to (3.7)).

The final term in (3.8) is handled, using the Leibniz and chain rules and Cauchy-Schwarz, as

$$\begin{aligned}
&\Big|\int_{I^*}\varphi_1 d\Gamma(f^2\psi_s^2,\varphi_0)\Big| \le 2\Big|\int_{I^*}\varphi_1 f\psi_s^2 d\Gamma(f,\varphi_0)\Big| + 2\Big|\int_{I^*}\varphi_1 f^2\psi_s d\Gamma(\psi_s,\varphi_0)\Big| \\
&\le c\Big\{\Big(\int_{I^*}\psi_s^2 d\Gamma(f,f)\Big)^{1/2} + \Big(\int_{I^*} f^2 d\Gamma(\psi_s,\psi_s)\Big)^{1/2}\Big\}\Big(\int_{I^*}\varphi_1^2 f^2\psi_s^2 d\Gamma(\varphi_0,\varphi_0)\Big)^{1/2} \\
&\le c' D^{1/2} L F^{1/2},
\end{aligned}$$

where we used Step 1 in the final line. Thus we obtain $A \le F \le cDL^2$ so that (3.4) holds.

Step 3. We finally consider the general case. When either (3.6) or (3.7) holds, the result is already proved in Step 2. So assume that neither of them hold. Then I^* must intersect both $B(x_0,\delta R/2)$ and $B(x_0,\delta R)^c$, so $s \ge \delta R/4$. We use Lemma 2.3 to cover I with balls $B_i = B(x_i, c_1 R)$, where $c_1 \in (0,\delta/4)$ has been chosen small enough so that each $B_i^* := B(x_i, c_1 R/\delta)$ satisfies at least one of (3.6) or (3.7). We can then apply (3.4) with I replaced by each ball B_i: writing $s' = c_1 R$ we have

$$\int_{B_i} f^2 d\Gamma(\varphi,\varphi) \le c_2 (s'/R)^{2\theta}\Big(\int_{B_i^*} d\Gamma(f,f) + \Psi(s')^{-1}\int_{B_i^*} f^2 d\mu\Big).$$

We then sum over i. Since no point of I^* is in more than L_0 (not depending on x_0 or R) of the B_i^*, and $c_1 s \le s' \le s$, we obtain (3.4) for I. □

References

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[BH] N. Bouleau, F. Hirsch. *Dirichlet forms and analysis on Wiener space.* de Gruyter, Berlin-New York, 1991.